Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

On a nonlocal model of non–isothermal phase separation

Herbert Gajewski[1]

submitted: 27th August 2001

[1] Weierstrass Institute for Applied
Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin
Germany
E-Mail: gajewski@wias-berlin.de

Preprint No. 671
Berlin 2001

1991 *Mathematics Subject Classification.* 35K45, 35K57, 35B40, 80A20, 80A22.

Key words and phrases. Coupled Cahn–Hilliard equations, binary alloys, segregation model, nonlocal interaction, free energy, entropy, Onsager relations, initial boundary value problem, global existence and uniqueness, Lyapunov function, asymptotic behaviour.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax:	+ 49 30 2044975
E-Mail (X.400):	c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet):	preprint@wias-berlin.de
World Wide Web:	http://www.wias-berlin.de/

Abstract

A nonlocal model of non-isothermal phase separation in binary alloys is presented. The model is deduced from a free energy with a nonconvex part taking into account nonlocal particle interaction. The model consists of a system of second order parabolic evolution equations for heat and mass, coupled by nonlinear drift terms and a state equation which involves a nonlocal interaction potential. The negative entropy turns out to be Lyapunov functional of the system and yields the key estimate for proving global existence and uniqueness results and for analyzing the asymptotic behaviour as time goes to infinity.

1. Introduction

We consider a binary alloy with components A and B occupying a spatial domain Ω. We denote by u and $1-u$ the (scaled) local concentrations of A and B, respectively and by T the (non constant) temperature.

For describing phase separation processes in such systems local models, so-called coupled Cahn-Hilliard equations, have been proposed in [20, 4]. These models extend the classical Cahn-Hilliard approach [6] to non–isothermal situations and are based on local free energy densities of Landau–Ginzburg type

$$f_{LG}(u,T) = (1+T)\ \phi(u) - \psi(T) + k\,u\,(1-u) + \frac{\lambda}{2}\left|\nabla u\right|^2 \tag{1.1}$$

with convex functions ϕ and ψ. Alt and Pavlov [4] proved the existence of weak solutions to initial boundary value problems associated to coupled Cahn-Hilliard equations. Uniqueness could be shown by Shen and Zheng [21] for one space dimension.

In this paper a nonlocal model of non–isothermal phase separation is proposed. We replace the local free energy density (1.1) by a nonlocal expression

$$f(u,T) = (1+T)\ \phi(u) - \psi(T) + u\int_\Omega \mathcal{K}(|x-y|)(1-u(y))\,dy, \tag{1.2}$$

where again the functions ϕ and ψ are convex and the kernel $\mathcal{K}$ of the integral term describes nonlocal interaction. (1.2) may be written in the form [7, 14]

$$f(u,T) = (1+T)\ \phi(u) - \psi(T) + ku(1-u) + \frac{1}{2}\int_\Omega \mathcal{K}(|x-y|)|u(x)-u(y)|^2\,dy$$

with

$$k = k(x) = \int_\Omega \mathcal{K}(|x-y|)dy,$$

in order to make more transparent the relation to (1.1).

Nonlocal free energies seem to be reasonable, if one takes a closer look to Cahn–Hilliard's arguments motivating (1.1), and have been rigorously derived in a stochastical setting in [14] for the isothermal case. That case leads to a second order parabolic equation with a nonlocal drift term such that global existence and uniqueness of solutions can be proved [15, 13].

In this paper we extend these results to the non–isothermal case. Starting from the free energy density (1.2), we derive in a thermodynamically consistent way a non–isothermal nonlocal phase separation model. This model consists of a system of second order parabolic equations for u and T, coupled by a nonlocal drift term. According to thermodynamics, the energy is conserved and the entropy is nondecreasing. These properties yield the key estimates for proving global existence and uniqueness results and for analyzing the asymptotic behaviour as time tends to infinity.

