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Calibration of LIBOR models to caps and swaptions: a way around intrinsic instabilities via parsimonious structures and a collateral market criterion

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Abstract

We expose an intrinsic stability problem in joint calibration of a LIBOR market model to caps and swaptions by direct least squares calibration. This problem typically encounters if one tries to identify jointly the volatility norm behaviour *and* the correlation structure of the forward LIBORs. As a remedy we propose collateral incorporation of a 'Market Swaption Formula', a rule-of-thumb formula which practitioners tend to use, in the calibration routine. It is shown by experiments with practical data that with this new calibration procedure and suitably parametrized volatility structures LIBOR model calibration to caps and swaptions is stable. The involved calibration routine is based on standard swaption approximation or its refinements by Hull & White, Jäckel & Rebonato. We deal with the issue of differently settled caps and swaptions by accordingly adapting the swap rate formula and give a respective modification of Jäckel and Rebonato's refined swaption approximation formula.

1 Introduction and summary

Since the development of the well-known LIBOR market model (e.g. Brace, Gatarek, Musiela [1997] and Jamshidian [1997]), joint calibration of this model to prices of caps and swaptions has been a perennial problem. For a clear discussion of particular thorny issues involved we introduce notations and specify the LIBOR market model as the forward LIBOR process L with respect to a given tenor structure $t_0 < T_1 < T_2 < \ldots < T_n$ in the terminal bond measure $\mathbb{P}_n$:

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^{(n)}, \tag{1}$$

where for $i = 1, \ldots, n-1$ the processes L_i are defined in the intervals $[t_0, T_i]$, $\delta_i = T_{i+1} - T_i$ are day count fractions, and, $\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,d})$ are given *deterministic* functions, called factor loadings, defined in $[t_0, T_i]$. Further in (1), $(W^{(n)}(t) \mid t_0 \leq t \leq T_{n-1})$ is a standard d-dimensional Brownian motion under $\mathbb{P}_n$.

As a matter of fact, a one-factor model, i.e. a model with only one Brownian motion $(n = 1)$, is generally considered too restrictive to describe the dynamics of the yield curve properly. Also, a LIBOR model with time independent volatility norms $|\gamma_i|$ is considered unrealistic as in practice LIBOR volatilities tend to increase short before they approach their maturity. However, as we will see by Observation 1.1 below, a more factor model (i.e. $d \geq 2$) with time dependent volatility norms $|\gamma_i|(t)$ has essentially too many degrees of freedom to be identified by the prices of caps and swaptions alone.

Observation 1.1 Let us fix some arbitrary, time independent, instantaneous correlation structure ρ (rank d, $1 \leq d \leq n-1$) and take a system of d unit vectors $e_i \in \mathbb{R}^d$ with $\rho_{ij} = e_i \cdot e_j$. Then, we consider volatility norms $g_i(t) := |\gamma_i|(t)$ of piece-wise constant functions on the tenor structure $\{t_0, T_1, \ldots, T_n\}$. We thus obtain a LIBOR market model (1) with $\gamma_i(t) = g_i(t)e_i$ which, obviously, has $n(n-1)/2$ free parameters in the g_i whereas we fixed the correlation structure in advance. Clearly, this number is exactly the same as the total amount of caps and swaptions and so, at least in principle, it would be possible to match the prices of these instruments by just fitting the piece-wise constant $g_i(t)$. We now emphasize that this observation holds regardless the ex-ante specified correlation structure ρ!

By Observation 1.1 we suspect that due to many degrees of freedom it may be possible to match a system of caps and swaptions closely by various LIBOR models of different nature. For instance, suppose a cap/swaption price system is calibrated well, in some sense, by a (multi-factor) model with a correlation structure of the form $\rho_{ij} = \exp(-\rho|i-j|)$ and rather flat volatility functions g_i. Then, it may be possible that the same prices can be matched within the same accuracy as well by a one factor model ($\rho \equiv 1$) together with stronger time varying $g_i(t)$. Needless to say that the latter model will have entirely different statistical properties. We stress that this is a problem of *model instability* which arises for non-parametric volatility structures as well as for parsimoniously structured volatilities. To explain, let's imagine the following situation. Suppose a system of market quotes can be fitted by a non-parametric structure with a mean relative accuracy of about 0.1%, involving a particular time independent instantaneous correlation structure combined with a family of piece-wise constant volatility norms g_i. OK, we then fix a completely different correlation structure and re-calibrate the piece-wise volatility norms g_i and see how close we can get. Knowing that we still have enough degrees of freedom we expect that we will attain a not too bad accuracy again. Indeed, we are not surprised to find a mean accuracy of 0.5% after re-calibration. However, in case the average bid-ask spread of the cap/swaption prices was 0.5% (typical bid-ask spreads in practice might be even higher), it is clearly not possible to say which model is better. For parsimonious models attainable accuracies are typically less (e.g. 2% -4%) but similar situations may occur as shown by experiments with practical data in Section 5.

The arguments above are supported by practical experiments in Section 5 and have led to the following main conclusion.

Conclusion 1.2 *For any LIBOR market model with more than one factor and time dependent volatility norms:*

1 *The information in the cap/swap market is not rich enough to identify jointly the instantaneous model correlations* **and** *the time dependence of the volatility norms, even if the correlation structure under consideration is assumed to be time independent.*

2 Any 'implied identification' of the instantaneous correlation structure of the forward LIBORs can be seen as the consequence of a particular parametrization of the volatility norms in the model. As an example, a natural and popular choice is $|\gamma_i|(t) = c_i g(T_i - t)$ with a common function g belonging to some pre-specified class (e.g. piece-wise constant) and different constants c_i for different LIBORs. However, this choice though reasonable, is hard to justify properly by economical arguments and so is any entailed 'implied' correlation structure.

3 Direct joint calibration of the instantaneous correlation structure **and** *the volatility norm behaviour to the cap/swap market suffers from model instability.*

4 For realistic LIBOR market models (more factors and time dependent volatility norms) we need to involve an additional economic concept to overcome model instability in the method of calibration to caps and swaptions.

As new economic concept suggested in Conclusion 1.2-[4] we propose in this paper the incorporation of a so called "Market Swaption Formula (MSF)" in the objective function of the calibration routine. Below we introduce this MSF as a "rule of thumb" formula (3) in accordance with the usual intuition of the market. This formula comes down to a natural link between implied Black volatilities of caps and swaptions and the *global* correlation structure of the LIBOR process.

Let for $i = 1, \ldots, n$, B_i be the value of a zero bond with face value \$1 at maturity time T_i, seen at the present calendar date t_0. Then, it is well-known that the swap rate $S_{p,q}$ over period $[T_p, T_q]$ with settlement dates $T_{p+1}, T_{p+2}, \ldots, T_q$, seen at t_0, may be written as

$$S_{p,q} = \frac{B_p - B_q}{B_{p,q}} = \sum_{k=p}^{q-1} w_k^{p,q} L_k, \tag{2}$$

where $B_{p,q} := \sum_{k=p}^{q-1} \delta_k B_{k+1}$ is the so called annuity numeraire and $w_k^{p,q} := \delta_k B_{k+1}/B_{p,q}$ are weights. Hence, the swap rate can be seen as a weighted sum of forward LIBOR rates.

