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Delayed exchange of stabilities in a class of singularly perturbed parabolic problems

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Abstract

We consider a class of singularly perturbed parabolic problems in case of exchange of stabilities, that is, the corresponding degenerate equation has two intersecting roots. By means of the technique of asymptotic lower and upper solutions we prove that the considered initial-boundary value problem has a unique solution exhibiting the phenomenon of delayed exchange of stabilities. Thus, the problem under consideration has a canard solution.

1 Introduction

Consider an autonomous dynamical system $\mathcal{S}$ depending on some parameter λ. The study of the influence of λ on the long-term behavior of the dynamical system $\mathcal{S}$ represents an essential part of the bifurcation theory. λ^* is called a bifurcation point for $\mathcal{S}$ concerning the region $\mathcal{G}$ in the phase space of $\mathcal{S}$ if in any neighborhood $\mathcal{N}$ of λ^* in the parameter space there exist two points λ_1 and λ_2 such that the phase portrait of $\mathcal{S}$ in $\mathcal{G}$ is not topologically equivalent for λ_1 and λ_2.
If we assume that λ is slowly changing in time then we arrive at the so-called dynamic bifurcation theory [1]. As an example we consider the scalar ordinary differential equation

$$\frac{dx}{dt} = f(x,\lambda), \tag{1.1}$$

where we assume $f(0,\lambda) \equiv 0$ for all λ. For definiteness we suppose that $\lambda^* = 0$ is an bifurcation point of (1.1), where $x = 0$ is stable (unstable) for $\lambda < 0$ ($\lambda > 0$). This assumption implies that the bifurcation point $\lambda = 0$ is generically related either to a transcritical bifurcation (see Fig 1.1) or to a pitchfork bifurcation (see Fig. 1.2).

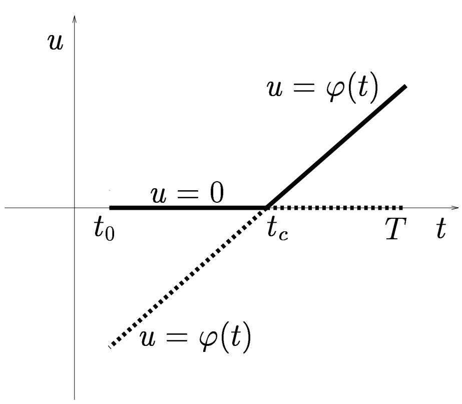


Fig. 1.1. Transcritical bifurcation

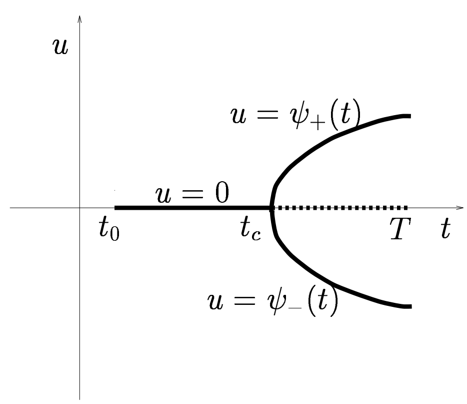


Fig. 1.2. Pitchfork bifurcation

Now we suppose that λ increases slowly with t. For simplicity we set

$$\lambda = \varepsilon t,$$

where ε is a small positive parameter. Introducing the slow time τ by $\tau = \varepsilon t$, the differential equation (1.1) takes the form

$$\varepsilon \frac{dx}{d\tau} = f(x, \tau), \tag{1.2}$$

that is, (1.2) is a singularly perturbed non-autonomous differential equation. Under our assumption, the solution set $f^{-1}(0)$ of the degenerate equation of (1.2)

$$0 = f(x, \tau) \tag{1.3}$$

consists in the $\tau - x$–plane of two curves intersecting for $\tau = 0$, as indicated in Fig. 1.1 and Fig. 1.2. All points of $f^{-1}(0)$ are equilibria of the associated equation to (1.2)

$$\frac{dx}{d\sigma} = f(x, \tau), \tag{1.4}$$

where τ has to be considered as a parameter. The curve $x = 0$ is an invariant manifold of (1.4) which is attracting for $\tau < 0$ and repelling for $\tau > 0$. We call this situation as exchange of stabilities (according to Lebovitz and Schaar [21]).
If we consider for equation (1.2) the initial value problem

$$x(\tau_0) = x_0, \quad \tau_0 < 0, \tag{1.5}$$

and if we assume that x_0 belongs to the region of attraction of the invariant manifold $x = 0$, then it follows from the standard theory of singularly perturbed systems (see, e.g., [32] - [34]) that the solution $x(\tau, \varepsilon)$ of the initial value problem (1.2),(1.5) exists at least for $\tau_0 < \tau < 0$.
For $\tau > 0$ there are the following possibilities for the behavior of the solution $x(\tau, \varepsilon)$:

(i). $x(\tau, \varepsilon)$ follows immediately the new stable branch emerging at $\tau = 0$.

