Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Existence and approximation of slow integral manifolds in some degenerate cases

Klaus R. Schneider, [1] Vladimir A. Sobolev [2]

submitted: 12th November 2002

[1] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany
E-Mail: schneider@wias-berlin.de

[2] Samara State University
P.O.Box 10902
Chapaevskaya 69–8
443099 Samara
Russia
E-Mail: sable@ssu.samara.ru

No. 782
Berlin 2002

2000 *Mathematics Subject Classification.* 34C45, 34D15, 34E15.

Key words and phrases. singular perturbations, integral manifolds, degenerate cases.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider singularly perturbed differential systems whose degenerate equations have an isolated but not simple solution. In that case, the standard theory to establish a slow integral manifold near this solution does not work. Applying scaling transformations and using the technique of gauge functions we reduce the original singularly perturbed problem to a regularized one such that the existence of slow integral manifolds can be established by means of the standard theory of singular perturbations. We illustrate our method by several examples.

1 Introduction

Singularly perturbed differential systems of the type

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,\varepsilon), \\ \varepsilon\,\frac{dy}{dt} &= g(x,y,t,\varepsilon) \end{aligned} \tag{1.1}$$

play an important role as mathematical models of numerous nonlinear phenomena in biology, chemistry, control theory, and in other fields (see, e.g., [4, 5, 7, 10, 15, 16, 17]). A usual approach in the qualitative study of (1.1) is to consider first the degenerate system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,0), \\ 0 &= g(x,y,t,0) \end{aligned} \tag{1.2}$$

and then to draw conclusions for the qualitative behavior of the full system (1.1) for sufficiently small ε. A special case of this approach is the quasi–steady state assumption. A mathematical justification of that method can be given by means of the theory of integral manifolds for singularly perturbed systems (1.1) (see, e.g., [3, 13, 15]).

In order to recall a basic result of the geometric theory of singularly perturbed systems we introduce the following notation and assumptions.

Let I_i be the interval $I_i := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon_i\}$, where $0 <\varepsilon_i \ll 1, i = 0, 1, \ldots$.

(A_1). $f : R^m \times R^n \times R \times \overline{I_0} \to R^m$, $g : R^m \times R^n \times R \times \overline{I_0} \to R^n$ are sufficiently smooth and uniformly bounded together with their derivatives.

(A_2). There are some region $G \in R^m$ and a map $h : G \times R \to R^n$ of the same smoothness as g such that

$$g(x, h(x,t), t, 0) \equiv 0 \quad \forall (x,t) \in G \times R.$$

(A_3). The spectrum of the Jacobian matrix $g_y(x, h(x,t), t, 0)$ is uniformly separated from the imaginary axis for all $(x,t) \in G \times R$.

Then the following result is valid (see, e.g., [3, 15]):

Proposition 1.1. *Under the assumptions $(A_1) - (A_3)$ there is a sufficiently small positive ε_1, $\varepsilon_1 \leq \varepsilon_0$, such that for $\varepsilon \in \overline{I_1}$ system (1.1) has a smooth integral manifold $\mathcal{M}_\varepsilon$ with the representation*

$$\mathcal{M}_\varepsilon := \{(x,y,t) \in R^{m+n+1} : y = \psi(x,t,\varepsilon),\ (x,t) \in G \times R\}$$

and with the asymptotic expansion

$$\psi(x,t,\varepsilon) = h(x,t) + \varepsilon \psi_1(x,t) + \ldots\ .$$

Remark 1.1. *The global boundedness assumption in (A_1) with respect to (x,y) can be relaxed by modifying f and g outside some bounded region of $R^m \times R^n$.*

Remark 1.2. *In applications it is usually assumed that the spectrum of the Jacobian matrix $g_y(x, h(x,t), t, 0)$ is located in the left half plane. Under this additional hypothesis the manifold $\mathcal{M}_\varepsilon$ is exponentially attracting for $\varepsilon \in I_1$.*

The case that assumption (A_3) is violated is called critical. We distinguish three sub-cases:

1. The Jacobian matrix $g_y(x,y,t,0)$ is singular on some subspace of $R^m \times R^n \times R$. In that case, system (1.1) is referred to as a singular singularly perturbed system. This sub-case has been treated in [6, 7, 13].