Definition 1.3 Market Swaption Formula (MSF) *The MSF poses that, given the Black volatilities γ_i^B of the caplets and the global correlations $Cor(L_i(T_p), L_j(T_p))$ of the LIBOR process, the implied Black volatilities $\sigma_{p,q}^{MSF}$ of the MSF swaption prices are given by*

$$S_{p,q}^2 (\sigma_{p,q}^{MSF})^2 \quad = \quad \sum_{i,j=p}^{q-1} w_i^{p,q} w_j^{p,q} L_i L_j \, \gamma_i^B \gamma_j^B \, Cor(L_i(T_p), L_j(T_p)). \tag{3}$$

Essentially, the ideas behind formula (3) originate from Rebonato [1996] and also Schoenmakers, Coffey [1998]. In fact, they are related to other approximation formulas discussed in Section 2. Now our central result in this paper is enhanced joint calibration to caps and swaptions by collateral use of the MSF in the calibration procedure. The key idea is basically *'Calibrate the LIBOR market model such that the prices of caps and swaptions are fitted 'as good as possible' while the MSF formula is matched 'fairly well'* and is implemented as a modification of the standard mean-squares objective function by the Market Swaption Formula. The details are given in Section 3. It turns out that incorporation of the MSF is a way to identify less ambiguously (de-)correlations in the market model. In other words, the MSF serves as an instrument which decides more or less the trade-off between the explanation power of the correlation structure and the volatility norms and as such is a remedy for the intrinsic instability of direct least squares calibration.

Having found a way around the intrinsic or model instability of the LIBOR market model we are still left with the problem of *parameter stability.* Non-parametric (even time independent) correlation structures as well as piece-wise constant volatility norms are difficult to identify in a stable way because of their large number of free parameters. That means, a small perturbation in the swaption prices may be reflected in wildly changing parameter sets. This is the classical problem of over fitting. To overcome this problem we will implement the parsimonious correlation structures by Schoenmakers, Coffey [2000] which are endogenously positive, have nice economical features, and are particularly designed for the LIBOR market model. Besides, we will use exponentially parametrized volatility norm functions as proposed by Rebonato [1999].

In Section 2 we outline a direct least squares method for calibration against caps and swaptions which is based on parsimonious correlation structures of Schoenmakers, Coffey [2000,2002] and a well known approximate relationship among implied Black-volatilities of caps and swaptions, see e.g. Rebonato [1996] and also Schoenmakers, Coffey [1998] where was already touched upon this calibration methodology via a ratio correlation structure. In particular, Jäckel and Rebonato [2000] show by case studies that the above mentioned approximate relation between caplet and swaption volatilities is usually quite good and, moreover, they give a refinement of this approximation which may be used instead. Here it should be mentioned that Hull and White [2000] derived a similar swaption approximation method with respect to a differently structured volatility matrix $\gamma(t)$. Further in Section 2 we argue in the spirit of Conclusion 1.2 that direct least squares calibration may be instable. Empirical confirmation of the stability problem by experiments with practical data will be presented in Section 5. We modify the mean squares objective function with the MSF in a suitable way in Section 3 and illustrate in practice the stability properties of the thus obtained new calibration procedure in Section 5.

Before we test our new method on market data, however, we have to deal with the fact that caps and swaps are settled differently in practice. A way to handle this issue is given in Section 4 where we adapt the swap rate formula accordingly. Moreover, in the Appendix we derive a respective modification of Jäckel and Rebonato's refined swaption formula which applies for differently settled caps and swaptions.

Finally, in Section 6 we show how to extract a low factor market model with an arbitrarily chosen number of factors (Brownian motions) from a once calibrated multi-factor model by principal component analysis. We underline that the here proposed way of calibrating low factor models is conceptually generic and most of all stable since the multi-factor calibration is done in a stable way.

2 Direct least squares calibration to caps and swaptions

As suggested in several studies, e.g. Schoenmakers, Coffey [1998], Rebonato [2000], rather than calibrating the market model (1) directly to market prices of swaptions, for instance by Monte Carlo simulation, we will take advantage of the following well known approximate relationship between (local) swap volatilities, LIBOR volatilities, and instantaneous LIBOR correlations (e.g. Rebonato [1996]),

$$S_{p,q}^2 \sigma_{p,q}^2 \approx \sum_{i,j=p}^{q-1} w_i^{p,q} w_j^{p,q} L_i L_j |\gamma_i||\gamma_j| \rho_{ij}, \tag{4}$$

which may be explained by studying the Itô differential of the swap rate (2), e.g. see [5, 6, 10, 15]. As in Section 1 we assume deterministic volatility norms g_i and time independent instantaneous correlations ρ_{ij}. By integrating (4) we thus obtain,

$$\frac{1}{T_p - t_0} \int_{t_0}^{T_p} \sigma_{p,q}^2(s) ds \approx \sum_{i,j=p}^{q-1} \frac{\rho_{ij}}{T_p - t_0} \int_{t_0}^{T_p} \frac{w_i^{p,q}(s) w_j^{p,q}(s) L_i(s) L_j(s)}{S_{p,q}^2(s)} g_i(s) g_j(s) ds, \tag{5}$$

where t_0 denotes the present calendar date. Next, we note that the (stochastic) fractions in the r.h.s. integrands of (5), which by (2) may be regarded as weights, tend to vary relatively slow in practice and therefore may be approximated by their values at t_0. Under this additional assumption instantaneous swap volatilities may be considered as deterministic (though model inconsistent) and as a well known consequence swaprate processes are log-normal martingales under their respective annuity numeraire measure. So the quantities in the l.h.s. of (5) may be seen as squares of implied Black volatilities $\sigma_{p,q}^B$ consistent with *model* swaption prices $Swpn_{p,q}$ and thus obtain the following swaption approximation,

$$(\sigma_{p,q}^B)^2 := \sum_{i,j=p}^{q-1} \frac{w_i^{p,q}(t_0) w_j^{p,q}(t_0) L_i(t_0) L_j(t_0)}{S_{p,q}^2(t_0)} \frac{\rho_{ij}}{T_p - t_0} \int_{t_0}^{T_p} g_i(s) g_j(s) ds; \tag{6}$$

$$\begin{aligned} Swpn_{p,q} &= B_{p,q}(t_0) \mathbb{E}_{p,q} \left(S_{p,q}(T_p) - K \right)^+ \\ &\approx B_{p,q} S_{p,q}(t_0) \mathcal{N}(d_+) - B_{p,q} K \mathcal{N}(d_-), \text{ with} \qquad (7) \\ d_\pm &:= \frac{\ln[S_{p,q}(t_0)/K] \pm (\sigma_{p,q}^B)^2 (T_p - t_0)/2}{\sigma_{p,q}^B \sqrt{T_p - t_0}} \end{aligned}$$

and K being the strike of the swaption. In (7), $\mathcal{N}$ denotes the cumulative standard normal distribution function. So, by (7) we get approximative model swaption prices which should be computed otherwise by tedious Monte Carlo simulation. Further, a nice feature of swaption approximation via (6) and (7) is that we may calibrate the market model as well by fitting the volatilities (6) directly to ATM swaption volatilities quoted in the market[1]

[1] Since we calibrate to at the money caps and swaptions this makes hardly any difference in practice.