In the next section we derive the model, formulate our assumptions and state the initial boundary value problem to be solved. In Section 3 we prove existence and uniqueness of a global solution in time. Finally, Section 4 is devoted to the asymptotic behaviour and the characterization of asymptotic states as maximizers of the entropy functional under the constraints of energy and mass conservation.

2. The model, assumptions, problem formulation

Let be $\Omega \subset \mathbb{R}^n$, $1 \le n \le 3$, a bounded open domain with piecewise smooth (comp. [8]) boundary $\Gamma = \partial\Omega$ and ν the outer unit normal on Γ. Denote by $L^p = L^p(\Omega)$, $H^{s,p} = H^{s,p}(\Omega)$, $0 \le s \le \frac{3}{2}$, $1 \le p \le \infty$, the usual function spaces on Ω, $H^1 = H^{1,2}(\Omega)$, $\|\cdot\|_p$, $\|\cdot\|_2 = \|\cdot\|$, the norm in L^p and by $(\cdot\,,\cdot)$ the pairing between H^1 and its dual $(H^1)^*$ [1, 11, 16]. For a time interval $(0,\tau)$, $\tau > 0$, and a Banach space X we denote by $L^p(0,\tau;X)$ the usual spaces of Bochner integrable functions with values in X. We set $\mathbb{R}^1_+ = (0,\infty)$ and $Q = (0,\tau)\times\Omega$, $\Sigma = (0,\tau)\times\Gamma$. "Generic" positive constants are denoted by C. For a function $u \in L^1$ we set

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u\ dx\ , \quad |\Omega| = meas\ \Omega.$$

We consider a binary alloy with components A and B occupying Ω. Let u and $1-u$ be the (scaled) local concentrations of A and B, respectively and let T denote the local (non constant) temperature.

We want to derive evolution equations for u and T and start from the free energy

$$F(u,T) = \int_\Omega f(u,T)\ dx = \int_\Omega \Big\{(1+T)\ \phi(u) - \psi(T) + u\mathcal{P}(1-u)\Big\}\ dx, \qquad (2.1)$$

where the operator $\mathcal{P} \in (L^2 \mapsto L^2)$ is defined by

$$(\mathcal{P}\varrho)(x) = \int_\Omega \mathcal{K}(|x-y|)(1-u(y))\,dy \;\;\forall \varrho \in L^2 \tag{2.2}$$

and the functions $\phi, \psi, \mathcal{K} \in (\mathbb{R}^1_+ \to \mathbb{R}^1)$ will be specified later on. According to the rules of thermodynamics, we introduce entropy

$$S = -\int_\Omega \partial_T F\,dx = \int_\Omega \Big(\psi'(T) - \phi(u)\Big)\,dx, \tag{2.3}$$

and energy

$$E = F - \int_\Omega T\partial_T F\,dx = \int_\Omega \Big\{\phi(u) + T\psi'(T) - \psi(T) + u\mathcal{P}(1-u)\Big\}\,dx. \tag{2.4}$$

To find equilibrium values for u and T, we maximize the entropy under the constraints

$$\overline{u} = \overline{u_0}, \quad E = E_0. \tag{2.5}$$

Applying Lagrange's method, we maximize the augmented entropy

$$S_\lambda = S + \lambda_1 \int_\Omega u\,dx + \lambda_2 E.$$

By means of the corresponding Euler-Lagrange equations

$$\partial_u S_\lambda = -\phi'(u) + \lambda_1 + \lambda_2 \partial_u E = 0, \qquad \partial_T S_\lambda = \psi''(T)(1+\lambda_2 T) = 0,$$

we identify the Lagrange multipliers λ_i as 'entropy variables'

$$\lambda_1 = \frac{v}{T}, \quad \lambda_2 = -\frac{1}{T}, \tag{2.6}$$

where the chemical potential v is given by

$$v = (1+T)\phi'(u) + w \tag{2.7}$$

with

$$w = \partial_u E = \mathcal{P}(1 - 2\,u). \tag{2.8}$$

Assuming that $\phi'^{-1} \in (\mathbb{R}^1 \to (0,1))$ exists, we get from (2.7) the state equation

