A mathematical model for impulse resistance welding

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Abstract

We present a mathematical model of impulse resistance welding. It accounts for electrical, thermal and mechanical effects, which are nonlinearly coupled by the balance laws, constitutive equations and boundary conditions. The electrical effects of the weld machine are incorporated by a discrete oscillator circuit which is coupled to the field equations by a boundary condition.

We prove the existence of weak solutions for a slightly simplified model which however still covers most of its essential features, e.g. the quadratic Joule heat term and a quadratic term due to non-elastic energy dissipation.

We discuss the numerical implementation in a 2D setting, present some numerical results and conclude with some remarks on future research.

Key words: *Joule heating, resistance welding, thermo-viscoelasticity*
AMS subject classifications: *35D05, 35Q72, 74F15*

1 Introduction

Compared to other resistance welding technologies, impulse welding is characterized by an extremely short weld time and a very small heat-affected zone (HAZ). Therefore, this process is well–suited for the welding of different or coated materials. Owing to the smallness of the HAZ, the joint is cooled down rapidly by the surrounding material, leading to fine grained structure of the melted zone.

The aim of this paper is to derive a mathematical model for this process, prove the existence of (weak) solutions and present some numerical results. The model accounts for electrical, thermal and mechanical effects, which are nonlinearly coupled by the balance laws, constitutive equations and boundary conditions. It can be viewed as an extension to the thermistor problem, which has been investigated by quite a number of authors (see e.g. [1], [12] and the references therein).

In two previous papers [7], [8] we investigated related electro-thermoelastic problems. Here, we pay special attention to the electrical properties of the weld machine. They are discribed by an oscillator circuit, which is coupled with the balance law of electrical charge by a non-standard Neumann boundary condition. Moreover, we have included an additional quadratic term in the energy balance equation, which stems from the energy dissipation due to the temperature dependent nonelastic part of the constitutive law.

On the other hand we do not consider the influence of the solid-liquid phase transition. To prove existence of weak solutions to the model, we also have to neglect the influence of thermal expansion. However, this assumption does not pose a big restriction, since the numerical simulations confirm that the influence of thermal expansion is indeed small compared to the viscoelastic contribution.

The paper is organized as follows. In the next section we explain the impulse resistance welding process and derive the balance laws and constitutive relationships. In Section 3 we state the mathematical model as well as assumptions on the data and formulate an existence result. Section 4 is devoted to the proof of the existence theorem. We conclude with some numerical results and comments on further research.

2 Process description and derivation of model equations

Figure 1 depicts a sketch of the impulse welding process. First, the capacitor (1) is charged by the transformer (2). For fixed capacity the charging voltage U_C defines the stored energy that can be supplied to the welding process. Then the welding process is started by applying a pressure f onto the electrode (3).

When the primary circuit between capacitor and impulse transformer (4) is closed, current flows through the contact area between the weld pieces (5), (6) in the secondary circuit. In this damped resonant circuit the capacitor energy is completely transformed into heat during a short period of time. Owing to the geometric singularity, the biggest increase in temperature takes place in the projection tip.

The impulse transformer is adjusted to acchieve a discharging current below the aperiodic limit. In this case the efficiency of the welding process is higher than in the case of a discharging current that develops as a damped oscillation, because the amount of energy

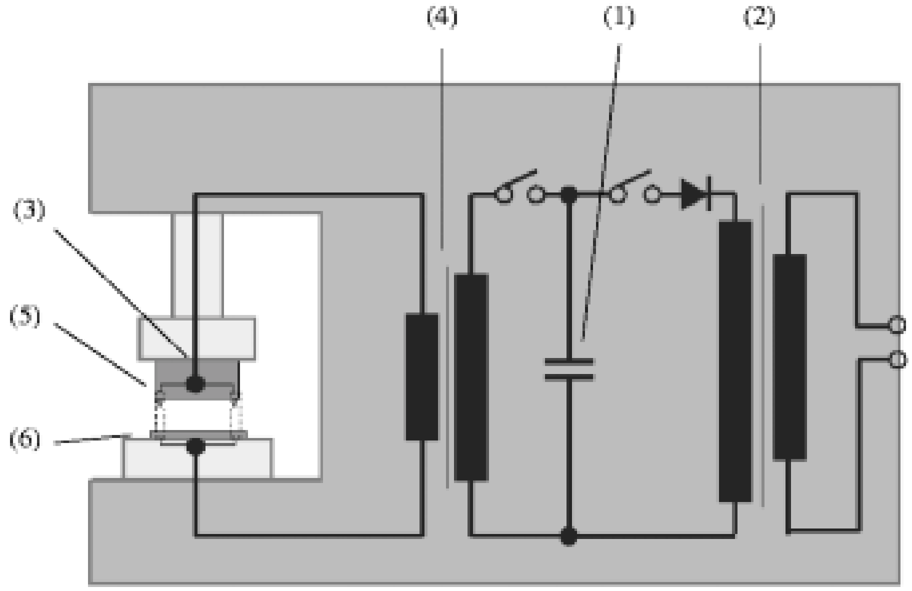

Figure 1: Sketch of a capacitor impulse weld machine.

that can dissipate by heat conduction is smaller, leading also to a smaller size of the heat affected zone.
When the melting temperature is reached in the contact area, the joint begins to grow. The increasing deformation of the projection corresponds to a growing contact area and a decreasing resistance at the joint. Finally, heat conduction and a decreasing discharging current prevent a further rise of temperature and the process stops with a fast cooling of the welded pieces.
To derive the physical model we make the following assumptions and simplifications:
The workpieces to be joined are assumed to be isotropic, deformable, electrically and thermally conducting materials. More precisely, we describe the melted zone as a Maxwellian thermo-viscoelastic body. The solid part, including the heat affected zone is assumed to behave thermoelastically.
To obtain basic features of the process, we make further simplifications. Assuming small deformations we formulate the balance laws in the undeformed region. Therefore one can only expect reasonable results from the simulations as long as the overall deformations, especially in the projection tip, remain small. Finally, we also neglect the initial plastic deformation of the projection tip.
To obtain the velocity $v(x,t)$, the electric potential $\phi(x,t)$ and the temperature $\theta(x,t)$, we evaluate the quasistatic balance laws of momentum and electrical charge as well as the balance law of internal energy:

$$\begin{aligned} \text{div }(\sigma) &= 0, && (2.1a)\\ \text{div }(J) &= 0, && (2.1b)\\ \rho\dot{e} + \text{ div } q &= \sigma : \varepsilon(v) + E \cdot J. && (2.1c) \end{aligned}$$

Here ρ is the mass density, σ the stress tensor, $E = -\nabla\phi$ the electric field, J the electric current density, q the heat flux, e the specific internal energy and $\varepsilon(v) = \frac{1}{2}(Dv + D^T v)$ the strain rate tensor. The scalar products in R^n and $R^{n\times n}$ are denoted by '$\cdot$' and ':', respectively.
The constitutive equations for J, q and σ are

$$
\begin{aligned}
J &= \hat{J}(\theta, \nabla\phi) = -\gamma(\theta)\nabla\phi, && (2.2a)\\
q &= \hat{q}(\theta, \nabla\theta) = -\lambda\nabla\theta, && (2.2b)\\
\sigma &= \hat{\sigma}(\theta, \varepsilon, \varepsilon^c) = K\Big(\varepsilon - \varepsilon^c - \alpha(\theta - \theta_R)I\Big). && (2.2c)
\end{aligned}
$$

Equations (2.2a) and (2.2b) are the laws of Ohm and Fourier, γ is the specific electric conductivity, and λ is the heat conductivity, (2.2c) is Hooke's law for small displacements of a thermoelastic maxwellian body and $K = \{K_{ijkl}\}$ is the isotropic stiffness tensor. The parameter θ_R is a reference temperature, and α denotes the thermal expansion coefficient. For the creep strain ε^c, which is the internal variable of the maxwellian body, we assume an evolution equation similar to the Norton creep law,

$$\dot{\varepsilon^c} = B(\theta, \sigma) = \eta_1(\theta)|S|^{\eta_2 - 2}S, \tag{2.3}$$

where $S = \sigma - \frac{1}{n}(tr\,\sigma)I$ is the trace–free part of the stress tensor and n is the space dimension.
Combining (2.2c) and (2.3), we obtain

$$\dot{\sigma} + KB(\sigma, \theta) = K\Big(\varepsilon(v) - \alpha\dot{\theta}I\Big). \tag{2.4}$$