We next proceed with choosing a particular form for the volatility norms g_i,

$$g_i(t) = c_i g(T_i - t), \tag{8}$$

where the i-independent function g takes care of the practically observed "hump shape" in the volatility behaviour as function of time to LIBOR maturity, and the c_i are (positive) constants for different LIBORs. As g has to act, in principle, on $[0,\infty[$ it is plausible to take a constant plus a linear combination of the first two Laguerre functions $e^{-s/2}$ and $(s-1)e^{-s/2}$, properly scaled. Without restriction we require $g(0) = 1$ in (8), then choose $g_\infty := \lim_{s\to\infty} g(s)$ as parameter and define

$$g(s) = g_{a,b,g_\infty}(s) := g_\infty + (1 - g_\infty + as)e^{-bs}, \quad a, b, g_\infty > 0. \tag{9}$$

See Figure 2 for a typical example. In fact, parametrization (9) is essentially the same as the one proposed in Rebonato [1999]. Now, the parameters a, b, g_∞ and c_i are to be determined consistent with the Black caplet volatilities γ_i^B, via

$$\begin{aligned}(\gamma_i^B)^2 &= \frac{1}{T_i - t_0}\int_{t_0}^{T_i} g_i^2(s)ds = \frac{c_i^2}{T_i - t_0}\int_{t_0}^{T_i} g^2(T_i - s)ds \\ &= \frac{c_i^2}{T_i - t_0}\int_0^{T_i - t_0} g_{a,b,g_\infty}^2(s)ds.\end{aligned} \tag{10}$$

Let us introduce for $p \le \min(i,j)$ the quantities

$$\begin{aligned}\alpha_{i,j,p}^{a,b,g_\infty} &:= \frac{1}{T_p - t_0}\int_{t_0}^{T_p} \frac{g_i(s)g_j(s)}{\gamma_i^B\gamma_j^B}ds = \frac{1}{T_p - t_0}\frac{c_i c_j}{\gamma_i^B\gamma_j^B}\int_{t_0}^{T_p} g(T_i - s)\, g(T_j - s)ds \\ &= \frac{\sqrt{T_i - t_0}\sqrt{T_j - t_0}}{T_p - t_0}\,\frac{\int_{t_0}^{T_p} g_{a,b,g_\infty}(T_i - s)\, g_{a,b,g_\infty}(T_j - s)ds}{\sqrt{\int_0^{T_i - t_0} g_{a,b,g_\infty}^2(s)ds}\sqrt{\int_0^{T_j - t_0} g_{a,b,g_\infty}^2(s)ds}}.\end{aligned} \tag{11}$$

Fortunately, expression (11) is easily evaluated analytically.[2] Hence, the coefficients c_i have dropped in (11) and from (6) we obtain

$$\sigma_{p,q}(a, b, g_\infty; \eta_1, \eta_2, \rho_\infty) := \sqrt{\sum_{i,j=p}^{q-1} \frac{w_i^{p,q}(t_0)w_j^{p,q}(t_0)L_i(t_0)L_j(t_0)}{S_{p,q}^2(t_0)}\gamma_i^B\gamma_j^B\alpha_{i,j,p}^{a,b,g_\infty}\ \rho_{ij}(\eta_1,\eta_2,\rho_\infty)}, \tag{12}$$

where as the next step, after parsimoniously parameterizing the volatility norms, we have chosen a full rank parsimonious correlation structure suitable for the LIBOR market model,

$$\begin{aligned}\rho_{ij}(\eta_1,\eta_2,\rho_\infty) \;&:=\; \exp\Bigg[-\frac{|j-i|}{m-1}\Bigg(-\ln\rho_\infty \\ &\quad +\eta_1\frac{i^2 + j^2 + ij - 3mi - 3mj + 3i + 3j + 2m^2 - m - 4}{(m-2)(m-3)} \\ &\quad -\eta_2\frac{i^2 + j^2 + ij - mi - mj - 3i - 3j + 3m + 2}{(m-2)(m-3)}\Bigg)\Bigg], \\ &\quad i,j = 1,\dots,m, \quad 3\eta_1 \ge \eta_2 \ge 0,\ 0 \le \eta_1 + \eta_2 \le -\ln\rho_\infty.\end{aligned} \tag{13}$$

[2] We omit the rather long expressions, to prevent errors in the tedious calculations one might produce the results easily with a program like, Mathematica or Maple, e.g..

For a motivation and systematic derivation of (13) and related correlation structures we refer to Schoenmakers, Coffey [2000] and its updated version Schoenmakers, Coffey [2002].

Now, inevitably, the following question arises. Suppose we are given a LIBOR model with certain volatility norms g_i consistent with (8) and (9) and correlation structure of the form (13). Then, for this particular model, how close are approximative swaption prices obtained via (6) and (7) to model swaption prices, for instance, obtained via Monte Carlo simulation? This issue is studied in Jäckel, Rebonato [2000] and from this paper we conclude the following:

i) For a flat initial yield curve, a typical volatility norm structure (8)-(9) and correlation structure (13) with $\eta_1 = \eta_2 = 0$, Monte Carlo simulated swaption prices agree with prices approximated via (6)-(7) up to an average error of about 0.3% relative. [3] So approximation (6) works out pretty well in this case.

ii) For a non-flat yield curve (a typical GBP curve) the pricing errors due to (6) are larger, approximately 2% relative on average.

iii) By taking into account terms with $\partial w_i^{p,q}/\partial L_j$ in the expanded Itô differential of (2) we may refine approximation (6). In fact, this refinement comes down to a suitable correction of the weights $w_i^{p,q}$ in (6). See Jäckel, Rebonato [2000] for further details. For a flat initial yield curve somewhat surprisingly it turns out that Jäckel & Rebonato's refined swaption approximation formula coincides with (6) again, but, for a typical GBP curve the average relative error between with this refined formula approximated swaption prices and (Monte Carlo simulated) model prices reduces to approximately 0.3%. In fact, Hull & White [2000] propose a similar refined swaption approximation formula, however, there the concerning expression is based on a differently parameterized volatility structure and therefore less convenient in our context.

Based on swaption approximation (6)-(7), where if need be (6) is refined by correcting the weights according to Jäckel, Rebonato [2000], we now aim to calibrate the (approximate) swaption prices (7) to a system of ATM market swaption prices. Equivalently, as we are dealing with ATM prices, we may calibrate the volatilities (6) to ATM Black swaption volatilities in the market.[4] As a first approach we therefore aim to fit (12) in least square sense to market quotes, i.e., we are going to minimize the 'root mean square' distance

$$RMS(a, b, g_\infty; \eta_1, \eta_2, \rho_\infty) :=$$
$$\sqrt{\frac{2}{(n-1)(n-2)} \sum_{1 \le p \le q-2,\ q \le n} \left(\frac{\sigma^B_{p,q} - \sigma_{p,q}(a, b, g_\infty; \eta_1, \eta_2, \rho_\infty)}{\sigma^B_{p,q}} \right)^2} \longrightarrow \min_{a,b,g_\infty;\eta_1,\eta_2,\rho_\infty}. \quad (14)$$

We thus have to carry out a least squares search for six parameters a, b, g_∞, η_1, η_2, ρ_∞. Then, the c_i are determined by (10) and the calibration of the multi factor LIBOR model is principally done. However, since we are trying to calibrate jointly the time dependence

[3] In Table 2, Jäckel, Rebonato [2000], only the subset $S_{p,41}$ of the swaption matrix is considered. We have carried out similar experiments and observed an overall relative RMS error of approximately 0.3% due to approximation (6)-(7) for the there used model data and a flat initial yield curve of 7% p.a.