2. The Jacobian matrix $g_y(x,y,t,0)$ has eigenvalues on the imaginary axis with non-vanishing imaginary parts. A similar case has been investigated in [12, 15].

3. The Jacobian matrix $g_y(x,y,t,0)$ is singular on the set $\mathcal{M}_0 := \{(x,y,t) \in R^m \times R^n \times R : y = h(x,t), (x,t) \in G \times R\}$. In that case, $y = h(x,t)$ is generically an isolated root of $g = 0$ but not a simple one. This case will be studied in the following.

The paper is organized as follows. In section 2 we formulate the problem and recall the method of gauge functions by considering a degenerate two-dimensional autonomous singularly perturbed differential system. Section 3 is devoted to the case of a quasi-homogeneous degenerate system, the case of an autonomous homogeneous singularly perturbed system is treated in section 4. Section 5 contains the case of a quasi-homogeneous degenerate system. Several examples are given, one example describes a partial cheap optimal control problem.

2 Formulation of the problem. Preliminaries

We consider system (1.1) under the assumptions (A_1) and (A_2). Instead of hypothesis (A_3) we suppose

$$det\ g_y(x, h(x,t), t, 0) \equiv 0 \quad \forall (x,t) \in G \times R, \tag{2.1}$$

that is, $y = h(x,t)$ is not a simple root of the degenerate equation

$$g(x, y, t, 0) = 0. \tag{2.2}$$

Under this assumption we cannot apply Proposition 1.1 to system (1.1) in order to establish the existence of a slow integral manifold near $\mathcal{M}_0$ for small ε. Our goal is to derive conditions which imply that for sufficiently small ε, system (1.1) has at least one integral manifold $\mathcal{M}_\varepsilon$ with the representation

$$y = \psi_i(x,t,\varepsilon) = h(x,t) + \varepsilon^{q_i} h_{1,i}(x,t) + \varepsilon^{2q_i} h_{2,i}(x,t) + \dots.$$

where $q_i, 0 < q_i < 1$, is a rational number.
The key idea to solve this problem consists in looking for scalings and transformations of the type

$$\varepsilon = \mu^r,\ y = \tilde{y}(\mu, z, x, t),\ t = \tilde{t}(\mu, \tau)$$

such that system (1.1) can be reduced to a system

$$\begin{aligned} \frac{dx}{d\tau} &= f(x, z, \tau, \mu), \\ \mu\frac{dz}{d\tau} &= g(x, z, \tau, \mu) \end{aligned} \tag{2.3}$$

to which Proposition 1.1 can be applied. In this process the method of gauge function plays an important role.

We illustrate our approach by considering a simple example, at the same time we recall the method of undetermined gauges.

Example 2.1. Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \varepsilon \frac{dy}{dt} &= -y^2 - y^3 + \varepsilon\phi^2(x,t), \end{aligned} \tag{2.4}$$

where ϕ is a smooth positive function. The degenerate equation to (2.4) reads

$$0 = -y^2 - y^3 \tag{2.5}$$

and has the isolated but multiple root $y = 0$. To find a transformation reducing system (2.4) to a system to which Proposition 1.1 can be applied we look for an approximation of the roots of the equation

$$0 = -y^2 - y^3 + \varepsilon\phi^2(x,t) \tag{2.6}$$

by means of the method of undetermined gauges (see, e.g., [9]). To this purpose we represent a solution of (2.6) in the form

$$y \cong \delta_1(\varepsilon)y_1(x,t) + \delta_2(\varepsilon)y_2(x,t) + \ldots. \tag{2.7}$$