To derive a constitutive equation for the internal energy we exploit the entropy inequality with the entropy flux $h = \frac{1}{\theta}q$,

$$\rho\dot{s} + \text{ div } \left(\frac{1}{\theta}q\right) \geq 0. \tag{2.5}$$

The evaluation of this inequality can be considerably simplified by introducing Lagrange multipliers Λ^v, Λ^θ and Λ^c and considering

$$
\begin{aligned}
&\rho\dot{s} + \text{ div } \left(\frac{1}{\theta}q\right) - \Lambda^v \cdot \Big(\rho\dot{v} - \text{ div } \sigma\Big)\\
&\qquad -\Lambda^\theta\Big(\rho\dot{e} + \text{ div } q - \sigma : \varepsilon(v)\Big) - \Lambda^c : \Big(\dot{\varepsilon^c} - B(\theta, \sigma)\Big) \geq 0. \qquad (2.6)
\end{aligned}
$$

Here we replaced the quasistatic momentum balance (2.1a) with the complete one. While (2.5) is valid only for solutions to the field equations (2.1a)-(2.2c), (2.6) must hold for all fields, according to I Shi Liu's theorem [11].
Using the Helmholtz free energy Ψ, we have

$$e = \Psi + s\theta.$$

Now we assume that s and Ψ only depend on ε, ε^c, θ, $\nabla\theta$ and that they are continuously differentiable with respect to these quantities, i.e.

$$
\begin{aligned}
\dot{s} &= \frac{\partial s}{\partial \varepsilon} : \varepsilon(v) + \frac{\partial s}{\partial \varepsilon^c} : \dot{\varepsilon^c} + \frac{\partial s}{\partial \theta}\dot{\theta} + \frac{\partial s}{\partial \nabla\theta} \cdot (\dot{\nabla\theta}), && (2.7)\\
\dot{\Psi} &= \frac{\partial \Psi}{\partial \varepsilon} : \varepsilon(v) + \frac{\partial \Psi}{\partial \varepsilon^c} : \dot{\varepsilon^c} + \frac{\partial \Psi}{\partial \theta}\dot{\theta} + \frac{\partial \Psi}{\partial \nabla\theta} \cdot (\dot{\nabla\theta}). && (2.8)
\end{aligned}
$$

Since (2.6) is linear in $\dot{v}$, and also in $\operatorname{div} q$, we obtain $\Lambda^v = 0$ and $\Lambda^\theta = \frac{1}{\theta}$, leading to the inequality

$$-q \cdot \nabla\theta - \frac{1}{\theta}\Big(\rho(\dot{\Psi} + s\dot{\theta}) - \sigma : \varepsilon(v)\Big) - \Lambda^c : \Big(\dot{\varepsilon^c} - B(\theta,\sigma)\Big) \geq 0. \tag{2.9}$$

Next, in view of (2.8) we see that (2.9) is linear with respect to $\dot{\theta}$ and obtain $s = -\frac{\partial\Psi}{\partial\theta}$. In the same way we conclude that ψ is independent of $\nabla\theta$, $\sigma = \rho\frac{\partial\psi}{\partial\varepsilon}$ as well as $\Lambda^c = -\frac{\rho}{\theta}\frac{\partial\Psi}{\partial\varepsilon^c}$. Invoking (2.2b) and (2.3) we finally arrive at

$$-\frac{\rho}{\theta}\eta_1(\theta)|S|^{\eta_2-1}\frac{\partial\Psi}{\partial\varepsilon^c} : S \geq 0, \tag{2.10}$$

which can only be satisfied if $\frac{\partial\Psi}{\partial\varepsilon^c} = \kappa S$, with a scalar $\kappa < 0$. Assuming ψ is twice continuously differentiable and using Hooke's law (2.2c), we obtain $\kappa = -1/\rho$. Finally, we gather up all these relationships to conclude

$$\rho\dot{e} = \rho c_V\dot{\theta} + (\sigma - \theta\sigma_{,\theta}) : \Big(\varepsilon(v) - B(\theta,\sigma)\Big). \tag{2.11}$$

Here, $c_V = e_{,\theta}$ is the specific heat capacity at constant volume and we have

$$\sigma_{,\theta} = -\alpha K I. \tag{2.12}$$

To prescribe boundary conditions, we distinguish between three different parts of the boundary $\Gamma = \partial\Omega$: Γ_0, where the workpiece is fixed, Γ_1, where the pressure is applied to the workpiece and the electric current from the oscillator circuit is supplied, and the remaining part $\Gamma_2 = \Gamma \setminus (\Gamma_0 \cup \Gamma_1)$ (cf. Fig. 2). Owing to the short weld time of few milliseconds, we neglect heat conduction across the boundary, i.e. we assume

$$q \cdot \nu = 0 \quad \text{on } \Gamma \times (0,T), \tag{2.13a}$$

where ν is the outer normal vector to Γ. For σ and v, we assume

$$v = 0 \quad \text{on } \Gamma_0 \times (0,T), \tag{2.13b}$$

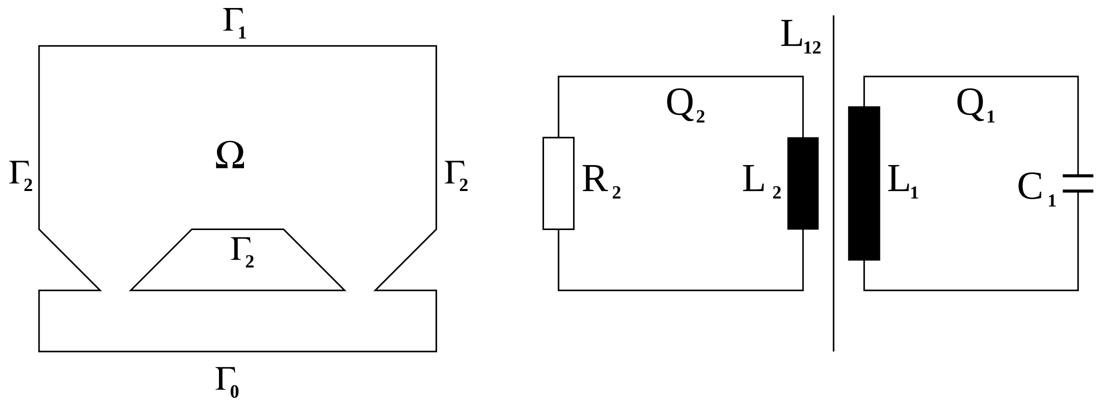

Figure 2: Definition of boundaries (left) and circuit diagram of the resonant ircuit during welding (right).

$$\sigma_{ij}\nu_j = \begin{cases} f_i & \text{on } \Gamma_1 \times (0,T) \\ 0 & \text{on } \Gamma_2 \times (0,T). \end{cases} \tag{2.13c}$$

For the electric potential we assume

$$\phi = 0 \quad \text{on } \Gamma_0 \times (0,T), \quad J \cdot \nu = 0 \quad \text{on } \Gamma_2 \times (0,T). \tag{2.13d}$$

On $\Gamma_1 \times (0,T)$ we assume

$$J \cdot \nu = -\frac{1}{|\Gamma_1|}\dot{Q}_2. \tag{2.13e}$$

Here, $|\Gamma_1|$ indicates the area of Γ_1 and $\dot{Q}_2$ is the current intensity in the secondary circuit as depicted in Fig. 2. $Q = (Q_1, Q_2)^T$ is obtained as the solution to the following system of ordinary differential equations

$$\begin{pmatrix} L_1 & L_{12} \\ L_{12} & L_2 \end{pmatrix} \begin{pmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \end{pmatrix} + \begin{pmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & R_2 \end{pmatrix} \begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{pmatrix} =: \begin{pmatrix} 0 \\ -U_\Omega \end{pmatrix},$$

or equivalently

$$\mathcal{L}\ddot{Q} + \mathcal{C}^{-1}Q = -\mathcal{R}\dot{Q}, \tag{2.14}$$

with initial conditions $\dot{Q}(0) = (0,0)^T$, $Q(0) = (C_1 U_C, 0)^T$. U_C is the charging voltage according to Fig. 1 and R_2 is the discrete equivalent resistance of the weld pieces Ω. The mutual inductivity L_{12} is given by $L_{12} = k\sqrt{L_1 L_2}$ with a coupling factor $k \in (0,1)$. This implies that the matrix $\mathcal{L}$ is not singular.
The voltage drop U_Ω on Ω can be derived from the identity

$$U_\Omega \dot{Q}_2 = \int_\Omega J \cdot E = \int_\Omega \gamma(\theta)|\nabla\phi|^2.$$