[4] The relative errors for swaption prices are in practice of the same order as for swaption volatilities.

of the volatility norms **and** the instantaneous correlation structure of the forward LIBORs we may be faced with

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as explained in general in Section 1. To get a feeling for this problem in our particular situation let us look at (12) again. In (12) we see that each term in the sum on the right-hand-side contains a product of an expression α which exclusively depends on the shape of g and a correlation factor ρ which only depends on three correlation parameters. Now, loosely speaking, one could state that a system of market swaption volatilities which in fact appear on the left-hand-side of (12) determines these products in a stable way, but, the determination of their individual factors may be instable! See Section 5 for an illustration of this phenomenon by a practical example.

Remark 2.1 As turns out in practice swaption approximation via (6)-(7), where if necessary (6) is refined by weight corrections, is good enough for our purposes in the sense that the average relative error of approximate ATM swaption prices (or volatilities (6)) with respect to ATM model prices (or implied model volatilities) is at most comparable but usually much less than the relative RMS error of the attainable calibration fit.

Remark 2.2 In a rougher approximation one might choose the volatility norms to be time independent, hence $a = b = 0$ in (9) and then $\alpha_{i,j,p}^{0,0,\cdot} \equiv 1$ in (11). However, generally, $\alpha_{i,j,p}^{a,b,g_\infty}$ may be less or greater than 1, depending on a, b and g_∞. See in Figure 1 an α-surface for a typical choice of g which is plotted in Figure 2.

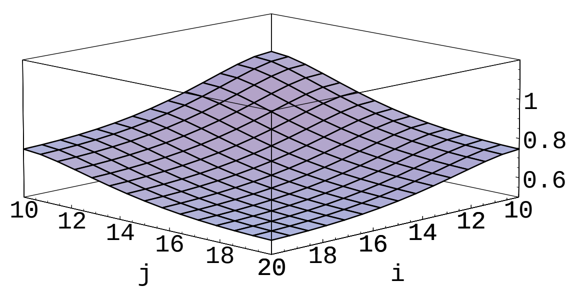

Figure 1: $\alpha_{i,j,p}^{0.5,0.4,0.6}$; $p = 10 \leq i, j \leq 20$

3 Stable calibration via the MSF

Experiments showed that for artificial data sets where swaption prices where simulated by our LIBOR model with typical pre-selected parameters a, b, g_∞ and $\eta_1, \eta_2, \rho_\infty$ the least squares minimization procedure returned the input parameters always fairly good though

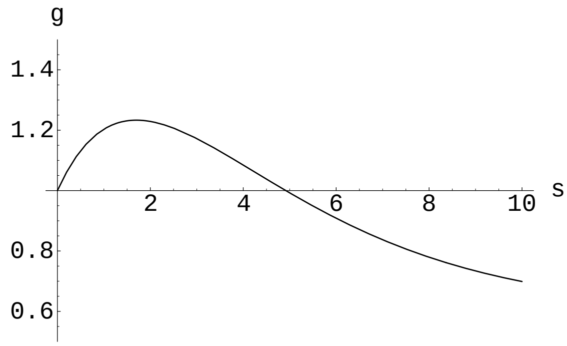

Figure 2: $s \to g_{0.5,0.4,0.6}(s)$

with small RMS errors due to the little inaccuracy of the involved swaption approximation. In contrast, from direct least squares calibration experiments with various Euro-market data we found out that for some data sets comparable fits may be achieved by, on one hand, a rather flat g-function combined with a correlation structure with ρ_∞ relatively close to zero and, on the other hand, a highly non-flat g-function combined with correlations $\rho_{ij} \equiv 1$, hence a one-factor model. This phenomenon particularly occured in situations where the attainable overall RMS fit was not too well, possibly caused by internal misalignments in the cap/swaption market data. See for an example Section 5. In Sections 1 and 2 we explained the cause of this stability problem and in this section we propose a new calibration strategy as a way around. Roughly, we propose the following:

- *Fit the LIBOR model to the swaption prices via minimizing (14) as close as possible, but, such that the 'rule-of-thumb' Market Swaption Formula (3) is still matched 'fairly well' by this model.*

- *In case an 'exact' fit is possible, the calibration procedure should return this fit.*

The MSF involves global correlations. However, there are generally no closed form expressions for these correlations of a LIBOR market model, but, by neglecting the stochastic nature of the log-LIBOR drifts (which are in magnitude of second order anyway) it is easy to derive the following approximation in terms of the model factor loadings determined by c_i, g and ρ,

$$\begin{aligned} Cor(L_i(T_p), L_j(T_p)) &\approx Cor(\ln L_i(T_p), \ln L_j(T_p)) \\ &\approx \frac{\int_{t_0}^{T_p} g_{a,b,g_\infty}(T_i - s)\, g_{a,b,g_\infty}(T_j - s) ds}{\sqrt{\int_{t_0}^{T_p} g^2_{a,b,g_\infty}(T_i - s) ds}\sqrt{\int_{t_0}^{T_p} g^2_{a,b,g_\infty}(T_j - s) ds}} \rho_{ij} \\ &=: \rho_{ij,p}^{\mathrm{global};\,\rho,g}. \end{aligned} \tag{15}$$

Hence, in terms of the LIBOR market model the MSF becomes by this approximation,

$$S_{p,q}^2(t_0)(\sigma_{p,q}^{MSF}(g;\rho))^2 \approx \sum_{i,j=p}^{q-1} w_i^{p,q}(t_0)w_j^{p,q}(t_0)L_i(t_0)L_j(t_0)\gamma_i^B\gamma_j^B\rho_{ij,p}^{\text{global};\,\rho,g}. \tag{16}$$

We now implement our new calibration strategy via minimization of the following objective function,

$$RMS(g;\rho)\ \max\left(RMS(g;\rho), RMS^{MSF}(g;\rho)\right), \tag{17}$$

where $RMS(g;\rho) := RMS(a,b,g_\infty;\eta_1,\eta_2,\rho_\infty)$ is given by (14) and

$$RMS^{MSF}(g;\rho) \ := \ \sqrt{\frac{2}{(n-1)(n-2)}\sum_{1\le p\le q-2,\ q\le n}\left(\frac{\sigma_{p,q}^B-\sigma_{p,q}^{MSF}(g;\rho)}{\sigma_{p,q}^B}\right)^2}.$$

The idea behind (18) is clear: For parameters g, ρ with $RMS^{MSF}(g;\rho)\le RMS(g;\rho)$, the objective function is just equal to the mean squares error $MS(g;\rho)$ of the (approximative) model swaption prices with respect to the market quotes and so disregards the precise value of the MSF fitting error. If $RMS(g;\rho)\le RMS^{MSF}(g;\rho)$, however, the objective function equals the geometric mean $\sqrt{MS(g;\rho)\,MS^{MSF}(g;\rho)}$ of the direct mean squares error and the mean squares $MS^{MSF}(g;\rho)$ of the MSF fit. Since search algorithms usually work better with differentiable objects we next replace in (18) the function $\max(x,y)$ by $\sqrt[4]{x^4+y^4}$ which is differentiable for $(x,y)\neq(0,0)$. Then, we square the objective function and thus obtain the following minimization problem,

$$MS(g;\rho)\sqrt{\{MS(g;\rho)\}^2+\{MS^{MSF}(g;\rho)\}^2} \longrightarrow \min_{g:\ a,b,g_\infty;\,\rho:\ \eta_1,\eta_2,\rho_\infty} \tag{18}$$

(MS=mean squares).

