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für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Longtime behavior of the traveling-wave model for semiconductor lasers

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submitted: 19th June 2002

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No. 743
Berlin 2002

2000 *Mathematics Subject Classification.* 78A60, 37L10, 35P10.

Key words and phrases. laser dynamics, invariant manifold theory, hyperbolic systems of partial differential equations.

Humboldt University of Berlin, Germany, and Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany. This work was supported by the Sonderforschungsbereich 555 "Komplexe Nichtlineare Prozesse" of the Deutsche Forschungsgemeinschaft.

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Abstract

The traveling-wave model is a popular tool for investigating longitudinal dynamical effects in semiconductor lasers, e.g., sensitivity to delayed optical feedback. This model consists of a hyperbolic linear system of partial differential equations (PDEs) with one spatial dimension which is nonlinearly coupled with a slow subsystem of ordinary differential equations (ODEs). Firstly, we prove the basic statements about the existence of solutions of the initial-boundary-value problem and their smooth dependence on initial values and parameters. Hence, the model constitutes a smooth infinite-dimensional dynamical system. Then, we exploit this fact and the particular slow-fast structure of the system to construct a low-dimensional attracting invariant manifold for certain parameter constellations. The flow on this invariant manifold is described by a system of ODEs which is accessible to classical bifurcation theory and numerical tools like, e.g., AUTO.

1 Introduction

Due to their inherent speed, semiconductor lasers are of great interest for modern optical data transmission and telecommunication technology. Typically, these applications utilize the laser in a non-stationary mode, e.g., to produce high-frequency oscillations or pulse trains. Multi-section lasers allow to cultivate and control these nonlinear effects by designing the longitudinal structure of the device; see, e.g., [19], [29]. This paper focusses on the basic properties of the *traveling-wave model* with gain dispersion describing longitudinal effects in semiconductor lasers as introduced in [2], [5], [15], [25].

Structure of the traveling-wave model This model describes the dynamics of the laser by the interaction of two physical variables: the complex electro-magnetic field E, roughly speaking the light amplitude, and the effective carrier density n within the active zone of the device. The system has the structure

$$\begin{aligned} \dot{E} &= H(n)E \\ \dot{n} &= \varepsilon(I - n - g(n)[E, E]) \end{aligned} \tag{1}$$

if we neglect noise and high-power effects. Here, E is complex and spatially resolved in the longitudinal direction of the laser, and n describes the spatially section-wise averaged carrier density. Hence, system (1) couples a linear system of partial differential equations (PDEs) for E with a system of ordinary differential equations (ODEs) for n. Furthermore, the variables E and n act on different time-scales implying a slow-fast structure of (1). This fact is expressed by the presence of the small parameter ε which is the ratio between the averaged lifetime of a photon and the averaged lifetime of a carrier. Finally, g is a hermitian form implying a symmetry of (1) with respect to rotation of the complex variable E. Consequently, (1) admits solutions of the type ($E(t) = E_0 e^{i\omega t}$, $n(t) = \text{const}$), i.e., rotating waves or *stationary lasing states*.

A remark about the relation of the traveling-wave model to other models concerned with semiconductor lasers: A very popular model for the simulation of delayed optical feedback effects in lasers is the *Lang-Kobayashi model*; see [28] and references therein. The Lang-Kobayashi system is a system of delay-differential equations which has also the structure (1). It turns out that all results of this paper extend to the Lang-Kobayashi system in an obvious manner (see §6).

Non-technical overview In §2 we introduce the system under consideration in detail and specify all conditions on the parameters assumed implicitly in the following sections.

In §3 we prove the basic statements about existence, boundedness and regularity of solutions of the initial-boundary value problem corresponding to (1) on arbitrarily large non-negative time intervals $[0, T]$. Furthermore, we prove that the solutions depend smoothly (C^∞) on initial values and all parameters. Hence, (1) constitutes a smooth infinite-dimensional dynamical system. In this section, we consider also inhomogeneous boundary conditions in (1) modeling optical injection into the laser. We permit the inhomogeneity to be discontinuous in time to allow modeling of rectangular-shape signals. This potential discontinuity prevents homogenization of the boundary conditions [18]. However, the introduction of the inhomogeneity as an infinite-dimensional variable (and part of E) transforms the system back into structure (1). Then, all statements of this section are a direct consequence of the theory of strongly continuous semigroups and an a-priori estimate exploiting the small dissipation in (1).

In §4 we investigate the spectral properties of the operator H for fixed n extending results of [21] and [20]. Although the cases of periodic boundary conditions and Dirichlet type boundary conditions have to be treated separately, the fundamental result is the same for both cases: The growth properties of the strongly continuous semigroup generated by H are determined by the dominating eigenvalues of H which are isolated and of finite algebraic multiplicity.

Section 5 is concerned with the construction of a finite-dimensional attracting invariant manifold utilizing the slow-fast structure of (1) and the results of §3 and §4. The result follows from the general theorems of [7], [8], [9] if we introduce appropriate coordinates and cut-off modifications.

Finally, in §6 we explain how the system of ODEs obtained in §5 can be made accessible to standard numerical bifurcation analysis tools like AUTO [11], and conclude that the model reduction theorem of §5 is also valid for the Lang-Kobayashi system.

The appendix explains the physical interpretation of the quantities appearing in the traveling-wave model, and lists possible ranges of the parameters.

2 The traveling-wave model with nonlinear gain dispersion

A well known model describing the longitudinal effects in narrow laser diodes is the *traveling wave model*, a hyperbolic system of PDEs coupled with a system of ODEs [2], [15], [25].

This model has been extended by adding polarization equations to include nonlinear gain dispersion effects [1], [2], [5], [23]. In this section we introduce the corresponding system of differential equations and specify the fundamental assumptions on its coefficients.

Let $\psi(t,z) \in \mathbb{C}^2$ describe the complex amplitude of the optical field split into a forward and a backward traveling wave. Let $p(t,z) \in \mathbb{C}^2$ be the corresponding nonlinear polarization (see appendix). Both quantities depend on time and the one-dimensional spatial variable $z \in [0,L]$ (the longitudinal direction within the laser). The vector $n(t) \in \mathbb{R}^m$ represents the spatially averaged carrier densities within the individual sections of the laser (see Fig. 1). The initial-boundary value problem

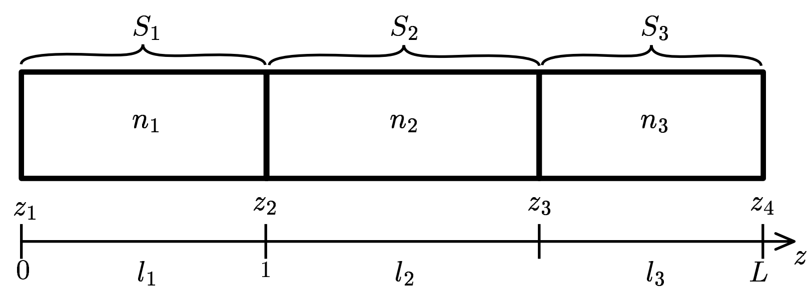

Figure 1: Typical geometric configuration of the domain in a laser with 3 sections.

reads as follows:

$$
\begin{aligned}
\partial_t \psi(t,z) &= \sigma \partial_z \psi(t,z) + \beta(n(t),z)\psi(t,z) - i\kappa(z)\sigma_c \psi(t,z) + \rho(n(t),z)p(t,z) \quad (2)\\
\partial_t p(t,z) &= (i\Omega_r(n(t),z) - \Gamma(n(t),z)) \cdot p(t,z) + \Gamma(n(t),z)\psi(t,z) \quad (3)\\
\frac{d}{dt} n_k(t) &= I_k - \frac{n_k(t)}{\tau_k} - \frac{P}{l_k}\left(G_k(n_k(t)) - \rho_k(n_k(t))\right) \int_{S_k} \psi(t,z)^* \psi(t,z) dz \\
&\quad - \frac{P}{l_k} \rho_k(n_k(t)) \operatorname{Re}\left(\int_{S_k} \psi(t,z)^* p(t,z) dz \right) \text{ for } k = 1 \ldots m \quad (4)
\end{aligned}
$$

accompanied by the inhomogeneous boundary conditions

$$\psi_1(t,0) = r_0\psi_2(t,0) + \alpha(t),\ \psi_2(t,L) = r_L\psi_1(t,L) \tag{5}$$

and the initial conditions

$$\psi(0,z) = \psi^0(z),\ p(0,z) = p^0(z),\ n(0) = n^0. \tag{6}$$

The Hermitian transpose of a $\mathbb{C}^2$-vector ψ is denoted by ψ^* in (4). We will define the appropriate function spaces and discuss the possible solution concepts in §3. The quantities and coefficients appearing above have the following sense (see also Tab. 1 and Fig. 1):

L is the length of the laser. The laser is subdivided into m sections S_k of length l_k with starting points z_k for $k = 1\dots m$. We scale the system such that $l_1 = 1$ and denote $z_{m+1} = L$. Thus, $S_k = [z_k, z_{k+1}]$. All coefficients are supposed to be spatially constant in each section, i.e. if $z \in S_k$, $\kappa(z) = \kappa_k$, $\Gamma(n,z) = \Gamma_k(n_k)$, $\beta(n,z) = \beta_k(n_k)$, $\rho(n,z) = \rho_k(n_k)$. The matrices σ and σ_c are defined by

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The model for $\beta(n,z) = \beta_k(n_k) \in \mathbb{C}$ $(z \in S_k)$ we use throughout the work reads

$$\beta_k(\nu) = d_k + (1 + i\alpha_{H,k})G_k(\nu) - \rho_k(\nu)$$

where $d_k \in \mathbb{C}$, $\alpha_{H,k} \in \mathbb{R}$, and $\operatorname{Re} d_k < 0$. A section S_k is either *passive*, then the functions G_k and ρ_k are identically zero, or S_k is *active*. In this case, $G_k : (\underline{n}, \infty) \to \mathbb{R}$ is a smooth[1] strictly monotone increasing function satisfying $G_k(1) = 0$, $G_k'(1) > 0$. Its limits are $\lim_{\nu\searrow\underline{n}} G_k(\nu) = -\infty$, $\lim_{\nu\to\infty} G_k(\nu) = \infty$ where $\underline{n} \le 0$. Typical models for G_k in active sections are

$$\begin{aligned} G_k(\nu) &= \tilde{g}_k \log\nu && (\underline{n} = 0) \text{ or} \\ G_k(\nu) &= \tilde{g}_k \cdot (\nu - 1) && (\underline{n} = -\infty). \end{aligned}$$

If $G_k \not\equiv 0$, the function $\rho(n,z) = \rho_k(n_k)$ is bounded for $n_k < 1$. Moreover, we suppose $\rho_k, \Omega_{r,k}, \Gamma_k : (\underline{n}, \infty) \to \mathbb{R}$ to be smooth and Lipschitz continuous, and $\Gamma_k(\nu) > 1$.

The coefficients r_0 and r_L in (5) are complex with modulus less than 1. The inhomogeneity $\alpha(t)$ is bounded but may be discontinuous in time. The variables and coefficients, their physical meanings, and their typical ranges are shown in Tab. 1.

Finally, we introduce the hermitian form

$$g_k(\nu)\left[\begin{pmatrix}\psi\\p\end{pmatrix}, \begin{pmatrix}\varphi\\q\end{pmatrix}\right] = \frac{1}{l_k}\int\limits_{S_k} (\psi^*(z), p^*(z)) \begin{pmatrix} G_k(\nu)-\rho_k(\nu) & \frac{1}{2}\rho_k(\nu) \\ \frac{1}{2}\rho_k(\nu) & 0 \end{pmatrix} \begin{pmatrix}\varphi(z)\\q(z)\end{pmatrix} dz \tag{7}$$

[1]The notation *smooth* refers to C^∞ throughout this paper.

and the notations

$$
\begin{aligned}
\|\psi\|_k^2 &= \int_{S_k} \psi^*(z)\psi(z)dz \\
f_k(\nu,(\psi,p)) &= I_k - \frac{\nu}{\tau_k} - P g_k(\nu)\left[\begin{pmatrix}\psi\\ p\end{pmatrix},\begin{pmatrix}\psi\\ p\end{pmatrix}\right]
\end{aligned} \tag{8}
$$

for $\nu \in [\underline{n},\infty)$ and $\psi, p \in \mathbb{L}^2([0,L];\mathbb{C}^2)$. Using these notations, (4) reads

$$
\frac{d}{dt} n_k = f_k(n_k,(\psi,p)) \text{ for } k = 1\ldots m. \tag{9}
$$

3 Existence and Uniqueness of Classical and Mild Solutions

In this section, we treat the inhomogeneous initial-boundary value problem (2)-(5) as an autonomous nonlinear evolution equation

$$
\frac{d}{dt}u(t) = Au(t) + g(u(t)), \quad u(0) = u_0 \tag{10}
$$

where $u(t)$ is an element of a Hilbert space V, A is a generator of a C_0 semigroup $S(t)$, and $g : U \subseteq V \to V$ is smooth and locally Lipschitz continuous in an open set $U \subseteq V$. The inhomogeneity in (5) is included in (10) as a component of u.

