Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Super-Brownian motion with extra birth at one point

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submitted: 26 November 2002

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No. 790
Berlin 2002

1991 *Mathematics Subject Classification.* 60J80 60K35.

Key words and phrases. Super-Brownian motion, measure-valued processes, heat equation, singular potential, one-point potential.

∗ Partly supported by the research program "Interacting Systems of High Complexity" of the German Science Foundation
∗∗ Supported in part by an NSA grant and the graduate program "Probabilistic Analysis and Stochastic Processes" of the German Science Foundation.

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Abstract A super-Brownian motion in two and three dimensions is constructed where "particles" give birth at a higher rate, if they approach the origin. Via a log-Laplace approach, the construction is based on Albeverio et al. (1995) who calculated the fundamental solutions of the heat equation with one-point potential in dimensions less than four.

Contents

1 Introduction

1.1 Motivation and background

Measure-valued branching processes, also called superprocesses, arise naturally as limits of particle branching Markov processes. There is an immense literature on this topic; see the expositions [Daw93], [Dyn94], [LG99], [Eth00], or [Per02], for example. Since these models involve mainly "non-interacting particles", many powerful tools are available, and many detailed properties of these processes are known. Building on this success, many probabilists have turned their attention to more complicated models, many of which are governed by singularities. For example, in 2 or more dimensions, with probability 1, continuous super-Brownian motion takes values in the space of measures whose closed support has Lebesgue measure 0. Nevertheless, in certain situations, pairs of such processes can kill each other when the corresponding "particles" meet (see, e.g. [EP94]).

Another example of singular behavior is catalytic branching. Here, the branching of the "particles" is controlled by a catalytic measure; the higher the "density" of this measure, the faster the "particles" branch or die. This catalytic measure can be supported on a set of Lebesgue measure 0, as long as it is not a polar set of Brownian motion. In other words, individual "particles" must have a positive probability of "hitting the measure". See [DF95, Del96, FK99, Kle00].

A further example of singular behavior is mass creation. One could imagine a "mass creation measure", which would give rise to new "particles" whenever the "particles" of the superprocess hit the support of the measure. For the extreme case of a single point source δ_0 in R, see [EF00, ET02]. In particular, a continuous super-Brownian motion in R with a point source makes sense. In higher dimensions, however, at first sight one would expect that a super-Brownian motion with single point mass creation degenerates to ordinary super-Brownian motion, since the Brownian particles do not hit a given point.

Our goal is to disprove this intuition. But first, we mention a deterministic, "one-particle model", which already gives a different picture. This model was developed by mathematical physicists starting in the 1930's; see Albeverio et. al. [AGHKH88] for historical background.

Consider the heat equation in R^d with a one-point potential:

$$\frac{\partial u}{\partial t} \;=\; \Delta u \,+\, \delta_0^{(\alpha)} u \;=:\; \Delta^{(\alpha)} u \tag{1}$$

(by $f := g$ or $g =: f$, we mean that f is defined to be equal to g). Heuristically, $\Delta^{(\alpha)} = \Delta + \delta_0^{(\alpha)}$ is the limit as $\varepsilon \downarrow 0$ of the operator

$$\Delta_\varepsilon^{(\alpha)} \;:=\; \Delta \,+\, h(d,\alpha,\varepsilon)\,\varepsilon^{-d}\mathbf{1}_{B_\varepsilon(0)}\,, \tag{2}$$

where $B_\varepsilon(y)$ denotes the open ball around $y \in \mathsf{R}^d$ with radius ε, and $h(d,\alpha,\varepsilon)$ is some additional rescaling factor, depending on a parameter α, at least.

For instance, in dimension $d = 3$,

$$h(3, \alpha, \varepsilon) := \left(k + \tfrac{1}{2}\right)^2 \pi^2 \varepsilon - 8\pi^2 \alpha \varepsilon^2 - \zeta \varepsilon^3, \qquad \alpha \in \mathsf{R}, \quad \varepsilon > 0, \tag{3}$$

where k is any integer and ζ any real number (we rely on [AGHKH88, (H.74)]). Then, in a sense, $\Delta_\varepsilon^{(\alpha)} \to \Delta^{(\alpha)}$ as $\varepsilon \downarrow 0$, where the limit operator $\Delta^{(\alpha)}$ is *independent* of k and ζ (so for simplification one could set $k = 0 = \zeta$).

Physically, the parameter α is related to the "scattering length" $-(4\pi\alpha)^{-1}$ (see, for instance, [AGHKH88, p.13]). In particular, $\Delta^{(\alpha)} \to \Delta$ as $\alpha \uparrow \infty$, giving the "free" case. We understand $\delta_0^{(\alpha)}$ as $\lambda_\alpha \delta_0$. In dimension $d = 3$ the coupling constant λ_α of the point source δ_0 has to be of the form $\lambda_\alpha = \varepsilon - \alpha\varepsilon^2$ with ε "infinitesimal" in a special way.

Even though the number of particles that hit the origin is infinitesimal, one can imagine that they give raise to a positive mass, provided that the birth rate is high enough. This explains, why the linear ε–term in (3) is not allowed to be too small, in particular, it cannot be negative. In the latter case, particles will simply die, and nothing else will happen. But since there are only infinitesimally many particles hitting the origin, their possible death will pass unnoticed. At this point we would like to understand certain questions from a probabilistic point of view. For instance, in dimensions $d = 3$, why don't all sufficiently high coefficients of the linear ε–term occur, and why is the limit operator $\Delta^{(\alpha)}$ independent of the integer k? Unfortunately, this is outside the scope of the present paper.

Strictly speaking, $\{\Delta^{(\alpha)} : \alpha \in \mathsf{R}\}$ is the family of all self-adjoint extensions on $\mathcal{L}^2(\dot{\mathsf{R}}^d, \mathrm{d}x)$, $d = 2, 3$, of the Laplacian Δ acting on $\mathcal{C}_0^{(\infty)}(\dot{\mathsf{R}}^d)$, where $\dot{\mathsf{R}}^d := \mathsf{R}^d \backslash \{0\}$. See, for instance, [AGHKH88, Chapters I.1 and I.5].

The *fundamental solutions* P^α to equation

$$\frac{\partial u}{\partial t} = \Delta^{(\alpha)} u \quad \text{on} \quad (0, \infty) \times \dot{\mathsf{R}}^d, \quad d = 2, 3, \tag{4}$$

have been computed in [ABD95]. P^α is different from the heat kernel, for each $\alpha \in \mathsf{R}$ (only for $\alpha \uparrow \infty$ one gets back the heat kernel, at least in $d = 3$, see Subsection 2.4 below). P^α is a basic object in the present paper.

1.2 Sketch of result

Based on the preceding analytical results from [ABD95], the *purpose* of the present paper is to construct a measure-valued super-Brownian motion $X = \{X_t : t \geq 0\}$ in $\dot{\mathsf{R}}^d = \mathsf{R}^d \backslash \{0\}$, $d = 2, 3$,[1] related to the formal log-Laplace

[1]) Recall that the one-dimensional super-Brownian motion with extra birth at 0, that is, related to the log-Laplace equation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + \delta_0 - v^2 \quad \text{on} \quad (0, \infty) \times \mathsf{R},$$

was introduced in [EF00].

equation

$$\left.\begin{aligned} \frac{\partial v}{\partial t} &= \Delta^{(\alpha)} v - \eta\, v^{1+\beta} \quad \text{on } (0,\infty) \times \dot{\mathsf{R}}^d, \\ v(0+,x) &= \varphi(x) \ \geq\ 0, \qquad x \in \dot{\mathsf{R}}^d, \end{aligned}\right\} \tag{5}$$

with constants $0 < \beta \leq 1,\ \eta \geq 0,$ and where the φ are appropriate test functions. Of course, X is related to (5) via the log-Laplace transition functional

$$-\log \mathbf{P}\big\{\mathrm{e}^{-\langle X_t\,,\varphi\rangle} \mid X_0\big\} \;=\; \big\langle X_0\,, v(t,\,\cdot\,)\big\rangle, \qquad t > 0, \tag{6}$$

of the Markov process X.

Roughly speaking, we have "many" independent Brownian "particles" which everywhere undergo critical branching with index $1+\beta$ and rate η, but additionally give birth to new particles if they "approach" the origin 0.

Here is a rough formulation of our main result; a more precise statement will be given in Theorem 29 below.

Theorem 1 (Existence of X). *If* $d = 2$, *let* $0 < \beta \leq 1$, *and if* $d = 3$, *let* $0 < \beta < 1$. *Then, for each* $\alpha \in \mathsf{R}$, *there is a non-degenerate measure-valued (time-homogeneous) Markov process* $X = X^\alpha$ *having log-Laplace transition functional* (6) *with* v *solving* (5).

We call X a *super-Brownian motion in* R^d *with extra birth at point* $x = 0$. Note that in the case $\eta = 0$, the process degenerates to the deterministic mass flow related to the kernels P^α, the fundamental solutions to (1). At the same time, this mass flow is identical with the expectation of X, for any η. In particular, $X = X^\alpha$ is different from ordinary super-Brownian motion (corresponding to $\alpha = \infty$).

Remark 2 (Open problem). The condition $\beta < 1$ in the three-dimensional case, which excludes finite variance branching as in continuous super-Brownian motion, looks a bit strange. We need this condition for technical reasons, to handle some singularities at the point $x = 0$ where extra birth occurs (see Remark 10 below). ◇

It would, of course, be interesting to reveal that this superprocess X has strange new properties. However, we leave this task for a future paper and present this construction result separately, since it seems to be interesting enough.

1.3 Outline

In Section 2 we give some estimates involving the basic solutions and the semigroup related to the linear equation (1). Then, in Section 3, we show that the log-Laplace equation (5) is well posed. Here we use Picard iteration, but for the non-negativity of solutions we go back to a linearized equation. For the construction of X in Section 4 we use a Trotter product formula, alternating between purely continuous state branching (Feller's branching diffusion if

$\beta = 1$) and deterministic mass with single point mass creation (related to the kernels P^{α}).

For background from a mathematical physics point of view we recommend [AGHKH88], and for basic facts on superprocesses we refer to one of the systematic treatments [Daw93, Dyn94, LG99, Eth00, Per02] which we have already mentioned.

2 The heat equation with birth at a single point

After introducing the set Φ of test functions, on which the heat flow acts continuously (Lemma 6), we define the kernel P^{α} in Subsection 2.4 and show the strong continuity of the related flow S^{α} on Φ (Corollary 12).

2.1 Preliminaries: test functions in $\Phi \subset \mathcal{H}_+$

The letter C denotes a constant which might change its value from occurrence to occurrence. $C_{\#}$ and $C_{(\#)}$ refer to specific constants which are defined around Lemma #, say, or formula (#), respectively.

Let ϕ denote the *weight and reference function*

$$\phi(x) \ := \ |x|^{-(d-1)/2}, \qquad x \in \dot{\mathsf{R}}^d = \mathsf{R}^d \backslash \{0\}. \tag{7}$$

For each fixed constant $\varrho \geq 1$, we introduce the Lebesgue space $\mathcal{H} = \mathcal{H}^{\varrho} = \mathcal{L}^{\varrho}(\dot{\mathsf{R}}^d, \phi(x)\mathrm{d}x)$ of equivalence classes φ of measurable functions on $\dot{\mathsf{R}}^d$ for which $\|\varphi\|_{\mathcal{H}} < \infty$, where [2]

$$\|\varphi\|_{\mathcal{H}} \ := \ \Big(\int_{\mathsf{R}^d} \mathrm{d}x \ \phi(x) \, |\varphi|^{\varrho}(x) \Big)^{1/\varrho}. \tag{8}$$

(As usual, we do not distinguish between an equivalence class and its representatives.)

For fixed $\varrho \geq 1$, let $\Phi = \Phi^{\varrho}$ denote the set of all *continuous* functions $\varphi : \dot{\mathsf{R}}^d \to \mathsf{R}$ such that $\varphi \in \mathcal{H}$ and

$$0 \ \leq \ \varphi \ \leq \ C_{(9)}\phi \quad \text{for some constant } C_{(9)} = C_{(9)}(\varphi). \tag{9}$$

We endow Φ with the topology inherited from $\mathcal{H}$. Note that the set $\mathcal{C}^+_{\mathrm{com}} = \mathcal{C}^+_{\mathrm{com}}(\dot{\mathsf{R}}^d)$ of all non-negative continuous functions on $\dot{\mathsf{R}}^d$ with compact support is contained in Φ. Note also that $\varphi \in \Phi$ might have a singularity at $x = 0$ of order $|x|^{-\xi}$ with $0 < \xi < \frac{d+3}{2}$. The functions in Φ will serve as test functions in log-Laplace representations.

Let $\mathcal{M} = \mathcal{M}(\dot{\mathsf{R}}^d)$ denote the set of all measures μ defined on $\dot{\mathsf{R}}^d$ such that $\langle \mu, \varphi \rangle := \int \mu(\mathrm{d}x) \, \varphi(x) < \infty$ for all $\varphi \in \Phi$. We equip $\mathcal{M}$ with the vague

[2] Here and in similar cases we use this simplified integration domain R^d, since in case of integration with respect Lebesgue measure there is no difference including or not including the point $0 \in \mathsf{R}^d$ of singularity.

topology (recall that $\mathcal{C}_{\mathrm{com}}^{+} \subset \Phi$). Of course, each measure $\mu \in \mathcal{M}$ can also be considered as a measure on R^d with zero mass at $0 \in \mathsf{R}^d$. But in our pairing $\langle \mu, \varphi \rangle$, $\varphi \in \Phi$, we cannot extend to work with measures μ on R^d allowing positive mass at 0 by the mentioned possible singularities of the $\varphi \in \Phi$.

If μ is a finite measure, we write $\|\mu\|$ for its total mass. The symbol ℓ denotes the Lebesgue measure, A^{c} the complement of A, and $a \vee b$ the maximum of a and b.