Testing (2.1b) with ϕ using (2.2a) and (2.12), we obtain

$$\int_\Omega \gamma(\theta)|\nabla\phi|^2 = \dot{Q}_2 \frac{1}{|\Gamma_1|} \int_{\Gamma_1} \phi.$$

In other words, we have

$$U_\Omega = \frac{1}{|\Gamma_1|} \int_{\Gamma_1} \phi. \tag{2.15}$$

3 Problem statement and main result

In the sequel, we neglect the effect of thermal expansion, i.e. we assume $\alpha \equiv 0$. Moreover, we consider a particular viscous law, i.e. we put $\eta_2 = 2$ in (2.3). Let $\Omega \subset \boldsymbol{R}^3$ be a bounded domain with smooth boundary Γ, $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, such that meas $\Gamma_0 > 0$ and $\Gamma_i \cap \Gamma_j = \emptyset$, if $i \neq j$, and $Q = \Omega \times (0, T)$.
Summing up the balance laws (2.1a-c), the constitutive equations (2.4), (2.11) and the boundary conditions (2.13a-e), we consider the following problem:

Find $v = (v_1, v_2, v_3)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$, θ, ϕ such that

$$\begin{aligned}
-\operatorname{div}\sigma &= 0 \quad \text{in } Q, && (3.16a)\\
\varepsilon_{ij}(v) &= C_{ijkl}\dot{\sigma}_{kl} + \eta(\theta)S_{ij} \quad \text{in } Q, && (3.16b)\\
\dot{\theta} - \Delta\theta &= \eta(\theta)|S|^2 + \gamma(\theta)|\nabla\phi|^2 \quad \text{in } Q, && (3.16c)\\
-\operatorname{div}(\gamma(\theta)\nabla\phi) &= 0 \quad \text{in } Q, && (3.16d)\\
\sigma(.,0) = \sigma_0, &\qquad \theta(.,0) = \theta_0 \quad \text{in } \Omega, && (3.16e)\\
\frac{\partial\theta}{\partial\nu} &= 0 \quad \text{on } \Gamma \times (0,T), && (3.16f)\\
v = 0, &\qquad \phi = 0 \quad \text{on } \Gamma_0 \times (0,T), && (3.16g)\\
\sigma_{ij}\nu_j = f_i, &\qquad \gamma(\theta)\frac{\partial\phi}{\partial\nu} = A\phi + a \quad \text{on } \Gamma_1 \times (0,T), && (3.16h)\\
\sigma_{ij}\nu_j = 0, &\qquad \gamma(\theta)\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \Gamma_2 \times (0,T). && (3.16i)
\end{aligned}$$

Here, $C = K^{-1}$ and we have normed all physical parameters to one that do not depend on temperature.
We make the following assumptions:

(A1) $\eta, \gamma \in C(\boldsymbol{R})$, such that $|\eta(\xi)| \leq \eta_0$, $0 < \gamma_1 < \gamma(\xi) < \gamma_2$ for all $\xi \in \boldsymbol{R}$, where η_0 and $\gamma_{1,2}$ are positive constants,

(A2) $f_i \in H^1(0, T, L^2(\Gamma_1))$, $i = 1, 2, 3$,

(A3) $\{\sigma_0\}_{ij} \in L^2(\Omega)$, $i, j = 1, 2, 3$, $\theta_0 \in L^1(\Omega)$,

(A4) $C_{ijkl} \in L^\infty(\Omega)$, $C_{ijkl} = C_{klji} = C_{jikl}$, $C_{ijkl}\xi_{ij}\xi_{kl} \geq C_0|\xi|^2$, for all $\xi_{ij} = \xi_{ji}$ and a positive constant C_0,

(A5) $A : L^2(0, T, L^2(\Gamma_1)) \to C[0, T]$ is a linear continuous operator satistying the condition

$$|(A\phi)(t)| \leq c_A \int_0^t \|\phi(\xi)\|_{L^2(\Gamma_1)} d\xi \ ,$$

where a, c_A are constants.

Moreover, we assume that all functions with two lower indices are symmetric in those indices.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space with $p \geq 1$ and

$$W^{1,p}_{\Gamma_0}(\Omega) = \{v \in W^{1,p}(\Omega)\Big| v = 0 \text{ on } \Gamma_0\}, \quad \text{with norm } \|v\|_{W^{1,p}_{\Gamma_0}(\Omega)} = \Big(\int_\Omega |\nabla v|^2\Big)^{1/2}.$$

We denote

$$\mathcal{W}_T = L^2(0,T; W^{1,2}_{\Gamma_0}(\Omega)),$$

$$\mathcal{V} = \{\psi \in L^{q'}(0,T; W^{1,q'}(\Omega)) \cap C([0,T]; L^\infty(\Omega)) | \dot{\psi} \in L^2(Q),\, \psi(T) = 0\},$$

where $q' > 5$ will be chosen such that $\frac{1}{q} + \frac{1}{q'} = 1$ for some $q \in (1, \frac{5}{4})$ to be defined later.

Remark 3.1 *Since the matrix $\mathcal{L}$ is not singular, the system (2.14) has a unique solution (Q_1, Q_2) for every given $\phi \in L^2(0,T,L^2(\Gamma_1))$. Hence, the boundary condition (2.13e) is well-defined. Testing (2.14) with $(\dot{Q}_1, \dot{Q}_2)$ it is an easy exercise to prove that (2.13e) is equivalent to* (3.16h)*, where the operator A satisfies (A5).*

Our main result is

Theorem 3.1 *Assume (A1)-(A5). Then, there exists a solution to problem* (3.16a) - (3.16i)*, such that*

$$v_i,\, \phi \in \mathcal{W}_T,\, \sigma_{ij}, \dot{\sigma}_{ij} \in L^2(Q) \quad \text{for } i,j = 1,2,3,$$

$$\theta \in L^q(0,T,W^{1,q}(\Omega)) \quad \text{for } q \in (1, \frac{5}{4}),$$

$$\int_Q \sigma : \varepsilon(u) = \int_{\Gamma_1 \times (0,T)} fu \quad \text{for all } u = (u_1,u_2,u_3) \in [\mathcal{W}_T]^3, \tag{3.17a}$$

$$\varepsilon(v) = C\dot{\sigma} + \eta(\theta) S \qquad \text{in } Q, \tag{3.17b}$$

$$-\int_Q \theta\dot{\psi} + \int_Q \nabla\theta\nabla\psi = \int_Q \Big(\eta(\theta)|S|^2 + \gamma(\theta)|\nabla\phi|^2\Big)\psi + \int_\Omega \theta_0\psi(0) \quad \text{for all } \psi \in \mathcal{V}, \tag{3.17c}$$

$$\int_Q \gamma(\theta)\nabla\phi\nabla\varphi = \int_{\Gamma_1\times(0,T)} \Big(A\phi + a\Big)\varphi \quad \text{for all } \varphi \in \mathcal{W}_T, \tag{3.17d}$$

and the initial condition (3.16e) *for σ holds.*

The proof will be detailed in the next section. The main difficulty lies in the quadratic terms on the right-hand side of (3.17c). However, it can be handled using an estimate developed in [2]. Thus we can proceed similar to the proof of the Main Theorem in [3], where an induction heating problem has been considered.
We first consider a regularized boundary value problem with a truncation operator and prove the existence of a solution for a fixed parameter δ. Then we obtain uniform estimates in δ and pass to the limit as $\delta \to 0$.

4 Proof of Theorem 3.1

4.1 The truncated problem

For $\delta > 0$ we introduce the function

$$\tau_\delta(\xi) = \begin{cases} \frac{1}{\delta}, & \xi > \frac{1}{\delta} \\ \xi, & |\xi| \leq \frac{1}{\delta} \\ -\frac{1}{\delta}, & \xi < -\frac{1}{\delta} \end{cases}$$

and consider the following boundary value problem.