Clearly, if a very close fit is possible with (14), for example, when we would calibrate to Monte Carlo simulated swaption prices for a particular choice of input parameters a,b,g_∞ and $\eta_1,\eta_2,\rho_\infty$ instead of calibrating to market swaption quotes, then due to the factor $MS(g;\rho)$ in (18) in front, the concerning parameters will be retrieved (as it should be) and the calibration will not be affected by the MSF. However, in practice the usual cases is that there is no very accurate fit via (14) possible and then the procedure will return the parameters such that $RMS(g;\rho)$ is as close as possible to zero while the average error $RMS^{MSF}(g;\rho)$ is not too large, in a sense. *In fact, one might consider then the (eventually refined) swaption price formula (6)-(7) with the calibrated parameter set as a model based correction of the more intuitive Market Swaption Formula (3)!*

4 Dealing with differently settled caps and swaptions

In the US, UK and Japanese market caps are quarterly and swaps are semi-annually settled. In the Euro market swaps are annual while semi-annual caplets are available. Clearly, this complicates a direct application of the method described in Sections 2 and 3 where caps and swaptions are assumed to be settled at the same tenors. We deal with this problem by taking a LIBOR[5] model with respect to an equidistant δ-period tenor structure $T_j = t_0 + j\delta$, $j \geq 0$, starting at the calendar date t_0 of the given market data and modify the swap rate formula (2) for 2δ-settled swaps into (19) below. Then, for the European market we may take $\delta = 0.5$ and for the other markets $\delta = 0.25$. We thus consider swap rates $\widehat{S}_{p,q}$ in connection with 2δ-settled swap contracts over periods $[T_p, T_q]$ with p and q even. By standard arguments it follows that

$$\widehat{S}_{p,q} = \frac{B_p - B_q}{\sum_{k=1}^{\frac{q-p}{2}} 2\delta B_{p+2k}} = \sum_{j=p}^{q-1} \widehat{w}_j^{p,q} L_j, \tag{19}$$

with $\widehat{w}_j^{p,q} = B_{j+1} / \sum_{k=1}^{(q-p)/2} 2B_{p+2k}$. Obviously, the whole calibration procedure in Sections 2,3 goes through with $S_{p,q}$, $w_j^{p,q}$ in (6)-(7), (12), and (16) replaced by $\widehat{S}_{p,q}$, $\widehat{w}_j^{p,q}$, thus yielding $\widehat{\sigma}_{p,q}(g;\rho)$ and $\widehat{\sigma}_{p,q}^{MSF}(g;\rho)$ as implied model and MSF volatilities, respectively. Then, the expression for MS in (18) modifies to

$$\widehat{MS}(g;\rho) := \frac{8}{(n-1)(n-3)} \sum_{1 \leq p \leq q-2,\ q \leq n,\ p.q\ even} \left(\frac{\sigma_{p,q}^B - \widehat{\sigma}_{p,q}(g;\rho)}{\sigma_{p,q}^B} \right)^2 \tag{20}$$

and a similar modification applies to the expression for $\widehat{MS}^{MSF}$.

For differently settled caps and swaptions the formula improving approximation (6) given by Jäckel, Rebonato [2000], which basically comes down to replacing the weights $w_j^{p,q}$ in (6) and (12) by $w_j^{p,q} + y_j^{p,q}$, where $y_j^{p,q}$ is a correction computed from the initial yield curve, needs to be modified as well. In the Appendix a refined swaption approximation formula in connection with (19) is derived and given via a correction term $\widehat{y}_j^{p,q}$ (see (23)) which needs to be added to $\widehat{w}_j^{p,q}$ in (19). For a calibration procedure based on this refined formula in the context of differently settled caps and swaptions we simply use $\widehat{w}_j^{p,q} + \widehat{y}_j^{p,q}$ in stead of $w_j^{p,q}$ in (6)-(7), (12), (16), and (20).

Whereas in Jäckel and Rebonato [2000] the correction term $y_j^{p,q}$ vanishes for a flat yield curve it turns out in the Appendix that, generally, the *modified* correction term (23) does *not* vanish when the yield curve is flat. This somewhat remarkable fact was confirmed by simulation tests which showed that in the case of a flat initial yield curve the "standard" swaption approximation via (6) modified for swaps defined by (19) was significantly less accurate compared with the case studies of Jäckel and Rebonato [2000]. However, we note that for typically "humpe shaped" functions g and correlation structures ρ our simulations showed that the mean model accuracy of the in the Appendix derived refined swaption formula lays within 0.5% relative, both for the flat initial yield curve and the GBP curve used in Jäckel and Rebonato [2000]. *Therefore, for differently settled caps and swaptions*

[5] For instance, in the European market we should speak of "EurIBOR" etc, but for convenience we just use one term "LIBOR" throughout.

we recommend to use the refinement (23) in any case, whether the initial yield curve is flat or not.

Remark 4.1 Unlike in (2), the modified coefficients $\widehat{w}_j^{p,q}$ do not necessarily sum up to 1 exactly, for $j = p, \ldots, q-1$.

5 An empirical case study; the Euro market 18.10.2001

As an example we demonstrate the calibration procedure for EURO-market quotes at October, 18, 2001. The yield curve is given by the discount factors (zero-bonds) in Table 1 and the available semi-annual caplet volatilities are given in Table 2. We thus take a tenor structure with $\delta = 0.5$ and $n = 41$ and compute the in Table 2 missing caplet volatilities by linear interpolation. For the (Black) swaption volatilities relevant for this tenor structure, see Table 3. All calibration experiments below will be based on interpolated caplet volatilities obtained from Table 2 and (exclusively) the swaption volatilities which are given in Table 3. In this respect we note that we don't apply any interpolation or smoothing procedure to the swaption data. Since EURO-swaptions are annually settled we use the refined approximation based on (23) in the Appendix. For the minimum search of different objective function (20) we use a version of the Powell algorithm, e.g. see Numerical Recipes in C, [9]. Experiments have shown that it is difficult in practice to identify jointly the three "hump shape" parameters a, b, g_∞ together in the sense that calibration results are very close if one fixes $a = 0$, hence a decaying exponential for g. In this respect one could argue that market data only identify that LIBOR volatilities start "low" and end up "high" when reaching maturity, rather than identifying more detailed behaviour of g. We will now test the following three calibration procedures:

I *Direct calibration under* $\rho \equiv 1$ *and* $a = 0$

II *Direct calibration under* $g \equiv 1$

III *Calibration via incorporating MSF by (18) under* $\eta_2 = 0$, $a = 0$

The different calibration procedures I, II, and III will be run in a sequential way. First we calibrate only to swaptions with maturity time 1 year, then the thus identified model parameters are taken as starting values for a calibration to swaptions with maturity 1 year and 2 year, and so on. We thus end up with a sequence of parameter sets where each set identifies a LIBOR model which is calibrated to the corresponding segment of Table 3. The results are given in Table 4-I, Table 4-II and Table 4-III, where we note that in the last row of these tables all swaptions of Table 3 are involved.