2.2 Heat flow estimates on $\mathcal{H}$

In this subsection we fix a dimension $d \geq 1$. Let $P = P(t; x, y)$ refer to the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{on} \quad (0, \infty) \times \mathsf{R}^d. \tag{10}$$

In other words,

$$P(t; x, y) = (4\pi t)^{-d/2} \, \mathrm{e}^{-|y-x|^2/4t}, \qquad t > 0, \quad x, y \in \mathsf{R}^d. \tag{11}$$

Let S denote the semigroup corresponding to this heat kernel P.

Here is our first estimate [with ϕ the weight and reference function from (7)]:

Lemma 3 (A heat flow estimate). *There is a constant $C_3 = C_3(d)$ such that*

$$S_t \phi \leq C_3 \, \phi, \qquad t \geq 0. \tag{12}$$

Proof. Without loss of generality, let $t > 0$ and $x \neq 0$. We have to show that

$$\frac{1}{\phi(x)} \, S_t \phi \, (x) = \frac{1}{\phi(x)} \int_{\mathsf{R}^d} \mathrm{d}y \; \phi(y) \, \frac{1}{(4\pi t)^{d/2}} \; \mathrm{e}^{-|y-x|^2/4t} \tag{13}$$

is bounded in $t > 0$ and $x \neq 0$. By the change of variables $y \to t^{1/2} y$ and with notation $x t^{-1/2} =: z \neq 0$, we get

$$\begin{aligned} \frac{1}{\phi(x)} \, S_t \phi(x) &= \frac{C}{\phi(z)} \int_{\mathsf{R}^d} \mathrm{d}y \; \phi(y) \; \mathrm{e}^{-|y-z|^2/4} \\ &\leq \frac{C}{\phi(z)} \int_{\mathsf{R}^d} \mathrm{d}y \; \phi(y) \; \mathrm{e}^{-\left[|y|-|z|\right]^2/4}. \end{aligned} \tag{14}$$

Setting $|z| =: s > 0$ and substituting $|y| = r$, we further obtain

$$\frac{1}{\phi(x)} \, S_t \phi(x) \leq C \, s^{(d-1)/2} \int_0^\infty \mathrm{d}r \; r^{(d-1)/2} \; \mathrm{e}^{-|r-s|^2/4}. \tag{15}$$

First we restrict the integration to $r \leq 2s$. For this part we get the bound

$$C \, s^{d-1} \int_0^{2s} \mathrm{d}r \;\; \mathrm{e}^{-|r-s|^2/4} = C \int_0^2 \mathrm{d}r \; s^d \, \mathrm{e}^{-s^2|r-1|^2/4}, \tag{16}$$

by the substitution $r \to sr$. The latter expression is continuous in $s > 0$ and converges to 0 as $s \downarrow 0$ and as $s \uparrow \infty$ (use bounded convergence for $r \neq 1$), thus it is bounded in $s > 0$. On the other hand, for the integration restricted to $r \geq 2s$ we get the bound

$$C \int_{2s}^{\infty} \mathrm{d}r\ r^{d-1}\ \mathrm{e}^{-|r-s|^2/4} \;=\; C \int_{s}^{\infty} \mathrm{d}r\ (r+s)^{d-1}\, \mathrm{e}^{-r^2/4}. \tag{17}$$

But $(r+s)/r = 1 + s/r \leq 2$ for $r \geq r \geq s > 0$, and

$$\int_{0}^{\infty} \mathrm{d}r\ r^{d-1}\, \mathrm{e}^{-r^2/4} \;<\; \infty. \tag{18}$$

This completes the proof. ∎

We finish this subsection with a simple maximization result.

Lemma 4 (Maximum in the center). *Fix a constant* $\varkappa > 0$. *Then*

$$S_t\phi^{\varkappa}(x) \;\leq\; S_t\phi^{\varkappa}(0), \qquad t > 0, \quad x \in \mathsf{R}^d. \tag{19}$$

Proof. We will use the fact that in the integral

$$\int_{\mathsf{R}^d} \mathrm{d}y\ P(t;x,y)\, \phi^{\varkappa}(y) \tag{20}$$

the mapping $y \mapsto \phi^{\varkappa}(y)$ is radially symmetric and decreasing in $|y|$. The same is true for $y \mapsto P(t;x,y)$, except a shift by x.

1° (*Simplification*). Let $a,b,c,d \geq 0$, then, by expanding,

$$(a+b)(c+d) + ac \;\geq\; (a+b)c + a(c+d). \tag{21}$$

2° (*Functions with n steps*). For $n \geq 2$, let

$$f_i \;:=\; \sum_{j=1}^{n} a_{i,j}\, 1_{B_j} \;\geq\; 0, \qquad i = 1,2, \tag{22}$$

be two step functions defined on $n \geq 2$ cubes $B_1, \ldots, B_n$ in R^d of equal volume, say v. For $i = 1,2$, let $\bar{f}_i$ be constructed from f_i by rearranging the $a_{i,j}$ to $\bar{a}_{i,1} \geq \cdots \geq \bar{a}_{i,n}$. Then

$$\int_{\mathsf{R}^d} \mathrm{d}x\ \bar{f}_1(x)\, \bar{f}_2(x) \;\geq\; \int_{\mathsf{R}^d} \mathrm{d}x\ f_1(x)\, f_2(x). \tag{23}$$

In fact,

$$\int_{\mathsf{R}^d} \mathrm{d}x\ f_1(x)\, f_2(x) \;=\; v \sum_{j=1}^{n} a_{1,j}\, a_{2,j}\,. \tag{24}$$

Rearranging if necessary, we may assume that $f_1 = \bar{f}_1$, that is $a_{1,j} = \bar{a}_{1,j}$, $1 \leq j \leq n$. Exploiting step 1°, we may switch from f_2 to $\bar{f}_2$ by a sequence of rearrangements which never decrease the integral in (24). This then gives the claim (23).

3° (*Approximation*). We may assume that the right hand side of (19) is finite. Then the "integrals" in (19) [recall (20)] can be approximated by using step functions as in (22). Then (19) follows from (23) by passing to the limit. ■

2.3 Strong continuity of the heat flow on $\mathcal{H}$

Next we will proof the following statement.

Lemma 5 (Estimate of S in case of an additional singularity). *Let $d \geq 1$, $0 \leq \beta \leq 1$, and assume that ϱ in (8) satisfies*

$$\varrho > \frac{1}{1-\beta(d-1)/2d}. \tag{25}$$

Then there is a constant $C_5 = C_5(d,\beta,\varrho)$ such that for all $\varphi \in \mathcal{H} = \mathcal{H}^\varrho$,

$$\left\| S_t(\varphi\phi^\beta)\right\|_{\mathcal{H}}^\varrho \leq C_5\, t^{-\beta\varrho(d-1)/4}\, \|\varphi\|_{\mathcal{H}}^\varrho, \qquad t>0. \tag{26}$$

Proof. Fix d,β,ϱ as in the lemma. For $t>0$ and $x \in \mathsf{R}^3$, we introduce the measures

$$\mu_{t,x}(\mathrm{d}y) := t^\kappa P(t;x,y)\,\phi^\lambda(y)\,\mathrm{d}y \tag{27}$$

with

$$\kappa := \frac{\beta\varrho(d-1)}{4(\varrho-1)} \quad \text{and} \quad \lambda := \frac{\beta\varrho}{\varrho-1}. \tag{28}$$

By Lemma 4,

$$\|\mu_{t,x}\| \leq \|\mu_{t,0}\| = \int_{\mathsf{R}^d} \mathrm{d}y\; P(1;0,y)\,\phi^\lambda(y) =: C_{(29)}, \tag{29}$$

where in the last step we used Brownian scaling and the identity $\kappa - \lambda(d-1)/4 = 0$. Note that $C_{(29)} = C_{(29)}(d,\beta,\varrho)$ is finite by our assumption (25). Therefore, the measures $\mu_{t,x}$ are finite with total mass at most $C_{(29)}$ independent of t and x. Now, for each finite measure μ on R^d, and measurable φ, by Hölder's inequality,

$$\Big[\int_{\mathsf{R}^d} |\varphi|(y)\,\mu(\mathrm{d}y)\Big]^\varrho \leq \|\mu\|^{\varrho-1}\int_{\mathsf{R}^d}\mu(\mathrm{d}y)\,|\varphi|^\varrho(y). \tag{30}$$

Applied to the measures $\mu_{t,x}$ we get

$$\begin{aligned}\left|S_t(\varphi\phi^\beta)(x)\right|^\varrho &= t^{-\kappa\varrho}\Big|\int_{\mathsf{R}^d}\mu_{t,x}(\mathrm{d}y)\,\phi^{\beta-\lambda}(y)\,\varphi(y)\Big|^\varrho \\ &\leq t^{-\kappa\varrho}\,\|\mu_{t,x}\|^{\varrho-1}\int_{\mathsf{R}^d}\mu_{t,x}(\mathrm{d}y)\,\phi^{(\beta-\lambda)\varrho}(y)\,|\varphi|^\varrho(y) \\ &\leq t^{-\kappa(\varrho-1)}\,C_{(29)}^{\varrho-1}\,S_t|\varphi|^\varrho(x), \end{aligned} \tag{31}$$

since $\lambda + (\beta - \lambda)\varrho = 0$ by (28). But by Lemma 3,

$$\begin{aligned} \int_{\mathsf{R}^d} \mathrm{d}x \; \phi(x)\, S_t|\varphi|^\varrho\,(x) &= \int_{\mathsf{R}^d} \mathrm{d}y \; |\varphi|^\varrho(y)\, S_t\phi\,(y) \\ &\leq \int_{\mathsf{R}^d} \mathrm{d}y \; |\varphi|^\varrho(y)\, C_3\, \phi(y) \;=\; C_3\, \|\varphi\|_{\mathcal{H}}^\varrho \,. \end{aligned} \tag{32}$$

Hence,

$$\|S_t(\varphi\phi^\beta)\|_{\mathcal{H}}^\varrho \;\leq\; t^{-\kappa(\varrho-1)}\, C_{(29)}^{\varrho-1}\, C_3\, \|\varphi\|_{\mathcal{H}}^\varrho \,, \tag{33}$$

and the claim follows since $\kappa(\varrho-1) = \beta\varrho(d-1)/4$. ■

Lemma 5 with $\beta = 0$ yields the following result.

Lemma 6 (Strong continuity of the heat flow on $\mathcal{H}$). *The semigroup S acting on $\mathcal{H} = \mathcal{H}^\varrho$ is strongly continuous.*

Proof. Fix $\varphi \in \mathcal{H}$. By linearity, we may assume that $\varphi \geq 0$. Consider $t \in (0,1]$.

1° (*Reducing to bounded functions on compact sets*). Fix $\varepsilon \in (0,1]$. We choose a compact set $K \subset \mathsf{R}^d$ so large that $\|\varphi\, 1_{K^c}\|_{\mathcal{H}} < \varepsilon$, and then a number $N \geq 1$ such that $\|\varphi\, 1_K\, 1_{\{\varphi > N\}}\|_{\mathcal{H}} < \varepsilon$. Then

$$\begin{aligned} \|S_t\varphi - \varphi\|_{\mathcal{H}} \;\leq\; & \big\|S_t(\varphi\, 1_{K^c}) - \varphi\, 1_{K^c}\big\|_{\mathcal{H}} + \big\|S_t(\varphi\, 1_K\, 1_{\{\varphi>N\}}) - \varphi\, 1_K\, 1_{\{\varphi>N\}}\big\|_{\mathcal{H}} \\ & + \big\|S_t(\varphi\, 1_K\, 1_{\{\varphi\leq N\}}) - \varphi\, 1_K\, 1_{\{\varphi\leq N\}}\big\|_{\mathcal{H}} \\ \leq\; & C\,\varepsilon + \big\|S_t(\varphi\, 1_K\, 1_{\{\varphi\leq N\}}) - \varphi\, 1_K\, 1_{\{\varphi\leq N\}}\big\|_{\mathcal{H}} \,, \end{aligned} \tag{34}$$

where in the last step we used twice Lemma 5 with $\beta = 0$. Thus, for the rest of the proof we may assume that φ is bounded by $N \geq 1$ and vanishes outside a compact set $K \subset \mathsf{R}^d$. That is, from now on in this proof we assume that R^d is replaced by K in the definition of $\mathcal{H}$.

2° (*Passing to a continuous function*). Fix $\varepsilon \in (0,1]$. Choose a continuous non-negative function $f_\varepsilon \leq N$ (on K) such that $\varphi = f_\varepsilon$ on a measurable set $A_\varepsilon \subseteq K$ satisfying $\ell(A_\varepsilon^c) \leq \varepsilon$. Then, again by twice applying Lemma 5 with $\beta = 0$,

$$\begin{aligned} \|S_t\varphi - \varphi\|_{\mathcal{H}} \;&\leq\; \big\|S_t(\varphi\, 1_{A_\varepsilon^c}) - \varphi\, 1_{A_\varepsilon^c}\big\|_{\mathcal{H}} + \big\|S_t(f_\varepsilon\, 1_{A_\varepsilon}) - f_\varepsilon\, 1_{A_\varepsilon}\big\|_{\mathcal{H}} \\ &\leq\; C\,\|\varphi\, 1_{A_\varepsilon^c}\|_{\mathcal{H}} + C\,\|f_\varepsilon\, 1_{A_\varepsilon^c}\|_{\mathcal{H}} + \|S_t f_\varepsilon - f_\varepsilon\|_{\mathcal{H}} \,. \end{aligned} \tag{35}$$

For $x \in K$ fixed, $S_t f_\varepsilon(x) \to f_\varepsilon(x)$ as $t \downarrow 0$,

$$\sup_{t\geq 0} \|S_t f_\varepsilon\|_\infty \;\leq\; \|f_\varepsilon\|_\infty \;<\; \infty \tag{36}$$

(with $\|\cdot\|_\infty$ denoting the supremum norm), and ϕ is integrable on K. Hence, by dominated convergence, the third term in (35) will vanish as $t \downarrow 0$, for fixed ε. On the other hand, $\|\varphi\, 1_{A_\varepsilon^c}\|_{\mathcal{H}}$ converges to 0 as $\varepsilon \downarrow 0$. Finally, the same is true for $\|f_\varepsilon\, 1_{A_\varepsilon^c}\|_{\mathcal{H}}$ since $f_\varepsilon \leq N$. This completes the proof. ■

2.4 The fundamental solutions P^α

Fix $\alpha \in \mathsf{R}$. We now introduce the fundamental solutions $P^\alpha = P^\alpha(t;x,y)$ of the heat equation with one-point potential $\delta_0^{(\alpha)}$, that is of equation (4).