3.1 Notation

The Hilbert space V is defined by

$$
V := \mathbb{L}^2([0,L];\mathbb{C}^4) \times \mathbb{R}^m \times \mathbb{L}^2_\eta([0,\infty);\mathbb{C}) \tag{11}
$$

where $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$ is the space of weighted square integrable functions. The scalar product of $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$ is defined by

$$
(v,w)_\eta := \mathrm{Re}\int_0^\infty \bar{v}(x)\cdot w(x)(1+x^2)^\eta dx.
$$

We choose $\eta < -1/2$ such that the space $\mathbb{L}^\infty([0,\infty);\mathbb{C})$ is continuously embedded in $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$. The complex plane is treated as two-dimensional real plane in the definition of the vector space V such that the standard $\mathbb{L}^2$ scalar product $(\cdot,\cdot)_V$ of V is differentiable. The corresponding components of $v \in V$ are denoted by

$$
v = (\psi_1, \psi_2, p_1, p_2, n, a).
$$

The spatial variable in ψ and p is denoted by $z \in [0,L]$ whereas the spatial variable in a is denoted by $x \in [0,\infty)$. The Hilbert space $\mathbb{H}^1_\eta([0,\infty);\mathbb{C})$ equipped with the scalar product

$$
(v,w)_{1,\eta} := (v,w)_\eta + (\partial_x v, \partial_x w)_\eta
$$

is densely and continuously embedded in $\mathbb{L}^2_\eta([0,\infty);\mathbb{C})$. Moreover, its elements are continuous [24]. Consequently, the Hilbert spaces

$$\begin{aligned} W &:= \mathbb{H}^1([0,L];\mathbb{C}^2) \times \mathbb{L}^2([0,L];\mathbb{C}^2) \times \mathbb{R}^m \times \mathbb{H}^1_\eta([0,\infty);\mathbb{C}), \text{ and} \\ W_{\mathrm{BC}} &:= \{(\psi,p,n,a) \in W : \psi_1(0) = r_0\psi_2(0) + a(0), \psi_2(L) = r_L\psi_1(L)\} \end{aligned}$$

are densely and continuously embedded in V. The linear functionals $\psi_1(0) - r_0\psi_2(0) - a(0)$ and $\psi_2(L) - r_L\psi_1(L)$ are continuous from $W \to \mathbb{R}$. We define the linear operator $A : W_{\mathrm{BC}} \to V$ by

$$A(\psi_1,\psi_2,p,n,a) := (-\partial_z\psi_1, \partial_z\psi_2, 0, 0, \partial_x a).$$

The definition of A and W_{BC} treat the inhomogeneity α in the boundary condition (5) as the boundary value at 0 of the variable a. We define the open set $U \subseteq V$ by

$$U := \{(\psi,p,n,a) \in V : n_k > \underline{n} \text{ for } k = 1\dots m\},$$

and the nonlinear function $g : U \to V$ by

$$g(\psi,p,n,a) = \begin{pmatrix} \beta(n)\psi - i\kappa\sigma_c\psi + \rho(n)p \\ (i\Omega_r(n) - \Gamma(n))p + \Gamma(n)\psi \\ (f_k(n_k,(\psi,p)))_{k=1}^m \\ 0 \end{pmatrix}. \tag{12}$$

The corresponding coefficients of (2)–(4) define the smooth maps $\beta : (\underline{n},\infty)^m \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$ and $\rho, \Omega_r, \Gamma : \mathbb{R}^m \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$. The function g is continuously differentiable to any order with respect to all arguments and its Frechet derivative is bounded in any closed bounded ball $B \subset U$ [12].

According to the theory of C_0 semigroups, there are two solution concepts [17]:

Definition 1 *Let $T > 0$. A solution $u : [0,T] \to V$ is a classical solution of* (10) *if $u(t) \in W_{\mathrm{BC}} \cap U$ for all $t \in [0,T]$, $u \in C^1([0,T];V)$, $u(0) = u_0$, and equation* (10) *is valid in V for all $t \in (0,T)$.*

The inhomogeneous initial-boundary value problem (2)-(6) and the autonomous evolution system (10) are equivalent in the following sense: Suppose $\alpha \in \mathbb{H}^1([0,T);\mathbb{C})$ in (5). Let $u = (\psi,p,n,a)$ be a classical solution of (10). Then, u satisfies (2)-(3), and (6) in $\mathbb{L}^2$ and (4), (5) for each $t \in [0,T]$ if and only if $a^0|_{[0,T]} = \alpha$. On the other hand, assume that (ψ,p,n) satisfies (2)-(3), and (6) in $\mathbb{L}^2$ and (4), (5) for each $t \in [0,T]$. Then, we can choose a $a^0 \in \mathbb{H}^1_\eta([0,\infty);\mathbb{C})$ such that $a^0|_{[0,T]} = \alpha$ and obtain that $u(t) = (\psi(t), p(t), n(t), a^0(t+\cdot))$ is a classical solution of (10) in $[0,T]$.

Definition 2 *Let $T > 0$, A be a generator of a C_0 semigroup $S(t)$ of bounded operators in V. A solution $u : [0,T] \to V$ is a mild solution of* (10) *if $u(t) \in U$ for all $t \in [0,T]$, and $u(t)$ satisfies the variation of constants formula in V*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)g(u(s))ds. \tag{13}$$

We prove in Lemma 3 that A generates a C_0 semigroup in V. Mild solutions of (10) are a reasonable generalization of the classical solution concept of (2)-(5) to boundary conditions including discontinuous inputs $\alpha \in \mathbb{L}^2_\eta([0,\infty);\mathbb{C})$.

3.2 Global Existence and Uniqueness of Solutions for the Truncated Problem

In order to prove uniqueness and global existence of solutions of (10), we apply the theory of strongly continuous semigroups [17].

Lemma 3 *$A : W_{\mathrm{BC}} \subset V \to V$ generates a C_0 semigroup $S(t)$ of bounded operators in V.*

Proof: We specify the C_0 semigroup $S(t)$ explicitly. Denote the components of $S(t)(\psi_1^0, \psi_2^0, p^0, n^0, a^0)$ by $(\psi_1(t,z), \psi_2(t,z), p(t,z), n(t), a(t,x))$ for $z \in [0,L]$, $x \in [0,\infty)$, and let $t \le L$.

$$\begin{aligned}
\psi_1(t,z) &= \begin{cases} \psi_1^0(z-t) & \text{for} \quad z > t \\ r_0\psi_2^0(t-z) + a^0(t-z) & \text{for} \quad z \le t \end{cases} \\
\psi_2(t,z) &= \begin{cases} \psi_2^0(z+t) & \text{for} \quad z < L-t \\ r_L\psi_1^0(2L-t-z) & \text{for} \quad z \ge L-t \end{cases} \\
p(t,z) &= 0 \\
n(t) &= 0 \\
a(t,x) &= a^0(x+t).
\end{aligned}$$

For $t > L$ we define inductively $S(t)u = S(L)S(t-L)u$. This procedure defines a semigroup of bounded operators in V since

$$\|\psi_1(t,\cdot)\|^2 + \|\psi_2(t,\cdot)\|^2 + \|a(t,\cdot)\|^2 \le 2(1+t^2)^{-\eta}\left(\|\psi_1^0\| + \|\psi_2^0\| + \|a^0\|\right)$$

for $t \le L$. The strong continuity of S is a direct consequence of the continuity in the mean in $\mathbb{L}^2$. It remains to be shown that S is generated by A.

Let $u = (\psi_1^0, \psi_2^0, p^0, n^0, a^0)$ satisfy $\lim_{t\to 0}\frac{1}{t}(S(t)u - u) \in V$, define $\varphi_t(z) := \frac{1}{t}(\psi_1(t,z) - \psi_1^0(z))$, $\varphi_0 = \lim_{t\to 0}\varphi_t$, and let $\delta > 0$ be small. Firstly, we prove that $u \in W_{\mathrm{BC}}$. φ_t coincides with the difference quotient $\frac{1}{t}(\psi_1^0(z-t) - \psi_1^0(z))$ for $t < \delta$ and $z \in [\delta, L]$. Thus, $\partial_z\psi_1^0 \in \mathbb{L}^2([\delta,L];\mathbb{C})$ exists. Furthermore, $\varphi_t(\cdot + t) \to \varphi_0$ in $\mathbb{L}^2([0,L-\delta];\mathbb{C})$. Since $\varphi_t(\cdot+t) = \frac{1}{t}(\psi_1^0(z) - \psi_1^0(z+t))$, $\partial_z\psi_1^0$ exists also in $\mathbb{L}^2([0,L-\delta];\mathbb{C})$. Consequently $\psi_1^0 \in \mathbb{H}^1([0,L];\mathbb{C})$. The same argument holds for $\psi_2^0 \in \mathbb{H}^1([0,L];\mathbb{C})$ and for $a^0 \in \mathbb{H}^1_\eta([0,\infty);\mathbb{C})$.

In order to verify that u satisfies the boundary conditions we write

$$\varphi_t(z) = \begin{cases} z \in [t,L]: & -\frac{1}{t}\int_{z-t}^{z}\partial_z\psi_1^0(\zeta)d\zeta \\ z \in [0,t]: & \frac{1}{t}\left(r_0\int_0^{t-z}\partial_z\psi_2^0(\zeta) + \partial_z a^0(\zeta)d\zeta - \int_0^z \partial_z\psi_1^0(\zeta)d\zeta\right) + \\ & +\frac{1}{t}\left(r_0\psi_2^0(0) + a^0(0) - \psi_1^0(0)\right) \end{cases} \tag{14}$$

Consequently, the limit φ_0 is in $\mathbb{L}^2([0,L];\mathbb{C})$ if and only if $r_0\psi_2^0(0)+a^0(0)-\psi_1^0(0)=0$. The same argument using $\frac{1}{t}(\psi_2(t,z)-\psi_2^0(z))$ implies $r_L\psi_1^0(L)-\psi_2^0(L)=0$.

Finally, we prove that for any $u\in W_{\mathrm{BC}}$ we have $\lim_{t\to 0}\frac{1}{t}(S(t)u-u)=Au$. Using the notation φ_t introduced above, we have $\int_0^t|\varphi_t(z)|^2dz\to_{t\to 0}0$ due to (14). Hence, $\varphi_t\to_{t\to 0}-\partial_z\psi_1^0$ on $[0,L]$. Again, we can use the same arguments to obtain the limits $\partial_z\psi_2^0$ and $\partial_x a^0$. □

The operators $S(t)$ have a uniform upper bound

$$\|S(t)\|\leq Ce^{\gamma t} \tag{15}$$

within finite intervals $[0,T]$. In order to apply the results of the C_0 semigroup theory [17], we truncate the nonlinearity g smoothly: For any bounded ball $B\subset U$ which is closed w.r.t. V, we choose $g_B:V\to V$ such that g_B is smooth, globally Lipschitz continuous, and $g_B(u)=g(u)$ for all $u\in B$. This is possible because the Frechet derivative of g is bounded in B and the scalar product in V is differentiable with respect to its arguments. We call

$$\frac{d}{dt}u(t)=Au(t)+g_B(u(t)),\quad u(0)=u_0 \tag{16}$$

the truncated problem (10). The following Lemma 4 is a consequence of the results in [17].

Lemma 4 (global existence for the truncated problem)
The truncated problem (16) *has a unique global mild solution $u(t)$ for any $u_0\in V$. If $u_0\in W_{\mathrm{BC}}$, $u(t)$ is a classical solution of* (16).

Corollary 5 (local existence) *Let $u_0\in U$. There exists a $t_{\mathrm{loc}}>0$ such that the evolution problem* (10) *has a unique mild solution $u(t)$ on the interval $[0,t_{\mathrm{loc}}]$. If $u_0\in W_{\mathrm{BC}}\cap U$, $u(t)$ is a classical solution of* (10) *in $[0,t_{\mathrm{loc}}]$.*

3.3 A-priori Estimate — Existence of Semiflow

In order to state the result of Lemma 4 for (10), we need the following a-priori estimate for the solutions of the truncated problem (16).