Conclusions

From Table 4-I we see that a reasonable RMS fit to each segment of the swaption matrix is already attainable by a one-factor model. However, we see that the market rule of thumb formula MSF is drastically violated by this model. In contrast, by our new developed calibration method we obtain in III a fit which has a comparable RMS accuracy, but, with much less violation of the MSF! By observing Table 4-I and Table 4-III the stability problem of direct calibration becomes clear. In particular, when the whole swaption matrix is involved the corresponding RMS calibration errors in Table 4-I and Table 4-III are, also in view of typical bid-ask spreads in the swaption market, not significantly different. Hence, an unambiguous identification of the model parameters based on the RMS objective function alone is hardly possible. Further in Table 4-III we see, as expected, that in case the whole swaption matrix is involved the attainable RMS fit is less and that in this situation the MSF error is not very much larger than the RMS error. In fact, the MSF then prevents the search routine for ending up with a comparable RMS fit with $\rho \equiv 1$ as in Table 1 and so forces stability of the calibration.
Finally, for flat volatility norms (II) we have basically $RMS = RMS^{MSF}$ but in Table 4-II we see that the attainable RMS accuracy is then particularly poor for calibration to short maturity swaptions. This may partially explain that flat volatilities are considered unrealistic in general.

6 Calibration of low factor models

For any desired number of factors (Brownian motions) d, it is possible to extract a d-factor model from a multi-factor model as follows.

(I) Carry out a stable multi-factor calibration as outlined in Section 3 and thus identify the correlation matrix ρ.

(II) Construct by principal component analysis an approximation $\tilde{\rho}$ of ρ with rank d :

- *(i)* determine Q and Λ such that $\rho = Q\Lambda Q^\top$ with $QQ^\top = I$, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $\lambda_1 > \cdots > \lambda_n > 0$;
- *(ii)* set $\tilde{\Lambda} := (\lambda_1, \dots, \lambda_d, 0, \dots, 0)$, $E := Q\tilde{\Lambda}^{1/2}$,

$$\tilde{E} := \left[\frac{1}{\sqrt{\sum_{l=1}^{d} \tilde{E}_{il}^2}} E_{ik} \right]_{1 \le i \le n-1,\ 1 \le k \le d} \quad \text{and then take } \tilde{\rho} := \tilde{E}\tilde{E}^\top.$$

(III) Substitute $\tilde{\rho}$ for ρ in (12) and re-calibrate a, b, g_∞, hence the volatility "hump" g, by (14) while keeping $\tilde{\rho}$ fixed and then re-compute the c_i according to (10).

Remark 6.1 Since now the correlation structure ρ is determined, the re-calibration of g may be done by direct least squares as there are no stability problems anymore. In fact, the price for the dimension reduction will be a larger violation of the market swaption formula MSF rather than a larger calibration error.

Remark 6.2 In the multi-factor calibration we have restricted g to $a = 0$ for the sake of stability. In the low-factor re-calibration, however, one may relax this restriction and thus opens up the possibility of identifying a really "humpe shaped" volatility function.

Tables

j	T_j (yr)	B_j	j	T_j (yr)	B_j
1	0.5	0.98260	22	11	0.57295
2	1	0.96675	23	11.5	0.55574
3	1.5	0.94967	24	12	0.53894
4	2	0.93160	25	12.5	0.52280
5	2.5	0.91248	26	13	0.50712
6	3	0.89262	27	13.5	0.49174
7	3.5	0.87222	28	14	0.47666
8	4	0.85132	29	14.5	0.46189
9	4.5	0.83017	30	15	0.44767
10	5	0.80875	31	15.5	0.43434
11	5.5	0.78748	32	16	0.42161
12	6	0.76618	33	16.5	0.40917
13	6.5	0.74526	34	17	0.39704
14	7	0.72449	35	17.5	0.38519
15	7.5	0.70415	36	18	0.37383
16	8	0.68409	37	18.5	0.36255
17	8.5	0.66450	38	19	0.35136
18	9	0.64527	39	19.5	0.34063
19	9.5	0.62656	40	20	0.33033
20	10	0.60826	41	20.5	0.32064
21	10.5	0.59043			

Tabel 1: Discount factors, 18.10.01

j	T_j (yr)	γ_j^B (%)
1	0.5	23.25
2	1	22.97
3	1.5	21.50
4	2	20.03
5	2.5	19.06
6	3	17.95
8	4	16.38
10	5	15.40
12	6	14.41
14	7	13.77
16	8	13.16
18	9	12.74
20	10	12.40
24	12	12.10
30	15	11.79
40	20	11.40

Tabel 2: Caplet ATM volatilities, 18.10.01

$Mat.\backslash Per.$	1 yr	2 yr	3 yr	4 yr	5 yr	6 yr	7 yr	8 yr	9 yr	10 yr	15 yr
1 yr	20.71	18.89	17.32	16.16	15.21	14.53	13.92	13.42	13.01	12.65	11.57
2 yr	18.12	16.59	15.49	14.71	14.11	13.65	13.22	12.87	12.58	12.28	11.28
3 yr	16.58	15.17	14.35	13.78	13.38	13.06	12.73	12.45	12.21	12.01	11.01
4 yr	15.39	14.13	13.48	13.11	12.83	12.58	12.33	12.14	11.94	11.77	10.74
5 yr	14.28	13.39	12.95	12.60	12.35	12.15	11.95	11.76	11.64	11.48	10.51
7 yr	12.86	12.16	11.84	11.54	11.34	11.22	11.02	10.90	10.80	10.69	
10 yr	11.66	10.93	10.65	10.43	10.28	10.17	10.05	9.98	9.89	9.80	
15 yr	10.87	10.19	9.95	9.70	9.60						

Table 3: Swaption ATM volatilities, 18.10.01

up to mat.	*# swpnts*	b	g_∞	RMS	*max. err.*	*mat.* $\times$ *per.*	RMS^{MSF}
1 yr	11	0.56	0.46	0.017	0.046	1×1 yr	0.19
2 yr	22	0.64	0.46	0.020	0.050	2×1 yr	0.18
3 yr	33	0.68	0.46	0.020	0.048	1×1 yr	0.17
4 yr	44	0.70	0.46	0.021	0.053	4×15 yr	0.16
5 yr	55	0.70	0.46	0.022	0.061	5×15 yr	0.16
7 yr	65	0.66	0.45	0.023	0.054	5×5 yr	0.16
10 yr	75	0.50	0.44	0.035	0.079	10×10 yr	0.16
15 yr	80	0.46	0.43	0.044	0.120	15×4 yr	0.16