(ii). $x(\tau, \varepsilon)$ follows for some O(1)-time interval (not depending on ε) the repelling part of the invariant manifold $x = 0$ and then jumps to the stable branch.

(iii). $x(\tau, \varepsilon)$ follows for some O(1)-time interval the repelling part of the invariant manifold $x = 0$ and then jumps away from this manifold (possibly blowing up).

The case (ii) is called delayed exchange of stabilities, case (iii) is called delayed loss of stability. The corresponding solutions are said to be canard solutions.
The case of exchange of stabilities for singularly perturbed ordinary differential equations has been treated by several authors using different methods (see, e.g., [11-22, 25, 26, 29-31]). In the papers [23, 24], the authors have applied the method of lower and upper solutions to derive conditions for an immediate and for a delayed

exchange of stabilities.

The same technique has been used in the papers [2, 5-10] to derive conditions for an immediate exchange of stabilities for different classes of partial differential equations. In the sequel, we will show that the same method can be used to establish the phenomenon of delayed exchange of stabilities for a class of singularly perturbed parabolic problems. Therefore, the technique of asymptotic differential inequalities provides an efficient way to establish canard solutions also for partial differential equations.

2 Formulation of the Problem

We consider the scalar singularly perturbed parabolic differential equation

$$\begin{aligned} &\varepsilon\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=g(u,x,t,\varepsilon),\\ &(x,t)\in Q=\{(x,t):0<x<1,\ 0<t\le T\}, \end{aligned} \tag{2.1}$$

where $\varepsilon>0$ is a small parameter, and study the initial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t,\varepsilon) &= \frac{\partial u}{\partial x}(1,t,\varepsilon)=0 \quad \text{for} \quad t\in(0,T],\\ u(x,0,\varepsilon) &= u^0(x) \quad \text{for} \quad x\in[0,1]. \end{aligned} \tag{2.2}$$

A root $u=\varphi(x,t)$ of the *degenerate equation*

$$g(u,x,t,0)=0 \tag{2.3}$$

represents a family of equilibria of the *associated equation* to (2.1)

$$\frac{du}{d\tau}=g(u,x,t,0), \tag{2.4}$$

where x and t have to be considered as parameters.

We recall that a root $u=\varphi(x,t)$ is referred to as *stable (unstable) in a region* G if $g_u(\varphi(x,t),x,t,0)<0\ (>0) \quad \forall(x,t)\in G$.

As in [10], we consider the case that the degenerate equation (2.3) has exactly two roots $u=\varphi_1(x,t)$ and $u=\varphi_2(x,t)$ intersecting in a curve such that an exchange of stabilities arises. In difference to [10], we treat in this chapter the phenomenon of delayed exchange of stabilities, that is, we derive conditions such that the solution $u(x,t,\varepsilon)$ of (2.1), (2.2) stays in the unstable region of $\varphi_1(x,t)$ arising for $t=t_c(x)$ for some O(1)-time interval near the unstable root $\varphi_1(x,t)$ and then either jumps to the stable root $\varphi_2(x,t)$ (delayed exchange of stability) or escapes from the unstable root (delayed loss of stability).

3 Assumptions

Let I_u be an open bounded interval containing the origin, let $I_{\varepsilon_0} = \{\varepsilon : 0 < \varepsilon < \varepsilon_0 \ll 1\}$, $D = Q \times I_u \times I_{\varepsilon_0}$. Let the functions g and u^0 satisfies the smoothness condition

(A$_0$). $g \in C^2(\overline{D}, R)$, $u^0 \in C^2([0,1], I_u)$.