1° $(d=3)$. Based on [ABD95, formula array (3.4)], for $d=3$, we can *define*

$$P^\alpha(t;x,y) \;:=\; P(t;x,y) + \frac{2t}{|x|\,|y|}\,P\bigl(t;|x|+|y|\bigr) - \frac{8\pi\alpha\, t}{|x|\,|y|}\int_0^\infty \mathrm{d}u\;\mathrm{e}^{-4\pi\alpha u} P\bigl(t;u+|x|+|y|\bigr), \tag{37}$$

$t>0,\ x,y\neq 0,$ where with a slight abuse of notation,

$$P(t;r) \;:=\; (4\pi t)^{-d/2}\exp\bigl(-r^2/4t\bigr), \qquad t,r>0. \tag{38}$$

Note that the term in (37) involving the integral is always finite and that it disappears for $\alpha=0$. Otherwise, using the substitution $|\alpha|u\to u$ (for $\alpha\neq 0$) one realizes that $P^\alpha(t;x,y)$ is continuous and decreasing in α, and that $P^\alpha \downarrow P$ pointwise as $\alpha\uparrow\infty$, whereas $P^\alpha\uparrow\infty$ pointwise as $\alpha\downarrow-\infty$.

2° $(d=2)$. On the other hand, by [ABD95, formula (3.15)], for $d=2$, we may *define*

$$P^\alpha(t;x,y) \;:=\; P(t;x,y) + \frac{\sqrt{4\pi t}}{\sqrt{|x|\,|y|}}\,P\bigl(t;|x|+|y|\bigr)\times \int_0^\infty \mathrm{d}u\;\frac{t^u\,\mathrm{e}^{-\alpha u}}{\Gamma(u)}\int_0^\infty \mathrm{d}r\;\frac{r^{u-1}\,\mathrm{e}^{-(|x|+|y|)^2/4tr}}{(r+1)^{u+1/2}}\,\tilde K_0\Bigl(\frac{|x|\,|y|}{2t}(r+1)\Bigr) \tag{39}$$

$t>0,\ x,y\neq 0,$ where Γ is the Gamma function,

$$\Gamma(u) \;:=\; \int_0^\infty \mathrm{d}s\; s^{u-1}\mathrm{e}^{-s}, \qquad u>0, \tag{40}$$

and

$$\tilde K_0(z) \;:=\; \mathrm{e}^z\,(2z/\pi)^{1/2}\,K_0(z), \qquad z\geq 0, \tag{41}$$

with $K_0\geq 0$ the Macdonald function of order 0. In other words, K_0 is the modified Bessel function of the third kind, of order 0. See [Leb65, p.109].

Recall that P^a $(d=2,3)$ is the family of fundamental solutions to equation (4), computed in [ABD95]. Since $\Delta^{(\alpha)}$ is a self-adjoint extension of Δ on $\dot{\mathsf{R}}^d$, the kernel P^α solves the heat equation:

Corollary 7 (Solutions of the heat equation). *Let $d=2,3$ and $\alpha\in\mathsf{R}$. Then*

$$\frac{\partial}{\partial t}P^\alpha(t;x,y) \;=\; \Delta P^\alpha(t;x,y) \quad on \quad (0,\infty)\times\dot{\mathsf{R}}^d, \tag{42}$$

where the Laplacian acts on x (or y, respectively). In particular, $(t,x,y)\mapsto P^\alpha(t;x,y)$ is jointly continuous on $(0,\infty)\times\dot{\mathsf{R}}^d$.

Let S^α denote the semigroup corresponding to the kernel P^α, $\alpha\in\mathsf{R}$.

2.5 Bounds of P^α

In this subsection we will derive some bounds for the kernels P^α introduced in (37) and (39), respectively. To this end, we set

$$\bar{P}(t;x,y) \ := \ t^{-1/2}\,\phi(x)\,\phi(y)\,\mathrm{e}^{-|x|^2/4t}\ \mathrm{e}^{-|y|^2/4t}\,, \tag{43}$$

for $t>0$ and $x,y\neq 0$ [recall the weight and reference function ϕ from (7)].

Lemma 8 (P^α bound). *Let $d=2,3$. For each $\alpha\in\mathsf{R}$ and $T>0$, there is a constant $C_8=C_8(d,\alpha,T)$ such that*

$$P(t;x,y)\ \le\ P^\alpha(t;x,y)\ \le\ P(t;x,y)+C_8\,\bar{P}(t;x,y) \tag{44}$$

for all $t\in(0,T]$ and $x,y\neq 0$.

Proof. 1° $(d=3)$. By the arguments after (38), for $\alpha\ge 0$,

$$P\ \le\ P^\alpha\ \le\ P^0\ \le\ P+2\,(4\pi)^{-3/2}\,\bar{P}, \tag{45}$$

since

$$\big(|x|+|y|\big)^2\ \ge\ |x|^2+|y|^2. \tag{46}$$

So we will restrict our attention to $\alpha<0$. Abbreviating

$$-4\pi\alpha\ =:\ \frac{r}{2}\ >\ 0 \quad\text{and}\quad |x|+|y|\ =:\ R\ \ge\ 0 \tag{47}$$

and using the last inequality in (45) and (46), it suffices to verify that

$$\int_0^\infty \mathrm{d}u\ \ \frac{r}{2}\,\mathrm{e}^{\frac{r}{2}u}\,P(t;u+R)\ \le\ C_{(48)}\,P(t;R) \tag{48}$$

[recall notation (38)] with a positive constant $C_{(48)}=C_{(48)}(T,r)$ independent of t and R. Fix any

$$u_0\ >\ rT \quad\text{and put}\quad u_1\ :=\ u_0-rT\ >\ 0. \tag{49}$$

Consider first the integral in (48) restricted to $u\in[0,u_0]$. Here we can use $P(t;u+R)\le P(t;R)$ and the fact that

$$\int_0^{u_0}\mathrm{d}u\ \frac{r}{2}\,\mathrm{e}^{\frac{r}{2}u}\ \le\ \mathrm{e}^{\frac{r}{2}u_0}, \tag{50}$$

resulting in a positive constant independent of t and R. It remains to deal with

$$\int_{u_0}^\infty \mathrm{d}u\ \frac{r}{2}\,\mathrm{e}^{\frac{r}{2}u}\,(4\pi t)^{-3/2}\ \mathrm{e}^{-(u+R)^2/4t} \tag{51a}$$

$$=\ \frac{r}{2}\,(4\pi t)^{-3/2}\ \mathrm{e}^{-(2rRt-r^2t^2)/4t}\int_{u_0}^\infty \mathrm{d}u\ \ \mathrm{e}^{-(u+R-rt)^2/4t} \tag{51b}$$

for $0 < t \leq T$. The exponential factor in front of the integral in (51b) is bounded by

$$\mathrm{e}^{r^2 T/4} \;=:\; C_{(52)} \tag{52}$$

which is a positive constant independent of t and R. Substituting $u+R-rt \to u$ and recalling notation (49), the integral in (51b) can be bounded by

$$\int_{R+u_1}^{\infty} \mathrm{d}u \;\, \mathrm{e}^{-u^2/4t} \;\leq\; \frac{2T}{u_1} \int_{R}^{\infty} \mathrm{d}u \;\frac{u}{2t}\, \mathrm{e}^{-u^2/4t} \;=\; \frac{2T}{u_1}\, \mathrm{e}^{-R^2/4t}. \tag{53}$$

Thus for the integral in (51a) we found the bound

$$C_{(52)}\, \frac{r}{2}\, \frac{2T}{u_1}\, P(t;R), \tag{54}$$

which finishes the proof in the case $d = 3$.

2° ($d = 2$). Recall definition (41) of $\tilde{K}_0$. According to [ABD95, after (3.14)],

$$\lim_{z\to\infty} \tilde{K}_0(z) \;=\; 1. \tag{55}$$

Consulting [Tra69, Section 1.15, equation (1.66)], we find that

$$K_0(z) \;\sim\; -\gamma - \log(z/2) \;\sim\; -\log z \quad \text{as } z \downarrow 0, \tag{56}$$

where γ is Euler's constant. Therefore,

$$\lim_{z\downarrow 0} \tilde{K}_0(z) \;=\; \lim_{z\downarrow 0} \left[\mathrm{e}^{z}(2z/\pi)^{1/2} \log z\right] \;=\; 0. \tag{57}$$

Since $\tilde{K}_0$ is continuous, relations (55) and (57) together give

$$\|\tilde{K}_0\|_\infty \;<\; \infty. \tag{58}$$

Fix $\alpha \in \mathsf{R}$ and consider $0 < t \leq T$. We may assume that $T \geq 1$. We start by estimating the inner integral appearing on the right hand side of definition (39) of P^α. For $u > 0$,

$$\begin{aligned} &\int_0^\infty \mathrm{d}r\; \frac{r^{u-1}\mathrm{e}^{-(|x|+|y|)^2/4tr}}{(r+1)^{u+1/2}}\, \tilde{K}_0\left(\frac{|x|\,|y|}{2t}(r+1)\right) \\ &\qquad \leq\; \|\tilde{K}_0\|_\infty \int_0^\infty \mathrm{d}r\; \frac{r^{u-1}}{(r+1)^{u+1/2}}\,. \end{aligned} \tag{59}$$

If $r \geq 1$, drop the 1 in the denominator, otherwise drop the r there. Thus, for the inner integral in (39) we find the bound

$$\|\tilde{K}_0\|_\infty\; [2 + 1/u]. \tag{60}$$

Using this bound, we turn to the outer integral of (39). For the Gamma function Γ of (40), Stirling's formula gives

$$\Gamma(u) \;\sim\; \sqrt{2\pi}\,(u-1)^{u-1/2}\, \mathrm{e}^{-u+1} \quad \text{as } u \uparrow \infty. \tag{61}$$

It follows that, for some constant $C_{(62)} = C_{(62)}(T,\alpha)$,

$$\int_1^\infty \mathrm{d}u \ \frac{t^u \,\mathrm{e}^{-\alpha u}}{\Gamma(u)} \ \le \ C_{(62)}, \qquad 0 \le t \le T. \tag{62}$$

Next, using integration by parts, we estimate $u\,\Gamma(u)$ for $u \in (0,1]$:

$$u\,\Gamma(u) \ = \ \int_0^\infty \mathrm{d}s \ u s^{u-1} \mathrm{e}^{-s} \ = \ \int_0^\infty \mathrm{d}s \ s^u \,\mathrm{e}^{-s} \ \ge \ \mathrm{e}^{-1} \int_0^1 \mathrm{d}s \ s \ =: \ C_{(63)}. \tag{63}$$

Finally, for some constant $C_{(64)} = C_{(64)}(T,\alpha)$, since $T \ge 1$,

$$\int_0^1 \mathrm{d}u \ \frac{t^u \,\mathrm{e}^{-\alpha u}}{u\,\Gamma(u)} \ \le \ \frac{1}{C_{(63)}} T \,\mathrm{e}^{|\alpha|} \ =: \ C_{(64)}. \tag{64}$$

Altogether, we found that the double integral appearing on the right hand side of definition (39) of P^α is bounded by a constant depending only on α, T. This gives estimate (44) also in the case $d=2$, since $\tilde{K}_0 \ge 0$, finishing the proof. ■

2.6 Strong continuity of S^α

We abbreviate

$$\bar{S}_t\varphi\,(x) \ := \ \int_{\mathsf{R}^d} \mathrm{d}y \ \varphi(y)\, \bar{P}(t;x,y), \qquad t>0, \quad x \ne 0, \tag{65}$$

with $\bar{P}$ from (43), as long as the right hand side expression makes sense. The estimates (44) and Minkowski's inequality then imply that

$$\|S_t\varphi\|_{\mathcal{H}} \ \le \ \|S_t^\alpha\varphi\|_{\mathcal{H}} \ \le \ \|S_t\varphi\|_{\mathcal{H}} + C_8 \,\|\bar{S}_t\varphi\|_{\mathcal{H}}\,, \qquad 0 < t \le T, \tag{66}$$

for those φ for which the right hand side of (66) is meaningful and finite.

Lemma 9 (Estimate of $\bar{S}$ in case of an additional singularity). *Let $d = 2,3$ as well as $0 \le \beta \le 1$, and assume*

$$\frac{1}{1-\beta(d-1)/(d+1)} \ < \ \varrho \ < \ \frac{d+1}{d-1}\,. \tag{67}$$

Then there is a constant $C_9 = C_9\,(d,\beta,\varrho)$ such that for all $\varphi \in \mathcal{H} = \mathcal{H}^\varrho$,

$$\left\|\bar{S}_t(\varphi\phi^\beta)\right\|_{\mathcal{H}}^\varrho \ \le \ C_9 \,\varepsilon(t,\varphi)\, t^{-\beta\varrho(d-1)/4} \,\|\varphi\|_{\mathcal{H}}^\varrho\,, \qquad t>0, \tag{68}$$

where $0 \le \varepsilon(t,\varphi) \le 1$ and $\varepsilon(t,\varphi) \to 0$ as $t \downarrow 0$.