$$u = \phi'^{-1}\left(\frac{v-w}{1+T}\right). \tag{2.9}$$

Equation (2.8) together with (2.5) and (2.9) can be seen as a system of nonlinear integral equations for determing equilibrium values of

$$w = w(x), \quad v = const., \quad T = const.$$

For describing nonequilibrium situations we suppose (2.5), (2.8) and (2.9) to remain true but with non constant

$$v = v(t,x), \quad T = T(t,x), \quad (t,x) \in Q,$$

and that the time evolution of v and T is governed by conservation equations for mass and energy:

$$\frac{\partial u}{\partial t} + \nabla \cdot J_u = 0 \text{ in } Q, \quad \nu \cdot J_u = 0 \text{ on } \Sigma, \tag{2.10}$$

$$\partial_u E \; u_t \; + \; \partial_T E \; T_t + \nabla \cdot J_e = 0 \text{ in } Q, \quad \nu \cdot J_e = 0 \text{ on } \Sigma. \tag{2.11}$$

We postulate (comp. the semiconductor energy model in [2]), the gradients of the entropy variables $\frac{v}{T}$ and $-\frac{1}{T}$ to be driving forces for mass and energy fluxes such that

$$J_u = -\mu\Big[\nabla\frac{v}{T} - (\phi' + w)\nabla\frac{1}{T}\Big], \tag{2.12}$$

$$J_e = -\kappa\nabla T + (\phi' + w) J_u. \tag{2.13}$$

Here μ and κ are nonnegative mobility and heat conduction parameters, respectively. Note that

$$J_e = [\kappa T^2 + \mu(\phi' + w)^2]\nabla\frac{1}{T} + \mu(\phi' + w)\nabla\frac{v}{T},$$

such that J_u and J_e satisfy Onsager's reciprocity relations with respect to $\frac{v}{T}$ and $-\frac{1}{T}$. Moreover, it is easy to check (comp. Lemma 4.1), that, according to the second law of thermodynamics, (2.10) - (2.13) together with (2.3) and (2.4) imply a Clausius–Duhem inequality of the form

$$\frac{dS}{dt} = \int_\Omega \Big\{\mu\Big|\nabla\frac{v}{T} - (\phi' + w)\nabla\frac{1}{T}\Big|^2 + \kappa|\nabla \log T|^2\Big\} \, dx \geq 0.$$

By (2.7), (2.10) and (2.13) we can replace (2.11) by the heat conduction equation

$$T\psi''(T)\frac{\partial T}{\partial t} - \nabla \cdot (\kappa\nabla T) + J_u \cdot \nabla w = 0 \quad \text{in } Q, \quad \nu \cdot (\kappa\nabla T) = 0 \text{ on } \Sigma. \tag{2.14}$$

Thus our nonlocal, non–isothermal phase separation model consists in the equations (2.7), (2.8), (2.10), (2.12) and (2.14) completed by initial conditions

$$u(0,x) = u_0(x), \quad T(0,x) = T_0(x), \quad x \in \Omega, \tag{2.15}$$

and specifications of the parameter functions $\phi, \psi, \mathcal{K}, \mu, \kappa$ given by the following assumptions:

(A_1) $\quad \phi(u) = u\log u + (1-u)\log(1-u),$

(A_2) $\quad \psi(T) = \displaystyle\int_1^T b(r)(\frac{T}{r} - 1)dr, \;\; \log b \in L^\infty_{loc}(\mathbb{R}^1_+), \;\; b(s) \geq b_0 = const. > 0,$