Lemma 6 *Let $T>0$, $u_0\in W_{\mathrm{BC}}\cap U$. If $\underline{n}>-\infty$, we suppose $I_k\tau_k>\underline{n}$ for all $k=1\ldots m$. There exists a closed bounded ball B such that $B\subset U$ and the solution $u(t)$ of the B-truncated problem* (16) *starting at u_0 stays in B for all $t\in[0,T]$.*

Proof: Let $u_0=(\psi^0,p^0,n^0,a^0)\in W_{\mathrm{BC}}\cap U$.

Preliminary consideration
Let $n_*\in(\underline{n},n_k^0)$ be such that $G_k(n_*)-\rho_k(n_*)<0$ for all $k=1\ldots m$ where $G_k\not\equiv 0$. Let $t_1>0$ be such that the solution of the non-truncated problem (10) $u(t)=$

$(\psi(t), p(t), n(t), a(t))$ exists in $[0, t_1]$, and $n_k(t) \geq n_*$ for all $k = 1 \ldots m$ and $t \in [0, t_1]$. We define the function

$$h(t) := \frac{P}{2}\|\psi(t)\|^2 + \sum_{k=1}^{m} l_k(n_k(t) - n_*).$$

Because of the structure of the nonlinearity g (linear in (ψ, p)), $u(t)$ is classical in $[0, t_1]$. Hence, $h(t)$ is differentiable and the differential equations (2) and (4) imply

$$\frac{d}{dt}h(t) \leq J - \sum_{k=1}^{m}\left(\frac{l_k}{\tau_k}n_k + P\,\mathrm{Re}\,d_k\|\psi\|_k^2\right) \leq J - \tilde{\tau}^{-1}n_* - \gamma h(t),$$

where

$$\begin{aligned}
\gamma &:= \min\left\{\tau_k^{-1}, -\frac{\mathrm{Re}\,d_k}{2} : k = 1 \ldots m\right\} > 0 \\
J &:= \sum_{k=1}^{m} l_k I_k + \sup\left\{|r_0 z + a^0(x)|^2 - |z|^2 : z \in \mathbb{C}, x \in [0, T]\right\} < \infty \\
\tilde{\tau}^{-1} &:= \sum_{k=1}^{m} l_k \tau_k^{-1}.
\end{aligned}$$

Consequently, $h(t) \leq \max\{h(0), \gamma^{-1}J - \gamma^{-1}\tilde{\tau}^{-1}n_*\}$ for all $t \in [0, t_1]$. Since $h(0) = \frac{P}{2}\|\psi^0\|^2 + \sum_{k=1}^{m} l_k n_k^0 - Ln_*$, we obtain the estimate

$$0 \leq h(t) \leq M - \xi \cdot n_* \tag{17}$$

where

$$M := \max\left\{\gamma^{-1}J, \frac{P}{2}\|\psi^0\|^2 + \sum_{k=1}^{m} l_k n_k^0\right\}, \text{ and } \qquad \xi := \min\left\{\gamma^{-1}\tilde{\tau}^{-1}, L\right\}$$

do not depend on n_*. Since $n_k(t) \geq n_*$ in $[0, t_1]$ for all $k = 1 \ldots m$, the estimate (17) for $h(t)$ and the differential equation (3) for p imply bounds for ψ, p and n in $[0, t_1]$:

$$\begin{aligned}
\|\psi(t)\|^2 &\leq S(n_*)^2 := 2P^{-1}(M - \xi \cdot n_*) \\
\|p(t)\| &\leq \|p^0\| + S(n_*) \\
n_k &\in \left[n_*, n_* + (2l_k)^{-1}PS(n_*)^2\right].
\end{aligned} \tag{18}$$

Hence, $f_k(n_*, (\psi(t), p(t)))$ is greater than

$$I_k - \frac{n_*}{\tau_k} - \frac{P}{l_k}\max_{\Theta \in \mathbb{R}}\left[(G_k(n_*) - \rho_k(n_*))\Theta^2 + |\rho_k(n_*)|(\|p^0\| + S(n_*))\Theta\right] \tag{19}$$

for all $k = 1 \ldots m$ and $t \in [0, t_1]$.

Construction of B

Since $G_k(\nu) \to_{\nu \to \underline{n}} -\infty$ and $\rho_k(\nu)$ bounded for $\nu \to \underline{n}$, or $G_k = \rho_k = 0$, we can find a n_* such that the expression (19) is greater than 0 for all $k = 1 \dots m$. Then, we choose B such that $(\psi, p, n, a) \in B$ if ψ, p and n satisfy (18) for this n_* and $a = a^0(t + \cdot)$ for $t \in [0, T]$.

Indirect proof of invariance of B
Assume that the solution $v(t) = (\psi(t), p(t), n(t), a(t))$ of the B-truncated problem leaves B. The preliminary consideration and the construction of B imply that there exists a t_1 such that $u(t)$ exists in $[0, t_1]$, and, for one $k \in \{1 \dots m\}$, $n_k(t_1) = n_*$ and $n_k(t) > n_*$ for all $t \in [0, t_1]$. Consequently, $\dot{n}_k(t_1) = f_k(n_k(t_1), (\psi(t_1), p(t_1))) < 0$. However, this contradicts to the construction of n_* such that (19) is greater than 0. $\square$

Moreover, a solution $u(t)$ starting at $u_0 \in W_{\mathrm{BC}} \cap U$ and staying in a bounded closed ball $B \subset U$ in $[0, T]$ is a classical solution in the whole interval $[0, T]$ because of the structure of the nonlinearity g.

The bounds (18) do not depend on the complete W_{BC}-norm of u_0 but on its V-norm and the $\mathbb{L}^\infty$-norm of $a^0|_{[0,T]}$. Hence, we can state the global existence theorem also for mild solutions:

Theorem 7 (global existence and uniqueness)
Let $T > 0$, $u_0 = (\psi^0, p^0, n^0, a^0) \in U$ and $\|a^0|_{[0,T]}\|_\infty < \infty$. If $\underline{n} > -\infty$, let $I_k \tau_k > \underline{n}$ for all $k = 1 \dots m$. There exists a unique mild solution $u(t)$ of (10) in $[0, T]$. Furthermore, if $u_0 \in W_{\mathrm{BC}} \cap U$, $u(t)$ is a classical solution of (10).

The bounds (18) do not depend on T explicitly, either. Thus, the solutions are globally bounded if a^0 is bounded:

Corollary 8 (global boundedness)
Let $u_0 = (\psi^0, p^0, n^0, a^0) \in U$ and $\|a^0\|_\infty < \infty$. There exists a constant C such that $\|u(t)\|_V \leq C$.

The next corollary is an immediate consequence of the general theory of C_0 semigroups [17]:

Corollary 9 (continuous dependence on initial values)
Let $T > 0$, $u_j^0 = (\psi^j, p^j, n^j, a^j) \in U$, $\|a^j|_{[0,T]}\|_\infty < \infty$ for $j = 1, 2$. There exists a constant C depending on $\|u_1^0\|_V$, $\|u_2^0\|_V$, $\|a^1|_{[0,T]}\|_\infty$, $\|a^2|_{[0,T]}\|_\infty$, and T such that $\|u_1(t) - u_2(t)\|_V \leq C \cdot \|u_1^0 - u_2^0\|_V$.

Therefore, the nonlinear equation defines a semiflow $S(t; u_0)$ for $t > 0$. S is even continuously differentiable with respect to its second argument in the following sense:

Corollary 10 (continuous differentiability of the semiflow)
Let $T > 0$, $u^0 = (\psi^0, p^0, n^0, a^0) \in U$, $\|a^0|_{[0,T]}\|_\infty < \infty$. Let

$$\mathcal{M}_{C,\varepsilon} := \left\{ (\psi, p, n, a) \in V : \|a|_{[0,T]}\|_\infty \leq C, \|(\psi, p, n, a)\|_V \leq \varepsilon \right\}.$$

Then, $\mathcal{M}_{C,\varepsilon}$ is a closed subset of V, and

$$S(t; u_0 + h) - S(t; u_0) = S_L(t, 0)h + o_C(\|h\|_V)$$

for $h \in \mathcal{M}_{C,\varepsilon}$ for arbitrary C and sufficiently small ε. $S_L(t,s)$ is the evolution operator of the linear evolution equation in V

$$\frac{d}{dt}v(t) = Av(t) + \frac{\partial}{\partial u}g(u(t))v(t),\ v(s) = v_0.$$

This follows from the C_0 semigroup theory [17] since we can choose a common ball B for all $u_0 + h$, $h \in \mathcal{M}_{C,\varepsilon}$. This result extends to C^k smoothness ($k > 1$) since the nonlinearity g is C^∞ with respect to all arguments.

The continuous dependence of the solution on all parameters within a bounded parameter region is also a direct consequence of the C_0 semigroup theory. In order to obtain a uniform a-priori estimate, we impose additional restrictions on the parameters: $1 - |r_0| > c > 0$, $I_k\tau_k - \underline{n} > c > 0$, $\operatorname{Re} d_k < -c < 0$ for $k = 1 \ldots m$, and for active sections ($g_k \neq 0$), $g_k > c > 0$, for a uniform constant c.

4 Asymptotic behavior of the linear part

4.1 Introduction of a small parameter

We restrict ourselves to the autonomous system (2)-(4) in the following. The boundary conditions are

$$\psi_1(t, 0) = r_0\psi_2(t, 0), \quad \psi_2(t, L) = r_L\psi_1(t, L) \tag{20}$$

in the autonomous case.

We reformulate (2)-(4) to exploit its particular structure. The space dependent subsystem is linear in ψ and p:

$$\partial_t \begin{pmatrix}\psi\\ p\end{pmatrix} = H(n)\begin{pmatrix}\psi\\ p\end{pmatrix}. \tag{21}$$

The linear operator

$$H(n) = \begin{pmatrix}\sigma\partial_z + \beta(n) - i\kappa\sigma_c & \rho(n)\\ \Gamma(n) & i\Omega_r(n) - \Gamma(n)\end{pmatrix} \tag{22}$$

acts from

$$Y := \{(\psi, p) \in \mathbb{H}^1([0, L]; \mathbb{C}^2) \times \mathbb{L}^2([0, L]; \mathbb{C}^2) : \psi \text{ satisfying } (20)\}$$

into $X = \mathbb{L}^2([0, L]; \mathbb{C}^4)$. $H(n)$ generates a C_0 semigroup $T_n(t)$ acting in X. Its coefficients κ, and for each $n \in \mathbb{R}^m$ $\beta(n)$, $\Omega_r(n)$, $\Gamma(n)$ and $\rho(n)$ are linear operators in $\mathbb{L}^2([0,L];\mathbb{C}^2)$ defined by the corresponding coefficients in (2), (3). The maps $\beta, \rho, \Gamma, \Omega_r : \mathbb{R}^m \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$ are smooth.

We observe that I_k and τ_k^{-1} in (8) are approximately two orders of magnitude smaller than 1 (see Tab. 1). Hence, we can introduce a small parameter ε and set $P = \varepsilon$ in (4) such that (9) reads:

$$\frac{d}{dt} n_k = f_k(n_k, E) = \varepsilon (F_k(n_k) - g_k(n_k)[E,E]) \tag{23}$$

for $E \in X$ where the coefficients in $F_k(n_k) = \varepsilon^{-1}(I_k - n_k \tau_k^{-1})$ are of order 1. Although ε is not directly accessible, we treat it as a parameter and consider the limit $\varepsilon \to 0$ while keeping F_k fixed. At $\varepsilon = 0$, the carrier density n is constant. It enters the linear subsystem (21) as a parameter. We will investigate the longtime behavior of this linear equation throughout the rest of this section. For brevity, we drop the argument n.

4.2 Spectral Properties of $H(n)$

In this section, we investigate the spectrum of the operator $H(n)$ treating n as a parameter.