Find $v = (v_1, v_2, v_3)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$, θ, ϕ such that

$$\begin{aligned}
-\operatorname{div} \sigma &= 0 \quad \text{in } Q, && (4.18a)\\
\varepsilon_{ij}(v) &= C_{ijkl}\dot{\sigma}_{kl} + \eta(\theta) S_{ij} \quad \text{in } Q, && (4.18b)\\
\dot{\theta} - \Delta\theta &= \eta(\theta)\tau_\delta\left(|S|^2\right) + \gamma(\theta)\tau_\delta\left(|\nabla\phi|^2\right) \quad \text{in } Q, && (4.18c)\\
-\operatorname{div}\left(\gamma(\theta)\nabla\phi\right) &= 0 \quad \text{in } Q, && (4.18d)\\
\sigma(.,0) = \sigma_0, &\qquad \theta(.,0) = \tau_\delta(\theta_0) \quad \text{in } \Omega, && (4.18e)\\
\frac{\partial\theta}{\partial\nu} &= 0 \quad \text{on } \Gamma \times (0,T), && (4.18f)\\
v = 0, &\qquad \phi = 0 \quad \text{on } \Gamma_0 \times (0,T), && (4.18g)\\
\sigma_{ij}\nu_j = f_i, &\qquad \gamma(\theta)\frac{\partial\phi}{\partial\nu} = A\phi + a \quad \text{on } \Gamma_1 \times (0,T), && (4.18h)\\
\sigma_{ij}\nu_j = 0, &\qquad \gamma(\theta)\frac{\partial\phi}{\partial\nu} = 0 \quad \text{on } \Gamma_2 \times (0,T). && (4.18i)
\end{aligned}$$

To prove the existence of a solution to (4.18a) - (4.18i) for a fixed δ, we apply the Schauder fixed point theorem.
First we consider the problem of finding ϕ with a given $\bar{\theta} \in L^2(Q)$, such that

$$\begin{aligned}
-\operatorname{div}\left(\gamma(\bar{\theta})\nabla\phi\right) &= 0 \quad \text{in } Q, && (4.19a)\\
\phi = 0 \quad \text{on } \Gamma_0 \times (0,T), &\qquad \gamma(\bar{\theta})\frac{\partial\phi}{\partial\nu} = A\phi + a \quad \text{on } \Gamma_1 \times (0,T), && (4.19b)\\
\gamma(\bar{\theta})\frac{\partial\phi}{\partial\nu} = 0 &\qquad \text{on } \Gamma_2 \times (0,T), && (4.19c)
\end{aligned}$$

and define $\phi = \phi(\bar{\theta}) \in \mathcal{W}_T$.
Then we consider the problem

$$\begin{aligned}
-\operatorname{div} \sigma &= 0 \quad \text{in } Q, && (4.20a)\\
\varepsilon(v) &= C\dot{\sigma} + \eta(\bar{\theta}) S \quad \text{in } Q, && (4.20b)\\
v = 0 &\qquad \text{on } \Gamma_0 \times (0,T), && (4.20c)\\
\sigma_{ij}\nu_j = f_i &\qquad \text{on } \Gamma_1 \times (0,T), && (4.20d)\\
\sigma_{ij}\nu_j = 0 &\qquad \text{on } \Gamma_2 \times (0,T), && (4.20e)\\
\sigma(.,0) = \sigma_0 &\qquad \text{in } \Omega, && (4.20f)
\end{aligned}$$

and find $S = S(\bar\theta) \in L^2(Q)$.
The next step is to consider the problem of finding θ with known functions $S = S(\bar\theta)$ and $\phi = \phi(\bar\theta)$, such that

$$\dot\theta - \Delta\theta = \eta(\bar\theta)\tau_\delta\Big(|S|^2\Big) + \gamma(\bar\theta)\tau_\delta\Big(|\nabla\phi|^2\Big) \quad \text{in } Q, \tag{4.21a}$$

$$\frac{\partial\theta}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{4.21b}$$

$$\theta(.,0) = \tau_\delta(\theta_0) \quad \text{in } \Omega. \tag{4.21c}$$

We can solve the problem (4.21a) - (4.21c) and find $\theta = \theta(\bar\theta)$ such that $\theta \in \mathcal{U}$,

$$\mathcal{U} = \{u \in L^2(0,T;W^{1,2}(\Omega)) \big| \dot u \in L^2(0,T;(W^{1,2}(\Omega))^*)\},$$

where $(W^{1,2}(\Omega))^*$ is the space dual of $W^{1,2}(\Omega)$. According to [13] the embedding $\mathcal{U} \subset L^2(Q)$ is compact.
Thus, we have constructed a compact map

$$B \,:\, L^2(Q) \ni \bar\theta \mapsto \theta \in L^2(Q), \tag{4.22}$$

and we want to find a fixed point of this map by using the Schauder theorem. This will be done in the following four Lemmas.

Lemma 4.1 *Let $\bar\theta \in L^2(Q)$ be given. Then, there exists a unique weak solution $\phi = \phi(\bar\theta) \in \mathcal{W}_T$ to* (4.19a) - (4.19c).

Proof:
We apply the Banach fixed point theorem. To this end, we introduce a mapping $F \,:\, \mathcal{W}_T \to \mathcal{W}_T$, $\hat\phi \mapsto \phi$, where ϕ is the solution to

$$\int\limits_Q \gamma(\bar\theta)\nabla\phi\nabla\varphi = \int\limits_0^T\int\limits_{\Gamma_1}(A\hat\phi + a)\varphi \quad \text{for all } \varphi \in \mathcal{W}_T. \tag{4.23}$$

Since this linear elliptic equation is uniquely solvable, F is well-defined. By the continuous embedding $W^{1,2}_{\Gamma_0}(\Omega) \subset L^2(\Gamma_1)$ and by (A5), there exists a constant c_1, such that

$$\|v\|_{L^2(\Gamma_1)} \le c_1\|v\|_{W^{1,2}_{\Gamma_0}(\Omega)} \quad \text{and } |(Av)(t)| \le c_1 c_A t^{1/2}\|v\|_{L^2(0,t;W^{1,2}_{\Gamma_0}(\Omega))} \quad \text{for } t \in [0,T].$$

Now let $\phi = \phi_1 - \phi_2$, where $\phi_i = F\hat\phi_i$ and $\hat\phi = \hat\phi_1 - \hat\phi_2$. From the identities similar to (4.24) below, written for ϕ and $\hat\phi$, we obtain

$$\gamma_1 \int\limits_0^t \|\phi\|^2_{W^{1,2}_{\Gamma_0}(\Omega)} \le t^{1/2}|\Gamma_1|^{1/2}\|A\hat\phi\|_{C[0,t]}\|\phi\|_{L^2(0,t;L^2(\Gamma_1))}$$

$$\le t|\Gamma_1|^{1/2}c_1^2 c_A\|\hat\phi\|_{L^2(0,t;W^{1,2}_{\Gamma_0}(\Omega))}\|\phi\|_{L^2(0,t;W^{1,2}_{\Gamma_0}(\Omega))}.$$

Applying Young's inequality, we can conclude that F is a contraction on $\mathcal{W}_{T^+}$, if T^+ has been chosen small enough. In the same way we can prove that F is a self-mapping on some $M \subset \mathcal{W}_{T^+}$. Hence, we can apply the Banach fixed point theorem on $\mathcal{W}_{T^+}$ and obtain the existence on the whole time interval $(0,T)$ by a bootstrap argument. It is important that we have the global estimate of the solution to (4.19a)-(4.19c). Indeed, this solution satisfies the following identity

$$\int_{\Omega} \gamma(\theta)\nabla\phi(t)\nabla\varphi = \int_{\Gamma_1} (A\phi(t) + a)\varphi \quad \text{for all } \varphi \in W^{1,2}_{\Gamma_0}(\Omega) \tag{4.24}$$

for almost all $t \in (0,T)$. Let $\varphi = \phi(t)$. Taking into account the inequality from (A5) the last relation implies

$$\|\phi(t)\|^2_{W^{1,2}_{\Gamma_0}(\Omega)} \leq c_3 \int_0^t \|\phi(\xi)\|^2_{W^{1,2}_{\Gamma_0}(\Omega)} d\xi + c_4 \ ,$$

where the constants c_3, c_4 are independent of ϕ. Hence by the Gronwall lemma the following estimate is obtained,

$$\|\phi\|_{\mathcal{W}_T} \leq c \ .$$

□

Lemma 4.2 *Problem* (4.20a) - (4.20f) *has a unique solution such that*

$$v_i \in \mathcal{W}_T, \quad \sigma_{ij},\, \dot{\sigma}_{ij} \in L^2(Q) \text{ for } i,j = 1,2,3,$$

$$\begin{aligned} \int_Q \sigma : \varepsilon(u) &= \int_{\Gamma_1 \times (0,T)} fu \quad \text{for all } u \in [\mathcal{W}_T]^3, \\ \varepsilon(v) &= C\dot{\sigma} + \eta(\theta)S \quad \text{a.e. in } Q, \\ \sigma(.,0) &= \sigma_0 \quad \text{in } \Omega. \end{aligned}$$

For the proof, we refer to [9]. Hence, we find a unique function $S = S(\bar{\theta})$.