With respect to the roots of the degenerate equation we suppose

(A$_1$). *The degenerate equation* (2.3) *has in* $\overline{I_u} \times \overline{Q}$ *exactly two roots:* $u \equiv 0$ *and* $u = \varphi(x,t)$, $\varphi(x,t) \in C^2(\overline{Q}, I_u)$. *The roots* $u \equiv 0$ *and* $u = \varphi(x,t)$ *intersect in some smooth curve* $\mathcal{K}$ *with the representation* $t = t_c(x) \in C^1([0,1],(0,T))$. *For definiteness we suppose*

$$\varphi(x,t) < 0 \quad \textit{for} \quad 0 \le t < t_c(x),\ 0 \le x \le 1,$$

$$\varphi(x,t) > 0 \quad \textit{for} \quad t_c(x) < t \le T,\ 0 \le x \le 1$$

(see Fig. 3.1).

From assumption (A$_1$) it follows

$$\varphi(x, t_c(x)) \equiv 0 \quad \text{for} \quad 0 \le x \le 1.$$

Concerning the stability of these roots we assume

(A$_2$).

$$\begin{array}{lll} g_u(0,x,t,0) < 0,\ g_u(\varphi(x,t),x,t,0) > 0 & \textit{for} & 0 \le t < t_c(x),\ 0 \le x \le 1, \\ g_u(0,x,t,0) > 0,\ g_u(\varphi(x,t),x,t,0) < 0 & \textit{for} & t_c(x) < t \le T,\ 0 \le x \le 1. \end{array}$$

Hypothesis (A$_2$) implies that the roots $u \equiv 0$ and $u = \varphi(x,t)$ of the degenerate equation (2.3) considered as families of equilibria of the associated equation (2.4) exchange their stabilities at the curve $\mathcal{K}$.

Furthermore, we suppose

(A$_3$). $g(0,x,t,\varepsilon) \equiv 0$ for $(x,t,\varepsilon) \in \overline{Q} \times \overline{I}_0$.

Assumption (A$_3$) is motivated by applications in reaction kinetics where we are looking for nonnegative solutions.

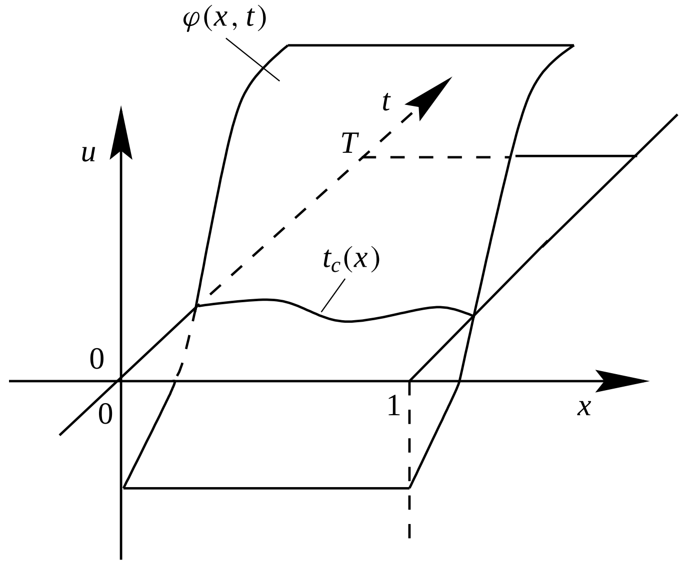


Fig. 3.1. Intersection of $u \equiv 0$ and $u = \varphi(x,t)$ in the curve $t = t_c(x)$.

Now we introduce the functions

$$g_u^{\min}(t) = \min_{x\in[0,1]} g_u(0,x,t,0), \quad g_u^{\max}(t) = \max_{x\in[0,1]} g_u(0,x,t,0) \quad \text{for} \quad 0 \le t \le T.$$

Obviously, we have for $(x,t) \in \overline{Q}$

$$g_u^{\min}(t) \le g_u(0,x,t,0) \le g_u^{\max}(t). \tag{3.5}$$

We need also the primitives of these functions:

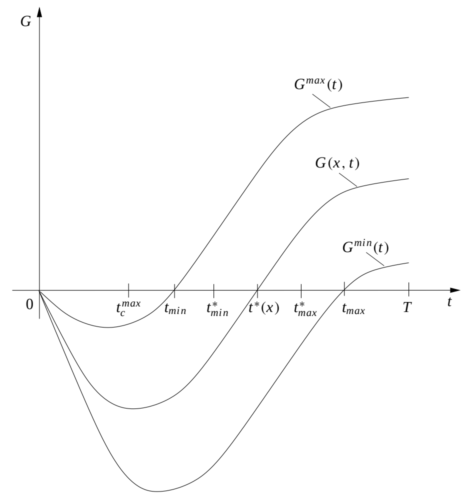
$$G^{\min}(t) = \int_0^t g_u^{\min}(s)ds,\ G(x,t) = \int_0^t g_u(0,x,s,0)ds,\ G^{\max}(t) = \int_0^t g_u^{\max}(s)ds.$$

By (3.5) the following inequalities hold for $(x,t) \in \overline{Q}$ (see Fig. 3.2)

$$G^{\min}(t) \ \le\ G(x,t) \ \le\ G^{\max}(t).$$

From assumption (A_2) we get that the equation $G^{\min}(t) = 0$ has at most one solution in the interval $(0,T)$. We assume that this solution exists.

(A_4). *The equation* $G^{\min}(t) = 0$ *has a solution* $t = t_{max}$ *in* $(0,T)$.



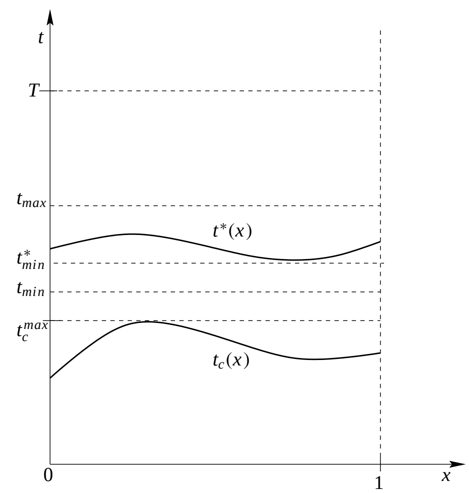


Fig. 3.2. Inclusion of $G(x,t)$ by G^{min} and G^{max} for given x

Fig. 3.3. Location of $t_c(x)$ and $t^*(x)$

From hypotheses (A_2) and (A_4) it follows that the equation $G^{\max}(t) = 0$ has a unique solution $t = t_{\min}$ in $(0, T)$, and that for each $x \in [0,1]$ the equation $G(x,t) = 0$ has a unique solution $t = t^*(x)$ in $(0,T)$ (see Fig. 3.2).

Obviously, for $x \in [0,1]$ we have

$$t_{\min} \le t^*(x) \le t_{\max}.$$

Finally we assume that the following conditions hold.

(A_5).

$$t_c^{max} = \max_{x\in[0,1]} t_c(x) < t_{min} \quad \text{(see Fig. 3.3).}$$

(A_6). *There is a positive number c_0 such that $(-c_0, c_0) \subset I_u$ where I_u is the interval from assumption (A_0), and*

$$g(u,x,t,\varepsilon) \le g_u(0,x,t,\varepsilon)u \quad \text{for } |u| \le c_0,\ x \in [0,1],\ 0 \le t \le t^*(x),\ \varepsilon \in I_{\varepsilon_0}.$$

We note that assumption (A_6) is satisfied if the second derivative $g_{uu}(0,x,t,\varepsilon)$ is negative for all (x,t,ε) under consideration.

(A_7). *$u^0(x)$ lies in the basin of attraction of the stable root $u \equiv 0$.*

4 Main results

Our main result is concerned with the estimate of the delay time in cases of delayed exchange or delayed loss of stabilities.

Theorem 4.1 *Assume the hypotheses* (A_1)–(A_7) *to be valid and* $u^0(x) > 0$. *Then, for sufficiently small* ε , *there exists a unique solution* $u(x,t,\varepsilon)$ *of* (2.1), (2.2) *which is positive and satisfies*

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = 0 \quad \textit{for} \quad (x,t) \in [0,1] \times (0, t_{min}), \tag{4.6}$$

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = \varphi(x,t) \quad \textit{for} \quad (x,t) \in [0,1] \times (t_{\max}, T]. \tag{4.7}$$

In case $u^0(x) < 0$, *the unique solution* $u(x,t,\varepsilon)$ *of* (2.1), (2.2) *is negative and and satisfies*

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = 0 \quad \textit{for} \quad (x,t) \in [0,1] \times (0, t_{\min}),$$

for $t > t_{\min}$ *the solution escapes from* $u \equiv 0$ *at some time* t_{esc} *(escaping time) which can be estimated by* $t_{esc} \leq t_{max}$.