Define the set of complex "resonance frequencies"

$$\mathcal{W} = \{c \in \mathbb{C} : c = i\Omega_{r,k} - \Gamma_k \text{ for at least one } k \in \{1 \dots m\}\} \subset \mathbb{C}$$

and $\chi : \mathbb{C} \setminus \mathcal{W} \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$ (see appendix for explanation and [5], [23] for details) by

$$\chi(\lambda) = \frac{\rho\Gamma}{\lambda - i\Omega_r + \Gamma} \in \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2)) \text{ for each } \lambda \in \mathbb{C} \setminus \mathcal{W}.$$

For $\lambda \in \mathbb{C} \setminus \mathcal{W}$, the following relation follows from (22): λ is in the resolvent set of H if and only if the boundary value problem

$$(\sigma \partial_z + \beta - i\kappa\sigma_c + \chi(\lambda) - \lambda)\varphi = 0 \quad \text{with b. c. (20)} \tag{24}$$

has only the trivial solution $\varphi = 0$ in $\mathbb{H}^1([0,L];\mathbb{C}^2)$. The transfer matrix corresponding to (24) is

$$T_k(z,\lambda) = \frac{e^{-\gamma_k z}}{2\gamma_k} \begin{pmatrix} \gamma_k + \mu_k + e^{2\gamma_k z}(\gamma_k - \mu_k) & i\kappa_k (1 - e^{2\gamma_k z}) \\ -i\kappa_k (1 - e^{2\gamma_k z}) & \gamma_k - \mu_k + e^{2\gamma_k z}(\gamma_k + \mu_k) \end{pmatrix} \tag{25}$$

for $z \in S_k$ where $\mu_k = \lambda - \chi_k(\lambda) - \beta_k$ and $\gamma_k = \sqrt{\mu_k^2 + \kappa_k^2}$ [2], [20]. The right-hand-side of (25) does not depend on the branch of the square root in γ_k since the expression is even with respect to γ_k. Denote the overall transfer matrix of (24) by $T(z_1, z_2; \lambda)$ for $z_1, z_2 \in [0,L]$. The function

$$h(\lambda) = \begin{pmatrix} r_L, & -1 \end{pmatrix} T(L,0;\lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} = \begin{pmatrix} r_L & -1 \end{pmatrix} \prod_{k=m}^{1} T_k(l_k;\lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} \tag{26}$$

defined in $\mathbb{C} \setminus \mathcal{W}$ is the characteristic function of H: Its roots are the eigenvalues of H and $\mathcal{R} := \{\lambda \in \mathbb{C} \setminus \mathcal{W} : h(\lambda) \neq 0\}$ is the resolvent set. Consequently, all $\lambda \in \mathbb{C} \setminus \mathcal{W}$ are either eigenvalues of H or in $\mathcal{R}$, i. e., there is no essential (continuous or residual) spectrum in $\mathbb{C} \setminus \mathcal{W}$. We note that $\max \operatorname{Re} \mathcal{W} \ll -1$ for physically sensible parameter constellations. Let $\zeta \in \mathbb{L}^2([0,L];\mathbb{C}^2)$. We denote the solution φ of the inhomogeneous boundary value problem

$$(\sigma\partial_z + \beta - i\kappa\sigma_c + \chi(\lambda) - \lambda)\varphi + \zeta = 0 \quad \text{with b. c. (20)} \tag{27}$$

by $R_1(\lambda)\zeta$. An expression for $R_1(\lambda)\zeta$ is

$$\begin{aligned}[R_1(\lambda)\zeta](z) = \; & \frac{1}{h(\lambda)} T(z,0;\lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} (r_L, -1) \int_0^L T(L,s;\lambda)\sigma\zeta(s)\, ds - \\ & \int_0^z T(z,s;\lambda)\sigma\zeta(s)\, ds. \end{aligned} \tag{28}$$

Hence, $R_1(\lambda) : \mathbb{L}^2([0,L];\mathbb{C}^2) \to \mathbb{L}^2([0,L];\mathbb{C}^2)$ is compact for $\lambda \in \mathcal{R}$. The resolvent of H, $R(\lambda) := (\lambda Id - H)^{-1} : \mathbb{L}^2([0,L];\mathbb{C}^4) \to \mathbb{L}^2([0,L];\mathbb{C}^4)$ for $\lambda \in \mathcal{R}$ is

$$R(\lambda)\begin{pmatrix} \psi \\ p \end{pmatrix} = \begin{pmatrix} R_1(\lambda)\left(\psi + \frac{\rho p}{\lambda - i\Omega_r + \Gamma}\right) \\ \frac{1}{\lambda - i\Omega_r + \Gamma}\left[p + \Gamma R_1(\lambda)\left(\psi + \frac{\rho p}{\lambda - i\Omega_r + \Gamma}\right)\right] \end{pmatrix} \tag{29}$$

which is a compact perturbation of the operator $(\psi, p) \to (0, (\lambda - i\Omega_r + \Gamma)^{-1}p)$.

The following lemma provides an approximate upper bound for the real parts of the eigenvalues.

Lemma 11 *Let $\lambda \in \mathbb{C} \setminus \mathcal{W}$ be in the point spectrum of H. Then, λ is geometrically simple, and its real part satisfies the estimate*

$$\operatorname{Re}\lambda \le \Lambda_u := \max_{k=1\ldots m} \left\{ -\frac{\Gamma_k}{2}, \operatorname{Re}\beta_k + 2\rho_k \right\}.$$

Proof: Let (ψ, p) be an eigenvector associated to λ. Then, ψ is a multiple of $T(z,0;\lambda)\binom{r_0}{1}$, and $p = \Gamma\psi/(\lambda - i\Omega_r + \Gamma)$. Thus, λ is geometrically simple. Partial integration of the eigenvalue equation (24) and its complex conjugate equation yields:

$$2\operatorname{Re}\lambda \le 2 \max_{k=1\ldots m} \left(\operatorname{Re}\beta_k + \operatorname{Re}\chi_k(\lambda)\right). \tag{30}$$

For $\operatorname{Re}\lambda > -\Gamma_k/2$, we get $\operatorname{Re}\chi_k(\lambda) \le |\chi_k(\lambda)| \le 2\rho$. $\square$

It turns out that we have to treat the cases $r_0 r_L = 0$ and $r_0 r_L \neq 0$ differently for more detailed analysis of the spectrum of H and the growth properties of the semigroup $T(t)$.

4.3 The differentiable case: $r_0 r_L = 0$

According to the notations in [17], [10] we denote:

Definition 12 *A C_0 semigroup $T(t)$ is called eventually differentiable if there exists a $t_0 \geq 0$ such that $t \to T(t)x$ is differentiable for all $x \in X$ and $t > t_0$. It is called eventually compact if there exists a $t_0 \geq 0$ such that $T(t)$ is a compact operator for all $t > t_0$.*

Theorem 13 *If $r_0 r_L = 0$ in (20), then the C_0 semigroup $T(t)$ generated by H is eventually differentiable.*

Proof: Let M, ω be such that $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$. According to [17], it is sufficient to find constants $a > 0$, $b > 0$, and $C > 0$ such that

1. $\mathcal{R} \supset \Sigma := \{\lambda : \; b \operatorname{Re} \lambda + \log |\operatorname{Im} \lambda| \geq a\}$, and
2. $\|R(\lambda)\| \leq C |\operatorname{Im} \lambda|$ for all $\lambda \in \Sigma$, $\operatorname{Re} \lambda \leq \omega$.

Firstly, we prove property 1. We know that $\mathbb{C}_\omega := \{\lambda : \operatorname{Re} \lambda > \omega\} \subset \mathcal{R}$ because of $\|T(t)\| \leq M e^{\omega t}$. Consider the following two sets

$$\begin{aligned} \mathcal{S}_1 &:= \{\lambda : \operatorname{Im} \lambda > 1\} \setminus \mathbb{C}_\omega \\ \mathcal{S}_2 &:= \{\lambda : \operatorname{Im} \lambda < -1\} \setminus \mathbb{C}_\omega . \end{aligned}$$

Within each of both sets, we can choose the branch of the square root for γ_k satisfying

$$\lim_{|\lambda| \to \infty} \gamma_k(\lambda) - \mu_k(\lambda) = \lim_{|\lambda| \to \infty} \gamma_k(\lambda) - \lambda = 0. \tag{31}$$

Consider the function

$$\begin{aligned} \tilde{h}(\lambda) &= h(\lambda) \exp\left(- \sum_{k=1}^{m} \gamma_k(\lambda) l_k \right) \\ &= (r_L, -1) \prod_{k=m}^{1} \left(T_k(l_k; \lambda) e^{-l_k \gamma_k(\lambda)} \right) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} \end{aligned} \tag{32}$$

which is a multiple of the characteristic function $h(\lambda)$ of H. (31) implies that the factor matrices $\tilde{T}_k(\lambda) = e^{-l_k \gamma_k(\lambda)} T_k(l_k; \lambda)$ of $\tilde{h}$ have the form

$$\tilde{T}_k(\lambda) = \begin{pmatrix} e^{-2 l_k \gamma_k(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} + A_k(\lambda)$$

where all coefficients of A_k satisfy the inequality

$$|A_{k,ij}(\lambda)| \leq c_k |\lambda|^{-1} e^{-2 l_k \operatorname{Re} \lambda} \tag{33}$$

for some $c_k > 0$ in $\mathcal{S}_1$ and in $\mathcal{S}_2$. Hence, we can expand the matrix product in (32) into a sum such that $\tilde{h}(\lambda)$ reads:

$$\tilde{h}(\lambda) = r_0 r_L \exp\left(\sum_{k=1}^{m} \gamma_k(\lambda) l_k\right) - 1 + r(\lambda).$$

The first summand is zero and the remainder $r(\lambda)$ is bounded by

$$|r(\lambda)| \le c|\lambda|^{-1} e^{-2L\,\mathrm{Re}\,\lambda} \tag{34}$$

for some $c > 0$ in $\mathcal{S}_1$ and $\mathcal{S}_2$. If we choose $b > 2L$, then

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Sigma}} |\lambda|^{-1} e^{-2L\,\mathrm{Re}\,\lambda} = 0 \qquad \text{for any } a > 0.$$

Thus, we can choose a sufficiently large such that $\Sigma \setminus \mathbb{C}_\omega \subset \mathcal{S}_1 \cup \mathcal{S}_2$ and

$$c|\lambda|^{-1} e^{-2L\,\mathrm{Re}\,\lambda} < 1/2 \quad \text{for all } \lambda \in \Sigma \setminus \mathbb{C}_\omega.$$

Hence, $|r(\lambda)| < 1/2$, and $|\tilde{h}(\lambda)| > 1/2$ for all $\lambda \in \Sigma \setminus \mathbb{C}_\omega$. Consequently, $\Sigma \subset \mathcal{R}$.

Concerning property 2: The only term which is unbounded w.r.t. λ for $|\lambda| \to \infty$ in the right-hand-side of (29) is $R_1(\lambda)$. We substitute $h(\lambda) = \tilde{h}(\lambda)\exp\left(\sum_{k=1}^{m} l_k \gamma_k(\lambda)\right)$ in (28) and estimate

$$|T_k(z;\lambda)| \le c e^{-l_k\,\mathrm{Re}\,\lambda} \tag{35}$$

for all $\lambda \in \mathcal{S}_1$ and $\mathcal{S}_2$ due to (31). (35) and $\tilde{h}(\lambda) > 1/2$ imply

$$\|R_1(\lambda)\| \le c e^{-3L\,\mathrm{Re}\,\lambda} \tag{36}$$

for all $\lambda \in \mathcal{S}_1$ and $\mathcal{S}_2$. Hence, if we choose $b > 3L$ in the definition of Σ, property 2 is also satisfied in Σ. $\square$

The next theorem establishes precisely how the growth properties of the semigroup $T(t)$ are related to the spectrum of H.