(A_3) $\quad$ the kernel $\mathcal{K} \in (\mathbb{R}^1_+ \mapsto \mathbb{R}^1)$ is such that the potential operator

$$\varrho \mapsto \mathcal{P}\varrho, \quad (\mathcal{P}\varrho)(x) = \int_\Omega \mathcal{K}(|x-y|)\varrho(y) \, dy,$$

satisfies $\Delta\mathcal{P} \in (L^2 \mapsto L^2)$ and

$$\|\mathcal{P}\varrho\|_p \le k_{0,p}\|\varrho\|_q, \quad \|\nabla\mathcal{P}\varrho\| \le k_{1,p}\|\varrho\|_q, \; 1 \le p \le \infty, \quad \frac{1}{p}+\frac{1}{q}=1;$$

(A_4) the mobility μ has the form $\mu = \dfrac{T\ \mu_0(x)}{(1+T)\phi''(u)}$, $\log\mu_0 \in L^\infty \cap H^1$;

(A_5) the heat conduction κ has the form $\kappa = \kappa_0(x)b(T)$, $\log\kappa_0 \in L^\infty$;

(A_6) the initial values $u_0,\ T_0$ satisfy

$$u_0 \in H^1,\ \phi'(u_0) \in L^\infty,\ 0 < \overline{u_0} < 1,\ \theta_0 := \int_0^{T_0} b(r)\ dr \in H^1,\ \psi'(T_0) \in L^1.$$

Note that

$$\phi \ \in\ C[0,1] \cap C^3(0,1), \quad \phi'(u) = \log\frac{u}{1-u}, \quad \phi''(u) = \frac{1}{u\ (1-u)} \ge 4, \quad 0 < u < 1,$$
$$0 \ \le\ \phi'^{-1}(r) = 1/(1+e^{-r}) \le 1,\ \forall r \in \mathbb{R}^1$$

and

$$|\phi(u_1)-\phi(u_2)|^2 \le c_\phi^2\ (u_1-u_2)(\phi'(u_1)-\phi'(u_2)) \quad \forall u_1, u_2 \in (0,1), \tag{2.16}$$

with $c_\phi^2 = \frac{4u^*(1-u^*)}{(2u^*-1)^2} \simeq 0.44$, where u^* solves the equation $(2u^*-1)\phi'(u^*) = 2$.

Remark 2.1. In order to keep our paper as transparent as possible we put technically simple assumptions. The most of our results could be proved for more general situations. That concerns especially the function ϕ, which must only satisfy: $\phi \in C[0,1] \cap C^3(0,1)$ is strongly convex, $\phi'^{-1} \in C(\mathbb{R}^1 \to (0,1))$, $1/\phi''$ is concave. An estimate like (2.16) is only used in our last theorem (Theorem 4.4) concerning uniqueness of equilibrium states.

Remark 2.2. Examples for kernels $\mathcal{K}$ satisfying A_3 are Newton potentials [18]

$$\mathcal{K}(|x|) = \kappa_n|x|^{2-n}, \quad n \ne 2; \quad \mathcal{K}(|x|) = -\kappa_2\log|x|, \quad n=2; \quad \kappa_n = \text{ const.} > 0,$$

and usual mollifiers like

$$\mathcal{K}(|x|) = \begin{cases} C\exp\left[-h^2/(h^2-|x|^2)\right] & \text{if } |x| < h, \\ 0 & \text{if } |x| \ge h, \end{cases}$$

where $h > 0$ characterizes the range of the interaction. Note also that arbitrary $\mathcal{K} \in C^2([0, diam\ \Omega])$ satisfy (A_3) with

$$k_{0,p} = \|\mathcal{K}\|_{L^p(\Omega\times\Omega)}, \quad k_{1,p} = \|\mathcal{K}'\|_{L^p(\Omega\times\Omega)}.$$

Remark 2.3. It is a well-known fact of potential theory that for Newton potentials $\mathcal{K}$ the potential w given by (2.8) satisfies Poisson's equation

$$-\Delta w = 1 - 2\,u,$$

such that in this special case our nonlocal phase separation model turns out to be local and, moreover, to be closely related to energy models of semiconductors (comp. [2] and the literature quoted therein), where the role of w is played by the electrostatic potential.

Remark 2.4. Mobilities of the form $\mu = \mu_0/f''(u)$ seem to be natural and have been considered e.g. in [9, 10, 14, 15].