Lemma 4.3 *For $\phi = \phi(\bar{\theta})$ and $S = S(\bar{\theta})$ problem* (4.21a) - (4.21c) *has a unique solution $\theta \in \mathcal{U}$.*

Proof:
For fixed $\delta > 0$, we have

$$h^\delta(\bar{\theta}) \equiv \eta(\bar{\theta})\tau_\delta\left(|S|^2\right) + \gamma(\bar{\theta})\tau_\delta\left(|\nabla\phi|^2\right) \in L^2(Q).$$

Defining the operator $\mathrm{L} : L^2(0,T;W^{1,2}(\Omega)) \longrightarrow L^2(0,T;(W^{1,2}(\Omega))^*)$ by the formula

$$\mathrm{L}(\theta)\tilde{\theta} = \int_Q \nabla\theta \cdot \nabla\tilde{\theta}, \quad \theta,\, \tilde{\theta} \in L^2(0,T;W^{1,2}(\Omega)),$$

problem (4.21a) - (4.21c) can be written as an abstract evolution problem

$$\begin{aligned} \dot{\theta} - \mathrm{L}(\theta) &= h^{\delta}(\bar{\theta}) \quad \text{in } L^2(0,T;(W^{1,2}(\Omega))^*) \ , & (4.25)\\ \theta(0) &= \tau_{\delta}(\theta_0) \ . & (4.26) \end{aligned}$$

Note that $L^2(0,T;W^{1,2}(\Omega)) \subset L^2(Q) \subset L^2(0,T;(W^{1,2}(\Omega))^*)$ with continuous injections. Due to general results for evolution equations (cf. e.g. [10]), there exists a unique solution $\theta \in \mathcal{U}$ to problem (4.25), (4.26). Observe that $\theta \in \mathcal{U}$ implies $\theta \in C([0,T];L^2(Q))$, and thus the initial condition (4.26) is fulfilled. □

Since $\mathcal{U}$ has compact embedding into $L^2(Q)$, we have

$$\theta = \theta(\bar{\theta}) = B(\bar{\theta}).$$

Lemma 4.4 *The operator B as defined in* (4.22) *is continuous.*

Proof:
Let $\bar{\theta}^n \to \bar{\theta}$ strongly in $L^2(Q)$ and denote $v^n = v(\bar{\theta}^n)$, $\sigma^n = \sigma(\bar{\theta}^n)$, $\phi^n = \phi(\bar{\theta}^n)$, $S^n = S(\bar{\theta}^n)$, $\theta^n = \theta(\bar{\theta}^n)$. We can assume that

$$\bar{\theta}^n \to \bar{\theta} \quad \text{a.e. in } Q;$$

$$\eta(\bar{\theta}^n) \to \eta(\bar{\theta}), \quad \gamma(\bar{\theta}^n) \to \gamma(\bar{\theta}) \quad \text{a.e. in } Q \text{ and strongly in } L^p(Q),\ p \in [1,\infty). \qquad (4.27)$$

In view of (A1) - (A4), for the boundary value problem

$$\int_Q \sigma^n : \varepsilon(u) = \int_{\Gamma_1 \times (0,T)} fu \quad \text{for all } u \in [\mathcal{W}_T]^3, \qquad (4.28)$$

$$\begin{aligned} \varepsilon(v^n) &= C\dot{\sigma}^n + \eta(\bar{\theta}^n)S^n \qquad \text{in } Q, & (4.29)\\ \sigma(.,0) &= \sigma_0 \qquad \text{in } \Omega, & (4.30) \end{aligned}$$

we can infer

$$\|\sigma^n_{ij}\|_{L^2(Q)} + \|\dot{\sigma}^n_{ij}\|_{L^2(Q)} + \|v^n_i\|_{\mathcal{W}_T} \le c,$$

uniformly in n and $i,j = 1,2,3$. Hence, we can assume as $n \to \infty$

$$\sigma^n_{ij},\ \dot{\sigma}^n_{ij},\ S^n_{ij} \longrightarrow \sigma_{ij},\ \dot{\sigma}_{ij},\ S_{ij} \quad \text{weakly in } L^2(Q),$$

$$v^n \longrightarrow v \quad \text{weakly in } [\mathcal{W}_T]^3.$$

Note that by (4.27),

$$\eta(\bar{\theta}^n)S^n_{ij} \longrightarrow \eta(\bar{\theta})S_{ij} \quad \text{weakly in } L^2(Q).$$

Consequently, relations (4.28) - (4.30) imply

$$\int_Q \sigma^n : \varepsilon(u) = \int_{\Gamma_1 \times (0,T)} fu, \quad \text{for all } u \in [\mathcal{W}_T]^3, \qquad (4.31)$$

$$\begin{aligned} \varepsilon(v) &= C\dot{\sigma} + \eta(\bar{\theta})S \qquad \text{in } Q, & (4.32)\\ \sigma(.,0) &= \sigma_0 \quad \text{in } \Omega. & (4.33) \end{aligned}$$

>From (4.31) - (4.33) it follows that

$$\begin{aligned}
\int_Q (\sigma^n - \sigma) : \varepsilon(u) &= \int_{\Gamma_1\times(0,T)} fu \quad \text{for all } u \in [\mathcal{W}_T]^3, \\
\varepsilon(v^n - v) &= C(\dot\sigma^n - \dot\sigma) + \eta(\bar\theta^n)S^n - \eta(\bar\theta)S \qquad \text{in } Q, \\
(\sigma^n - \sigma)(.,0) &= 0 \qquad \text{in } \Omega,
\end{aligned}$$

which provides the relation

$$\frac{1}{2}\frac{d}{dt}\int_\Omega C(\sigma^n - \sigma) : (\sigma^n - \sigma) = \int_\Omega \eta(\bar\theta^n)(S^n - S) : (\sigma - \sigma^n) + \int_\Omega (\eta(\bar\theta^n) - \eta(\bar\theta))S : (\sigma - \sigma^n).$$

Integrating in time we derive

$$\|(\sigma^n - \sigma)(t)\|^2_{L^2(\Omega)} \le c_1 \int_0^t \|\sigma^n - \sigma\|^2_{L^2(\Omega)} + c_2 \int_0^t \int_\Omega (\eta(\bar\theta^n) - \eta(\bar\theta))^2 |S|^2,$$

with positive constants $c_{1,2}$, independent of n. Hence, by the Gronwall's lemma and Lebesgue's convergence theorem, this inequality implies

$$\|\sigma^n - \sigma\|^2_{L^2(Q)} \le \lambda_n \longrightarrow 0, \quad n \to \infty,$$

and we conclude as $n \to \infty$

$$\begin{aligned}
\sigma^n &\longrightarrow \sigma \qquad \text{strongly in } L^2(Q), \\
|S^n|^2 &\longrightarrow |S|^2 \quad \text{strongly in } L^1(Q) \text{ and a.e. in } Q
\end{aligned}$$

as well as

$$\tau_\delta(|S^n|^2) \longrightarrow \tau_\delta(|S|^2) \quad \text{strongly in } L^p(Q),\ p \in [1,\infty) \text{ and a.e. in } Q\ , \tag{4.34}$$

$$\eta(\bar\theta^n)\tau_\delta(|S^n|^2) \longrightarrow \eta(\bar\theta)\tau_\delta(|S|^2) \quad \text{strongly in } L^2(Q). \tag{4.35}$$