Theorem 14 *Let $\xi > \max \mathrm{Re}\,\mathcal{W}$, and denote $\mathbb{C}_\xi := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \ge \xi\}$, and $\sigma_+ := \mathrm{spec}\,H \cap \mathbb{C}_\xi$. Then, σ_+ consists of at most finitely many eigenvalues of H. All eigenvalues $\lambda \in \sigma_+$ have only finite algebraic multiplicity. The space X can be decomposed into two closed subspaces $X_1 \oplus X_2$ invariant with respect to H and $T(t)$ such that*

1. $\dim X_1 < \infty$, $\mathrm{spec}\,H|_{X_1} = \sigma_+$ *and X_1 is spanned by the finitely many generalized eigenvectors of H associated to the eigenvalues of H in σ_+.*

2. *There exists a $M > 0$ such that $\|T(t)|_{X_2}\| \le M e^{\xi t}$ for all $t > 0$.*

Proof: Let $\gamma \in \mathbb{C} \setminus \mathbb{C}_\xi$ be a smooth closed path around $\mathcal{W}$. Since the spectrum of H is discrete in $\mathbb{C} \setminus \mathcal{W}$, we can choose γ such that $\gamma \subset \mathcal{R}$. Define the projectors

$$\begin{aligned} P &:= \frac{1}{2\pi i} \oint_\gamma R(\lambda)\, d\lambda \\ Q &:= Id - P. \end{aligned}$$

These projectors decompose X into two closed subspaces $X_P = \operatorname{Im} P$, and $X_Q = \operatorname{Im} Q$ which are invariant with respect to H. The resolvent of $H|_{X_Q}$, $QR(\lambda)$, is compact since

$$Q \begin{pmatrix} 0 \\ (\lambda - i\Omega_r + \Gamma)^{-1} p \end{pmatrix} = 0,$$

and $R_1(\lambda)$ is compact. Since $T(t)$ is eventually differentiable, there exists a t_0 such that $T(t)$ is continuous with respect to t in the uniform operator topology for all $t \geq t_0$, i.e., $\|T(t+h) - T(t)\| \to_{h\to 0} 0$ for all $t \geq t_0$ [17]. Thus, $T(t)|_{X_Q}$ is continuous with respect to t in the uniform operator topology for all $t \geq t_0$. Consequently, $T(t)|_{X_Q}$ is eventually compact, i.e., compact for $t \geq t_0$ [17]. This permits us to split the closed subspace X_Q further: At most finitely many eigenvalues of $H|_{X_Q}$, the generator of $T(t)|_{X_Q}$, are situated in $\mathbb{C}_\xi$, and they have at most finite algebraic multiplicity [10]. We denote the corresponding finite-dimensional eigenspace by X_1, and its invariant closed complement by $X_{2,Q}$. Then, the spaces X_1 and $X_2 = X_P \oplus X_{2,Q}$ satisfy the assertions of the theorem: H_{X_P} is a bounded operator, and its spectrum outside the discrete set $\mathcal{W}$ is discrete. Hence, the growth of $T(t)|_{X_P}$ is restricted by $\|T(t)|_{X_P}\| \leq Me^{\xi t}$ for some $M > 1$ as the path γ is contained in $\mathbb{C} \setminus \mathbb{C}_\xi$. Likewise, the growth of the eventually compact semigroup $T(t)|_{X_{2,Q}}$ is bounded by the spectral bound of $H|_{X_{2,Q}}$ which is less than ξ: $\|T(t)|_{X_{2,Q}}\| \leq Me^{\xi t}$ for some $M > 1$ [10].
□

4.4 The hyperbolic case: $r_0 r_L \neq 0$

In order to prove a theorem similar to Theorem 14 for the case $r_0 r_L \neq 0$, we treat the operator H as a perturbation of the operator

$$H_0 = \begin{pmatrix} \sigma\partial_z + \beta & 0 \\ 0 & i\Omega_r - \Gamma \end{pmatrix}$$

defined in $Y \subset X$ (see also [12], [20], [21]). The spectrum of H_0 consists of $\mathcal{W}$ and the sequence of simple eigenvalues

$$\lambda_j^0 := \frac{1}{L}\left[\sum_{k=1}^{m} \beta_k l_k + \frac{1}{2}\log(r_0 r_L) + j\pi i\right] \text{ for } j \in \mathbb{Z}.$$

The eigenvector of H_0 associated to λ_j^0 is

$$b_j^0 := \left(e^{\left(-\lambda_j^0 z + \int_0^z \beta(z)\,dz\right)} r_0, e^{\left(\lambda_j^0 z - \int_0^z \beta(z)\,dz\right)}, 0, 0\right)^T.$$

The sequence $\{b_j^0 : j \in \mathbb{Z}\}$ establishes a basis of $\mathbb{L}^2([0,L];\mathbb{C}^2) \times \{0\}$, i.e., there exists an automorphism of X mapping an orthonormal basis of $\mathbb{L}^2([0,L];\mathbb{C}^2) \times \{0\}$ onto $\{b_j^0 : j \in \mathbb{Z}\}$.

Firstly, we prove an estimate for the location of the eigenvalues of H:

Lemma 15 *Let $r_0 r_l \neq 0$. Then, there exists a vertical strip $\mathcal{S} := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \in [\Lambda_l, \Lambda_u]$ such that $\operatorname{spec} H \subset \mathcal{S}$. There exist constants $R > 0$ and $C > 0$ such that the following holds:*

1. *If λ is an eigenvalue of H and $|\lambda| > R$, then λ is algebraically simple and there exists a $j \in \mathbb{Z}$ such that $|\lambda - \lambda_j^0| < C/|j| < \pi/(2L)$.*
2. *If $|\lambda_j^0| > R$, then there is exactly one eigenvalue of H in the ball B_j of radius $\pi/(2L)$ around λ_j^0.*

Proof: We choose the branch of the square root such that $\gamma_k(\lambda) - \mu_k(\lambda) \to 0$ and $\gamma_k(\lambda) - \lambda \to 0$ for $|\lambda| \to \infty$ in the negative half-plane of $\mathbb{C}$. Hence, $e^{2l_k\gamma_k(\lambda)} \to_{\operatorname{Re}\lambda\to-\infty} 0$. Consequently, the matrices

$$e^{l_k\gamma_k(\lambda)} T_k(l_k;\lambda) \to_{\operatorname{Re}\lambda\to-\infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Accordingly, the multiple of the characteristic function of H converges for $\operatorname{Re}\lambda \to -\infty$:

$$\exp\left(\sum_{k=1}^{m} l_k\gamma_k(\lambda)\right) h(\lambda) \to_{\operatorname{Re}\lambda\to-\infty} r_0 r_L \neq 0,$$

and this limit is uniform for $\operatorname{Im}\lambda$. Consequently, there exists a $\Lambda_l < 0$ such that $h(\lambda) \neq 0$ if $\operatorname{Re}\lambda < \Lambda_l$. The upper limit for the strip $\mathcal{S}$ has been constructed in Lemma 11.

Consider the function

$$h_0(\lambda) = r_0 r_L \exp\left(\sum_{k=1}^{m} \beta_k l_k - \lambda L\right) - \exp\left(-\sum_{k=1}^{m} \beta_k l_k + \lambda L\right).$$

The characteristic function h converges to h_0 within the vertical strip $\mathcal{S}$ for $|\operatorname{Im}\lambda| \to \infty$:

$$|h(\lambda) - h_0(\lambda)| \leq C/|\operatorname{Im}\lambda| \qquad \text{for } \lambda \in \mathcal{S} \text{ and some } C > 0. \tag{37}$$

The function h_0 has the period 2π with respect to $\operatorname{Im}\lambda$, and its roots are λ_j^0 ($j \in \mathbb{Z}$). Outside of the neighborhood of the roots λ_j^0, $|h_0|$ is uniformly bounded from below within $\mathcal{S}$: $|h_0| > c > 0$. Furthermore,

$$h_0'(\lambda_j^0) = (-1)^{j+1} 2L\sqrt{r_0 r_L} \neq 0.$$

Hence, all λ_j^0 are uniformly simple roots of h_0. Since h and h_0 are analytic in $\mathcal{S} \setminus \mathcal{W}$, the convergence (37) implies the assertions 1 and 2 of the lemma. $\square$

Corollary 16 *There exists a ball B, and constants $j_0 \geq 0$ and $C > 0$ such that there is a one-to-one correspondence between eigenvalues of H in $\mathbb{C} \setminus B$ and the elements of $\{\lambda_j^0 : |j| \geq j_0\}$. If we denote the eigenvalue corresponding to λ_j^0 by λ_j, then the eigenvector b_j associated to λ_j satisfies*

$$\left\|b_j - b_j^0\right\| \leq \frac{C}{|j|}$$

if b_j is scaled appropriately.

Proof: If we choose B around 0 of radius R according to Lemma 15, then we can associate the eigenvalue of H located in the ball $B_{\pi/(2L)}(\lambda_j^0)$ to λ_j^0.

The eigenvector b of H associated to λ can be scaled such that it has the form

$$b(z) = \begin{pmatrix} T(z,0;\lambda)\begin{pmatrix} r_0 \\ 1 \end{pmatrix} \\ \frac{\Gamma(z)}{\lambda - i\Omega_r(z) + \Gamma(z)} T(z,0;\lambda)\begin{pmatrix} r_0 \\ 1 \end{pmatrix} \end{pmatrix}. \tag{38}$$

Within the strip $\mathcal{S}$, the expressions $e^{\pm l_k \gamma_k(\lambda)}$ are uniformly bounded, and we can choose a branch of the square root such that $\gamma_k(\lambda) - \lambda \to_{\operatorname{Im}\lambda\to\infty} 0$, and $\gamma_k(\lambda) - \mu_k(\lambda) \to_{\operatorname{Im}\lambda\to\infty} 0$. Hence, the off-diagonal terms of each matrix T_k are of order $O(|\operatorname{Im}\lambda|^{-1})$, and the diagonal terms have the form $e^{\pm(\beta_k - \lambda)z} + O(|\operatorname{Im}\lambda|^{-1})$. $\square$

We can now state a theorem similar to Theorem 14:

Theorem 17 *Let $r_0 r_L \neq 0$, and $\xi > \max\{\max \operatorname{Re}\mathcal{W}, \operatorname{Re}\lambda_0^0\}$. Then, the space X can be decomposed into two closed subspaces $X_1 \oplus X_2$ which are invariant with respect to H and have the following properties:*

1. $\dim X_1 < \infty$, *and X_1 is spanned by at most finitely many generalized eigenvectors of H.*

2. *There exists a $M > 0$ such that $\|T(t)|_{X_2}\| \leq M e^{\xi t}$ for all $t \geq 0$.*

Proof: We define the family of operators $Y \to X$

$$H_\theta = \begin{pmatrix} \sigma\partial_z + \beta - i\sigma_c\theta\kappa & \theta\rho \\ \theta\Gamma & i\Omega_r - \Gamma \end{pmatrix}.$$

The operator H corresponds to $\theta = 1$ and H_0 to $\theta = 0$. The strip $\mathcal{S}$, the ball B and the constants j_0 and C from Lemma 15 and Corollary 16 can be chosen uniformly for the family of operators H_θ.

Since $\{b_j^0 : j \in \mathbb{Z}\}$ is a basis of $\mathbb{L}^2([0,L];\mathbb{C}^2) \times \{0\}$ [12], [21], there exists a constant c such that for any sequence $(x_j) \in \ell^2$ the inequality $c\sum_{j\in\mathbb{Z}} |x_j|^2 \leq \|\sum_{j\in\mathbb{Z}} x_j b_j^0\|^2$ holds.

We choose the constant j_0 sufficiently large such that Lemma 15 and Corollary 16 hold, $\operatorname{Re}\lambda_j < \xi$ for all $|j| > j_0$, and such that

$$\sum_{|j|>j_0} \|b_j - b_j^0\|^2 < c. \tag{39}$$

Next, we define the rectifiable path γ_1 as the border of the rectangle $[\Lambda_l + i(\operatorname{Im}\lambda^0_{j_0} + \pi/(2L)),\ \Lambda_l + i(\operatorname{Im}\lambda^0_{-j_0} - \pi/(2L)),\ \Lambda_u + i(\operatorname{Im}\lambda^0_{-j_0} - \pi/(2L)),\ \Lambda_u + i(\operatorname{Im}\lambda^0_{j_0} + \pi/(2L))]$. Thus, γ_1 is located in the resolvent set of H_θ for all $\theta \in [0,1]$. The spectral projections

$$P_\theta := \frac{1}{2\pi i}\oint_{\gamma_1} (\lambda Id - H_\theta)^{-1}\, d\lambda \qquad Q_\theta := Id - P_\theta$$

split X into the closed subspaces $X_{P,\theta} = \operatorname{Im} P_\theta$ and $X_{Q,\theta} = \operatorname{Im} Q_\theta$ which are invariant with respect to H_θ.

Next, we will construct a map $B : X \to X$ which is injective, a compact perturbation of Id in X and maps $X_{Q,0}$ into $X_{Q,1}$ by mapping $b_j^0 \to b_j$ for $|j| > j_0$:

The projections P_θ and Q_θ depend continuously on θ. Define a sufficiently fine mesh $\{\theta_l : l = 0 \dots N\}$ such that $\|P_{\theta_l} - P_{\theta_{l-1}}\| < 1$ for all $l = 1 \dots N$. Then $P_l + Q_{l-1}$ and $P_{l-1} + Q_l$ are automorphisms of X. Moreover, they are compact perturbations of Id since the resolvent $(\lambda Id - H_\theta)^{-1}$ is a compact perturbation of the operator $(\psi, p) \to (0, (\lambda - i\Omega_r + \Gamma)^{-1} p)$. Let $J := \prod_{l=N}^{1} (P_{\theta_l} + Q_{\theta_{l-1}})$, and $\tilde{J} := \prod_{l=1}^{N} (Q_{\theta_l} + P_{\theta_{l-1}})$. J and $\tilde{J}$ are automorphisms of X, and compact perturbations of Id. J maps injectively $X_{P,0}$ into $X_{P,1}$, and $\tilde{J}$ maps injectively $X_{P,1}$ into $X_{P,0}$. Thus, J is an isomorphism from $X_{P,0}$ onto $X_{P,1}$. We define B in the following way: Let $x = \sum_{|j|>j_0} x_j b_j^0 + x_P$ where $x_P \in X_{P,0}$. Then, $Bx := \sum_{|j|>j_0} x_j b_j + J x_P$. B is injective due to (39) and since J is injective, and B is a compact perturbation of Id [13].