Remark 10 (Restriction to infinite variance branching if $d=3$). Note that in dimension $d=3$ condition (67) can only be satisfied for some ϱ if $\beta < 1$ holds. ◇

Proof of Lemma 9. This time we work with the measures

$$\mu_t(\mathrm{d}y) \;:=\; t^{-\kappa}\,\mathrm{e}^{-|y|^2/4t}\,\phi^{\lambda}(y)\,\mathrm{d}y, \qquad t>0, \tag{69}$$

on R^d, where

$$\kappa \;:=\; \frac{d+1}{4} - \frac{(d-1)\beta\varrho}{4(\varrho-1)} > 0 \quad \text{and} \quad \lambda \;:=\; 1+\beta\varrho/(\varrho-1). \tag{70}$$

Note that the measures μ_t have a t–independent total mass

$$\|\mu_t\| \;=\; \int_{\mathsf{R}^d} \mathrm{d}y\;\mathrm{e}^{-|y|^2/4}\,\phi^{\lambda}(y) \;=:\; C_{(71)} = C_{(71)}(d,\beta,\varrho), \tag{71}$$

which is finite by the left hand inequality in assumption (67). Then, by our definition (43) of $\bar{P}$, for $t>0$ and $x\neq 0$,

$$\begin{aligned} &\left|\bar{S}_t(\varphi\phi^{\beta})\,(x)\right|^{\varrho} \\ &=\; t^{-\varrho/2+\kappa\varrho}\,\phi^{\varrho}(x)\,\mathrm{e}^{-\varrho|x|^2/4t}\left|\int_{\mathsf{R}^d}\mu_t(\mathrm{d}y)\,\phi^{-\lambda+\beta+1}(y)\,\varphi(y)\right|^{\varrho}. \end{aligned} \tag{72}$$

By (30) and the definition of μ_t we may continue with

$$\begin{aligned} &\left|\bar{S}_t(\varphi\phi^{\beta})(x)\right|^{\varrho} \\ &\leq\; t^{-\varrho/2+\kappa\varrho}\,\phi^{\varrho}(x)\,\mathrm{e}^{-\varrho|x|^2/4t}\,C_{(71)}^{\varrho-1}\,t^{-\kappa}\int_{\mathsf{R}^d}\mathrm{d}y\;\mathrm{e}^{-|y|^2/4t}\,\phi(y)\,|\varphi|^{\varrho}(y), \end{aligned} \tag{73}$$

since $(-\lambda+\beta+1)\varrho+\lambda = 1$. We may assume that $\varphi\neq 0$. Define

$$\varepsilon(t,\varphi) \;:=\; \frac{1}{\|\varphi\|_{\mathcal{H}}^{\varrho}}\int_{\mathsf{R}^d}\mathrm{d}y\;\phi(y)\,\mathrm{e}^{-|y|^2/4t}\,|\varphi|^{\varrho}(y). \tag{74}$$

Note that $0 < \varepsilon(t,\varphi) \leq 1$ and that $\varepsilon(t,\varphi) \to 0$ as $t\downarrow 0$, by dominated convergence. Consequently,

$$\left|\bar{S}_t(\varphi\phi^{\beta})(x)\right|^{\varrho} \;\leq\; t^{-\varrho/2+\kappa\varrho}\,\phi^{\varrho}(x)\,\mathrm{e}^{-\varrho|x|^2/4t}\,C_{(71)}^{\varrho-1}\,t^{-\kappa}\,\varepsilon(t,\varphi)\,\|\varphi\|_{\mathcal{H}}^{\varrho}. \tag{75}$$

Therefore,

$$\left\|\bar{S}_t(\varphi\phi^{\beta})\right\|_{\mathcal{H}}^{\varrho} \;\leq\; C_{(71)}^{\varrho-1}\,\varepsilon(t,\varphi)\,t^{-\varrho/2+\kappa\varrho-\kappa}\,\|\varphi\|_{\mathcal{H}}^{\varrho}\int_{\mathsf{R}^d}\mathrm{d}x\;\phi^{\varrho+1}(x)\,\mathrm{e}^{-\varrho|x|^2/4t}.$$

But the latter integral is finite since $-(\varrho+1)(d-1)/2+d>0$ by the right hand inequality in assumption (67). Moreover, using a change of variables, the integral gives an additional factor $t^{d/2-(\varrho+1)(d-1)/4}$, so that the whole t–term equals $t^{-\beta\varrho(d-1)/4}$. This finishes the proof. ■

Since condition (67) is stronger than (25), combining Lemmas 5 and 9 with inequality (66) gives the following result.

Corollary 11 (Estimate of S^α in case of an additional singularity). *Let $d=2,3$ as well as $0\le\beta\le 1$. Suppose (67). To each $T>0$ there is a constant $C_{11}=C_{11}(d,T,\alpha,\beta,\varrho)$ such that for all $\varphi\in\mathcal{H}=\mathcal{H}^\varrho$,*

$$\left\|S_t^\alpha(\varphi\phi^\beta)\right\|_{\mathcal{H}} \;\le\; C_{11}\,t^{-\beta(d-1)/4}\,\|\varphi\|_{\mathcal{H}}\,,\qquad 0<t\le T. \tag{76}$$

In particular,

$$\sup_{t\le T}\|S_t^\alpha\varphi\|_{\mathcal{H}} \;\le\; C_{11}\,\|\varphi\|_{\mathcal{H}} \;<\; \infty,\qquad \varphi\in\mathcal{H}. \tag{77}$$

Here is another consequence of Lemma 9:

Corollary 12 (Strong continuity of S^α). *Let $d=2,3$. For each $\alpha\in\mathsf{R}$, the semigroup S^α acting on $\mathcal{H}=\mathcal{H}^\varrho$ with $\varrho\in\big(1,\,(d+1)/(d-1)\big)$ is strongly continuous.*

Proof. Fix $\varphi\in\mathcal{H}$. By linearity, we may additionally assume that $\varphi\ge 0$. Consider $0<t\le T$. Decompose

$$S_t^\alpha\varphi \;=\; S_t\varphi+(S_t^\alpha-S_t)\varphi, \tag{78}$$

where by Lemma 8,

$$0\;\le\;(S_t^\alpha-S_t)\varphi\;\;\le\;C_8\,\bar{S}_t\varphi, \tag{79}$$

implying

$$\left\|(S_t^\alpha-S_t)\varphi\right\|_{\mathcal{H}}\;\;\le\;C_8\,\|\bar{S}_t\varphi\|_{\mathcal{H}}\,. \tag{80}$$

From (78) and (80)

$$\begin{aligned}\|S_t^\alpha\varphi-\varphi\|_{\mathcal{H}} \;&\le\; \|S_t\varphi-\varphi\|_{\mathcal{H}}+\left\|(S_t^\alpha-S_t)\varphi\right\|_{\mathcal{H}}\\ &\le\; \|S_t\varphi-\varphi\|_{\mathcal{H}}+C_8\,\|\bar{S}_t\varphi\|_{\mathcal{H}}\,.\end{aligned} \tag{81}$$

But by Lemma 9 with $\beta=0$, the second term in (81) goes to 0 as $t\downarrow 0$, whereas the first term does by Lemma 6. By (77), this finishes the proof. ∎

2.7 S^α as a flow on Φ

Recall our set Φ of continuous non-negative test functions introduced in Subsection 2.1. From the proof of Lemma 9 we also get the following result:

Corollary 13 (S^α bound). *Let $d=2,3$, assume $\varrho\in\big(1,\,(d+1)/(d-1)\big)$, and that $\varphi\in\mathcal{H}^\varrho$ satisfies[3] (9). Then, to each $T>0$, there is a constant $C_{13}=C_{13}(d,T,\alpha,\varrho,\varphi)$ such that*

$$0\;\le\;S_t^\alpha\varphi\;\le\;C_{13}\left(1+t^{-1/2+(d+1)(\varrho-1)/4\varrho}\right)\phi,\qquad 0<t\le T. \tag{82}$$

In particular, $S_t^\alpha\varphi\in\Phi$, for all $t>0$.

[3] Of course, an inequality on an element $\varphi\in\mathcal{H}$ means that the inequality holds for each representative in Lebesgue almost all points.

Proof. From Lemma 8,

$$0 \le S_t^\alpha \varphi \le S_t \varphi + C_8 \bar{S}_t \varphi. \tag{83}$$

Moreover, by assumption (9) on φ and by Lemma 3,

$$S_t \varphi \le C_{(9)} S_t \phi \le C\phi. \tag{84}$$

On the other hand, raising estimate (75) (with $\beta = 0$ there, implying $\kappa = (d+1)/4$ and $\lambda = 1$) into the power $1/\varrho$ gives

$$\bar{S}_t \varphi \le C\, t^{-1/2 + (d+1)(\varrho-1)/4\varrho}\, \phi\, \|\varphi\|_{\mathcal{H}}. \tag{85}$$

Putting together (83) – (85) yields (82). Finally, $(t,x) \mapsto S_t^\alpha \varphi$ is continuous on $(0,\infty) \times \dot{\mathsf{R}}^d$, since it solves the heat equation, recall Corollary 7. This finishes the proof. ■

Combining Corollaries 12 and 13, we get the following result.

Corollary 14 (S^α acting on Φ). *Let $d = 2,3$ and $\varrho \in \bigl(1, (d+1)/(d-1)\bigr)$. Then S^α is a strongly continuous linear semigroup acting on $\Phi = \Phi^\varrho$.*

3 Analysis of the log-Laplace equation

The main result of this section is the well-posedness of the log-Laplace equation (Theorem 17). Uniqueness follows from a contraction argument (Lemma 21). Existence is shown via a Picard iteration (Lemmas 22, 23, and 25), whereas non-negativity follows using a linearized equation (Lemma 24).

3.1 Preliminaries and purpose

Formally, we can rewrite the log-Laplace equation (5) as the following *integral equation*[4]:

$$v(t,x) = S_t^\alpha \varphi\,(x) - \eta \int_0^t \mathrm{d}s\; S_{t-s}^\alpha \bigl(v^{1+\beta}(s)\bigr)(x), \tag{86}$$

≥ 0, $x \ne 0$, (with constants $\alpha \in \mathsf{R}$, $\eta \ge 0$, $0 < \beta \le 1$, and where $\varphi \ge 0$ has still to be specified). Here in writing $v^{1+\beta}$ we have in mind that $v \ge 0$. Note also that this non-negativity implies the following *domination:*

$$0 \le v(t) \le S_t^\alpha \varphi, \qquad t \ge 0. \tag{87}$$

The task of this section is to verify that the log-Laplace equation (86) is well-posed in Φ.

Definition 15 (Φ–valued solution). Let $\varphi \in \Phi$. A measurable map $t \mapsto v(t) = V_t \varphi$ of R_+ into Φ is called a *solution* of (86), if (86) is true for all $x \ne 0$ and $t > 0$. ◇

[4] We often use notation as $v(s) := v(s,\,\cdot\,)$.

For convenience, we introduce the following hypothesis:

Hypothesis 16 (Choice of parameters). Let $\alpha \in \mathsf{R}$, $\eta \geq 0$, and

$$d = 2,3, \quad 0 < \beta \leq 1 \quad \text{and} \quad \frac{1}{1-\beta(d-1)/(d+1)} \; < \; \varrho \; < \; \frac{d+1}{d-1}. \tag{88}$$

Recall that for $d = 3$ this requires that $\beta < 1$. ◇

Now we are ready to state to state the main result of this section.

Theorem 17 (Well-posedness of the log-Laplace equation). *If Hypothesis 16 holds, and if $\varphi \in \Phi$, then equation (86) has a unique Φ–valued solution $v = V\varphi = \{V_t\varphi : t \geq 0\}$. Moreover, $V = \{V_t : t \geq 0\}$ is a non-linear strongly continuous semigroup acting on Φ.*

The rest of this section is devoted to the proof of this theorem.

3.2 First properties of solutions

Now we prepare for the uniqueness proof. Impose Hypothesis 16. Fix an integer $T > 0$ for a while, and $\varphi \in \Phi$. We will fix also a measurable function ψ on $(0, T] \times \dot{\mathsf{R}}^d$ such that

$$0 \; \leq \; \psi(t,x) \; \leq \; M\,(1 + t^{-\kappa})\,\phi^{\beta}(x), \tag{89}$$

with constants $M = M(T, \psi) > 0$ and

$$\kappa \; := \; \beta/2 \; - \; \beta(d+1)(\varrho - 1)/4\varrho \; \in \; (0,1). \tag{90}$$

Lemma 18 (Properties of the non-linear term). *There is a constant $C_{18} = C_{18}(d, M, T, \alpha, \beta, \varrho)$ such that*

$$\Big\| \int_0^t \mathrm{d}s \; S^{\alpha}_{t-s}(\psi(s)\, S^{\alpha}_s \varphi) \Big\|_{\mathcal{H}} \; \leq \; C_{18}\, \|\varphi\|_{\mathcal{H}}\, I(t), \qquad 0 < t \leq T, \tag{91}$$

where

$$\infty \; > \; I(t) \; := \; \int_0^t \mathrm{d}s \; (1 - s^{-\kappa})\,(t-s)^{-\lambda} \underset{t \downarrow 0}{\searrow} 0. \tag{92}$$

Moreover, if for fixed $t \in (0, T]$,

$$N_t(x) \; := \; \int_0^t \mathrm{d}s \; S^{\alpha}_{t-s}\big(\psi(s)\, S^{\alpha}_s \varphi\big)\,(x), \qquad x \in \dot{\mathsf{R}}^d, \tag{93}$$

satisfies

$$N_t(x) \; \leq \; S^{\alpha}_t \varphi\,(x), \qquad x \in \dot{\mathsf{R}}^d, \tag{94}$$

Then $N_t \in \Phi$.