The functions $\delta_i(\varepsilon)$, called gauges, must be determined along with the functions $y_i(x,t)$. Concerning the gauge functions $\delta_i(\varepsilon)$ we suppose that they are monotone in the interval I_0 and satisfy $\delta_i(\varepsilon) \to 0$ and $\delta_{i+1}(\varepsilon)/\delta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ for all i.
Substituting

$$y \cong \delta_1(\varepsilon)y_1$$

into (2.6) leads to the equation

$$0 \cong -y_1^2\delta_1^2(\varepsilon) + y_1^3\delta_1^3(\varepsilon) + \varepsilon\phi^2(x,t). \tag{2.8}$$

As $\delta_1^3(\varepsilon) \ll \delta_1^2(\varepsilon)$ for sufficiently small ε we simplify (2.8) to

$$0 \cong -y_1^2\delta_1^2(\varepsilon) + \varepsilon\phi^2(x,t). \tag{2.9}$$

Now we have to compare the order functions $\delta_1(\varepsilon)$ and ε. Supposing that $y^2\delta_1^2(\varepsilon)$ is the leading term in (2.9), we get $y_1 = 0$; if we assume that $\varepsilon\phi^2(x,t)$ is the leading term, then we are not able to determine y_1. If we suppose that $\delta_1^2(\varepsilon)$ and ε have the same order, then we can set

$$\delta_1(\varepsilon) := \sqrt{\varepsilon}. \tag{2.10}$$

We note that this is not the only possible choice for $\delta_1(\varepsilon)$ (see [9]). Putting (2.10) into (2.9) we obtain

$$y_1(x,t) = \pm\phi(x,t). \tag{2.11}$$

Similarly we can determine higher order gauges and coefficients.
Now we use the relations (2.7) and (2.10) to scale the parameter ε and the variable y by $\varepsilon = \mu^2$, $y = \mu z$. Substituting these relations into (2.4) we get

$$\begin{aligned} \frac{dx}{dt} &= \mu z \\ \mu \frac{dz}{dt} &= -z^2 + \phi^2(x,t) - \mu z^3. \end{aligned} \tag{2.12}$$

Taking into account that the degenerate equation to (2.12) has the two isolated simple solutions $z = \pm\phi(x,t)$ we can apply Proposition 1.1 to system (2.12) with respect to these roots and get that system (2.4) has two integral manifolds with the representation

$$y = \pm\phi(x,t)\sqrt{\varepsilon} + O(\varepsilon).$$

In the following sections we study the existence and approximation of slow integral manifolds of system (1.1) in some degenerate cases.

3 Quasi-homogeneous degenerate equations

We study system (1.1) under the assumption (A_1). We replace the assumptions (A_2) and (A_3) by the following hypotheses.

(H_1). The function $g(x,y,t,0)$ can be represented in the form

$$g(x,y,t,0) \equiv g_1(x,y,t) + g_2(x,y,t), \tag{3.1}$$

where the functions g_1 and g_2 have the following properties

(i) g_1 is homogeneous in y of degree $r \geq 2$, i.e., $\forall \lambda \in R$ we have

$$g_1(x,\lambda y,t) = \lambda^r g_1(x,y,t) \;\; \forall (x,y,t) \in R^m \times R^n \times R. \tag{3.2}$$

(ii) g_2 satisfies

$$g_2(x,y,t) = O(|y|^{r+1}) \text{ as } y \to 0 \tag{3.3}$$

uniformly in $(x,t) \in R^m \times R$.

Hypothesis (H_1) implies that $y = h(x,t) \equiv 0$ is a solution of the degenerate equation (2.2) satisfying (2.1).