Consequently, B is also surjective. Hence, it maps $X_{Q,0}$ *onto* $X_{Q,1}$, i. e. the set $\{b_j : |j| > j_0\}$ establishes a $\mathbb{L}^2$ basis of $X_{Q,1}$. This implies that there exists a $M > 0$ such that $\|T(t)|_{X_{Q,1}}\| \le M^{\xi t}$ since $\operatorname{Re}\lambda_j < \xi$ for all $|j| > j_0$.

Let γ_2 be a smooth closed path in $\mathcal{R}$ encircling $\mathcal{W}$, and situated in the half-plane $\{\lambda : \operatorname{Re}\lambda < \xi\}$ and in the interior of γ_1. Define the spectral projection

$$P_2 := \frac{1}{2\pi i}\oint_{\gamma_2} R(\lambda)\, d\lambda,$$

and its image by $X_{\mathcal{W}}$. $H|_{X_{\mathcal{W}}}$ is a bounded operator which has a discrete spectrum outside of $\mathcal{W}$. Hence, there exists a $M > 0$ such that $\|T(t)|_{X_{\mathcal{W}}}\| \le M e^{\xi t}$. Moreover, the projections P_1 and P_2 commute, and the image of $P_1 - P_2$ is finite-dimensional since the spectrum of H is discrete between the paths γ_1 and γ_2.

Consequently, we can define $X_1 = \operatorname{Im}(P_1 - P_2)$, and $X_2 = X_{Q,1} \oplus X_{\mathcal{W}}$ to meet the assertions of the theorem. $\square$

The Theorems 14 and 17 assert basically the same growth properties for the semigroup $T(t)$ despite the different constructions. We collect both results in the following corollary.

Corollary 18 *Denote*

$$\xi_0 := \begin{cases} \max\{\operatorname{Re}\lambda_0^0, \max\operatorname{Re}\mathcal{W}\} & \text{if } r_0 r_L \neq 0, \\ \max\operatorname{Re}\mathcal{W} & \text{if } r_0 r_L = 0. \end{cases}$$

Let $\xi > \xi_0$. Then, there are at most finitely many eigenvalues of H of finite algebraic multiplicity in the right half-plane $\mathbb{C}_\xi := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \xi\}$. Moreover, X can be decomposed into two $T(t)$-invariant subspaces

$$X = X_+ \oplus X_-$$

where X_+ is at most finite-dimensional and spanned by the generalized eigenvectors associated to the eigenvalues of H in $\mathbb{C}_\xi$. There exists a constant M such that the restriction of $T(t)$ to X_- is bounded according to

$$\|T(t)|_{X_-}\| \leq M e^{\xi t} \tag{40}$$

in any norm which is equivalent to the X-norm.

Remark: The eigenvalues of H can be computed numerically by solving the complex equation $h(\lambda) = 0$. The eigenvalues of H_0 in $\mathbb{C} \setminus \mathcal{W}$ form the sequence λ_j^0 for $\kappa = 0$, $\rho = 0$, $r_0^0 r_L^0 \neq 0$ (see Theorem 17). The roots of the characteristic function h can be obtained by continuing along the parameter path $\theta\kappa$, $\theta\rho$, $r_0^0 + \theta(r_0 - r_0^0)$, $r_L^0 + \theta(r_L - r_L^0)$ for $\theta \in [0, 1]$.

5 Existence and properties of the finite-dimensional center manifold

The results of §4 permit the application of theorems about the persistence and properties of normally hyperbolic invariant manifolds in Banach spaces [7], [8], [9] to the semiflow $S(t, \cdot)$ generated by system (21), (23) in the following situation:

Assumption 19 *Assume there exist a $\xi \in (\xi_0, 0)$ according to Corollary 18 and a simple connected compact set $\mathcal{K} \subset \mathbb{R}^m$ with the following property:*

The spectrum of $H(n)$ can be split for all $n \in \mathcal{K}$ in the following manner:

$$\begin{aligned} \operatorname{spec} H(n) &= \sigma_c(n) \cup \sigma_s(n) \qquad \text{where} \\ \operatorname{Re}\sigma_c(n) &\geq 0 \\ \operatorname{Re}\sigma_s(n) &< \xi < 0. \end{aligned}$$

Due to Corollary 18, the number of elements of $\sigma_c(n)$ is finite and, hence, constant in $\mathcal{K}$ if the eigenvalues are counted according to their algebraic multiplicity. We denote this number by q. Moreover, for each $\gamma \in [\xi, 0)$, there exists a bounded simple connected open set $U_\gamma \supset \mathcal{K}$ such that the splitting of $\operatorname{spec} H(n)$ can be can be extended to U_γ:

$$\begin{aligned} \operatorname{spec} H(n) = & \ \sigma_c(n) \cup \sigma_s(n) & & \text{where} \\ \operatorname{Re} \sigma_c(n) > & \ \gamma, & & \\ \operatorname{Re} \sigma_s(n) < & \ \xi & & \text{for all } n \in U_\gamma. \end{aligned}$$

There exist spectral projections of $H(n)$, $P_c(n)$ and $P_s(n) \in \mathcal{L}(X)$, corresponding to this splitting. They are well defined and unique for all $n \in U_\xi$ and depend smoothly on n. We define the corresponding closed invariant subspaces of X by $X_c(n) = \operatorname{Im} P_c(n) = \ker P_s(n)$ and $X_s(n) = \operatorname{Im} P_s(n) = \ker P_c(n)$. The complex dimension of $X_c(n)$ is q. Let $B(n) : \mathbb{C}^q \to X$ be a basis of $X_c(n)$ which depends smoothly on n. $B(\cdot)$ is well defined in U_ξ. Using these notations, we can state the following theorem:

Theorem 20 (Model reduction)
Let $k > 2$ be an integer number and $E_{\max} > 0$. Then, there exist a $\varepsilon_0 > 0$ and an open neighborhood $U \subset U_\xi$ of $\mathcal{K}$ such that the following statements hold. Define $b := \max_{n \in \operatorname{cl} U} \|B(n)^{-1} P_c(n)\|$, and the sets

$$\begin{aligned} \mathcal{B} &= \{(E_c, n) \in \mathbb{C}^q \times \mathbb{R}^m : \|E_c\| < bE_{\max} + 1, n \in U\} \subset \mathbb{C}^q \times \mathbb{R}^m, \text{ and} \\ \mathcal{N} &= \{(E, n) \in X \times \mathbb{R}^m : \|E\| < E_{\max}, n \in \Upsilon\} \subset X \times \mathbb{R}^m \end{aligned}$$

where Υ is an arbitrary closed subset of U. For all $\varepsilon \in (0, \varepsilon_0)$, there exists a C^k manifold $\mathcal{C}$ satisfying:

i. (Invariance) *$\mathcal{C}$ is $S(t, \cdot)$-invariant relative to $\mathcal{N}$ if $\varepsilon \in (0, \varepsilon_0)$.*

ii. (Representation) *$\mathcal{C}$ can be represented as the graph of a map which maps*

$$(E_c, n, \varepsilon) \in \mathcal{B} \times (0, \varepsilon_0) \to ([B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c, n) \in X \times \mathbb{R}^m$$

where $\nu : \mathcal{B} \times (0, \varepsilon_0) \to \mathcal{L}(\mathbb{C}^q; X)$ is C^{k-2} with respect to all arguments. Denote the E-component of $\mathcal{C}$ by

$$E_X(E_c, n, \varepsilon) = [B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c \in X.$$

iii. (Exponential attraction) *Let (E, n) be such that $S(t; (E, n)) \in \mathcal{N}$ for all $t \geq 0$. Then, there exist $(E_c, n_c) \in \mathcal{B}$, $M > 0$ and $t_c \geq 0$ such that*

$$\|S(t + t_c; (E, n)) - S(t; (E_X(E_c, n_c, \varepsilon), n_c))\| \leq Me^{\xi t} \text{ for all } t \geq 0. \qquad (41)$$

iv. (Flow) *The values $\nu(E_c, n, \varepsilon)E_c$ are in Y and their $P_c(n)$-component is 0 for all $(E_c, n, \varepsilon) \in \mathcal{B} \times (0, \varepsilon_0)$. The flow on $\mathcal{C} \cap \mathcal{N}$ is differentiable with respect* *to t and governed by the following system of ordinary differential equations (ODEs):*

$$\begin{aligned}\frac{d}{dt}E_c = & \left[H_c(n) + \varepsilon a_1(E_c, n, \varepsilon) + \varepsilon^2 a_2(E_c, n, \varepsilon)\nu(E_c, n, \varepsilon)\right] E_c \\ \frac{d}{dt}n = & \varepsilon F(E_c, n, \varepsilon)\end{aligned} \tag{42}$$

where

$$\begin{aligned}H_c(n) = & B(n)^{-1}H(n)P_c(n)B(n) \\ a_1(E_c, n, \varepsilon) = & -B(n)^{-1}P_c(n)\partial_n B(n)F(E_c, n, \varepsilon) \\ a_2(E_c, n, \varepsilon) = & B(n)^{-1}\partial_n P_c(n)F(E_c, n, \varepsilon)(Id - P_c(n)) \\ F(E_c, n, \varepsilon) = & (f_k(n_k) - g_k(n_k)[E_X(E_c, n_c, \varepsilon), E_X(E_c, n_c, \varepsilon)])_{k=1}^m .\end{aligned}$$

System (42) is symmetric with respect to rotation $E_c \to E_c e^{i\varphi}$ and ν satisfies the relation $\nu(e^{i\varphi}E_c, n, \varepsilon) = \nu(E_c, n, \varepsilon)$ for all $\varphi \in [0, 2\pi)$.

Remark: The theorem is a direct consequence of the general results of [7], [8], [9]. In this case, the invariant manifold is even finite-dimensional and exponentially stable. The proof is mostly concerned with the proper definition of the coordinates and describes in detail the appropriate cut-off modification of the system outside of the region of interest to make the unperturbed invariant manifold compact. A similar result about model reduction for systems of ODEs with the structure (1) has been presented already by [27].

Proof:
Existence, representation, and smoothness
Firstly, we introduce a splitting of $E \in X$ which is valid for $n \in U_\xi$. Let $n \in U_\xi$. For any $E \in X$, we define $E_c = B(n)^{-1}P_c(n)E \in \mathbb{C}^q$ and $E_s = P_s(n)E \in X_s(n)$. Then, $E = B(n)E_c + E_s$, and a decomposition of (21) by $B(n)^{-1}P_c(n)$ and $P_s(n)$ implies that $E_c \in \mathbb{C}^q$, $E_s \in X_s(n) \subset X$, and $n \in \mathbb{R}^m$ satisfy the system

$$\frac{d}{dt}E_c = H_c(n)E_c + a_{11}(E_c, E_s, n)E_c + a_{12}(E_c, E_s, n)E_s \tag{43}$$

$$\frac{d}{dt}E_s = H_s(n)E_s + a_{21}(E_c, E_s, n)E_c + a_{22}(E_c, E_s, n)E_s \tag{44}$$

$$\frac{d}{dt}n_k = f_k(E_c, E_s, n) \qquad \text{for } k = 1 \ldots m \tag{45}$$

where $H_c, a_{11} : \mathbb{C}^q \to \mathbb{C}^q$, $a_{12} : X \to \mathbb{C}^q$, $a_{21} : \mathbb{C}^q \to X$, $a_{22} : X \to X$, and $H_s : Y \to X$ are linear operators defined by

$$\begin{aligned}H_c(n) &= B^{-1}HP_cB & H_s(n) &= HP_s - 2\xi P_c \\ a_{11}(E_c, E_s, n) &= -B^{-1}P_c\partial_n Bf & a_{12}(E_c, E_s, n) &= B^{-1}\partial_n P_c f P_s \\ a_{21}(E_c, E_s, n) &= -P_s\partial_n Bf & a_{22}(E_c, E_s, n) &= -P_c\partial_n P_c f P_s\end{aligned}$$

$$f_k(E_c, E_s, n) = \varepsilon\left(F_k(n_k) - g_k(n_k)[B(n)E_c + E_s, B(n)E_c + E_s]\right)$$

for $k = 1 \dots m$. We introduced the term $-2\xi P_c E_s$ which is 0 artificially in (44). System (43)–(45) couples a system of ODEs in $\mathbb{C}^q$, an evolution equation in X, and a system of ODEs in $\mathbb{R}^m$. The right-hand-side of (43)–(45) is only properly defined as long as n stays in U_ξ.