Proof. First, by Corollary 12, we see that

$$\|S_s^\alpha \varphi\|_{\mathcal{H}} \;\le\; C\,\|\varphi\|_{\mathcal{H}}\,, \qquad 0 \le s \le T, \tag{95}$$

where $C = C(T)$. Now, Corollary 11 states that

$$\left\|S_t^\alpha(\varphi\,\phi^\beta)\right\|_{\mathcal{H}} \;\le\; C_{11}\,t^{-\lambda}\,\|\varphi\|_{\mathcal{H}}\,, \qquad 0 < t \le T, \tag{96}$$

with

$$\lambda \;:=\; \beta(d-1)/4. \tag{97}$$

Applying first (96) and then (95), we obtain

$$\left\|S_{t-s}^\alpha(\phi^\beta\,S_s^\alpha\varphi)\right\|_{\mathcal{H}} \;\le\; C_{11}\,(t-s)^{-\lambda}\,\|\varphi\|_{\mathcal{H}}\,, \qquad 0 \le s < t \le T. \tag{98}$$

Using the fact that S_t^α is an integral operator with a non-negative kernel, and exploiting assumption (89), we find

$$\left\|S_{t-s}^\alpha(\psi(s)\,S_s^\alpha\varphi)\right\|_{\mathcal{H}} \;\le\; C\,(1+s^{-\kappa})\,(t-s)^{-\lambda}\,\|\varphi\|_{\mathcal{H}}\,. \tag{99}$$

However, $I(t)$ from (92) can be written as

$$I(t) \;=\; \frac{t^{1-\lambda}}{1-\lambda} + t^{1-\lambda-\kappa}\int_0^1 \mathrm{d}s\; s^{-\kappa}\,(1-s)^{-\lambda}. \tag{100}$$

But the positive numbers κ and λ defined in (90) and (97), respectively, satisfy $\kappa+\lambda<1$, hence (91) and (92) follow. Thus, the integrals in (93) are finite for almost all x. By assumption (94), it remains to show that $N_t(x)$ from (93) is continuous in x.

Let $\delta \in (0,t)$. Then

$$\int_0^{t-\delta} \mathrm{d}s\; S_{t-s}^\alpha(\psi(s)\,S_s^\alpha\varphi)\,(x) \;=\; S_\delta^\alpha\int_0^{t-\delta}\mathrm{d}s\; S_{t-\delta-s}^\alpha\big(\psi(s)\,S_s^\alpha\varphi\big)\,(x). \tag{101}$$

We already showed that the latter integral term belongs to $\mathcal{H}_+$. Then by Corollary 13, the left hand side in (101) belongs to Φ, hence is continuous in x, for each δ. To complete the proof, it suffices to show that

$$\int_{t-\delta}^t \mathrm{d}s\; S_{t-s}^\alpha(\psi(s)\,S_s^\alpha\varphi)\,(x) \;\underset{\delta\downarrow 0}{\longrightarrow}\; 0 \quad \text{uniformly in } x\in K, \tag{102}$$

where K is any compact subset of $\dot{\mathsf{R}}^d$, we fix from now on. Next apply Corollary 13 to $S_s^\alpha\varphi$ together with the definition (90) of κ to get

$$0 \;\le\; S_s^\alpha\varphi \;\le\; C_{13}\,(1-s^{-\kappa/\beta})\,\phi \;\le\; C\,\phi, \tag{103}$$

since s in (102) is bounded away from 0. Inserting the assumed upper bound (89) on ψ, for the integral in (102) we find the estimate

$$C\int_{t-\delta}^t \mathrm{d}s\; S_{t-s}^\alpha\phi^{\beta+1}\,(x) \;=\; C\int_0^\delta \mathrm{d}s\; S_s^\alpha\,\phi^{\beta+1}\,(x). \tag{104}$$

Hence, it suffices to show that

$$s \mapsto \max_{x \in K} S_s^\alpha \phi^{\beta+1}(x) \quad \text{is integrable on } [0,\delta]. \tag{105}$$

By Lemma 8,

$$S_s^\alpha \phi^{\beta+1} \;\le\; S_s \phi^{\beta+1} + C_8\, \bar{S}_s \phi^{\beta+1}. \tag{106}$$

Now, $(s,x) \mapsto S_s\phi^{\beta+1}(x)$ is finite and satisfies the heat equation on $[0,\delta] \times K$, implying

$$\sup_{(s,x)\,\in\,[0,\delta]} S_s\phi^{\beta+1}(x) \;<\; \infty. \tag{107}$$

Turning to the second term in (106), by definition (43),

$$\bar{S}_s\phi^{\beta+1}(x) \;=\; s^{-1/2}\,\phi(x)\,\mathrm{e}^{-|x|/4s} \int_{\mathsf{R}^d} \mathrm{d}y\; \mathrm{e}^{-|y|/4s}\, \phi^{\beta+2}(y). \tag{108}$$

By the substitution $y \mapsto y\sqrt{s}$, the latter integral gives an additional power contribution to $s^{-1/2}$. Moreover,

$$\sup_{x \in K} \phi(x)\,\mathrm{e}^{-|x|/4s} \;\le\; C\,\mathrm{e}^{-C/s}, \tag{109}$$

which together with $s^{-\lambda}$ is integrable on $[0,\delta]$, for each λ. This finishes the proof. ■

Lemma 19 (Continuity at $t=0$). *Let $\varphi \in \Phi = \Phi^\varrho$ and $v = V\varphi$ a Φ–valued solution to* (86). *Under Hypothesis* 16, *for $T \ge 0$ fixed, there is a constant $C_{19} = C_{19}(d,T,\alpha,\varrho)$ such that*

$$\|V_t\varphi\|_{\mathcal{H}} \;\le\; C_{19}\,\|\varphi\|_{\mathcal{H}}, \qquad 0 \le t \le T. \tag{110}$$

Moreover, $V\varphi$ is strongly continuous at $t=0$, where $V_0\varphi = \varphi$.

Proof. By domination (87),

$$\|V_t\varphi\|_{\mathcal{H}} \;\le\; \|S_t^\alpha\varphi\|_{\mathcal{H}}. \tag{111}$$

Now (110) follows from Corollary 14. It remains to verify the continuity claim. Clearly, for $t \in (0,T]$,

$$|V_t\varphi - \varphi| \;\le\; |V_t\varphi - S_t^\alpha\varphi| + |S_t^\alpha\varphi - \varphi|. \tag{112}$$

By Corollary 12, it suffices to deal with the first term at the right hand side. By equation (86), we have to look at

$$\Big|\int_0^t \mathrm{d}s\; S_{t-s}^\alpha v^\beta(s)\,v(s)\Big|. \tag{113}$$

But from domination (87) and Corollary 13,

$$0 \;\le\; v^\beta(s) \;\le\; C\,(1+s^{-\kappa})\,\phi^\beta, \qquad 0 < s \le T, \tag{114}$$

with κ from (90) and a constant $C = C(T)$ (note that other dependencies are not important in the present proof). Thus, we can apply Lemma 18 to finish the proof. ■

3.3 Uniqueness of solutions

The following lemma will be useful when we estimate the difference of solutions to (86).

Lemma 20 (An elementary observation). *Let* $\beta > 0$ *and* $a, b \in \mathsf{R}$. *Then*

$$\left| a\,(a \vee 0)^{\beta} - b\,(b \vee 0)^{\beta} \right| \;\leq\; (1+\beta)\,\bigl(|a| + |b|\bigr)^{\beta}\; |a-b|. \tag{115}$$

Proof. First assume that $a, b \geq 0$. By the mean value theorem, there exists a number c between a and b such that

$$\left| a^{1+\beta} - b^{1+\beta} \right| \;=\; (1+\beta)\, c^{\beta}\, |a-b| \;\leq\; (1+\beta)\,(a+b)^{\beta}\, |a-b|. \tag{116}$$

This proves (115) for $a, b \geq 0$.

Now suppose that $a, b < 0$. In that case the left hand side in (115) disappears, hence (115) holds trivially.

Finally, it remains to consider the case $a < 0 \leq b$. Then,

$$\left| a\,(a \vee 0)^{\beta} - b\,(b \vee 0)^{\beta} \right| \;=\; b^{1+\beta} \;\leq\; (1+\beta)\, b^{\beta}\, b \;\leq\; (1+\beta)\,\bigl(|a| + |b|\bigr)^{\beta}\; |a-b|,$$

and the proof is finished. ■

We are ready to prove uniqueness for solutions to (86).

Lemma 21 (Uniqueness). *Fix* $\varphi \in \Phi$. *Suppose that* u, v *are* Φ*–valued solutions of equation* (86). *Then* $u = v$.

Proof. Let

$$D(t,x) \;:=\; u(t,x) - v(t,x), \qquad t > 0,\;\; x \neq 0. \tag{117}$$

Note that by Lemma 19, for $T > 0$ fixed,

$$\bigl\| D(t) \bigr\|_{\mathcal{H}} \;\leq\; 2\, C_{19}\, \|\varphi\|_{\mathcal{H}}, \qquad 0 \leq t \leq T. \tag{118}$$

By the elementary inequality (115),

$$\begin{aligned} \bigl| D(t,x) \bigr| \;&=\; \eta \left| \int_0^t \mathrm{d}s\; S^{\alpha}_{t-s}\bigl(u^{1+\beta}(s) - v^{1+\beta}(s)\bigr)(x) \right| \\ &\leq\; \eta \int_0^t \mathrm{d}s\; S^{\alpha}_{t-s} \bigl| u^{1+\beta}(s) - v^{1+\beta}(s) \bigr| (x) \\ &\leq\; 2\eta \int_0^t \mathrm{d}s\; S^{\alpha}_{t-s}\Bigl(\bigl[u^{\beta}(s) + v^{\beta}(s) \bigr]\, \bigl| D(s) \bigr| \Bigr)(x). \end{aligned} \tag{119}$$

From (114), we get

$$\bigl| D(t,x) \bigr| \;\leq\; C \int_0^t \mathrm{d}s\; (1 + s^{-\kappa})\, S^{\alpha}_{t-s}\, \bigl(\bigl| D(s) \bigr|\, \phi^{\beta} \bigr)(x). \tag{120}$$

Thus

$$\begin{aligned} \left\|D(t)\right\|_{\mathcal{H}} &\le C\int_0^t \mathrm{d}s\;(1+s^{-\kappa})\left\|S^\alpha_{t-s}\left(\left|D(s)\right|\phi^\beta\right)\right\|_{\mathcal{H}} \\ &\le C\int_0^t \mathrm{d}s\;(1+s^{-\kappa})\,C_{11}\,(t-s)^{-\lambda}\left\|D(s)\right\|_{\mathcal{H}}, \end{aligned} \tag{121}$$

$0<t\le T,$ where we used Corollary 11 and notation (97). Setting

$$D_t \;:=\; \sup_{0<s\le t}\left\|D(s)\right\|_{\mathcal{H}}, \qquad 0<t\le T, \tag{122}$$

[for finiteness, recall (118)], since $I(t)$ from (92) is increasing in t [recall (100)], we find

$$D_t \;\le\; C\,D_t\,I(t), \qquad 0<t\le T, \tag{123}$$

with some constant $C=C(T)$. Therefore, by (92), $D_t=0$ for all sufficiently small t. Since the model is time-homogeneous, we can repeat the argument finitely often to extent to the whole interval $[0,T]$. (For a more complicated time-inhomogeneous situation, see the proof of Lemma 22 below.) Because T is arbitrary, and since u and v are Φ–valued, we found $u=v$, and the proof is complete. ∎

3.4 Auxiliary functions $v_{N,n}$

For fixed integer $N\ge 2$ set

$$\psi_N \;:=\; \psi\wedge N. \tag{124}$$

For the fixed $T,N,\varphi,\psi,$ we inductively define functions $v_{N,n}$. First of all,

$$v_{N,0}(t) \;:=\; S^\alpha_t\varphi \;\in\; \Phi, \qquad 0\le t\le T. \tag{125}$$

Assuming that we have defined $v_{N,n}$ for some $n,$ let

$$v_{N,n+1}(t,x) \;:=\; S^\alpha_t\varphi\,(x) - \int_0^t \mathrm{d}s\;S^\alpha_{t-s}\big(\psi_N(s)\,v_{N,n}(s)\big)\,(x), \tag{126}$$

$0\le t\le T,$ $x\in\dot{\mathsf{R}}^d,$ provided the latter integral makes sense.

Lemma 22 (Properties of $v_{N,n}$). *For all* $n\ge 0$ *and* $t\in[0,T]$,

$$0\;\le\; v_{N,n}(t)\;\le\; S^\alpha_t\varphi, \quad \textit{and} \quad x\mapsto v_{N,n}(t,x) \quad \textit{is continuous.} \tag{127}$$

Proof. For $n=0,$ the claim is true by (125). Suppose that we have verified (127) for some $n\ge 0$. Then the integral in (126) is non-negative, hence

$$v_{N,n+1}(t)\;\le\; S^\alpha_t\varphi, \qquad t\in[0,T]. \tag{128}$$

Assume for the moment that $v_{N,n+1} \geq 0$ under our induction hypothesis. Then by Lemma 18,

$$v_{N,n+1}(t) \in \Phi, \qquad t \in [0,T], \tag{129}$$

and the proof would be finished.