(H_2).

$$g_\varepsilon(x,0,t,0) \not\equiv 0 \quad \text{in} \quad R^m \times R.$$

By means of the scaling

$$\varepsilon = \mu^r, y = \mu z \tag{3.4}$$

we get from (3.1) - (3.3)

$$g(x, \mu z, t, \mu^r) = \mu^r \Big(g_1(x, z, t) + g_\varepsilon(x, 0, t, 0) + \mu \bar{g}(x, z, t, \mu)\Big), \tag{3.5}$$

where $\bar{g}(x, z, t, \mu)$ is smooth. Substituting (3.4) into (1.1) and taking into account (3.5) we obtain

$$\begin{aligned} \dot{x} &= f(x, \mu z, t, \mu^r), \\ \mu \, \dot{z} &= g_1(x, z, t) + g_\varepsilon(x, 0, t, 0) + \mu \bar{g}(x, z, t, \mu). \end{aligned} \tag{3.6}$$

The degenerate equation of (3.6) reads

$$g_1(x, z, t) + g_\varepsilon(x, 0, t, 0) = 0. \tag{3.7}$$

Concerning this equation we assume:

(H_3). There is a smooth function $\bar{h} : R^m \times R \to R$ such that

(i) $z = \bar{h}(x, t)$ is a root of (3.7).

(ii) The spectrum of the Jacobian matrix $\frac{\partial g_1}{\partial z}(x, \bar{h}(x, t), t)$ is uniformly separated from the imaginary axis for $(x, t) \in G \times R$.

Applying Proposition 1.1 to system (3.6) we get

Theorem 2.1. *Under the hypotheses* $(A_1), (H_1), (H_2)$, *and* (H_3) *there is a sufficiently small positive* $\varepsilon_2, \varepsilon_2 \leq \varepsilon_1$, *such that for* $\varepsilon \in I_2$ *system* (1.1) *has the integral manifold*

$$\mathcal{M}_\varepsilon := \{(x, y, t) \in R^{m+n+1} : y = \bar{\psi}(x, t, \varepsilon), (x, t, \varepsilon) \in G \times R \times I_2\}$$

with the asymptotic representation

$$y = \bar{\psi}(x, t, \varepsilon) = \varepsilon^{1/r} \bar{h}(x, t) + \varepsilon^{2/r} \bar{h}_1(x, t) + \ldots.$$

Remark 2.1. *From Theorem* 2.1 *it follows that the integral manifold* $\mathcal{M}_\varepsilon$ *converges to the root* $y = 0$ *of the degenerate equation* (2.2) *as* ε *tends to* 0. *If equation* (3.6) *has more than one simple solutions then several integral manifolds branch from the non-simple solution* $y = 0$.

To illustrate Theorem 2.1 we may consider Example 2.1 from section 2. In that case we have $g_1(x, y, t) \equiv -y^2$, $g_2(x, y, t) \equiv -y^3$, $g_\varepsilon(x, y, t) \equiv \phi^2(x, t)$ such that the degenerate system (3.6) reads

$$y^2 - \phi^2(x, t) = 0.$$

Here, we have two slow integral manifolds of system (2.4) branching from the multiple solution $y = 0$.

4 Homogeneous systems

Consider the autonomous system

$$
\begin{aligned}
\frac{dx}{dt} &= f(x,y,\varepsilon), \\
\varepsilon\,\frac{dy}{dt} &= g(x,y,\varepsilon)
\end{aligned}
\tag{4.1}
$$

under the assumption

(H). f and g are homogeneous polynomials in x and y of degree $r, r \geq 2$, with coefficients smoothly depending on ε.

It follows from hypothesis (H) that $\forall \lambda \in R$ and $\forall\ (x,y,\varepsilon) \in R^m \times R^n \times I_0$

$$
\begin{aligned}
f(\lambda x, \lambda y, \varepsilon) &= \lambda^r f(x,y,\varepsilon), \\
g(\lambda x, \lambda y, \varepsilon) &= \lambda^r g(x,y,\varepsilon).
\end{aligned}
\tag{4.2}
$$