In the next step, we modify system (43)–(45) such that it is globally defined and generates a semiflow. Beforehand, we introduce some notation.

Let $d : \mathbb{R} \to [0,1]$ be a smooth monotone function such that

$$d(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1. \end{cases}$$

Let $\gamma \in (\xi/k, 0)$, and U be an open neighborhood of $\mathcal{K}$ such that $\mathrm{cl}\, U \subset U_\gamma$. Then, the borders of U and U_γ have a positive distance, and there exists a smooth and globally Lipschitz continuous map $N : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$N(n) = \begin{cases} n & \text{for } n \in U \\ \in U_\gamma & \text{for } n \notin U. \end{cases}$$

Let Υ be an arbitrary closed subset of U, $\sigma > 0$ and

$$\begin{aligned} n_{\max} &:= \max_{n \in U_\gamma} |n| \\ R &:= \sqrt{6 + (bE_{\max} + 1)^2 + n_{\max}^2}, \\ s(x, E_c, n) &:= |E_c|^2 + |n|^2 + x^2 - R^2 \quad \text{for } x \in \mathbb{R}, E_c \in \mathbb{C}^q, n \in \mathbb{R}^m, \\ \Delta(E_c, n) &:= d\left(|E_c|^2 + |n|^2 - (bE_{\max} + 1)^2 - n_{\max}^2\right). \end{aligned}$$

The functions s and Δ are smooth with respect to their arguments.

Consider the following modification of system (43)–(45):

$$\begin{aligned} \frac{d}{dt} E_c &= H_c(N(n))E_c + \tilde{a}_{11} E_c + \tilde{a}_{12} E_s \qquad (46) \\ &\quad - \Delta(E_c, n) \left[H_c(N(n))E_c + \tilde{a}_{11} E_c + \tilde{a}_{12} E_s + \sigma s(x, E_c, n) E_c \right] \\ \frac{d}{dt} E_s &= H_s(N(n))E_s + \tilde{a}_{21} E_c + \tilde{a}_{22} E_s \qquad (47) \\ \frac{d}{dt} n_k &= \tilde{f}_k(E_c, E_s, n) - \Delta(E_c, n) \left[\tilde{f}_k(E_c, E_s, n) + \sigma s(x, E_c, n) n_k \right] \qquad (48) \end{aligned}$$

for $k = 1 \dots m$, augmented by a differential equation for the dummy real variable x:

$$\frac{d}{dt} x = \tilde{g}(x, E_c) - \sigma s(x, E_c, n) x \qquad (49)$$

where

$$
\begin{aligned}
\tilde{a}_{11}(E_c, E_s, n) &= -B(N(n))^{-1} P_c(N(n)) \partial_n B(N(n)) \partial_n N(n) \tilde{f}(E_c, E_s, n) \\
\tilde{a}_{12}(E_c, E_s, n) &= B(N(n))^{-1} \partial_n P_c(N(n)) \partial_n N(n) \tilde{f}(E_c, E_s, n) P_s(N(n)) \\
\tilde{a}_{21}(E_c, E_s, n) &= -P_s(N(n)) \partial_n B(N(n)) \partial_n N(n) \tilde{f}(E_c, E_s, n) \\
\tilde{a}_{22}(E_c, E_s, n) &= -P_c(N(n)) \partial_n P_c(N(n)) \partial_n N(n) \tilde{f}(E_c, E_s, n) P_s(N(n)) \\
\tilde{f}_k(E_c, E_s, n) &= \ f_k(E_c, E_s, N(n)) \text{ for } k = 1 \dots m, \\
\tilde{g}(x, E_c) &= \begin{cases} \left[-\frac{1}{2x} \frac{d}{dt}\left(|E_c|^2 + |n|^2\right)\right] d(|x| - 1) & \text{for } |x| > 1 \\ 0 & \text{for } |x| \leq 1. \end{cases}
\end{aligned}
$$

The right-hand-side of system (46)–(49) is smooth and globally defined. It generates a semiflow $\tilde{S}_0(t; (E_c, E_s, n, x))$ on $\mathbb{C}^q \times X \times \mathbb{R}^m \times \mathbb{R}$. The modification has no effect if $(E_c, n) \in \mathcal{B}$. The equation for $\dot{x}$ implies

$$
\dot{s} = \begin{cases} -2\sigma s x^2 & \text{for } |x| \geq 2 \\ -2\sigma s \left[(1 - d(|x| - 1))(|E_c|^2 + |n|^2) + x^2\right] & \text{for } |x| < 2 \end{cases}
$$

in the vicinity of $\mathcal{M}_0 := \{(E_c, E_s, n, x) : s(x, E_c, n) = 0\}$. Thus $\mathcal{M}_0$ is an invariant manifold of $\tilde{S}_0$ which has an exponential attraction rate greater than 2σ. Moreover, system (46)–(49) implies:

$$
\frac{d}{dt}(P_c(N(n))E_s) = (\partial_n P_c \partial_n N \tilde{f} - 2\xi Id)(P_c(N(n))E_s).
$$

Hence, the manifold $\mathcal{M}_1 := \{(E_c, E_s, n, x) : P_c(N(n))E_s = 0\}$ is invariant with respect to (46)–(49). For bounded E_c and E_s, the rate of attraction towards $\mathcal{M}_1$ is close to $2|\xi|$.

There is a one-to-one correspondence between the semiflows $S(t; \cdot)$ and $\tilde{S}_0(t, \cdot)$ in the following sense: The map acting from

$$
\begin{aligned}
&\{(E_c, E_s, n, x) \in \mathcal{M}_0 \cap \mathcal{M}_1 : (E_c, n) \in \mathcal{B}\} \to \ X \times U \text{ defined by} \\
&(E_c, E_s, n, x) \to \ (B(n)E_c + E_s, n)
\end{aligned}
$$

is injective and maps $\tilde{S}_0$ onto S. The inverse

$$
(E, n) \to \left(B(n)^{-1}P_c(n)E, P_s(n)E, n, \sqrt{R^2 - |B(n)^{-1}P_c(n)E|^2 - |n|^2}\right)
$$

is properly defined in $\mathcal{N}$.

At $\varepsilon = 0$, $\tilde{f}$ and all $\tilde{a}_{ij}$ vanish. Hence,

$$
\tilde{\mathcal{C}} := \{(E_c, E_s, n, x) \in \mathbb{C}^q \times X \times \mathbb{R}^m : E_s = 0, s(x, E_c, n) = 0\}
$$

is a smooth compact invariant manifold of (46)–(49). E_s decays with a rate greater than $|\xi|$. Hence, if $2\sigma > |\xi|$, the attraction rate transversal to $\tilde{\mathcal{C}}$ is greater than

$|\xi|$. The generalized Lyapunov numbers for the component of the linearization of $\tilde{S}_0$ tangent to $\mathcal{C}$ are greater or equal than $\gamma > \xi/k$. The perturbation to nonzero ε is C^1 small, and all derivatives of the perturbation with respect to (E_c, E_s, n, x), and ε up to order k are bounded uniformly for small ε in the vicinity of $\tilde{\mathcal{C}}$. Consequently, the general theorems of [7], [8], [9] imply:

There exists an ε_0 such that for all $\varepsilon \in [0, \varepsilon_0)$ there exists a compact invariant C^k manifold $\tilde{\mathcal{C}}_0$ for $\tilde{S}_0(t, \cdot)$. $\tilde{\mathcal{C}}_0$ is a C^1 small perturbation of $\tilde{\mathcal{C}}$. Hence, its E_s-component can be represented as a C^k graph

$$E_s = \eta_0(E_c, n, x, \varepsilon).$$

The contraction rates towards $\mathcal{M}_0$ and $\mathcal{M}_1$ are greater than $|\xi|$ close to $\tilde{\mathcal{C}}$. Consequently, $\tilde{\mathcal{C}}_0 \subset \mathcal{M}_0 \cap \mathcal{M}_1$. The evolution of E_c, E_s and n does not depend on x if $(E_c, n) \in \mathcal{B}$. Hence, $\eta_0(E_c, n, x, \varepsilon)$ does not depend on x if $(E_c, n) \in \mathcal{B}$.

The existence of $\tilde{\mathcal{C}}_0$ and the one-to-one correspondence between S and $\tilde{S}_0$ imply that the manifold

$$\mathcal{C} := \{(B(n)E_c + \eta_0(E_c, n, \varepsilon), n) : (E_c, n) \in \mathcal{B}\}$$

is an invariant C^k manifold of S relative to $\mathcal{N}$. The flow on $\mathcal{C}$ is governed by

$$\begin{aligned} \frac{d}{dt}E_c = {} & [H_c(n) + a_{11}(E_c, \eta_0(E_c, n, \varepsilon), n, \varepsilon)]\, E_c \\ & + a_{21}(E_c, \eta_0(E_c, n, \varepsilon), n, \varepsilon)\eta_0(E_c, n, \varepsilon) \\ \frac{d}{dt}n_k = {} & f_k(E_c, \eta_0(E_c, n, \varepsilon), n). \end{aligned} \tag{50}$$

The rotational symmetry of the semiflow S implies

$$\eta_0(e^{i\varphi}E_c, n, \varepsilon) = e^{i\varphi}\eta_0(E_c, n, \varepsilon) \tag{51}$$

for all $(E_c, n, \varepsilon) \in \mathcal{B} \times [0, \varepsilon)$ and $\varphi \in [0, 2\pi)$.

Expansion of the graph η_0
The graph η_0 satisfies

$$\eta_0(E_c, n, 0) = 0 \qquad \text{for all } (E_c, n) \in \mathcal{B}. \tag{52}$$

Furthermore, the manifold $\mathcal{E} := \{(E, n) \in X \times U : E = 0\}$ is invariant with respect to S for positive ε. On $\mathcal{E}$, $\dot{E} = 0$, and $\dot{n}_k = \varepsilon F_k(n_k)$ for $k = 1 \ldots m$. Consequently, $\mathcal{E} \cap \mathcal{N} \subset \mathcal{C}$, i.e.,

$$\eta_0(0, n, \varepsilon) = 0 \qquad \text{for } n \in U,\ \varepsilon \in [0, \varepsilon_0). \tag{53}$$

Finally, we observe that the right-hand-side of (46)–(49) depends smoothly on E_c and ε. Exploiting the identities (52) and (53), we may expand

$$\begin{aligned} \eta_0(E_c, n, \varepsilon) &= \int_0^1 \partial_1\eta_0(sE_c, n, \varepsilon)\, ds E_c \\ &= \varepsilon \int_0^1 \int_0^1 \partial_1\partial_3\eta_0(sE_c, n, r\varepsilon)\, dr\, ds E_c. \end{aligned} \tag{54}$$

Denoting the double integral term in (54) by ν, we obtain

$$\eta_0(E_c, n, \varepsilon) = \varepsilon\nu(E_c, n, \varepsilon)E_c. \tag{55}$$

We obtain the assertion iv of the theorem by inserting (55) into system (50) for the flow on $\mathcal{C}$. The invariance of ν with respect to rotation of E_c is a direct consequence of (51).