Table 4-I: Sequential direct calibration, $\rho \equiv 1$, $a = 0$

up to mat.	*# swpnts*	η_1	η_2	ρ_∞	RMS	*max. err.*	*mat.* $\times$ *per.*	RMS^{MSF}
1 yr	11	0.26	0.00	0.06	0.045	0.083	1×15 yr	0.045
2 yr	22	0.68	0.00	0.09	0.042	0.069	2×2 yr	0.042
3 yr	33	1.30	0.52	0.16	0.035	0.064	3×2 yr	0.035
4 yr	44	1.30	0.00	0.13	0.034	0.067	4×2 yr	0.034
5 yr	55	1.89	0.00	0.15	0.031	0.061	4×2 yr	0.031
7 yr	65	1.54	0.00	0.12	0.037	0.071	7×2 yr	0.037
10 yr	75	0.86	0.00	0.08	0.049	0.10	10×3 yr	0.049
15 yr	80	0.40	0.00	0.08	0.057	0.13	15×4 yr	0.057

Table 4-II: Sequential direct calibration, $g \equiv 1$

up to mat.	*# swpnts*	η_1	ρ_∞	b	g_∞	*RMS err.*	*max. err.*	*mat.* $\times$ *per.*	RMS^{MSF}
1 yr	11	1.29	0.28	4.05	0.62	0.005	0.014	1×1 yr	0.045
2 yr	22	1.43	0.24	6.88	0.54	0.015	0.034	2×1 yr	0.040
3 yr	33	1.43	0.22	6.18	0.55	0.019	0.038	3×2 yr	0.039
4 yr	44	1.56	0.20	6.25	0.58	0.023	0.049	4×2 yr	0.035
5 yr	55	1.35	0.18	6.02	0.56	0.024	0.048	5×2 yr	0.037
7 yr	65	0.92	0.15	5.65	0.52	0.028	0.057	7×2 yr	0.044
10 yr	75	0.32	0.10	5.48	0.52	0.040	0.089	10×3 yr	0.052
15 yr	80	0.00	0.11	5.14	0.47	0.045	0.117	15×4 yr	0.061

Table 4-III: Sequential calibration by new method via MSF, $\eta_2 = 0$, $a = 0$

Appendix: Modification of Jäckel & Rebonatos refined swaption approximation

In the standard swaption approximation (6) (or (12)), as well as in its modification for 2δ settled swaptions in connection with δ settled LIBORs obtained in Section 4 by using (19) instead of (2), terms involving the derivatives of weights with respect to LIBORs are neglected. By taking these terms into account, one may derive an improvement like in Hull, White [2000] and Jäckel, Rebonato [2000]. We here derive a refined swaption approximation which is, in fact, a modification of Jäckel & Rebonato's formula, which applies in the case where swaptions are 2δ settled (e.g. annually), while caps are δ settled (e.g. semi-annually).

By (19) we have

$$\frac{\partial \widehat{S}_{p,q}}{\partial L_i} = \widehat{w}_i^{p,q} + \sum_{j=p}^{q-1} L_j \frac{\partial}{\partial L_i} \widehat{w}_j^{p,q} =: \widehat{w}_i^{p,q} + \widehat{y}_i^{p,q}, \tag{21}$$

with p and q even and

$$\widehat{w}_j^{p,q} = \frac{B_{j+1}}{2\sum_{k=1}^{(q-p)/2} B_{p+2k}} = \frac{\prod_{l=j+1}^{q-1}(1+\delta L_l)}{\sum_{k=1}^{(q-p)/2} 2\prod_{l=p+2k}^{q-1}(1+\delta L_l)}.$$

For derivation of the correction term $\widehat{y}_i^{p,q}$ in (21) its convenient to work with logarithms,

$$\ln \widehat{w}_j^{p,q} = \sum_{l=j+1}^{q-1} \ln(1+\delta L_l) - \ln \sum_{k=1}^{(q-p)/2} 2 \prod_{l=p+2k}^{q-1} (1+\delta L_l).$$

So $\partial \widehat{w}_j^{p,q}/\partial L_p = 0$, and for $p < i < q$,

$$\frac{1}{\widehat{w}_j^{p,q}} \frac{\partial \widehat{w}_j^{p,q}}{\partial L_i} = \frac{\delta}{1+\delta L_i} 1_{\{i>j\}} - \frac{1}{\sum_{k=1}^{(q-p)/2} B_{p+2k}} \sum_{k=1}^{[(i-p)/2]} \frac{\delta}{1+\delta L_i} B_{p+2k},$$

where $[x]$ denotes the largest integer less than or equal to x. Hence, we get by a little algebra

$$\begin{aligned}
\widehat{y}_i^{p,q} &= \sum_{j=p}^{q-1} L_j \frac{\partial}{\partial L_i} \widehat{w}_j^{p,q} = \frac{1}{1+\delta L_i} \frac{1}{2\sum_{k=1}^{(q-p)/2} B_{p+2k}} \sum_{j=p}^{i-1} \delta B_{j+1} L_j \\
&\quad - \frac{1}{1+\delta L_i} \sum_{j=p}^{q-1} \frac{\delta B_{j+1} L_j}{2\left\{\sum_{k=1}^{(q-p)/2} B_{p+2k}\right\}^2} \sum_{k=1}^{[(i-p)/2]} B_{p+2k}.
\end{aligned} \tag{22}$$

In the spirit of Jäckel, Rebonato [2000], we introduce the notations,

$$\begin{aligned}
F_i^q &:= \sum_{j=i}^{q-1} \delta B_{j+1} L_j \\
G_s^q &:= 2 \sum_{k=0}^{(q-s)/2} B_{s+2k}; \qquad G_s^q := 0, \text{ if } s > q.
\end{aligned}$$

Then, from (22) it follows that

$$\begin{aligned}
\widehat{y}_i^{p,q} &= \frac{1}{1+\delta L_i} \frac{F_p^q - F_i^q}{G_{p+2}^q} - \frac{1}{1+\delta L_i} \frac{F_p^q}{[G_{p+2}^q]^2} (G_{p+2}^q - 2 \sum_{k=[(i-p)/2]+1}^{(q-p)/2} B_{p+2k}) \\
&= \frac{F_p^q G_{2[i/2]+2}^q - F_i^q G_{p+2}^q}{[G_{p+2}^q]^2 (1+\delta L_i)}.
\end{aligned} \tag{23}$$

Resuming:

The refined swaption formula for differently settled caps and swaptions follows from (6)-(7) by simply replacing there $S_{p,q}$ and $w_j^{p,q}$ by $\widehat{S}_{p,q}$ and $\widehat{w}_i^{p,q} + \widehat{y}_i^{p,q}$, where $\widehat{w}_i^{p,q}$ and $\widehat{y}_i^{p,q}$ are given in (19) and (23), respectively.