Now we want to prove that as $n \to \infty$

$$\gamma(\bar\theta^n)\tau_\delta(|\nabla\phi^n|^2) \longrightarrow \gamma(\bar\theta)\tau_\delta(|\nabla\phi|^2) \quad \text{strongly in } L^2(Q), \tag{4.36}$$

where ϕ is a limit point of ϕ^n. Indeed, from the equations

$$\int_Q \gamma(\bar\theta^n)\nabla\phi^n \cdot \nabla\varphi = \int_{\Gamma_1\times(0,T)} (A\phi^n + a)\varphi \quad \text{for all } \varphi \in \mathcal{W}_T, \tag{4.37}$$

it follows uniformly in n

$$\|\phi^n\|_{\mathcal{W}_T} \le c,$$

with a positive constant c, and we can assume as $n \to \infty$

$$\phi^n \longrightarrow \phi \quad \text{weakly in } \mathcal{W}_T.$$

Then, by (4.27)

$$\gamma(\bar{\theta}^n)\nabla\phi^n \longrightarrow \gamma(\bar{\theta})\nabla\phi \quad \text{weakly in } L^2(Q). \tag{4.38}$$

In view of (4.38) and the weak convergence of ϕ^n, equations (4.37) provide

$$\int_Q \gamma(\bar{\theta})\nabla\phi\cdot\nabla\varphi = \int_{\Gamma_1\times(0,T)} (A\phi + a)\varphi \quad \text{for all } \varphi\in\mathcal{W}_T. \tag{4.39}$$

Note that relations (4.37), (4.39) are equivalent to the following identities

$$\int_\Omega \gamma(\theta^n)\nabla\phi^n(t)\nabla\varphi = \int_{\Gamma_1} (A\phi^n(t)+a)\varphi \quad \text{for all } \varphi\in W^{1,2}_{\Gamma_0}(\Omega),$$

$$\int_\Omega \gamma(\theta)\nabla\phi(t)\nabla\varphi = \int_{\Gamma_1} (A\phi(t)+a)\varphi \quad \text{for all } \varphi\in W^{1,2}_{\Gamma_0}(\Omega)$$

holding for almost all $t\in(0,T)$. Hence we obtain

$$\int_\Omega \gamma(\bar{\theta}^n)|\nabla\phi^n-\nabla\phi|^2 = \int_\Omega (\gamma(\bar{\theta}^n)-\gamma(\bar{\theta}))\nabla\phi\cdot\nabla(\phi-\phi^n) + \int_{\Gamma_1} A(\phi^n-\phi)(\phi^n-\phi).$$

Reasoning as in the proof of Lemma 4.1, we have

$$\|\phi^n(t)-\phi(t)\|^2_{W^{1,2}_{\Gamma_0}(\Omega)} \le c\int_0^t \|\phi^n(\xi)-\phi(\xi)\|^2_{W^{1,2}_{\Gamma_0}(\Omega)}d\xi + c\int_\Omega (\gamma(\bar{\theta}^n)-\gamma(\bar{\theta}))^2|\nabla\phi|^2.$$

Thus, we can apply Gronwall's lemma to conclude

$$\|\phi^n-\phi\|^2_{\mathcal{W}_T} \le \mu_n \longrightarrow 0,\ n\to\infty.$$

In addition, we have as $n\to\infty$

$$\begin{aligned} |\nabla\phi^n|^2 &\longrightarrow |\nabla\phi|^2 \quad \text{strongly in } L^1(Q) \text{ and a.e. in } Q,\\ \tau_\delta(|\nabla\phi^n|^2) &\longrightarrow \tau_\delta(|\nabla\phi|^2) \quad \text{strongly in } L^2(Q), \end{aligned}$$

and we have derived (4.36).
Testing the equations

$$\dot{\theta}^n - \Delta\theta^n = \eta(\bar{\theta}^n)\tau_\delta(|S^n|^2) + \gamma(\bar{\theta}^n)\tau_\delta(|\nabla\phi^n|^2), \tag{4.40}$$

$$\theta^n(0) = \tau_\delta(\theta_0), \tag{4.41}$$

with $\dot{\theta}^n$ we obtain uniformly in n,

$$\|\theta^n\|_{L^2(0,T;W^{1,2}(\Omega))\cap H^1(0,T;L^2(\Omega))} \le c.$$

Hence we can extract a subsequence still indicated by n such that

$$\theta^n \longrightarrow \theta \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega))\cap H^1(0,T;L^2(\Omega)). \tag{4.42}$$

By the compact embedding $L^2(0,T;W^{1,2}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \subset L^2(Q)$, we also have

$$\theta^n \longrightarrow \theta \quad \text{strongly in } L^2(Q). \tag{4.43}$$

In view of (4.42), (4.36) and (4.35) we can pass to the limit in (4.40), (4.41) and obtain $\theta = \theta(\bar\theta) = B(\bar\theta)$. Moreover, since the solution to all subproblems is unique, the limit does not depend on the extracted subsequence, hence the convergence holds for the whole sequence. □

To conclude with the fixed point argument let $\|\bar\theta\|_{L^2(Q)} \le \rho$ for some $\rho > 0$. Since the estimate

$$\|h^\delta(\bar\theta)\|_{L^2(Q)} + \|\tau_\delta(\theta_0)\|_{L^2(\Omega)} \le c$$

is uniform in ρ, from (4.25), (4.26) the uniform in ρ estimate

$$\|\theta\|_{\mathcal{U}} \le c$$

follows. Consequently,

$$\|B(\bar\theta)\|_{L^2(Q)} \le \rho$$

for sufficiently large ρ. Hence B is also a self-mapping. According to the Schauder fixed point theorem there exists a function $\bar\theta \in L^2(Q)$ such that $B(\bar\theta) = \bar\theta$.

4.2 Passing to the limit with $\delta \to 0$

Now we want to pass to the limit as $\delta \to 0$ in (4.18a) - (4.18i). To this end a solution to problem (4.18a) - (4.18i) corresponding to the parameter $\delta > 0$ will be denoted by $(v^\delta,\ \sigma^\delta,\ \theta^\delta,\ \phi^\delta)$ and it satisfies the following relations:

$$v_i^\delta,\ \phi^\delta \in \mathcal{W}_T, \quad \sigma_{ij}^\delta,\ \dot\sigma_{ij}^\delta \in L^2(Q), \quad i,j = 1,2,3, \quad \theta^\delta \in \mathcal{U}\ ,$$

$$\int_Q \sigma^\delta : \varepsilon(u) = \int_{\Gamma_1 \times (0,T)} fu \quad \text{for all } u = (u_1,u_2,u_3) \in [\mathcal{W}_T]^3, \tag{4.44a}$$

$$\varepsilon(v^\delta) = C\dot\sigma^\delta + \eta(\theta^\delta)S^\delta \qquad \text{in } Q, \tag{4.44b}$$

$$\begin{aligned} -\int_Q \theta^\delta\dot\psi + \int_Q \nabla\theta^\delta\nabla\psi &= \int_Q \Big(\eta(\theta^\delta)\tau_\delta(|S^\delta|^2) + \gamma(\theta^\delta)\tau_\delta(|\nabla\phi^\delta|^2)\Big)\psi \\ &\quad + \int_\Omega \tau_\delta(\theta_0)\psi(0) \quad \text{for all } \psi \in \mathcal{V}, \end{aligned} \tag{4.44c}$$

$$\int_Q \gamma(\theta)\nabla\phi^\delta\nabla\varphi = \int_{\Gamma_1\times(0,T)} \Big(A\phi^\delta + a\Big)\varphi \quad \text{for all } \varphi \in \mathcal{W}_T, \tag{4.44d}$$

$$\sigma^\delta(.,0) = \sigma_0 \quad \text{in } \Omega. \tag{4.44e}$$

>From (4.44a), (4.44b), (4.44d), (4.44e) it follows uniformly in δ and $1 \le i, j \le 3$,

$$\|v_i^\delta\|_{\mathcal{W}_T} + \|\sigma_{ij}^\delta\|_{L^2(Q)} + \|\dot\sigma_{ij}^\delta\|_{L^2(Q)} + \|\phi^\delta\|_{\mathcal{W}_T} \le c. \tag{4.45}$$