Remark 4.2 *From Theorem* 4.1 *it follows that the solution* $u(x,t,\varepsilon)$ *stays near the unstable root* $u = 0$ *of the degenerate equation at least for the time interval* $(t_c(x), t_{min})$,

Remark 4.3 *In case* $u^0(x) < 0$, *the solution* $u(x,t,\varepsilon)$ *may not exist for all* t *in* $[0,T]$.

Proof. We apply the method of differential inequalities. To this end, we recall the definition of ordered lower and upper solutions.

Definition 4.4 *Let* $\underline{U}(x,t,\varepsilon)$ *and* $\overline{U}(x,t,\varepsilon)$ *be functions continuously mapping* $\overline{Q} \times I_{\varepsilon_1}$ $(I_{\varepsilon_1} \subset I_{\varepsilon_0})$ *into* R, *twice continuously differentiable with respect to* x *and continuously differentiable in* t. *Then* $\underline{U}$ *and* $\overline{U}$ *are called ordered lower and upper solutions of* (2.1), (2.2) *for* $\varepsilon \in I_{\varepsilon_1}$, *if they satisfy for* $\varepsilon \in I_{\varepsilon_1}$

1°. $$\underline{U}(x,t,\varepsilon) \leq \overline{U}(x,t,\varepsilon) \quad \textit{for} \quad (x,t) \in \overline{Q},$$

2°. $$\varepsilon\left(\frac{\partial \underline{U}}{\partial t} - \frac{\partial^2 \underline{U}}{\partial x^2}\right) - g(\underline{U},x,t,\varepsilon) \leq 0 \leq \varepsilon\left(\frac{\partial \overline{U}}{\partial t} - \frac{\partial^2 \overline{U}}{\partial x^2}\right) - g(\overline{U},x,t,\varepsilon)$$
$$\textit{for } (x,t) \in Q,$$

3°. $$\frac{\partial \underline{U}}{\partial x}(0,t,\varepsilon) \geq 0 \geq \frac{\partial \overline{U}}{\partial x}(0,t,\varepsilon), \qquad \frac{\partial \underline{U}}{\partial x}(1,t,\varepsilon) \leq 0 \leq \frac{\partial \overline{U}}{\partial x}(1,t,\varepsilon)$$
$$\textit{for } t \in [0,T],$$

4°. $$\underline{U}(x,0,\varepsilon) \leq u^0(x) \leq \overline{U}(x,0,\varepsilon) \quad \textit{for} \quad x \in [0,1].$$

It is known (see, e.g., [28]) that the existence of ordered lower and upper solutions of (2.1), (2.2) implies the existence of a unique solution $u(x,t,\varepsilon)$ of (2.1), (2.2) satisfying

$$\underline{U}(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \overline{U}(x,t,\varepsilon).$$

Without loss of generality we may assume that $|u^0(x)| \le c_0$ for $0 \le x \le 1$, where c_0 is the constant from hypothesis (A_6).

Let $\nu > 0$ be any number independent of ε such that $t_{min} - \nu > t_c^{max}$ (see Fig. 3.3). It follows from assumption (A_4) that to given ν there is a constant $\delta_a(\nu) > 0$ such that the function $a(t,\nu)$ defined by

$$a(t,\nu) = g_u^{\max}(t) + \delta_a(\nu) \quad \text{for} \quad t \in [0,T] \tag{4.8}$$

satisfies

$$\int_0^{t_{\min}-\nu/2} a(t,\nu)dt = 0. \tag{4.9}$$

In what follows we consider the case $u^0(x) > 0$.