It turns out to be convenient to introduce a new variable

$$\theta = B(T), \quad B(s) = \int_0^s b(r)\, dr. \tag{2.17}$$

So we can express partial derivatives of the 'entropy' variables $\frac{v}{T}, \frac{1}{T}$ in terms of the 'dual' variables $(u,\ \theta)$:

$$\begin{aligned}
J_u &= -\mu\Big(\nabla\frac{v}{T} - (\phi' + w)\nabla\frac{1}{T}\Big) = -\frac{\mu}{T}\Big(\nabla v + \frac{\phi' + w - v}{T}\nabla T\Big) \\
&= -\frac{\mu}{T}\Big(\nabla v - \phi'\nabla T\Big) = -\frac{\mu}{T}\Big((T+1)\nabla\phi' + \nabla w\Big) \\
&= -\frac{\mu_0}{\phi''}\Big(\nabla\phi' + \frac{\nabla w}{1+T}\Big) = -\mu_0\Big(\nabla u + \frac{\nabla w}{\phi''(u)[1 + B^{-1}(\theta)]}\Big), \\
T\psi''(T)\frac{\partial T}{\partial t} &= \frac{\partial \theta}{\partial t}, \quad \kappa\nabla T = \kappa_0\nabla\theta.
\end{aligned}$$

Now we are ready to state (the weak form of) the initial boundary value problem to be solved.

Definition 2.1. A function triple $\{u, \theta, w\}$ is called solution of *Problem P*, if for all $t \in [0, \tau]$, almost all $x \in \Omega$ and all $h \in L^2(0, \tau; H^1)$ following relations hold:

$$\begin{aligned}
u &\in C(0,\tau;L^2) \cap L^2(0,\tau;H^1), \quad u_t \in L^2(0,\tau;(H^1)^*), \quad 0 \le u(t,x) \le 1, \\
\theta &\in C(0,\tau;L^2) \cap L^2(0,\tau;H^1), \quad \theta_t \in L^2(0,\tau;(H^1)^*), \quad \theta(t,x) \ge 0, \\
w &\in C(0,\tau;H^{1,\infty});
\end{aligned}$$

$$\int_0^\tau \Big\{(u_t, h) - \int_\Omega J_u \cdot \nabla h\, dx\Big\}\, dt = 0, \quad u(0,t) = u_0(x), \tag{2.18}$$

$$\int_0^\tau \Big\{(\theta_t, h) + \int_\Omega \Big[\kappa_0\nabla\theta\cdot\nabla h + J_u\cdot\nabla(\phi'(u) + w)\, h\Big]\, dx\Big\}\, dt = 0, \quad \theta(0,x) = \theta_0(x), \tag{2.19}$$

$$w(t) = \mathcal{P}(1 - 2u(t)), \tag{2.20}$$

$$J_u = -\mu_0\Big[\nabla u + \alpha(u)\beta(\theta)\nabla w\Big], \quad \alpha(u) = \frac{1}{\phi''(u)} \text{ if } u \in [0,1], \ \alpha(u) = 0 \text{ else}, \tag{2.21}$$

$$\beta(\theta) = \frac{1}{1 + B^{-1}(\theta)} \text{ if } \theta \ge 0, \ \beta(\theta) = 1 \text{ else}. \tag{2.22}$$

If $\{u, \theta, w\}$ is a solution of *Problem P*, the associated entropy variables can be calculated by

$$T = B^{-1}(\theta), \quad v = (1 + T)\phi'(u) + w. \tag{2.23}$$

Our main results to be proved in the next sections can be summarized as follows:

Theorem 2.2. *Let the assumptions $(A_1)-(A_6)$ be satisfied. Then*
(i) Problem P has a unique solution $\{u,\theta,w\}$ for arbitrary $\tau>0$;
(ii) there exists an equilibrium state $\{u^*,v^*,w^*,T^*\}$ such that

$$\begin{aligned} w^* &= \mathcal{P}(1-2u^*), \quad u^* = \frac{1}{1+\exp\frac{w^*-v^*}{1+T^*}}, \quad v^*,T^* = const.\ \ in\ \Omega, \\ \overline{u^*} &= \overline{u_0}, \quad E(u^*,T^*) = E(u_0,T_0), \quad \lim_{t\to\infty} S(u(t),T(t)) = S(u^*,T^*), \quad T = B^{-1}(\theta); \end{aligned}$$

(iii) $\{u^*,T^*\}$ maximizes the entropy S under the constraints
$$\overline{u^*} = \overline{u_0}, \quad E(u^*,T^*) = E(u_0,T_0);$$
(iv) the equilibrium state is unique, provided the additional assumption

$$(A_7) \qquad \gamma := \frac{k_{0,2}\ (b_0+c_\phi)}{2b_0} < 1 \quad or \quad E(u_0,T_0) > B(\gamma-1)\ |\Omega| + k_{0,1}$$

is fulfilled.

3. Existence and uniqueness

In this section we prove existence and uniqueness of a solution to *Problem P* by means of an iteration procedure. We start with two lemmas which allow us to solve the equations (2.18) and (2.19) successively.

Lemma 3.1. *Let $U\in L^\infty(Q)$ and T be given such that* $\log(1+T)\in L^2(0,\tau;H^1)$ *and $T\geq 0$ a. e. in Q . Then the problem*

$$\int_0^\tau \Big\{(u_t,h) + \int_\Omega \mu_0\Big[\nabla u + \alpha(u)\beta(\vartheta)\nabla w\Big]\cdot\nabla h\ dx\Big\}\ dt = 0,\ \ \forall h\in L^2(0,\tau;H^1), \tag{3.1}$$
$$u(0) = u_0, \quad w = \mathcal{P}(1-2\,U), \quad \vartheta = B(T),$$

has a unique solution $u\in C(0,\tau;L^2)\cap L^2(0,\tau;H^1)$ with $u_t\in L^2(0,\tau;(H^1)^*)$ and $0\leq u\leq 1$ a. e. in Q. Moreover, there exists a constant $C=C(\tau)$, independent of U and T, such that:

$$\begin{aligned} &(i) && \|\nabla u\|_{L^2(Q)} \leq C, \\ &(ii) && \|u_t\|_{L^2(0,\tau;(H^1)^*)} \leq C, \\ &(iii) && \|\phi'(u)\|_{L^\infty(Q)} + \|\phi''(u)\|_{L^\infty(Q)} + \|\nabla\phi'(u)\|_{L^2(Q)} \leq C, \\ &(iv) && |\nabla\phi'(u)| \in L^2(0,\tau;L^p), \quad for\ \ p<\frac{2n}{n-1}. \end{aligned}$$

Proof. Since by (A_3)

$$\|\beta(\vartheta)\nabla w\|_{L^\infty(Q)} = \Big\|\frac{\nabla w}{1+T}\Big\|_{L^\infty(Q)} \leq k_{1,\infty}|\Omega|, \tag{3.2}$$

existence and uniqueness of a solution u with $(i)-(ii)$ follow from standard results on parabolic equations [11, 19]. Further, choosing $h = \min(u, 0)$, resp. $h = \max(u, 1)$ in (3.1) and using that $0 \le u_0 \le 1$ and $\alpha(h) = 0$ if $h \notin [0,1]$, we see that $h = 0$, that means $u \ge 0$, resp. $u \le 1$.
(iii) We test (3.1) with the functions

$$\begin{aligned} h &= \frac{\varrho_+^r}{\phi''(u)}, \quad \varrho_+ = \quad \max[0, \phi'(u)], \quad r \ge 0, \\ g &= \frac{\varrho_-^r}{\phi''(u)}, \quad \varrho_- = -\min[0, \phi'(u)], \quad r \ge 0. \end{aligned}$$