Hence, we have uniformly in δ,

$$\| \, |S^\delta|^2 \|_{L^1(Q)} + \| \, |\nabla \phi^\delta|^2 \|_{L^1(Q)} \le c,$$

and, by the definition of τ^δ, uniformly in δ,

$$\|\tau_\delta(|S^\delta|^2)\|_{L^1(Q)} + \|\tau_\delta(|\nabla\phi^\delta|^2)\|_{L^1(Q)} \le c,$$

as well as

$$\|\tau_\delta(\theta_0)\|_{L^1(\Omega)} \le \|\theta_0\|_{L^1(\Omega)}.$$

Now we can use known results for linear parabolic equations with initial data and right-hand sides belonging to $L^1(\Omega)$ and $L^1(Q)$, respectively, which provide the existence of solutions in the space $L^q(0,T;W^{1,q}(\Omega))$, $1 < q < \frac{5}{4}$. In the case of Dirichlet boundary conditions, the result can be found in [2]. The case of Neumann conditions is considered in [4]. The parameter choice $q \in (1, \frac{5}{4})$ corresponds to the case $\Omega \subset \boldsymbol{R}^3$, considered here. Consequently, from (4.44c) it follows that

$$\|\theta^\delta\|_{L^q(0,T;W^{1,q}(\Omega))} \le c, \tag{4.46}$$

where the constant c is independent of δ. Also, from (4.44c), we have

$$\|\theta_t^\delta\|_{L^q(0,T;(W^{1,q}(\Omega))^*)+L^1(Q)} \le c, \tag{4.47}$$

uniformly in δ.
The space $\{\theta \in L^q(0,T;W^{1,q}(\Omega)); \theta_t \in L^q(0,T;(W^{1,q}(\Omega))^*) + L^1(Q)\}$ has compact embedding in $L^1(Q)$ [13, page 85].
Hence, by (4.46), (4.47), as $\delta \to 0$, we have

$$\theta^\delta \longrightarrow \theta \quad \text{strongly in } L^1(Q) \text{ and a.e. in } Q, \tag{4.48}$$

$$\eta(\theta^\delta) \to \eta(\theta), \quad \gamma(\theta^\delta) \to \gamma(\theta) \quad \text{strongly in } L^p(Q) \text{ for all } p \in [1,\infty) \text{ and a.e. in } Q. \tag{4.49}$$

By (4.45), one can assume that as $\delta \to 0$

$$\sigma_{ij}^\delta,\, \dot\sigma_{ij}^\delta \longrightarrow \sigma_{ij},\, \dot\sigma_{ij} \quad \text{weakly in } L^2(Q),$$

$$v_i^\delta \to v_i, \quad \phi^\delta \to \phi \quad \text{weakly in } \mathcal{W}_T,$$

for $1 \le i, j \le 3$. Similar to the proof of strong convergence of σ^n, ϕ^n in the proof of Lemma 4.4, we can verify that

$$\sigma_{ij}^\delta \longrightarrow \sigma_{ij} \quad \text{strongly in } L^2(Q)\,, \tag{4.50}$$

$$\phi^\delta \longrightarrow \phi \quad \text{strongly in } \mathcal{W}_T\,, \tag{4.51}$$

and hence

$$|S^\delta|^2 \longrightarrow |S|^2 \quad \text{strongly in } L^1(Q)\,, \tag{4.52}$$

$$|\nabla\phi^\delta|^2 \longrightarrow |\nabla\phi|^2 \quad \text{strongly in } L^1(Q). \tag{4.53}$$

By (4.48) - (4.53) and by the definition of τ_δ, we obtain

$$\begin{aligned}
\eta(\theta^\delta)S^\delta_{ij} &\longrightarrow \eta(\theta)S_{ij} \quad \text{strongly in } L^2(Q)\ , \\
\eta(\theta^\delta)\tau_\delta(|S^\delta|^2) &\longrightarrow \eta(\theta)|S|^2 \quad \text{strongly in } L^1(Q)\ , \\
\gamma(\theta^\delta)\tau_\delta(|\nabla\phi^\delta|^2) &\longrightarrow \gamma(\theta)|\nabla\phi|^2 \quad \text{strongly in } L^1(Q)\ , \\
\tau_\delta(\theta_0) &\longrightarrow \theta_0 \quad \text{strongly in } L^1(\Omega).
\end{aligned}$$

The above convergences allow us to pass to the limit as $\delta \to 0$ in (4.44a) - (4.44e), which provides (3.17a)-(3.17d) and the initial condition (3.16e) for σ_0. To conclude the proof of the theorem we note that

$$L^1(\Omega) \subset (W^{1,p}(\Omega))^* \quad \text{for } p > 3.$$

Indeed, we have

$$\int\limits_\Omega hg \le \|h\|_{L^1(\Omega)}\|g\|_{L^\infty(\Omega)} \le c\|h\|_{L^1(\Omega)}\|g\|_{W^{1,p}(\Omega)} \quad \text{for } p > 3. \tag{4.54}$$

In particular, from (3.17c) it follows $\theta \in L^q(0,T;W^{1,q}(\Omega))$ and

$$\theta_t \in L^q(0,T;(W^{1,q}(\Omega))^*) + L^1(Q) \subset L^1(0,T;(W^{1,p}(\Omega))^*) \quad \text{for } p>3,\ 1<q<\frac{5}{4}.$$

Hence, $\theta \in C([0,T];(W^{1,l}(\Omega))^*)$ for some $l > 1$ and the initial condition (3.16e) for θ_0 holds.

5 Numerical approximation and results

In this section we present some numerical results for the full model (2.1)–(2.4) and boundary conditions (2.13), including the effect of thermal expansion, however we restrict ourselves to the case of two space-dimensions. The computational domain is one half of the domain Ω as depicted in Fig. 2.

We descretize the weak formulation of the balance laws (1a-c) using simplicial finite elements. In each time-step the system is decoupled using a semi-implicite Euler scheme. The resulting subsystems are solved by Newton's method, where the stress increment is eliminated to avoid saddle point problems. For the implementation we used the toolbox *pdelib* (cf. [6]), which has been developed at WIAS, and extended it by components for the treatment of multidimensional mechanical problems. The material parameters correspond to the structural steel *ST35.8*. For details, we refer to [5].

Fig. 3 depicts the movement of the upper electrode in the course of time or different applied pressures f. If no pressure is applied (red curve), only he effect of thermal expansion can be observed. In the other cases, one bserves first a (linear) elastic sink in, then a period where thermal expansion revails, followed by a deep sink in when the weld temperature is reached. Fig. 4 depicts the evolution of shear stresses and Fig. 5 the evolution of normal stresses. Especially the relaxation of shear stresses fter weld begin at time t_3 can be observed nicely. These figures show that the welding process can be decently described by our odel during a long time period. However, when the deformations in the rojection tip become too big the assumption of small deformations is no longer alid and the numerical results become unrealistic. The beginning of this ffect can be seen in the last picture of Fig. 4.

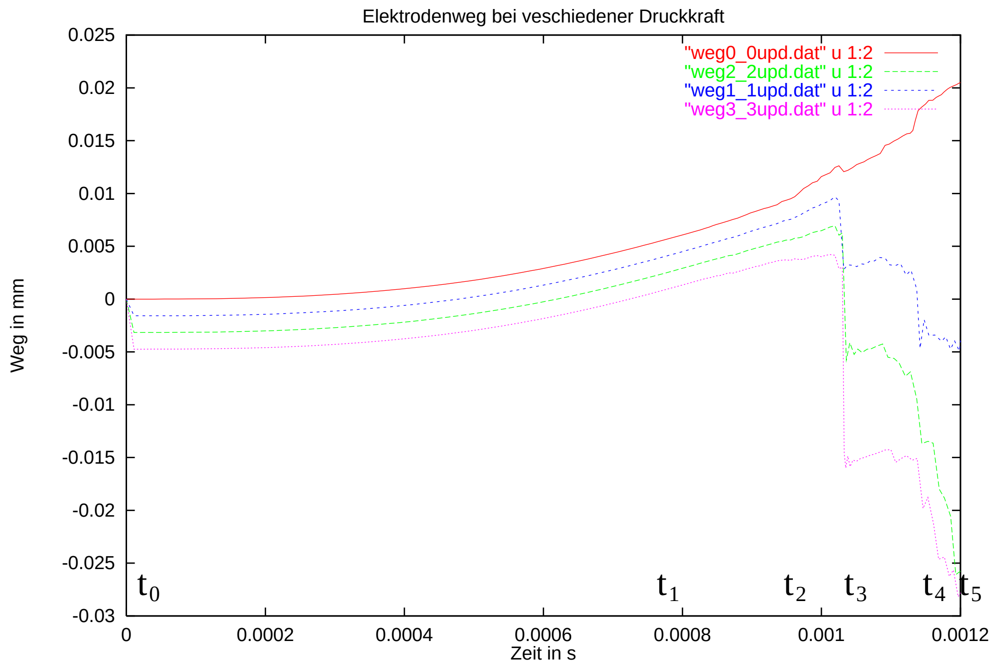

Figure 3: Top: Sink in of the upper electrode for different appplied pressures f and definition of characteristic times: until t_0 elastic, until t_1 thermoelastic, t_2 creep start, t_3 weld start.