In order to prove relation (4.6) we construct an upper solution $\overline{U}(x,t,\varepsilon)$ to (2.1), (2.2) for $(x,t) \in [0,1] \times [0, t_{\min} - \nu]$ in the form

$$\overline{U}(x,t,\varepsilon) = c_0 \exp\left\{ \frac{1}{\varepsilon} \int_0^t a(s,\nu) ds \right\}.$$

By (4.9) it holds

$$\int_0^t a(\tau,\nu)d\tau < 0 \quad \text{for} \quad t \in (0, t_{min} - \nu].$$

Therefore, we have

$$\lim_{\varepsilon \to 0} \overline{U}(x,t,\varepsilon) = 0 \quad \text{for} \quad (x,t) \in [0,1] \times (0, t_{min} - \nu]. \tag{4.10}$$

Since $\overline{U}(x,t,\varepsilon)$ does not depend on x and $|u^0(x)| < c_0$, the inequalities 3° and 4° for $\overline{U}$ in Definition 4.1 are satisfied trivially.
Next we verify that $\overline{U}(x,t,\varepsilon)$ satisfies the second inequality in 2°. It is easy to check that $\overline{U}(x,t,\varepsilon)$ obeys

$$\varepsilon \left(\frac{\partial \overline{U}}{\partial t} - \frac{\partial^2 \overline{U}}{\partial x^2} \right) = a(t,\nu)\, \overline{U}. \tag{4.11}$$

From (4.11) we get

$$\begin{aligned} &\varepsilon \left(\frac{\partial \overline{U}}{\partial t} - \frac{\partial^2 \overline{U}}{\partial x^2} \right) - g(\overline{U},x,t,\varepsilon) \\ &= g_u(0,x,t,\varepsilon)\overline{U} - g(\overline{U},x,t,\varepsilon) + (a(t,\nu) - g_u(0,x,t,\varepsilon))\overline{U}. \end{aligned} \tag{4.12}$$

By assumption (A_6) we have for $(x,t) \in [0,1] \times [0, t_{min} - \nu]$ and $\varepsilon \in I_{\varepsilon_0}$

$$g_u(0,x,t,\varepsilon)\, \overline{U} - g(\overline{U},x,t,\varepsilon) \ge 0. \tag{4.13}$$

From (3.5) and (4.8) we obtain for sufficiently small ε ($\varepsilon \in I_{\varepsilon_1} \subset I_{\varepsilon_0}$)

$$a(t,\nu) - g_u(0,x,t,\varepsilon) = g_u^{max}(t) + \delta_a(\nu) - g_u(0,x,t,\varepsilon) \geq 0. \tag{4.14}$$

From (4.12)–(4.14) it follows that $\overline{U}(x,t,\varepsilon)$ satisfies the second inequality from condition 2° of Definition 4.4 and, therefore, is for $\varepsilon \in I_{\varepsilon_1}$ an upper solution of (2.1), (2.2) in $[0,1] \times [0, t_{min} - \nu]$.
Since $u^0(x) > 0$, assumption (A_3) implies that $\underline{U} \equiv 0$ is for $\varepsilon \in I_{\varepsilon_1}$ a trivial lower solution of (2.1), (2.2). Hence, for $\varepsilon \in I_{\varepsilon_1}$, (2.1), (2.2) has a unique solution $u(x,t,\varepsilon)$ for $t \in [0, t_{\min} - \nu]$ satisfying by (4.10) the limit relation

$$\lim_{\varepsilon \to 0} u(x,t,\varepsilon) = 0 \quad \text{for} \quad (x,t) \in [0,1] \times (0, t_{\min} - \nu]. \tag{4.15}$$

Since ν is any small positive number, relation (4.15) is valid for $0 < t < t_{\min}$. Thus, relation (4.6) has been proven. Note that $t_{\min}$ is a lower bound for the escaping time t_{esc} of the solution $u(x,t,\varepsilon)$ from the unstable root $u = 0$, that is, $t_{\min} - t_c(x)$ yields a lower bound for the delay of exchange of stabilities.

Now we prove relation (4.7). Let $u^1(x,\varepsilon) = u(x, t_{min} - \nu, \varepsilon)$. Obviously we have $0 < u^1(x,\varepsilon) = o(\varepsilon)$.

We consider equation (2.1) for $t \in (t_{min} - \nu, T]$ with the initial-boundary conditions

$$\begin{aligned} u(x, t_{min} - \nu, \varepsilon) &= u^1(x,\varepsilon) \quad \text{for} \quad x \in [0,1], \\ \frac{\partial u}{\partial x}(0,t,\varepsilon) &= \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0 \quad \text{for} \quad t \in [t_{min} - \nu, T]. \end{aligned} \tag{4.16}$$

We note that $\underline{U} \equiv 0$ is a lower solution to that problem. In order to prove the existence of a solution of (2.1), (4.16), we construct an upper solution in $[0,1] \times [t_{min} - \nu, T]$ in the form

$$\overline{U}(x,t,\varepsilon) \equiv \varphi(x,t) + \sqrt{\varepsilon}(\gamma + z(x,\varepsilon)), \tag{4.17}$$

where z is defined by

$$z(x,\varepsilon) = \exp\Big(-\frac{\kappa x}{\sqrt{\varepsilon}}\Big) + \exp\Big(-\frac{\kappa(1-x)}{\sqrt{\varepsilon}}\Big), \tag{4.18}$$

the positive constants γ and κ will be chosen in an appropriate way later.