Thus, $y=0$ is a non-simple root of the degenerate equation $0 = g(x,y,0)$. Furthermore, if we replace in (4.1) x by λx, y by λy and t by $\lambda^{1-r}t$, then system (4.1) is invariant under this transformation. Thus, if $(x(t), y(t))$ is a solution of (4.1) then $(\lambda x(\lambda^{r-1}t), \lambda y(\lambda^{r-1}t))$ is also a solution of (4.1). This property implies that any slow invariant manifold $y = \psi(x,\varepsilon)$ of (4.1) has the form

$$
y \;= L(\varepsilon)x, \tag{4.3}
$$

where $L(\varepsilon)$ is a $(n \times m)$–matrix. Thus, under our conditions, any slow invariant manifold of system (4.1) is a linear manifold.
Exploiting the invariance of $y = L(\varepsilon)x$ with respect to system (4.1) we get the relation

$$
\varepsilon\ L(\varepsilon)\ f(x, L(\varepsilon)x, \varepsilon) \;\equiv\; g(x, L(\varepsilon)x, \varepsilon) \quad \forall\ x \in R^m. \tag{4.4}
$$

We consider (4.4) as an equation to determine the entries of the matrix $L(\varepsilon)$. Since that equation can have more than one solution we call (4.4) as bifurcation equation. Thus, we have the following result:

Theorem 4.1. *Under the assumption* (H), *any slow invariant manifold of* (4.1) *is a linear manifold* (4.3) *where the matrix* $L(\varepsilon)$ *is determined by the bifurcation equation* (4.4).

To illustrate Theorem 4.1 we consider the following examples.

Example 4.1.

$$\frac{dx}{dt} = 3x^3, \qquad \varepsilon\frac{dy}{dt} = y^3 + \varepsilon^3 x^3. \tag{4.5}$$

The corresponding degenerate equation is $y^3 = 0$. According to Theorem 4.1, any slow invariant manifold of (4.5) has the form $y = L(\varepsilon)x$, where L is a scalar function. By (4.4) the corresponding bifurcation equation reads

$$L^3 - 3\varepsilon L + \varepsilon^3 = 0.$$

This equation possesses three solutions. For small ε we find by means of the method of undetermined gauges the representations

$$\begin{aligned}
L_1(\varepsilon) &= \frac{1}{3}\varepsilon^2 + \frac{1}{81}\varepsilon^5 + o(\varepsilon^5),\\
L_2(\varepsilon) &= -\varepsilon^{1/2}\sqrt{3} - \frac{1}{6}\varepsilon^2 + o(\varepsilon^2),\\
L_3(\varepsilon) &= \varepsilon^{1/2}\sqrt{3} - \frac{1}{6}\varepsilon^2 + o(\varepsilon^2).
\end{aligned}$$

Thus, the differential system (4.5) under consideration has three slow invariant manifolds $y = L_k(\varepsilon)x, k = 1, 2, 3$.

5 Quasi-polynomial degenerate equations

Consider the system

$$\begin{aligned}
\frac{dx}{dt} &= f(x, y, t, \varepsilon),\\
\varepsilon\,\frac{dy}{dt} &= g(x, y, t, \varepsilon)
\end{aligned} \tag{5.1}$$

with $x \in R^m, y \in R, t \in R, \varepsilon \in I_0$. In what follows we assume

(V_1). f and g satisfy assumption (A_1), additionally we suppose that g is a polynomial with respect to y and ε.

By assumption (V_1), g can be represented in the form

$$g(x, y, t, \varepsilon) \equiv \sum_{i=k_0}^{n_0} a_{0i}(x, t)y^i + \varepsilon \sum_{i=k_1}^{n_1} a_{1i}(x, t)y^i + \ldots + \varepsilon^m \sum_{i=k_m}^{n_m} a_{mi}(x, t)y^i. \tag{5.2}$$

Furthermore, we suppose

(V_2).

$$k_0 \geq 2, \quad a_{0k_0}(x, t) \neq 0 \qquad \forall\,(x, t) \in R^m \times R.$$

It follows from hypothesis (V_2) that $y = 0$ is a multiple root of the degenerate equation of (5.1)

$$g(x,y,t,0) \equiv \sum_{i=k_0}^{n_0} a_{0i}(x,t)y^i = 0. \tag{5.3}$$

As in the previous sections we scale the parameter ε and the variable y by

$$\varepsilon = \mu^q, y = \mu^p z \tag{5.4}$$

and look for conditions on the coefficients $a_{jk_j}(x,t)$ such that the equation

$$\varepsilon \frac{dy}{dt} = g(x,y,t,\varepsilon) \tag{5.5}$$

can be transformed into an equation of the type

$$\mu \frac{dz}{dt} = \tilde{g}(x,z,t,\mu) \tag{5.6}$$

whose corresponding degenerate equation

$$0 = \tilde{g}(x,z,t,0) \tag{5.7}$$

has a simple root $z = \tilde{h}(x,t)$ to which Proposition 1.1 can be applied.