Next we will verify that $v_{N,n+1}$ is non-negative on $[0,1/N]$. Since $\psi_N \leq N$, and using the induction assumption, it follows that

$$S^\alpha_{t-s}\big(\psi_N(s)\, v_{N,n}(s)\big) \;\leq\; S^\alpha_{t-s}\,(N\, S^\alpha_s \varphi) \;\leq\; N\, S^\alpha_t \varphi. \tag{130}$$

Therefore, if $0 \leq t \leq 1/N$,

$$v_{N,n+1}(t) \;\geq\; S^\alpha_t \varphi \,-\, N \int_0^{1/N} \mathrm{d}s\; S^\alpha_t \varphi \;=\; 0. \tag{131}$$

Now we prove that $v_{N,n+1}$ is non-negative on $[0,T]$. We use induction on the time intervals $\big[k/N,\, (k+1)/N\big]$, $0 \leq k < NT$. To begin with, we have already shown that $v_{N,n+1}$ is non-negative on $[0,1/N]$. Also, we know already that $v_{N,n+1}(1/N) \in \Phi$. Suppose that we have shown $v_{N,n+1}$ is non-negative on $\big[(k-1)/N,\, k/N\big]$ for some $0 \leq k < NT-1$, and that $v_{N,n+1}(k/N) \in \Phi$. We will shift time, and define

$$v^{(k)}_{N,n+1}(t) \;:=\; v_{N,n+1}(t + k/N), \tag{132a}$$

$$\varphi^{(k)}_{N,n+1}(t) \;:=\; v_{N,n+1}(k/N), \tag{132b}$$

$$\psi^{(k)}_{N}(t) \;:=\; \psi_N(t + k/N), \tag{132c}$$

$0 \leq t \leq 1/N$. We claim that

$$v^{(k)}_{N,n+1}(t) \;:=\; S^\alpha_t \varphi^{(k)}_{N,n+1} - \int_0^t \mathrm{d}s\; S^\alpha_{t-s}\big(\psi^{(k)}_N(s)\, v^{(k)}_{N,n}(s)\big), \quad 0 \leq t \leq \frac{1}{N}. \tag{133}$$

Assume for the moment that (133) is true. Then the proof that $v_{N,n+1} \geq 0$ on $\big[k/N,\, (k+1)/N\big]$ reduces to showing that $v^{(k)}_{N,n+1} \geq 0$ on $[0,1/N]$. But this follows from the step we have already done. We are left with showing (133).

Using definition (126), we get

$$S^\alpha_t v_{N,n+1}(r) \;:=\; S^\alpha_{t+r}\varphi \,-\, \int_0^r \mathrm{d}s\; S^\alpha_{t+r-s}\big(\psi_N(s)\, v_{N,n}(s)\big). \tag{134}$$

Let $r = k/N$. Then, for $0 \leq t \leq 1/N$,

$$S^\alpha_t \varphi^{(k)}_{N,n+1} \;=\; S^\alpha_{t+k/N}\varphi \,-\, \int_0^{k/N} \mathrm{d}s\; S^\alpha_{t+k/N-s}\big(\psi_N(s)\, v_{N,n}(s)\big). \tag{135}$$

Also, by a change of variables, for $0 \leq t \leq 1/N$, we get

$$\int_0^t \mathrm{d}s\; S^\alpha_{t-s}\Big(\psi^{(k)}_N(s)\, v^{(k)}_{N,n}(s)\Big) \tag{136}$$

$$= \int_{k/N}^{t+k/N} \mathrm{d}s\; S^\alpha_{t+k/N-s}\big(\psi_N(s)\, v_{N,n}(s)\big).$$

Inserting (135) and (136) into the right hand side of (133), and then using (126), we get

$$S_t^\alpha \varphi_{N,n+1}^{(k)} - \int_0^t \mathrm{d}s\, S_{t-s}^\alpha\big(\psi_N^{(k)}(s)\, v_{N,n}^{(k)}(s)\big) \tag{137}$$

$$= S_{t+k/N}^\alpha \varphi - \int_0^{t+k/N} \mathrm{d}s\, S_{t+k/N-s}^\alpha\big(\psi_N(s)\, v_{N,n}(s)\big) \tag{138}$$

$$= v_{N,n+1}(t+k/N) = v_{N,n+1}^{(k)}(t),$$

which proves (133). This finishes the proof. ■

3.5 Auxiliary functions v_n

Recall that we fixed $\varphi \in \Phi_+$. For $n \geq 0$, we inductively define functions v_n as follows. Let

$$v_0(t) := S_t^\alpha \varphi, \qquad 0 \leq t \leq T, \tag{139}$$

and, given v_n, set

$$v_{n+1}(t,x) := S_t^\alpha \varphi(x) - \int_0^t \mathrm{d}s\, S_{t-s}^\alpha\big(\psi(s)\, v_n(s)\big)(x), \tag{140}$$

$0 \leq t \leq T,\ x \in \dot{\mathsf{R}}^d$.

Lemma 23 (Properties of v_n). *For each $n \geq 0$ and $t \in [0,T]$,*

$$\lim_{N\uparrow\infty} \big|v_{N,n}(t) - v_n(t)\big| = 0. \tag{141}$$

Moreover,

$$0 \leq v_n(t) \leq S_t^\alpha \varphi, \quad \textit{and} \quad x \mapsto v_n(t,x) \quad \textit{is continuous.} \tag{142}$$

Proof. Again, we use induction on n. The claims are trivially true for $n = 0$. Suppose they hold for n. By definitions (126) and (140),

$$\begin{aligned} &\big|v_{n+1}(t) - v_{N,n+1}(t)\big| \\ &= \Big|\int_0^t \mathrm{d}s\, S_{t-s}^\alpha\big(\psi(s)\, v_n(s) - \psi_N(s)\, v_{N,n}(s)\big)\Big| \\ &\leq \Big|\int_0^t \mathrm{d}s\, S_{t-s}^\alpha\Big(\big[\psi(s) - \psi_N(s)\big]\, v_n(s)\Big)\Big| \\ &\quad + \int_0^t \mathrm{d}s\, S_{t-s}^\alpha\Big(\psi(s)\,\big|v_n(s) - v_{N,n}(s)\big|\Big) =: A_{N,n} + B_{N,n} \end{aligned} \tag{143}$$

with the obvious correspondence. We will show that both $A_{N+1,n}$ and $B_{N+1,n}$ tend to 0 as $N \uparrow \infty$, giving (141) for $n+1$. This then yields also the remaining claims in Lemma 23 for $n+1$. In fact, by Lemma 22 then the inequalities hold in (142), and Lemma 18 gives the continuity claim.

First note that by the induction hypothesis,

$$\begin{aligned} A_{N,n} &\leq \int_0^t \mathrm{d}s \, S^\alpha_{t-s}\Big(\big[\psi(s) - \psi_N(s)\big]\, S^\alpha_s \varphi\Big) \\ &\leq \int_0^t \mathrm{d}s \, S^\alpha_{t-s}\big(\psi(s)\, S^\alpha_s \varphi\big) \;<\; \infty, \end{aligned} \tag{144}$$

by Lemma 18. Thus, by monotone convergence,

$$\lim_{N\uparrow\infty} A_{N,n} \;=\; 0. \tag{145}$$

By Lemma 22 and the induction hypothesis,

$$\big|v_{N,n}(t) - v_n(t)\big| \;\leq\; 2\, S^\alpha_t \varphi. \tag{146}$$

Moreover, by the induction assumption, (141) holds. Then, by Lemma 18, the dominated convergence theorem implies that

$$\lim_{N\uparrow\infty} B_{N,n} \;=\; 0, \tag{147}$$

finishing the proof. ■

3.6 A linearized equation

Next we show that v_n converges as $n \uparrow \infty$ to a solution of a linearized equation.

Lemma 24 (Linearized equation). *Fix again* $T \geq 1$, $\varphi \in \Phi$, *and* $\psi : \mathsf{R}_+ \times \mathsf{R}^d \to \mathsf{R}_+$ *satisfying* (89). *Then, for* $0 \leq t \leq T$, *the (non-negative) limit*

$$v(t) \;:=\; \lim_{n\uparrow\infty} v_n(t) \tag{148}$$

exists pointwise and is the unique Φ*–valued solution to*

$$v(t) \;=\; S^\alpha_t \varphi - \int_0^t \mathrm{d}s \, S^\alpha_{t-s}\big(\psi(s)\, v(s)\big), \qquad 0 \leq t \leq T, \tag{149}$$

implying

$$0 \;\leq\; v(t) \;\leq\; S^\alpha_t \varphi, \qquad 0 \leq t \leq T. \tag{150}$$

Proof. This follows from the usual Picard iteration argument, in along the same lines as in our uniqueness proof (Lemma 21). Continuity again follows from Lemma 18. ■

3.7 Existence of solutions

Our next goal is to use Lemma 24 to prove existence of a Φ–valued solution for equation (86). Hypothesis 16 is still in force.

Lemma 25 (Existence). *To each* $\varphi \in \Phi$, *there exists a* Φ*–valued solution* v *to the log-Laplace equation* (86).

Proof. We want to construct a sequence of Φ–valued functions v_n satisfying

$$v_n(t) \ \leq \ S_t^{\alpha}\varphi. \tag{151}$$

In fact, if $n=0$, set $v_0 := S^{\alpha}\varphi$. Assume that we have already defined v_n for some $n \geq 0$. Note that by Corollary 13,

$$\left|v_n^{\beta}(t)\right| \ \leq \ M\,(1+t^{-\kappa})\,\phi^{\beta} \tag{152}$$

with $\varkappa$ from (90). Let v_{n+1} be the unique Φ–valued solution to

$$v_{n+1}(t,x) \ = \ S_t^{\alpha}\varphi\,(x) - \int_0^t \mathrm{d}s\ S_{t-s}^{\alpha}\bigl(v_n^{\beta}(s)\,v_{n+1}(s)\bigr)\,(x) \tag{153}$$

according to Lemma 24, implying (151) for $n+1$. Altogether, by induction we defined Φ–valued functions v_n satisfying (153), (151), and (152).

For $n \geq 0$, let

$$D_n \ := \ v_{n+1} - v_n\,. \tag{154}$$

Then, as in the proof of the uniqueness Lemma 21, using Lemma 20, for fixed $T>0$, we find

$$\left|D_{n+1}(t)\right| \ \leq \ C\int_0^t \mathrm{d}s\ (1+s^{-\kappa})\,S_{t-s}^{\alpha}\bigl(\left|D_n(s)\right|\phi^{\beta}\bigr)\,, \tag{155}$$

$0 \leq t \leq T$, with a constant $C = C(T)$, and

$$\begin{aligned} \left\|D_{n+1}(t)\right\|_{\mathcal{H}} \ &\leq \ C\int_0^t \mathrm{d}s\ (1+s^{-\kappa})\left\|S_{t-s}^{\alpha}\bigl(\left|D_n(s)\right|\phi^{\beta}\bigr)\right\|_{\mathcal{H}} \\ &\leq \ C\int_0^t \mathrm{d}s\ (1+s^{-\kappa})\,C_{11}\,(t-s)^{-\lambda}\left\|D_n(s)\right\|_{\mathcal{H}}\,. \end{aligned} \tag{156}$$

Setting

$$D_{n,t} \ := \ \sup_{0\leq s\leq t}\left\|D_n(s)\right\|_{\mathcal{H}}\,, \qquad 0 \leq t \leq T, \tag{157}$$

we found that

$$D_{n+1,t} \ \leq \ \varepsilon_t\,D_{n,t}\,, \tag{158}$$

where the ε_t are independent of n, and $\varepsilon_t \to 0$ as $t \downarrow 0$. Thus, if our $T>0$ is small enough, then there exists a constant $0<\gamma<1$ such that if $0 \leq t \leq T$, then

$$D_{n+1,t} \ \leq \ \gamma\,D_{n,t}\,, \tag{159}$$

and so

$$D_{n,t} \;\le\; \gamma^n \, D_{0,t} \,. \tag{160}$$

Therefore, we can define

$$v(t,x) \;:=\; \sum_{n=0}^{\infty} D_n(t,x) \;=\; \lim_{n\uparrow\infty} v_n(t,x), \qquad t \le 0, \;\; x \ne 0 \tag{161}$$

where the limit is taken in $\mathcal{H}_+$.

From our construction, it follows that

$$0 \;\le\; v(t) \;\le\; S_t^\alpha \varphi \quad \text{and} \quad \left|v^\beta(t)\right| \;\le\; M\,(1+t^{-\kappa})\,\phi^\beta. \tag{162}$$

Now we want to show that v satisfies equation (86) for $0 \le t \le T$. We start from definition (153). First, by (161),

$$\lim_{n\to\infty} \left\|v_{n+1}(t) - v(t)\right\|_{\mathcal{H}} = 0. \tag{163}$$

As for the integral terms, we first note that for $a,b,c \ge 0$, by (115) we have

$$\begin{aligned} \left|ab^\beta - c^{1+\beta}\right| &\le \left|a^{1+\beta} - c^{1+\beta}\right| + \left|b^{1+\beta} - c^{1+\beta}\right| \\ &\le (1+\beta)(a+c)^\beta |a-c| + (1+\beta)(b+c)^\beta \, |b-c|. \end{aligned} \tag{164}$$

Therefore, using the second part of (162), we have

$$\begin{aligned} &\left\| \int_0^t \mathrm{d}s \; S_{t-s}^\alpha\big(v_{n+1}(s)\, v_n^\beta(s)\big) \;-\; \int_0^t \mathrm{d}s \; S_{t-s}^\alpha v^{1+\beta}(s) \right\|_{\mathcal{H}} \\ &\le\; \int_0^t \mathrm{d}s \;\; \left\| S_{t-s}^\alpha \left|v_{n+1}(s)\, v_n^\beta(s) - v^{1+\beta}(s)\right| \right\|_{\mathcal{H}} \\ &\le\; C \int_0^t \mathrm{d}s \; (1+s^{-\kappa}) \left\| S_{t-s}^\alpha \Big(\phi^\beta \left|v_{n+1}(s) - v(s)\right| + \phi^\beta \left|v_n(s) - v(s)\right| \Big) \right\|_{\mathcal{H}} . \end{aligned} \tag{165}$$