By (4.17) and assumption (A_3) we have

$$\begin{aligned} &\varepsilon\Big(\frac{\partial \overline{U}}{\partial t} - \frac{\partial^2 \overline{U}}{\partial x^2}\Big) - g(\overline{U},x,t,\varepsilon) = \varepsilon\left(\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2}\right) \\ &-\sqrt{\varepsilon}\kappa^2\, z(x,\varepsilon) - g(\varphi(x,t) + \sqrt{\varepsilon}(\gamma + z(x,\varepsilon)), x, t, \varepsilon) \\ &\geq -2\sqrt{\varepsilon}\kappa^2 - \sqrt{\varepsilon} g_u(\varphi(x,t),x,t,0)\ (\gamma + z(x,\varepsilon)) + o(\sqrt{\varepsilon}). \end{aligned}$$

By hypothesis (A_2) there is a positive constant σ such that

$$g_u(\varphi(x,t),x,t,0) \leq -\sigma < 0 \quad \text{for} \quad (x,t) \in [0,1] \times [t_{\min} - \nu, T].$$

Hence, we can conclude that for sufficiently large γ and for sufficiently small ε $\overline{U}(x,t,\varepsilon)$ satisfies the second inequality from condition 2° of Definition 4.4.

Now we check that $\overline{U}(x,t,\varepsilon)$ satisfies the inequalities in 3°. From (4.17) and (4.18) we obtain

$$\begin{aligned}\frac{\partial \overline{U}}{\partial x}(0,t,\varepsilon) &= \frac{\partial \varphi}{\partial x}(0,t) - \kappa\left(1 - \exp\left(-\frac{\kappa}{\sqrt{\varepsilon}}\right)\right),\\ \frac{\partial \overline{U}}{\partial x}(1,t,\varepsilon) &= \frac{\partial \varphi}{\partial x}(1,t) + \kappa\left(1 - \exp\left(-\frac{\kappa}{\sqrt{\varepsilon}}\right)\right).\end{aligned}$$

If we choose κ sufficiently large, then the inequalities for $\overline{U}$ in condition 3° of Definition 4.4 are satisfied. Therefore, $\overline{U}(x,t,\varepsilon)$ defined by (4.17) is for sufficiently small ε an upper solution of the problem (2.1), (4.16) in $[0,1]\times[t_{\min}-\nu,T]$ and we can conclude that problem (2.1), (2.2) has a unique solution $u(x,t,\varepsilon)$.

To obtain an upper estimate for the escaping time t_{esc} of the solution $u(x,t,\varepsilon)$ from the unstable root $u=0$, we construct for sufficiently small ε a nontrivial lower solution of (2.1), (2.2) in $[0,1]\times[0,t_{\max}+\nu]$, where $\nu>0$ is any number independent of ε satisfying $t_{max}+\nu<T$.

By hypothesis (A_4), there is to any given small $\nu>0$ a constant $\delta_b(\nu)>0$ such that the function $b(t,\nu)$ defined by

$$b(t,\nu) = g_u^{\min}(t) - \delta_b(\nu) \quad \text{for} \quad 0 \le t \le t_{max}+\nu \tag{4.19}$$

satisfies

$$\int_0^{t_{\max}+\nu} b(s,\nu)ds = 0. \tag{4.20}$$

Now we construct a lower solution in the form

$$\underline{U}(x,t,\varepsilon) = \eta \exp\left\{\frac{1}{\varepsilon}\int_0^t b(s,\nu)ds\right\}, \tag{4.21}$$

where $0<\eta<\min(\min_{0\le x\le 1}\varphi(x,t_{\max}), \min_{0\le x\le 1}u^0(x))$. The constant η will be more specified later.