Substituting (5.4) into the right hand side of (5.1), where we take into account (5.2), and rewrite the last equation of (5.1) in the form

$$\mu^{p+q} \frac{dz}{dt} = a_{0k_0}(x,t)\mu^{k_0 p}z^{k_0} + \sum_{j=1}^{m} a_{jk_j}(x,t)\mu^{jq+k_j p}z^{k_j} + h.o.t., \tag{5.8}$$

where h.o.t. means terms that are of higher order in μ compared with the leading order of the proceeding terms.

Let r be the leading order of the right hand side of (5.8). Then equation (5.8) can be reduced to the form (5.6) if it holds

$$p + q = r + 1. \tag{5.9}$$

To eliminate $z = 0$ as a multiple root of (5.7) we look for a scaling (5.4) such that the first term on the right hand side of (5.8) determines the leading order, that is

$$r = k_0 p, \tag{5.10}$$

and that there exist at least two terms of the leading order on the right hand side of (5.8). If we require that the $j-th$ term on the right hand side of (5.8) has the same order as the first term, then we get the relation

$$jq = (k_0 - k_j)p. \tag{5.11}$$

From (5.9) – (5.11) we obtain

$$p = \frac{j}{j(1-k_0)+k_0-k_j}, \quad 1 \le j \le m. \tag{5.12}$$

Since $k_j, j = 1, ..., m$, are non-negative integers, where $k_0 \ge 2$, and since p is a positive integer, it is easy to check that (5.12) defines only for $j=1$ and for $k_1 = 0$ an positive integer, namely $p=1$. Hence, we have $q \le k_0$. Thus, in order to be able to reduce (5.8) to an equation of type (5.6) we have to require $|a_{1,0}(x,t)| \ge a_1 > 0$ $\forall (x,t)$. This implies that g can be represented in the form

$$g(x,y,t,\varepsilon) = a_{0,k_0}(x,t)y^{k_0} + \varepsilon a_{1,0}(x,t) + \text{ h.o.t. in } y + \text{ h.o.t. in } \varepsilon. \tag{5.13}$$

But this representation is the same as treated in Theorem 2.1. Therefore, we have the following result

Theorem 5.1. *Suppose the hypotheses* (V_1) *and* (V_2) *to be valid. Then, under the additional condition* $|a_{1,0}(x,t)| \ge a_1 > 0 \quad \forall (x,t)$, *there exists in case of odd* k_0 *a slow integral manifold of system* (1.1), *in case of even* k_0 *we have additionally to assume* $a_{0,k_0}(x,t)a_{1,0}(x,t) < 0 \quad \forall (x,t)$.

Remark 5.1. *If* (5.1) *is an autonomous system with some special structure such that after some scaling of* y, ε *and* t *it can be represented in the form*

$$\begin{aligned} \frac{dx}{dt} &= f(x,z,\nu), \\ \nu^k \frac{dz}{dt} &= g(x,z,\nu), \end{aligned} \tag{5.14}$$

with $k \ge 2$, *then the existence of a slow integral manifold can be established under relaxed conditions.*

We illustrate Remark 5.1 by the following example.