Exponential attraction of $\mathcal{C}$

The theorems of [7], [8], [9] imply that the set of all points x which stay in a small tubular neighborhood of a compact normally hyperbolic invariant manifold $\mathcal{M}$ for all $t \geq 0$ form a center-stable manifold which is foliated by stable fibers of attraction rate according to the generalized Lyapunov numbers in the stable part of the linearization of the semiflow along $\mathcal{M}$. In order to map $\mathcal{N}$ into a small neighborhood of $\mathcal{C}$, we have to go again through the first part of the proof using a different scaling of the coordinate E_s: Redefine $E_s = \sqrt[4]{\varepsilon}P_s(n)E \in X_s(n)$. Then, $E = B(n)E_c + E_s/\sqrt[4]{\varepsilon}$, and a decomposition of (21) by $B(n)^{-1}P_c(n)$ and $P_s(n)$ implies that E_c, E_s and n satisfy system (43)–(45) where the coefficients a_{ij}, and the functions f_k $(k = 1, \ldots m)$ are slightly modified:

$$\begin{aligned} a_{11}(E_c, E_s, n) &= -B^{-1}P_c\partial_n Bf & \qquad a_{12}(E_c, E_s, n) &= B^{-1}\partial_n P_c f P_s/\sqrt[4]{\varepsilon} \\ a_{21}(E_c, E_s, n) &= -\sqrt[4]{\varepsilon}P_s\partial_n Bf & \qquad a_{22}(E_c, E_s, n) &= -P_c\partial_n P_c f P_s \end{aligned}$$

$$f_k(E_c, E_s, n) = \sqrt{\varepsilon}\left(\sqrt{\varepsilon}F_k(n_k) - g_k(n_k)[\sqrt[4]{\varepsilon}B(n)E_c + E_s, \sqrt[4]{\varepsilon}B(n)E_c + E_s]\right)$$

The modifications applied to system (43)–(45) to extend its domain of definition and make it generate a semiflow can be applied to the rescaled system as well. The rescaling changes only the coefficients $\tilde{a}_{ij}$, and the functions $\tilde{f}_k$ $(k = 1, \ldots m)$ of system (46)–(49):

$$\begin{aligned} \tilde{a}_{11}(E_c, E_s, n) &= -B(N(n))^{-1}P_c(N(n))\partial_n B(N(n))\partial_n N(n)\tilde{f}(E_c, E_s, n) \\ \tilde{a}_{12}(E_c, E_s, n) &= B(N(n))^{-1}\partial_n P_c(N(n))\partial_n N(n)\tilde{f}(E_c, E_s, n)P_s(N(n))/\sqrt[4]{\varepsilon} \\ \tilde{a}_{21}(E_c, E_s, n) &= -\sqrt[4]{\varepsilon}P_s(N(n))\partial_n B(N(n))\partial_n N(n)\tilde{f}(E_c, E_s, n) \\ \tilde{a}_{22}(E_c, E_s, n) &= -P_c(N(n))\partial_n P_c(N(n))\partial_n N(n)\tilde{f}(E_c, E_s, n)P_s(N(n)) \\ \tilde{f}_k(E_c, E_s, n) &= \ f_k(E_c, E_s, N(n)) \text{ for } k = 1\ldots m. \end{aligned}$$

This rescaled version of system (46)–(49) generates a semiflow $\tilde{S}_{1/4}(t;\cdot)$ which is equivalent to $\tilde{S}_0$ for $\varepsilon \neq 0$ At $\varepsilon = 0$, $\tilde{a}_{ij}$ and $\tilde{f}$ still vanish such that $\tilde{S}_{1/4}$ has also the exponentially attractive invariant manifold $\tilde{\mathcal{C}}$ for $\varepsilon = 0$. The perturbation to nonzero ε is C^1 small, too. (However, it is of lower order of ε.) Hence, we may adjust ε_0 such that the manifold $\tilde{\mathcal{C}}$ persists under perturbation to $\varepsilon \in (0, \varepsilon_0)$ for $\tilde{S}_0$ and $\tilde{S}_{1/4}$. Denote the perturbed invariant manifold for $\tilde{S}_{1/4}$ by $\tilde{\mathcal{C}}_{1/4}$. The graph $\eta_{1/4}$ representing the E_s component of $\tilde{\mathcal{C}}_{1/4}$ as a function of (E_c, n) in $\mathcal{B}$ satisfies

$$\eta_{1/4}(E_c, n, \varepsilon) = \sqrt[4]{\varepsilon}\eta_0(E_c, n, \varepsilon) \text{ for } \varepsilon \in (0, \varepsilon_0) \tag{56}$$

because the persisting invariant manifold is unique in a neighborhood of $\tilde{\mathcal{C}}$, and $\tilde{S}_0$ and $\tilde{S}_{1/4}$ are equivalent. The manifold $\mathcal{C}$ satisfies: $(E,n) \in \mathcal{C}$ if and only if $(E_c, n) \in \mathcal{B}$ and $E = B(n)E_c + \eta_{1/4}(E_c, n, \varepsilon)/\sqrt[4]{\varepsilon}$.

Let (E,n) be such that $S(t;(E,n)) \in \mathcal{N}$ for all $t \geq 0$. Then, the corresponding trajectory of $\tilde{S}_{1/4}$ is

$$\Big(E_c(t) = B(n(t))^{-1}P_c(n(t))E(t), E_s(t) = \sqrt[4]{\varepsilon}P_s(n(t))E(t), n(t),$$
$$x(t) = \sqrt{R^2 - |B(n(t))^{-1}P_c(n(t))E(t)|^2 - |n(t)|^2}\Big).$$

It satisfies $|E_c(t)| < bE_{\max}$, $n(t) \in \Upsilon$, and $\|E_s(t)\| < \sqrt[4]{\varepsilon}\max_{n\in\Upsilon}\|P_s(n)\|E_{\max}$ for all $t \geq 0$. Consequently, $(E_c(0), E_s(0), n(0), x(0))$ is in a small tubular neighborhood of $\tilde{\mathcal{C}}_{1/4}$ for all $t \geq 0$. Hence, it is in the center-stable manifold of $\tilde{\mathcal{C}}_{1/4}$ if ε_0 is sufficiently small. The existence of stable fibers for the center-stable manifold of $\tilde{\mathcal{C}}_{1/4}$ and the contraction rate greater than $|\xi|$ transversal to $\tilde{\mathcal{C}}_{1/4}$ imply that there exist a constant $M > 0$ and a trajectory $(E_c^*(t), E_s^*(t), n^*(t), x^*(t)) \in \tilde{\mathcal{C}}_{1/4}$ such that

$$\|(E_c(t), E_s(t), n(t), x(t)) - (E_c^*(t), E_s^*(t), n^*(t), x^*(t))\| < Me^{\xi t}.$$

Denote the distance between Υ and the border of U by δ ($\delta > 0$). Let $t_c \geq 0$ be such that $Me^{\xi t} < \min\{\delta, 1\}$ for all $t \geq t_c$. Then, $\|E_c^*(t)\| \leq bE_{\max} + 1$ and $n^*(t) \in U$ for all $t \geq t_c$. Consequently $(E_c^*(t), n^*(t)) \in \mathcal{B}$, and $E_s^*(t) = \eta_{1/4}(E_c^*(t), n^*(t), \varepsilon)$ for all $t \geq t_c$. Hence, we may choose $E_c = E_c^*(t_c)$ and $n_c = n^*(t_c)$ to meet assertion iii of the theorem. $\square$

6 Conclusions and generalizations

Mode approximation The graph of the center manifold enters the description (42) of the flow on $\mathcal{C}$ only in the form $O(\varepsilon^2)\nu$. All other terms appearing in (42) can be expressed analytically as functions of the eigenvalues of $H(n)$. Systems of the form (42) but replacing ν by 0 are called *Mode approximation models.* These models are implicit systems of ordinary differential equations because the eigenvalues of H are given only implicitly as roots of the characteristic function h of H. The consideration of mode approximations has proven to be extremely useful for numerical and analytical investigations of longitudinal effects in multi-section semiconductor lasers because the dimension of system (42) is typically low (q is often either 1 or 2); see, e.g., [2], [3], [4], [6], [22], [26], [29], [31].

The Lang-Kobayashi system There is an obvious generalization of Theorem 20 to another class of laser models. A very popular model for the investigation of delayed optical feedback effects in semiconductor lasers is the Lang-Kobayashi system [14]; see, e.g., [28] and references therein. It reads

$$\begin{aligned} \frac{d}{dt}E(t) &= (1+i\alpha)nE(t) + \eta e^{i\varphi}E(t-1) \\ \frac{d}{dt}n(t) &= \varepsilon\left(F(n) - g(n)|E(t)|^2\right) \end{aligned} \tag{57}$$

if its scaling is appropriate to the situation of a short external cavity [30]. System (57) generates a semiflow on the Banach space $C([-1,0];\mathbb{C}) \times \mathbb{R}$ and has also the structure (1). The parameters have the same sense as in (2)–(4) (we have dropped the indices since there is only one section). The parameter ε is small if the external cavity is short. The operator H is a delay operator in (57). According to [10], Corollary 18 is also valid for the delay operator H (ξ_0 is $-\infty$ in Corollary 18). Moreover, the cut-off modification performed in the proof of Theorem 20 manipulates only the finite-dimensional components E_c and n. Hence, the proof does not rely on the ability to cut-off a smooth map smoothly in the infinite-dimensional space X which is the Hilbert space $\mathbb{L}^2([0,L];\mathbb{C}^4)$ in §5 but a Banach space for system (57). The only property of the operator $H(n)$ used in the proof is the existence of a spectral splitting according to Assumption 19 accompanied by the results of Corollary 18, and the smooth dependence of the dominating subspace X_c on n. Consequently, if Assumption 19 is satisfied, Theorem 20 applies to (57) as well. The set $\mathcal{K}$ supposed to exist in Assumption 19 is a point n_0 in $\mathbb{R}$ (typically referred to as *threshold carrier density*) in the case of a scalar n.

There are other models in the spirit of (57) for different experimental situations, e.g., for lasers subject to dispersive feedback or for two lasers interacting with each other. All have the structure of (1) where H is a delay operator smoothly depending on n, and ε is small if the external cavity is short. Hence, Theorem 20 allows to reduce these models locally to low-dimensional systems of ODEs.

A Physical background of the traveling-wave equations and discussion of typical parameter ranges

System (2)–(4) is well-known as traveling wave-model describing longitudinal dynamical effects in semiconductor lasers (see [5], [15], [25] for further references). Results of numerical simulations have been presented in [2], [4], [5], [6], [19].

The quantities ψ and p describe the complex optical field E in a spatially modulated waveguide:

$$E(\vec{r},t) = E(x,y)\cdot(\psi_1(t,z)e^{i\omega_0 t - \frac{\pi}{\Lambda}z} + \psi_2(t,z)e^{i\omega_0 t + \frac{\pi}{\Lambda}z}).$$

The complex amplitudes $\psi_{1,2}(t,z)$ are the longitudinally slowly varying envelopes of E. The transversal space directions are x and y, the longitudinal direction is z, and $\vec{r} = (x,y,z)$. For periodically modulated waveguides, Λ is longitudinal modulation wavelength. The central frequency is $\omega_0/(2\pi)$, and $E(x,y)$ is the dominant transversal mode of the waveguide.

	typical range	explanation
$\psi(t,z)$	$\mathbb{C}^2$	optical field, forward and backward traveling wave
$i \cdot p(t,z)$	$\mathbb{C}^2$	nonlinear polarization
$n(t)$	$(\underline{n}, \infty)$	spatially averaged carrier density in section S_1
$\operatorname{Im} \beta_k^0$	$\mathbb{R}$	frequency detuning
$\operatorname{Re} \beta_k^0$	< 0, $(-10, 0)$	decay rate due to internal losses
α_H	$(0, 10)$	negative of line-width enhancement factor
g_1	≈ 1	differential gain in S_1
κ_k	$(-10, 10)$	real coupling coefficients for the optical field ψ
ρ_k	$[0, 1)$	maximum of the gain curve
Γ_k	$O(10^2)$	half width of half maximum of the gain curve
$\Omega_{r,k}$	$O(10)$	resonance frequency
I_k	$O(10^{-2})$	current injection
τ_k	$O(10^2)$	spontaneous lifetime for the carriers
P	$(0, \infty)$	scale of (ψ, p) (can be chosen arbitrarily)
r_0, r_L	$\mathbb{C}$, $\lvert r_0 \rvert, \lvert r_L \rvert < 1$	facet reflectivities

Table 1: Ranges and explanations of the variables and coefficients appearing in (2)-(12). See also [5], [23] to inspect their relations to the originally used physical quantities and scales.

The equation $\dot{E} = H(n)E$ (see §1) for an uncoupled waveguide ($\kappa = 0$), a monochromatic light-wave in forward direction $e^{i\omega t}\psi_1(z)$ and a constant carrier density n imply a spatial shape of the power $|\psi_1|^2$ according to

$$\partial_z |\psi_1(z)|^2 = (2 \operatorname{Re} \beta(z) + 2 \operatorname{Re} \chi(i\omega, z)) |\psi_1(z)|^2 \tag{58}$$

where

$$\chi(i\omega, z) = \frac{\rho(z)\Gamma(z)}{i\omega - i\Omega_r(z) + \Gamma(z)}. \tag{59}$$

$2 \operatorname{Re} \chi(i\omega, z))$ is a Lorentzian intended to fit the gain curve of the waveguide material. Hence, $\dot{E} = HE$ produces gain dispersion, i. e., the spatial growth rate of the wave $e^{i\omega t}\psi(z)$ depends on its frequency ω. The variable $p(t,z)$ reports the internal state of the gain filter. See [5], [23] for more details. The Lorentzian gain filter is also used by [1], [15], and [16]. Since the coefficients ρ, Γ, and Ω are supposed to be spatially section-wise constant, $\chi(\lambda, z) = \chi_k(\lambda)$ for z in section S_k for $k = 1 \ldots m$.

The equation (4) is a rate equation for the spatially section-wise averaged carrier density. It accounts for the current I_k, the spontaneous recombination $-n_k/\tau_k$, and the stimulated recombination. See table 1 for typical ranges of the quantities.

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