By Corollary 11, this chain of inequalities can be continued with

$$\begin{aligned} &\le\; C \int_0^t \mathrm{d}s \; (1+s^{-\kappa})\,(t-s)^{-\lambda} \left\|v_{n+1}(s) - v(s)\right\|_{\mathcal{H}} \\ &\quad + C \int_0^t \mathrm{d}s \; (1+s^{-\kappa})\,(t-s)^{-\lambda} \left\|v_n(s) - v(s)\right\|_{\mathcal{H}} \;\underset{n\uparrow\infty}{\longrightarrow}\; 0, \end{aligned} \tag{166}$$

by (163), domination, Corollary 12, and dominated convergence. Thus, v satisfies (86) in $\mathcal{H}_+$, for $t \le T$ and for sufficiently small T. By induction on intervals, as in the proof of Lemma 22, we extend the solution from $[0,T]$ to all times. By Lemma 18, the constructed solution is Φ–valued. ■

Completion of the proof of Theorem 17. With Lemmas 21 and 25, we already proved the uniqueness and existence claims, respectively. The semigroup property follows from uniqueness of solutions, and the strong continuity of V from Lemma 19. ■

4 Construction of X

4.1 Approximating equation

Based on Theorem 17, we will prove Theorem 29 via a Trotter product approach. Fix $n \geq 1$ and $\varphi \in \Phi$. We inductively define measurable functions v_n on $\mathsf{R}_+ \times \dot{\mathsf{R}}^d$. First of all,

$$v_n(0) \;:=\; S^\alpha_{1/n}\varphi. \tag{167}$$

Assume for the moment $v_n(\frac{k}{n})$ is defined for some $k \geq 0$. For $\frac{k}{n} \leq t < \frac{k+1}{n}$, set

$$v_n(t,x) \;:=\; \frac{v_n(\frac{k}{n},x)}{\Big[1+\eta\,\beta\, v_n^\beta(\frac{k}{n},x)\,(t-\frac{k}{n})\Big]^{1/\beta}}, \qquad x \neq 0. \tag{168}$$

Note that

$$\frac{\partial}{\partial t} v_n(t,x) \;=\; -\eta\, v_n^{1+\beta}(t,x) \quad \text{on} \quad \Big(\tfrac{k}{n}, \tfrac{k+1}{n}\Big) \times \dot{\mathsf{R}}^d, \tag{169}$$

that $v_n(\frac{k}{n}+\,,x) = v_n(\frac{k}{n},x)$, and that also the limit $v_n(\frac{k+1}{n}-\,,x)$ exists. Put

$$v_n\Big(\tfrac{k+1}{n}\,,x\Big) \;:=\; S^\alpha_{1/n} v_n\Big(\tfrac{k+1}{n}-\,,\,\cdot\,\Big)(x), \qquad x \neq 0. \tag{170}$$

Lemma 26 (Approximating log-Laplace equation). *The function $v_n \geq 0$ we have just defined satisfies*

$$v_n(t,x) \;=\; S^\alpha_{(1+[tn])/n}\varphi(x) \;-\; \eta \int_0^t \mathrm{d}s\; S^\alpha_{([tn]-[sn])/n}\big(v_n^{1+\beta}(s)\big)\,(x) \tag{171}$$

on $\mathsf{R}_+ \times \dot{\mathsf{R}}^d$.

Proof. Differentiating equation (171) to $t \neq \frac{k}{n}$, $k \geq 0$, gives the true statement (169). On the other hand, for $t = \frac{k}{n}$, $k \geq 0$,

$$v_n\big(\tfrac{k}{n}\big) \;=\; S^\alpha_{(1+k)/n}\varphi \;-\; \eta \sum_{i=1}^k \Big[S^\alpha_{(k-(i-1))/n} \int_{(i-1)/n}^{i/n} \mathrm{d}s\; v_n^{1+\beta}(s)\Big]. \tag{172}$$

By (169) and the fundamental theorem of calculus, the right hand side of the latter equation equals $v_n(\frac{k}{n})$, finishing the proof. ∎

Set

$$t_n := [tn]/n. \tag{173}$$

Since v_n is non-negative, from equation (171) we get the *domination*

$$0 \;\leq\; v_n(t) \;\leq\; S^\alpha_{1/n+t_n}\varphi, \qquad t \geq 0, \quad n \geq 1, \tag{174}$$

implying by Corollary 11 with $\beta = 0$,

$$\big\|v_n(t)\big\|_{\mathcal{H}} \;\leq\; C_{(175)}\,\|\varphi\|, \qquad 0 \leq t \leq T, \quad n \geq 1, \tag{175}$$

for each $T > 0$, and where $C_{(175)} = C_{(175)}(T)$. In particular, v_n is $\mathcal{H}_+$–valued. Our aim is to show that the v_n converge to the unique solution to (86). For this purpose, we will need the following estimate.

Lemma 27 (Pointwise bound). *Impose Hypothesis* 16. *To each* $\varphi \in \mathcal{H}_+$ *and* $T > 0$, *there is a* $\varphi_0 = \varphi_0(d, T, \alpha, \varphi, \varrho)$ *in* $\mathcal{H}_+$, *such that*

$$\sup_{T/2 \le t \le T} S_t^\alpha \varphi \ \le\ \varphi_0 \,. \tag{176}$$

Proof. Recall from (44) that

$$P^\alpha(t; x, y) \ \le\ P(t; x, y) + C_8\, \bar{P}(t; x, y). \tag{177}$$

Choose a constant $C_{(178)} = C_{(178)}(d, T)$ such that, for $T/2 \le t \le T$,

$$P(t; x, y) \ \le\ C_{(178)} P(T; x, y) \tag{178}$$

and

$$\bar{P}(t; x, y) \ =\ t^{-1/2}\, \phi(x)\, \phi(y)\, \mathrm{e}^{-|x|^2/4t}\ \mathrm{e}^{-|y|^2/4t} \ \le\ C_{(178)}\, \bar{P}(T; x, y) \tag{179}$$

[recall (43)]. From Lemma 5 (with $\beta = 0$) we conclude that $S_T\varphi$ belongs to $\mathcal{H}_+$, whereas Lemma 9 (with $\beta = 0$) gives $\bar{S}_T\varphi \in \mathcal{H}_+$. Therefore, we may set

$$\varphi_0 \ :=\ C_{(178)}\, \big(S_T\varphi + C_8\, \bar{S}_T\varphi\big) \tag{180}$$

to finish the proof. ∎

4.2 Convergence to the limit equation

With the function v_n we may pass to the limit.

Lemma 28 (Convergence to the limit equation). *Let* $\varphi \in \Phi$. *Define* v_n *as in* (167)–(170). *Let* v *be the unique* Φ*–valued solution to* (86) *according to Theorem* 17. *Then, for each* $t \ge 0$,

$$\lim_{n\uparrow\infty} \big\| v(t) - v_n(t) \big\|_{\mathcal{H}} \ =\ 0. \tag{181}$$

Proof. We may restrict our attention to $t \in [0, T]$ for any $T > 1$. From equations (86) and (171) we have

$$\begin{aligned} \big\| v(t) - v_n(t) \big\|_{\mathcal{H}} \ &\le\ \Big\| S_t^\alpha \varphi - S_{1/n + t_n}^\alpha \varphi \Big\|_{\mathcal{H}} \\ &+\ \eta \int_0^{t_n} \mathrm{d}s\ \big\| S_{t-s}^\alpha v^{1+\beta}(s) - S_{t_n - s_n}^\alpha v^{1+\beta}(s) \big\|_{\mathcal{H}} \\ &+\ \eta \int_0^{t_n} \mathrm{d}s\ \Big\| S_{t_n - s_n}^\alpha \big| v^{1+\beta}(s) - v_n^{1+\beta}(s) \big| \Big\|_{\mathcal{H}} \\ &+\ \eta\ \Big\| \int_{t_n}^t \mathrm{d}s\ S_{t-s}^\alpha v^{1+\beta}(s) \Big\|_{\mathcal{H}} + \eta\ \Big\| \int_{t_n}^t \mathrm{d}s\ S_{t_n - s_n}^\alpha v_n^{1+\beta}(s) \Big\|_{\mathcal{H}} \\ =:&\ A_n(t) + B_n(t) + C_n(t) + D_n(t) + E_n(t) \end{aligned} \tag{182}$$

with the obvious correspondence. We will deal with each of these terms separately.

1° $\big(A_n(t)\big)$. From the semigroup property and boundedness (77),

$$A_n(t) \;=\; \left\|S_t^{\alpha}\varphi - S_{1/n+t_n}^{\alpha}\varphi\right\|_{\mathcal{H}} \;\le\; C_{11}\,\left\|S_{|t-1/n-t_n|}^{\alpha}\varphi - \varphi\right\|_{\mathcal{H}}. \tag{183}$$

But

$$\left|t-\frac{1}{n}-t_n\right| \;\le\; \frac{2}{n}, \qquad t\ge 0. \tag{184}$$

Hence,

$$\sup_{0\le t\le T} A_n(t) \;\le\; C_{11} \sup_{0\le s\le 2/n} \|S_s^{\alpha}\varphi-\varphi\|_{\mathcal{H}} \underset{n\uparrow\infty}{\longrightarrow} 0 \tag{185}$$

by strong continuity according to Corollary 12.

2° $\big(D_n(t)\big)$. Clearly, by our estimates,

$$D_n(t) \;\le\; \eta\int_{t_n}^{t} \mathrm{d}s\;(1+s^{-\kappa})\,(t-s)^{-\lambda}\,\|\varphi\|_{\mathcal{H}}. \tag{186}$$

By scaling, the integral equals

$$t^{1-\lambda}\int_{t_n/t}^{1}\mathrm{d}s\;(1-s)^{-\lambda} + t^{1-\kappa-\lambda}\int_{t_n/t}^{1}\mathrm{d}s\;s^{-\kappa}\,(1-s)^{-\lambda} \;=:\; I_n(t)+I\!I_n(t)$$

with the obvious correspondence. Take $\varepsilon\in(0,T)$ and let $n>1/\varepsilon$. Since

$$\frac{t_n}{t} \;\ge\; 1-\frac{1}{tn} \;\ge\; 1-\frac{1}{\varepsilon n}, \qquad \varepsilon\le t\le T, \tag{187}$$

we get

$$\sup_{\varepsilon\le t\le T} I_n(t) \;\le\; T^{1-\lambda}\int_{1-1/\varepsilon n}^{1}\mathrm{d}s\;(1-s)^{-\lambda} \underset{n\uparrow\infty}{\longrightarrow} 0, \tag{188}$$

whereas

$$\sup_{0\le t\le \varepsilon} I_n(t) \;\le\; \varepsilon^{1-\lambda}\int_{0}^{1}\mathrm{d}s\;(1-s)^{-\lambda} \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{189}$$

Consequently,

$$\sup_{0\le t\le T} I_n(t) \underset{n\uparrow\infty}{\longrightarrow} 0, \tag{190}$$

and the same reasoning leads to the analogous statement on $I\!I_n(t)$. Summarizing,

$$\sup_{0\le t\le T} D_n(t) \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{191}$$

3° $\big(E_n(t)\big)$. Assume that $t>t_n$. By (171),

$$E_n(t) \;=\; \big\|v_n(t)-v_n(t_n)\big\|_{\mathcal{H}}. \tag{192}$$

According to the definition (168) of $v_n(t)$,

$$\begin{aligned} 0 \ &\leq \ v_n(t_n\,,x) - v_n(t,x) \\ &= \ v_n(t_n,x)\left(1 - \frac{1}{\left[1+\eta\,\beta\,v_n^{\beta}(t_n,x)\,(t-t_n)\right]^{1/\beta}}\right). \end{aligned} \tag{193}$$

Using domination and Lemma 27, there is a $\varphi_0 = \varphi_0(\varphi,t,\alpha,d,\varrho) \in \mathcal{H}_+$ such that

$$v_n(t_n\,,x) \ \leq \ S^{\alpha}_{1/n+t_n}\varphi\,(x) \ \leq \ \varphi_0\,, \tag{194}$$

since $t \leq 1/n + t_n \leq 1+t$. But (193) is increasing in $v_n(t_n\,,x)$, so we may insert (194) to obtain

$$0 \ \leq \ v_n(t_n\,,x) - v_n(t,x) \ \leq \ \varphi_0(x)\left(1 - \frac{1}{\left[1+\varphi_0^{\beta}(x)\right]^{1/\beta}}\right) \ \leq \ \varphi_0(x),$$

since $0 \leq t - t_n \leq 1/n$. Then, from dominated convergence we get

$$\sup_{0\leq t\leq T} E_n(t) \ \underset{n\uparrow\infty}{\longrightarrow} \ 0. \tag{195}$$

4° $\bigl(B_n(t)\bigr)$. First of all, we want to deal with $B_n(t)$ for small t. Clearly,

$$\left\|S^{\alpha}_{t-s}v^{1+\beta}(s)\right\|_{\mathcal{H}} \ \leq \ C\,(1+s^{-\kappa})\,(t-s)^{-\lambda}\,\|\varphi\|_{\mathcal{H}} \tag{196}$$

and

$$\left\|S^{\alpha}_{t_n-s_n}v^{1+\beta}(s)\right\|_{\mathcal{H}} \ \leq \ C\,(1+s^{-\kappa})\,(t_n-s)^{-\lambda}\,\|\varphi\|_{\mathcal{H}}\,. \tag{197}$$

Let $0<\varepsilon<T$. Using notation (92), from (196) and (197), since

$$t_n - s_n \ \geq \ t_n - s \ \geq \ 0, \tag{198}$$

for $0 \leq t \leq \varepsilon$,

$$B_n(t) \ \leq \ C\left[I(t) + I(t_n)\right]. \tag{199}$$

Moreover, since I is increasing [recall (100)],

$$\sup_{n\geq 1}\ \sup_{0\leq t\leq\varepsilon} B_n(t) \ \leq \ C \sup_{0\leq t\leq\varepsilon} I(t) \ \underset{\varepsilon\downarrow 0}{\longrightarrow} \ 0. \tag{200}$$

Now we may restrict to $t\in[\varepsilon,T]$. We want to exploit the strong continuity of the semigroup S^{α} acting on $\mathcal{H}_+$ (Corollary 12). To this end, we truncate $v^{1+\beta}$ to a function in $\mathcal{H}_+$, and consider a small time interval around t_n separately to get rid of the varying upper integration bound. Here are the details.