Example 5.1. We consider the two-dimensional system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \varepsilon \frac{dy}{dt} &= \alpha(x)y^3 + \varepsilon\beta(x)y + \varepsilon^2\gamma(x), \end{aligned} \tag{5.15}$$

where all coefficients are sufficiently smooth, and α and β satisfy for all x the relation $\alpha(x)\beta(x) < 0$.
Using the scaling

$$y = \mu^p z,\ \varepsilon = \mu^q$$

we obtain from (5.15)

$$
\begin{aligned}
\frac{dx}{dt} &= \mu^p z, \\
q^{p+q}\,\frac{dz}{dt} &= \alpha(x)\mu^{3p}z^3 + \beta(x)\mu^{q+p}z + \gamma(x)\mu^{2q}.
\end{aligned} \tag{5.16}
$$

As it can be verified, only the choice $q = 2p$ provides two terms on the right hand side of (5.16) with leading order $3p$. Thus, we get

$$
\begin{aligned}
\frac{dx}{dt} &= \mu^p z, \\
\mu^{3p}\,\frac{dz}{dt} &= \mu^{3p}(\alpha(x)z^3 + \beta(x)z + \mu\gamma(x)).
\end{aligned} \tag{5.17}
$$

If we cancel the factor μ^{3p} in the last equation, we do not obtain a singularly perturbed equation. But after introducing the scaled time $\tau = t\mu^p$ and setting $p = 1$ we get

$$
\begin{aligned}
\frac{dx}{d\tau} &= z, \\
\mu\,\frac{dz}{d\tau} &= \alpha(x)z^3 + \beta(x)z + \mu\gamma(x).
\end{aligned} \tag{5.18}
$$

The degenerate equation of (5.18) is

$$\alpha(x)z^3 + \beta(x)z = 0$$

and has the three simple roots $z = 0,\ z = \pm\sqrt{-\frac{\beta(x)}{\alpha(x)}}$.

Thus, the original system (5.15) has three slow invariant manifolds

$$
\begin{aligned}
y &= O(\varepsilon), \\
y &= \pm\sqrt{-\frac{\beta(x)}{\alpha(x)}}\,\sqrt{\varepsilon} + O(\varepsilon).
\end{aligned}
$$

The case that the variable y in (5.1) is a n-vector can be treated similarly. As an example we consider the following "partial cheap control" problem.

Example 5.2.

We investigate the optimal control problem

$$
\begin{aligned}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= x_1 + x_2 + u_2
\end{aligned}
$$

with the cost functional

$$J = \frac{1}{2}\int_0^T [x_1^2(t) + \varepsilon x_2^2(t) + u_1^2(t) + \varepsilon^2 u_2^2(t)]dt \to \ \min \ .$$

This problem is called a partial cheap control problem because one of the control terms in the cost functional is multiplied by a small parameter [11]. It is well known that the optimal control is given by the formula

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1} K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where the matrix K is a nonnegative solution of matrix Riccati equation

$$\frac{dK}{dt} \quad = \quad -K\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} K + K \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}^{-1} K - \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$$

satisfying the condition $K(T) = 0$. Using the ansatz

$$K = \begin{pmatrix} k_1 & \varepsilon k_2 \\ \varepsilon k_2 & \varepsilon k_3 \end{pmatrix}$$

we obtain the differential system

$$\begin{aligned} \frac{dk_1}{dt} &= k_1^2 + k_2^2 - 2\varepsilon k_2 - 1, \\ \varepsilon \frac{dk_2}{dt} &= \varepsilon k_1 k_2 + k_2 k_3 - \varepsilon(k_2 + k_3), \\ \varepsilon \frac{k_3}{dt} &= k_3^2 + \varepsilon^2 k_2^2 - 2\varepsilon k_3 - \varepsilon. \end{aligned} \tag{5.19}$$

The corresponding degenerate system

$$k_2 k_3 = 0, \ k_3^2 = 0$$

has the solution $k_2 = k_3 = 0$, but this solution is not simple. In order to get simple roots we apply the scaling

$$k_3 = \mu \ \kappa_3, \ \varepsilon = \mu^2. \tag{5.20}$$

Substituting (5.20) into (5.19) we obtain

$$\begin{aligned} \frac{dk_1}{dt} &= k_1^2 + k_2^2 - 2\mu^2 k_2 - 1, \\ \mu \frac{dk_2}{dt} &= \mu k_1 k_2 + k_2 \kappa_3 - \mu(k_2 + \mu\kappa_3), \\ \mu \frac{d\kappa_3}{dt} &= \kappa_3^2 - 1 + \mu^2 k_2^2 - 2\mu\kappa_3. \end{aligned} \tag{5.21}$$