It is obvious that $\underline{U}(x,t,\varepsilon)$ obeys conditions 3° and 4° of Definition 4.4 and satisfies the equation

$$\varepsilon\Big(\frac{\partial \underline{U}}{\partial t} - \frac{\partial^2 \underline{U}}{\partial x^2}\Big) = b(t,\nu)\underline{U}.$$

Using this equation we have

$$\begin{aligned}&\varepsilon\left(\frac{\partial \underline{U}}{\partial t} - \frac{\partial^2 \underline{U}}{\partial x^2}\right) - g(\underline{U},x,t,\varepsilon)\\ &= (b(t,\nu) - g_u(0,x,t,\varepsilon))\underline{U} + g_u(0,x,t,\varepsilon)\underline{U} - g(\underline{U},x,t,\varepsilon).\end{aligned} \tag{4.22}$$

From (3.5) and (4.19) it follows that for sufficiently small ε

$$(b(t,\nu) - g_u(0,x,t,\varepsilon))\underline{U} \le -\delta_b(\nu)\underline{U}/2. \tag{4.23}$$

Since $g(0,x,t,\varepsilon)=0$ (see assumptions (A_3)) we have

$$g(u,x,t,\varepsilon)=g_u(0,x,t,\varepsilon)u+\frac{1}{2}g_{uu}(u_*,x,t,\varepsilon)u^2.$$

and, therefore, for $|u|\le c_0$ (c_0 is the constant from assumption (A_6)) the inequality holds

$$g_u(0,x,t,\varepsilon)-g(u,x,t,\varepsilon)\le\kappa_1 u^2, \tag{4.24}$$

where κ_1 is some positive number. Thus, it follows from (4.22)–(4.24)

$$\varepsilon\left(\frac{\partial\underline{U}}{\partial t}-\frac{\partial^2\underline{U}}{\partial x^2}\right)-g(\underline{U},x,t,\varepsilon)\le\underline{U}(-\delta_b(\nu)/2+\kappa_1\underline{U}). \tag{4.25}$$

If we choose η such that $\eta\le\delta_b(\nu)/(2\kappa_1)$, then we get from (4.25)

$$\varepsilon\left(\frac{\partial\underline{U}}{\partial t}-\frac{\partial^2\underline{U}}{\partial x^2}\right)-g(\underline{U},x,t,\varepsilon)\le 0.$$

Thus, $\underline{U}(x,t,\varepsilon)$ defined by (4.21) is a nontrivial lower solution of (2.1), (2.2) for $t\in[0,t_{\max}+\nu]$ and, consequently, it holds $u(x,t,\varepsilon\ge\underline{U}(x,t,\varepsilon)$ for this time interval. By (4.21) and (4.20) we have $\underline{U}(x,t_{\max}+\nu,\varepsilon)=\eta$, thus it holds

$$u(x,t_{\max}+\nu,\varepsilon)\ge\eta \tag{4.26}$$

From this inequality we get the validity of the relation

$$\lim_{\varepsilon\to 0}u(x,t,\varepsilon)=\varphi(x,t)\quad\text{for}\quad(x,t)\in[0,1]\times(t_{\max}+\nu,T]. \tag{4.27}$$

This follows from the fact that for $t\ge t_{\max}+\nu$ the root $u=\varphi(x,t)$ of degenerate equation is stable, and that the positive function $u(x,t_{\max}+\nu,\varepsilon)$ lies in the basin of attraction of this root.

As ν does not depend on ε and can be chosen arbitrarily small,

relation (4.27) is valid for all t from the interval $(t_{\max},T]$. This completes the proof of relation (4.7), and consequently, the proof of Theorem 6.1 is completed for the case $u^0(x)>0$.

In the case $u^0(x)<0$ the proof is based on the same scheme with the following changes: $\overline{U}\equiv 0$ is a trivial upper solution

of problem (2.1), (2.2) for $0\le t\le T$,

$$\underline{U}(x,t,\varepsilon)=-c_0\exp\left\{\frac{1}{\varepsilon}\int_0^t a(s,\nu)ds\right\}$$

is a lower solution for $0\le t\le t_{\min}-\nu$, and

$$\overline{U}(x,t,\varepsilon)=-\eta\exp\left\{\frac{1}{\varepsilon}\int_0^t b(s,\nu)ds\right\}$$

is an nontrivial upper solution for $0\le t\le t_{\max}+\nu$. □

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