Take $\delta\in(0,\varepsilon)$ and $N \geq 1$. Set

$$v_{1,N}(t) \ := \ \bigl(v(t)\wedge N\bigr)\,1_{B_N(0)} \tag{201a}$$

$$v_{2,N}(t) \ := \ v(t) - v_{1,N}(t). \tag{201b}$$

Then for $\varepsilon \le t \le T$,

$$B_n(t) \;\le\; \int_0^{t-\delta} \mathrm{d}s\, \left\| S^\alpha_{t-s} v^{1+\beta}_{1,N}(s) - S^\alpha_{t_n-s_n} v^{1+\beta}_{1,N}(s) \right\|_{\mathcal{H}} \tag{202}$$

$$+ \int_{t-\delta}^{t} \mathrm{d}s\, \left\| S^\alpha_{t-s} v^{1+\beta}(s) \right\|_{\mathcal{H}} + \int_{t-\delta}^{t_n} \mathrm{d}s\, \left\| S^\alpha_{t_n-s_n} v^{1+\beta}(s) \right\|_{\mathcal{H}}$$

$$+ \int_0^{t-\delta} \mathrm{d}s\, \left\| S^\alpha_{t-s} v^{1+\beta}_{2,N}(s) \right\|_{\mathcal{H}} + \int_0^{t-\delta} \mathrm{d}s\, \left\| S^\alpha_{t_n-s_n} v^{1+\beta}_{2,N}(s) \right\|_{\mathcal{H}}$$

$$=: \; B^{(1)}_n(t) + \cdots + B^{(5)}_n(t)$$

in the obvious correspondence. Again, we deal with all terms separately.

4.1° $\big(B^{(2)}_n(t)\big)$. From (196) and scaling,

$$B^{(2)}_n(t) \;\le\; C \int_{1-\delta/t}^{1} \mathrm{d}s\, (1-s)^{-\lambda} + C \int_{1-\delta/t}^{1} \mathrm{d}s\, s^{-\kappa} (1-s)^{-\lambda}. \tag{203}$$

But $1-\delta/t \ge 1-\delta/\varepsilon$, hence

$$\sup_n \sup_{\varepsilon \le t \le T} B^{(2)}_n(t) \underset{\delta\downarrow 0}{\longrightarrow} 0. \tag{204}$$

4.2° $\big(B^{(3)}_n(t)\big)$. Similarly, from (197) and (198),

$$B^{(3)}_n(t) \;\le\; C \int_{t-\delta}^{t_n} \mathrm{d}s\, (1+s^{-\kappa})\, (t_n - s)^{-\lambda}. \tag{205}$$

Now

$$t \ge t_n \ge t - \frac{1}{n} \ge \varepsilon - \frac{1}{n} \ge \delta \tag{206}$$

provided that $n \ge 1/(\varepsilon-\delta)$. Thus, the lower integration bound can be replaced by $t_n - \delta$, and by scaling,

$$B^{(3)}_n(t) \;\le\; C \int_{1-\delta/t_n}^{1} \mathrm{d}s\, (1-s)^{-\lambda} + C \int_{1-\delta/t_n}^{1} \mathrm{d}s\, s^{-\kappa} (1-s)^{-\lambda}. \tag{207}$$

By (206), the lower integration bounds can be changed to $1-\delta/(\varepsilon - 1/n)$, implying

$$\limsup_{n\uparrow\infty} \sup_{\varepsilon \le t \le T} B^{(3)}_n(t) \underset{\delta\downarrow 0}{\longrightarrow} 0. \tag{208}$$

4.3° $\big(B^{(4)}_n(t)$ and $B^{(5)}_n(t)\big)$. First note that for $s \in [0, t-\delta]$,

$$t-s \quad \text{and} \quad t_n - s_n \quad \text{belong to} \quad [\delta/2, t\} \quad \text{if} \quad n > 2/\delta \tag{209}$$

(for instance, $t_n - s_n \ge t - \frac{1}{n} - s \ge -\frac{1}{n} + \delta \ge \delta$). Then, by domination and Corollary 13,

$$v^\beta_{2,N}(s) \;\le\; v^\beta(s) \;\le\; (S^\alpha_s \varphi)^\beta \;\le\; C\, (1+s)^{-\kappa}\, \phi^\beta, \tag{210}$$

hence, by Lemma 9, for $r \in [\delta/2, t]$ and $n > 2/\delta, s \le T$

$$\left\| S_r^\alpha v_{2,N}^{1+\beta}(s) \right\|_{\mathcal{H}} \;\le\; C\,(1+s)^{-\kappa} \left\| S_r^\alpha \bigl(v_{2,N}(s)\,\phi^\beta\bigr) \right\|_{\mathcal{H}} \;\le\; C\,(1+s)^{-\kappa} \left\| v_{2,N}(s) \right\|_{\mathcal{H}}.$$

Now by domination and boundedness, for $0 \le s \le T$,

$$\left\| v_{2,N}(s) \right\|_{\mathcal{H}} \;\le\; \left\| S_s^\alpha \varphi \right\|_{\mathcal{H}} \;\le\; \left\| \varphi \right\|_{\mathcal{H}} \tag{211}$$

and

$$\left\| v_{2,N}(s) \right\|_{\mathcal{H}} \underset{N\uparrow\infty}{\searrow} 0. \tag{212}$$

Therefore,

$$\begin{aligned} &\limsup_{n\uparrow\infty} \sup_{\varepsilon \le t \le T} \bigl(B_n^{(4)}(t) + B_n^{(5)}(t) \bigr) \\ &\le\; C \int_0^T \mathrm{d}s\; (1+s)^{-\kappa} \left\| v_{2,N}(s) \right\|_{\mathcal{H}} \underset{N\uparrow\infty}{\searrow} 0, \end{aligned} \tag{213}$$

by monotone convergence, for the fixed ε and δ.

4.4° $\bigl(B_n^{(1)}(t)\bigr)$. It remains to deal with $B_n^{(1)}(t)$. By the semigroup property, boundedness, and strong continuity,

$$\begin{aligned} &\left\| S_{t-s}^\alpha v_{1,N}^{1+\beta}(s) - S_{t_n - s_n}^\alpha v_{1,N}^{1+\beta}(s) \right\|_{\mathcal{H}} \\ &\le\; \sup_{0 \le r \le 2/n} \left\| S_r^\alpha v_{1,N}^{1+\beta}(s) - v_{1,N}^{1+\beta}(s) \right\|_{\mathcal{H}} \underset{N\uparrow\infty}{\searrow} 0 \end{aligned} \tag{214}$$

for all s and N, since by definition

$$v_{1,N}^{1+\beta}(s) \;\le\; N^{1+\beta}\, 1_{B_N(0)} \;\in\; \mathcal{H}_+. \tag{215}$$

Moreover, the supremum in (214) is bounded form above by

$$2\,N^{1+\beta} \left\| 1_{B_N(0)} \right\|_{\mathcal{H}}. \tag{216}$$

Therefore,

$$\sup_{\varepsilon \le t \le T} B_n^{(1)}(t) \;\le\; \int_0^T \mathrm{d}s \sup_{0 \le r \le 2/n} \left\| S_r^\alpha v_{1,N}^{1+\beta}(s) - v_{1,N}^{1+\beta}(s) \right\|_{\mathcal{H}} \underset{n\uparrow\infty}{\searrow} 0, \tag{217}$$

by monotone convergence, for all our N, ε, δ.

4.5° (*Conclusion*). Putting together (200), (204), (208), (213), and (217),

$$\sup_{0 \le t \le T} B_n(t) \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{218}$$

5° $\bigl(C_n(t)\bigr)$. First note that

$$C_n(t) \;=\; 0 \quad \text{for} \quad t \le 1/n. \tag{219}$$

So we may assume that $t \geq 1/n$. Next we apply Lemma 20 to get for the term in abstract value sign in the definition of $C_n(t)$ the bound

$$C\left[|v|^\beta(s)+|v_n|^\beta(s)\right]\left|v(s)-v_n(s)\right|. \tag{220}$$

From domination, the expression in square brackets is bounded by

$$(S_s^\alpha\varphi)^\beta+(S_{1/n+s_n}^\alpha\varphi)^\beta \;\leq\; C\,(1+s^{-\kappa})\,\phi^\beta, \tag{221}$$

where we used Corollary 13 and $1/n+s_n \geq s$. But by Corollary 11,

$$\left\|S_{t_n-s_n}^\alpha \left|v(s)-v_n(s)\right|\phi^\beta\right\|_{\mathcal{H}} \;\leq\; C\,(t_n-s_n)^{-\lambda}\left\|v(s)-v_n(s)\right\|_{\mathcal{H}}. \tag{222}$$

Setting

$$F_n(t) \;:=\; \sup_{s\leq t}\left\|v(s)-v_n(s)\right\|_{\mathcal{H}}, \tag{223}$$

we found for $\frac{1}{n}\leq t\leq T$,

$$\begin{aligned}&\int_0^{t_n}\mathrm{d}s\,\left\|S_{t_n-s_n}^\alpha\left|v^{1+\beta}(s)-v_n^{1+\beta}(s)\right|\right\|_{\mathcal{H}}\\ &\leq\; C\,F_n(t)\int_0^{t_n}\mathrm{d}s\,(1+s^{-\kappa})\,(t_n-s)^{-\lambda} \;\leq\; C\,F_n(t)\,I(t)\end{aligned} \tag{224}$$

[recall (92)].

6° (*Completion of the proof*). Inserting (185), (191), (195), (218), (224) into (182), we obtain

$$\left\|v(t)-v_n(t)\right\|_{\mathcal{H}} \;\leq\; \varepsilon_n+C\,F_n(t)\,I(t), \tag{225}$$

where $C=C(T)$ and where $\varepsilon_n=\varepsilon_n(T)$ is independent of t and tends to 0 as $n\uparrow\infty$. Restrict further to t such that $C\,I(t)<\frac{1}{2}$ [recall (92)] to get

$$F_n(t) \;\leq\; \varepsilon_n+\frac{1}{2}\,F_n(t), \quad \text{that is} \quad F_n(t) \;\leq\; 2\,\varepsilon_n \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{226}$$

Consequently,

$$\left\|v(t)-v_n(t)\right\|_{\mathcal{H}} \underset{n\uparrow\infty}{\longrightarrow} 0 \quad \text{for all sufficiently small } t. \tag{227}$$

Repeating the argument, we can lift up for $t\in[0,T]$. Since T was arbitrary, the proof of Lemma 28 is finished altogether. ∎

4.3 Construction of the process

Here is now the more precise formulation of our main result, announced in Theorem 1:

Theorem 29 (Existence of X). *Under Hypothesis* 16, *there is a non-degenerate* $\mathcal{M}(\dot{\mathsf{R}}^d)$*–valued (time-homogeneous) Markov process* $X = (X, P_\mu, \mu \in \mathcal{M})$ *with log-Laplace transition functional* (6), *where* v *solves* (5) *with initial state* $\varphi \in \Phi$ *satisfying* (9).

Remark 30 (Non-degeneracy). It is easy to see that the following expectation formula holds:

$$P_\mu \langle X_t, \varphi\rangle \;=\; \langle \mu, S_t^\alpha \varphi\rangle, \qquad \mu \in \mathcal{M}, \quad t \geq 0, \quad \varphi \in \Phi. \tag{228}$$

But

$$V_t\varphi \;\neq\; S_t^\alpha \varphi, \qquad t > 0, \quad \varphi \in \Phi, \quad \varphi \neq 0. \tag{229}$$

Hence, the log-Laplace formula (6) shows that X is different from its expectation, that is, it is non-degenerate. ◇

Proof of Theorem 29. Under Hypothesis 16, for $\mu \in \mathcal{M} = \mathcal{M}(\dot{\mathsf{R}}^d)$, from definitions (167)–(170) of $v_n =: V^n\varphi$, $\varphi \in \Phi$, for fixed $t > 0$ there is a random measure X_t^n in $\mathcal{M}$ satisfying

$$-\log \mathbf{P}\mathrm{e}^{-\langle X_t^n, \varphi\rangle} \;=\; \big\langle \mu, v_n(t)\big\rangle, \qquad \varphi \in \Phi. \tag{230}$$

In fact, X_t^n arises in the following way: Start at time 0 with μ. During $t \in [0, \frac{1}{n})$, let X_t^n evolve independently in each point according to continuous state branching with index $1+\beta$, starting from μ, that is, related to definition (168). Then, at time $\frac{1}{n}$ let $X_{1/n-}^n$ jump to $S_{1/n}^\alpha X_{1/n\,-}^n =: X_{1/n}^n$. Then continue with continuous state branching as before and up to time $\frac{2}{n}-$, etc., until t is reached.

From the convergence $v_n(t) \to v(t)$ according to Lemma 28 and domination as in (87), implying $v(t) \downarrow 0$ as $\varphi \downarrow 0$, we get the existence of a random measure X_t in $\mathcal{M}$ such that $X_t^n \to X_t$ in law as $n \uparrow \infty$. Since the map $\mu \mapsto \langle \mu, V_t\varphi\rangle$ is measurable, via $\mu \mapsto X_t$ we get a probability kernel Q_t in $\mathcal{M}$. From the semigroup property of $V\varphi$ follows that $Q = \{Q_t : t > 0\}$ satisfies Chapman-Kolmogorov. Hence, Q is the transition kernel of a Markov process X in $\mathcal{M}$, which is the desired superprocess. This finishes the proof. ■

Acknowledgment We would like to thank Sergio Albeverio, Zdzisław Brzeźniak, Werner Kirsch, and Karl-Theodor Sturm for discussions on the operators $\Delta^{(\alpha)}$.

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