We recall that in Jäckel, Rebonato [2000] swap and LIBOR rates are assumed to be settled at the same tenors and is shown that for a flat yield curve the correction term for the refined swaption approximation vanishes. Indeed, their experiments confirm that the standard approximation (6)-(7) is very good for a flat initial yield curve. So, we are now interested in the quality of the standard approximation modified for differently settled caps and swaptions. By similarly comparing Monte Carlo prices with approximated prices under a flat yield curve we found out that the accuracy of (6)-(7) modified for differently settled caps and swaptions via (19) was significantly less. The explanation is the following: *For a flat yield curve, the correction term (23) in the modified swaption refinement formula does not vanish in general.* Let us consider this phenomenon in more detail and assume the initial yield curve is flat, i.e. $L_i \equiv: L$. Then, the swap rate is flat also and in particular we have for $0 \le i \le q-p-2, \;\; i$ even,

$$\begin{aligned} S_{p+i,q} &= \sum_{j=p+i}^{q-1} \widehat{w}_j^{p+i,q} L_j = L \frac{\sum_{j=p+i}^{q-1} B_{j+1}}{2\sum_{k=1}^{(q-p-i)/2} B_{p+i+2k}} = L \frac{\sum_{l=1,\text{ odd}}^{q-p-i} B_{p+l+i} + \sum_{l=1,\text{ even}}^{q-p-i} B_{p+l+i}}{2\sum_{k=1}^{(q-p-i)/2} B_{p+i+2k}} \\ &= L \frac{\sum_{l=1,l\text{ odd}}^{q-p-i} (1+\delta L) B_{p+l+i+1} + \sum_{l=1,\text{ even}}^{q-p-i} B_{p+l+i}}{2\sum_{k=1}^{(q-p-i)/2} B_{p+i+2k}} = L(1+\frac{1}{2}\delta L). \end{aligned}$$

Let $i = 2l + p$ with $0 \le l \le (q-p-2)/2$, it then follows that

$$\begin{aligned} F_i^q &= F_{2l+p}^q = \sum_{j=2l+p}^{q-1} \delta B_{j+1} L = \sum_{k=l}^{(q-p-2)/2} \delta B_{2k+p+1} L + \sum_{k=l}^{(q-p-2)/2} \delta B_{2k+p+2} L \\ &= \sum_{k=l}^{(q-p-2)/2} \delta B_{2k+p+2} L(1+\delta L) + \sum_{k=l+1}^{(q-p)/2} \delta B_{2k+p} L = \delta(L + \frac{1}{2}\delta L^2) G_{p+2+2l}^q, \end{aligned}$$

and, for $i = 2l + p + 1$ with $0 \le l \le (q-p-2)/2$, we get

$$\begin{aligned} F_i^q &= F_{2l+p+1}^q = \sum_{j=2l+p+1}^{q-1} \delta B_{j+1} L \\ &= \sum_{k=l+1}^{(q-p-2)/2} \delta B_{2k+p+1} L + \sum_{k=l}^{(q-p-2)/2} \delta B_{2k+p+2} L \\ &= \sum_{k=l+1}^{(q-p-2)/2} \delta B_{2k+p+2} L(1+\delta L) + \sum_{k=l+1}^{(q-p)/2} \delta B_{2k+p} L \\ &= \frac{1}{2}\delta L(1+\delta L) G_{p+2l+4}^q + \frac{1}{2}\delta L G_{p+2l+2}^q. \end{aligned}$$

So, for $i = 2l + p$ with $0 \le l \le (q-p-2)/2$, we have

$$\begin{aligned} & F_p^q G_{2[i/2]+2}^q - F_i^q G_{p+2}^q \\ = \ & \delta(L + \frac{1}{2}\delta L^2) G_{p+2}^q G_{p+2l+2}^q - \delta(L + \frac{1}{2}\delta L^2) G_{p+2+2l}^q G_{p+2}^q = 0, \end{aligned}$$

hence $\widehat{y}_i^{p,q} = 0$. However, for $i = 2l + p + 1$ with $0 \le l \le (q-p-2)/2$, we obtain

$$\begin{aligned} F_p^q G_{2[i/2]+2}^q - F_i^q G_{p+2}^q &= \delta(L + \frac{1}{2}\delta L^2) G_{p+2}^q G_{2l+p+2}^q - \frac{1}{2}\delta L(1+\delta L) G_{p+2l+4}^q G_{p+2}^q - \frac{1}{2}\delta L G_{p+2l+2}^q G_{p+2}^q \\ &= \frac{1}{2}\delta L(1+\delta L)(G_{p+2l+2}^q - G_{p+2l+4}^q) G_{p+2}^q = \frac{1}{2}\delta L(1+\delta L) B_{p+2l+2} G_{p+2}^q \end{aligned}$$

and thus $\widehat{y}_i^{p,q} = \frac{B_{p+2l+2}}{2G_{p+2}^q}\delta L \neq 0$.

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References

[1] Brace, A., Gatarek, D., Musiela, M. (1997). The Market Model of Interest Rate Dynamics. *Mathematical Finance,* 7 (2), 127-155.

[2] Brigo, D. Mercurio, F. (2001). Joint calibration of the LIBOR market model to caps and swaptions volatilities. Working paper.

[3] Curnow, R.N., Dunnett, C.W. (1962). The numerical evaluation of certain multivariate normal integrals. *Ann. Math. Statist.,* (33), 571-579.

[4] Heath, D., Jarrow, R., Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation. *Econometrica,* Vol. 60, No. 1, 77-105.

[5] Hull, J., White, A. (2000). Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the Libor Market Model. *Journal of Fixed Income,* 10, 2, 46-62.

[6] Jäckel, P., Rebonato, R. (2000). Linking Caplet and Swaption Volatilities in a BGM/J Framework: Approximate Solutions. Working paper, www.rebonato.com, to appear in *Journal of Computational Finance.*

[7] Jamshidan, F. (1997). LIBOR and swap market models and measures. *Finance and Stochastics,* (1), 293-330.

[8] Kurbanmuradov, O., Sabelfeld, K., Schoenmakers, J. (1999). Lognormal approximations to LIBOR market models. Preprint no. 481, Weierstrass Institute Berlin. Revised version (2000), www.wias-berlin.de/private/schoenma, to appear in *Journal of Computational Finance.*

[9] Press, W., Teukolsky, S., Vetterling, W., Flannery, B. (1999). *Numerical Recipes in C.* Cambridge University Press.

[10] Rebonato, R. (1996). Interest-rate Option Models. John Wiley & Sons, Chichester.

[11] Rebonato, R. (1999). Volatility and Correlation. John Wiley & Sons.

[12] Rebonato, R. (1999a). On the simultaneous calibration of multifactor lognormal interest rate models to Black volatilities and to the correlation matrix. *Journal of Computational Finance,* Vol.2, Number 4.

[13] Schoenmakers, J.G.M., Coffey, B.: Systematic generation of parametric correlation structures for the LIBOR market model. Working paper, (2002).

[14] Schoenmakers, J.G.M., Coffey, B.: Stable implied calibration of a multi-factor LIBOR model via a semi-parametric correlation structure. Preprint no. 611, Weierstrass Institute Berlin, (2000).

[15] Schoenmakers, J.G.M., Coffey, B. (1998). LIBOR rate models, related derivatives and model calibration. Working paper, QMF 98 Sydney, Preprint no. 480 (1999), Weierstrass Institute Berlin.