The corresponding degenerate system

$$k_2\kappa_3 = 0,\ \kappa_3^2 = 1$$

has the simple nonnegative solution $k_2 = 0, \kappa_3 = 1$. Applying Proposition 1.1 we get that the original system (5.19) has the invariant manifold

$$k_2 = O(\sqrt{\varepsilon}),\ k_3 = \sqrt{\varepsilon} + O(\varepsilon). \tag{5.22}$$

Substituting (5.22) into (5.19) leads to the initial value problem

$$\frac{dk_1}{dt} = k_1^2 - 1 + O(\varepsilon),\ k_1(T) = 0.$$

By this way, the elements of the matrix K can be determined approximately.

References

[1] E. BENOIT (Ed.), *Dynamic Bifurcations.* Lect. Notes in Math., 1493, 1991.

[2] E. BENOIT, J.L. CALLOT, F. DIENER, M. DIENER, *Chasse au canard*, Collect. Math., **32**, (1981), 37–119.

[3] N. FENICHEL, *Geometric singular perturbation theory for ordinary differential equations*, J. Diff. Eq., **31**, (1979), 53–98.

[4] V.M. GOL'DSHTEIN, V.A. SOBOLEV, *A Qualitative Analysis of Singularly Perturbed Systems* (in Russian) Novosibirsk. Institute of Mathematics, 1988.

[5] J. GRASMAN, *Asymptotic Methods for Relaxation Oscillations and Applications* Appl. Math. Sci., 63, Springer–Verlag, New-York, 1987.

[6] Z. GU, N.N. NEFEDOV, R.E. O'MALLEY, *On singular singularly perturbed initial values problems*, SIAM J. Appl. Math., **49**, (1989), 1–25.

[7] L.I. KONONENKO, V.A. SOBOLEV, *Asymptotic expansion of slow integral manifolds*, Sib. Math. J., **35**, (1994), 1119–1132.

[8] M.A. KRASNOSEL'SKII et al., *Approximate Solution of Operator Equations*, (in Russian), Moscow, Nauka, 1969.

[9] J.A. MURDOCK, *Perturbations. Theory and Methods*, John Wiley & Sons Inc., 1991.

[10] R.E. O'MALLEY, *Singular Perturbation Methods for Ordinary Differential Equations*, Appl. Math. Sci., 89, Springer–Verlag, New-York, 1991.

[11] R.E. O'MALLEY, A. JAMESON, *Singular perturbations and singular arcs*, I, II. Trans. Autom. Control AC-20, (1975) 218–226 ; AC-22, (1977) 328–337.

[12] K.R. SCHNEIDER, *Perturbed center submanifolds and application to Hopf bifurcation*, Proc. 9th Int. Conf. Nonlin. Oscill., vol. 2, 450–455, Kiev 1984.

[13] K.R. SCHNEIDER, T. WILHELM, *Model reduction by extended quasi-steady state assumption*, J. Math. Biology, **40**, (2000), 443–450.

[14] V.A. SOBOLEV, *Singular perturbations in linearly quadratic optimal control problems*, Autom. Rem. Control, **52**, (1991), 180–189.

[15] V.V. STRYGIN, V.A. SOBOLEV, *Separation of Motions by the Integral Manifolds Method*, (in Russian), Moscow, Nauka, 1988.

[16] A.B. VASILl'EVA, V.F. BUTUZOV, *Singularly Perturbed Equations in the Critical Case*, Tech. Report MRC–TSR 2039, University of Visconsin, Madison, WI, 1980.

[17] A.B. VASILl'EVA, V.F. BUTUZOV, L.V. KALACHEV, *The Boundary Function Method for Singular Perturbation Problems*, SIAM Studies in Appl. Math., 1995.