6 Conclusions

We have investigated a mathematical model for impulse welding. The model accounts for thermal, mechanical and electrical effects. Moreover, we have taken care of the electrical effects of the weld machine by a discrete oscillator circuit.
The numerical results confirm that the model is capable of describing the weld process adequately as long as the assumption of small displacements remains valid.
In our mathematical analysis we have to neglect the effect of thermal expansion, which indeed is not big as the numerical simulations show. Moreover, we have only considered a particular inelastic constitutive law.
Thus a challenging mathematical problem is to prove existence of solutions without these restrictions. Another direction of future research is the question of dealing with the free boundary between liquid and solid part. Although this is a minor problem from practical point of view, it is an interesting mathematical task to derive interface conditions and investigate the resulting model.

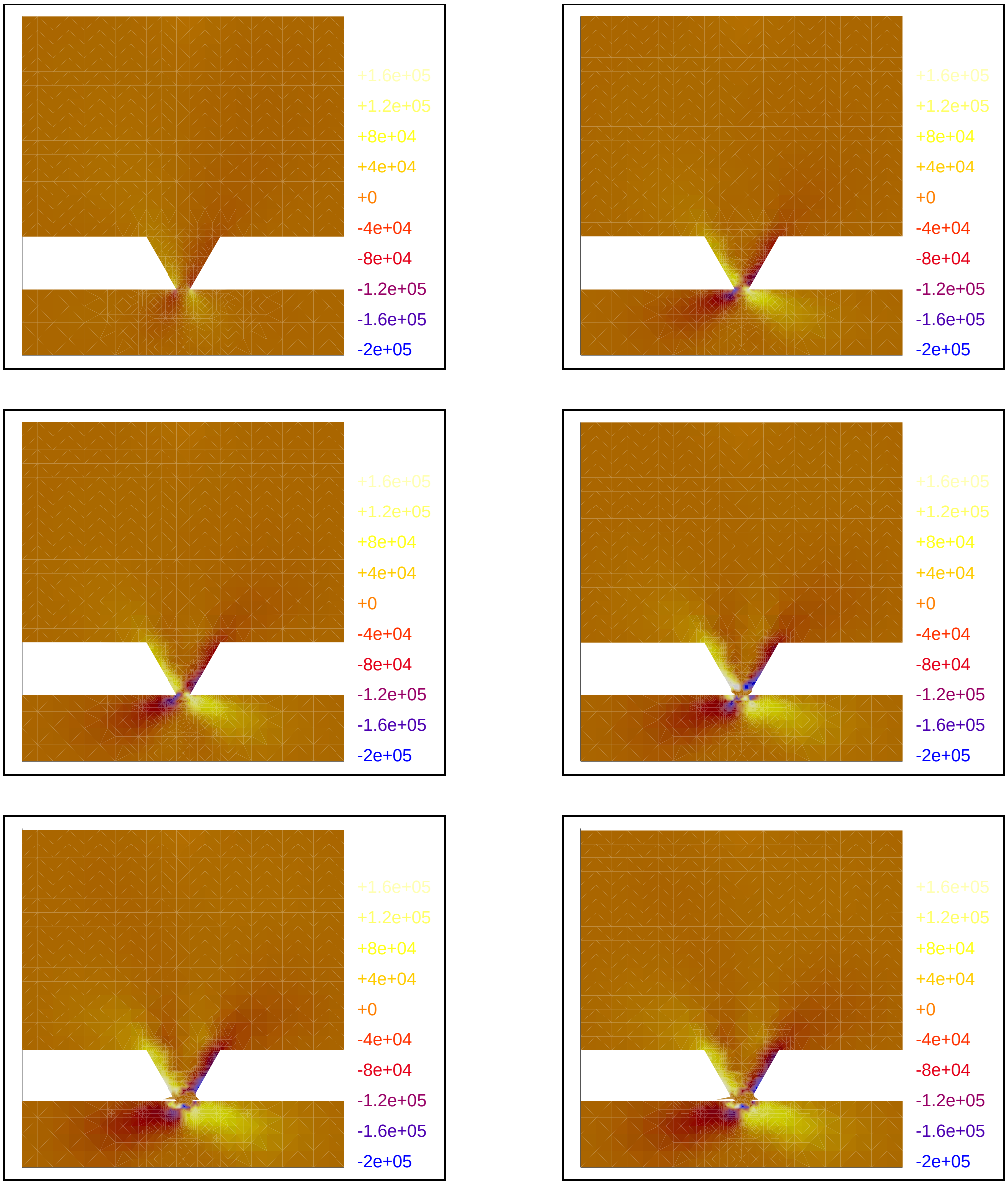

Figure 4: Evolution of shear stresses at times t_0 to t_5 from Fig. 3

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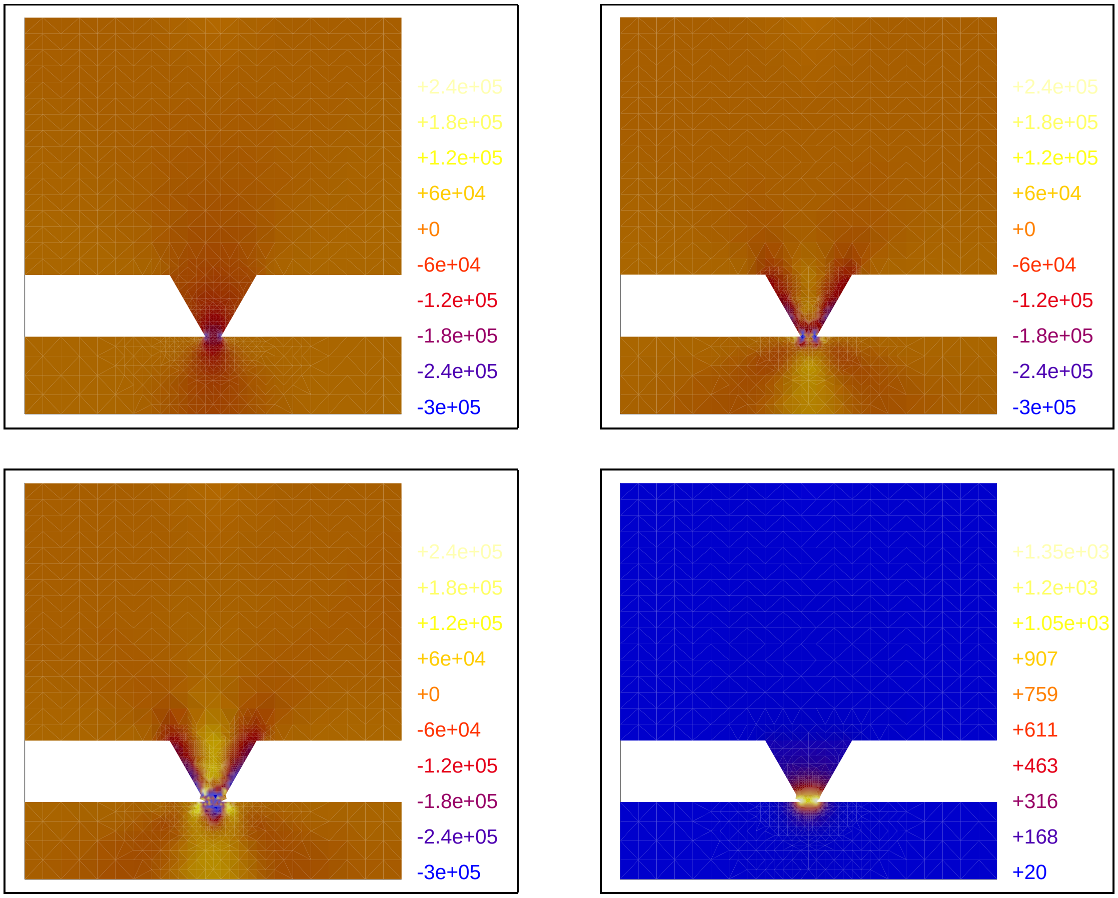

Figure 5: Normal stresses at times t_0 t_2 and t_3 and temperature distributuion at time t_3 (cf. Fig. 3).

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