Then, using once more (3.2) and that

$$\phi'''(u) \ge 0 \text{ if } \varrho_+ > 0 \text{ and } \phi'''(u) \le 0 \text{ if } \varrho_- > 0,$$

we can (comp. the proof of Theorem 3.6 in [13]) apply Alikakos' [3] version of Moser's iteration technique in order to prove $\|\phi'(u)\|_{L^\infty(Q)} \le C(\tau)$. This implies

$$\|\phi''(u)\|_{L^\infty(Q)} \le C(\tau) \tag{3.3}$$

and by (i)

$$\|\nabla\phi'(u)\|_{L^2(Q)} = \|\phi''(u)\nabla u\|_{L^2(Q)} \le C(\tau).$$

(iv) By (A_3), (A_4) and (3.2) we have $\nabla \cdot [\mu_0\alpha(u)\beta(\vartheta)\nabla w] \in L^2(Q)$. This means $u_t \in L^2(Q)$ by well-known results on parabolic equations (comp. [17], chapter III, Theorem 6.1). Thus from a regularity result on elliptic equations ([8], Theorem 2.1) we can deduce that $u \in L^2(0, \tau; H^s)$ for $s < \frac{3}{2}$ and consequently by Sobolew's embedding theorem $|\nabla u| \in L^2(0, \tau; L^p)$, $p < \frac{2n}{n-1}$.□

Lemma 3.2. *Let U be given as in Lemma* 3.1. *Let u be the solution of* (3.1). *Then the problem*

$$\begin{aligned} \int_0^\tau \Big\{(\theta_t, h) \;&+\; \int_\Omega \Big[\kappa_0 \nabla\theta \cdot \nabla h - (\beta(\theta)g + f)\; h\Big]\, dx\Big\}\, dt = 0, \;\; \forall h \in L^2(0,\tau; H^1), \\ \theta(0) \;&=\; \theta_0, \;\; w = \mathcal{P}(1 - 2U), \;\; g = \mu_0\alpha(u)\nabla w \cdot \nabla(\phi'(u) + w), \\ f \;&=\; \mu_0 \nabla u \cdot \nabla(\phi'(u) + w) \end{aligned} \tag{3.4}$$

has a unique solution $\theta \in C(0, \tau; L^2) \cap L^2(0, \tau; H^1)$ with $\theta_t \in L^2(0, \tau; (H^1)^*)$ and $\theta \ge 0$ a. e. in Q. Moreover, there exists a constant $C = C(\tau)$, independent of U, such that

$$\|\theta\|_{C(0,\tau;L^2)} + \|\nabla\theta\|_{L^2(Q)} \le C. \tag{3.5}$$

Proof. Since $\frac{n}{n-1} > \frac{2n}{n+2}$ for $n \le 3$, Lemma 3.1(iv) guarantees that

$$g \in L^2(Q) \text{ and } f \in L^2(0, \tau; L^{\frac{2n}{n+2}}) \subset L^2(0, \tau; (H^1)^*).$$

Thus the existence and uniqueness assertion follows from standard results on parabolic equations [11, 19]. Moreover, testing (3.4) with $h = \min[0, \theta]$ and noting that $\beta(\theta) = 1$ if $\theta \leq 0$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|h\|^2 + \int_\Omega \kappa_0 |\nabla h|^2 \, dx &= \int_\Omega (\beta(\theta) g + f) \; h \; dx \\ &= \int_\Omega \frac{\mu_0}{\phi''(u)} |\nabla \phi' + \nabla w|^2 \; h \; dx \leq 0.\end{aligned}$$

Since $\theta_0 \in L^2$ and $\theta_0 \geq 0$ a. e. in Q by (A_6), this means $\theta \geq 0$ a. e. in Q.
(ii) Using Lemma 3.1(iii), we obtain from (3.1) and (3.4)

$$\begin{aligned}\frac{d}{dt}\Big[(\phi, \theta) + \frac{1}{2}\|\theta\|^2\Big] &= (u_t, \phi'\theta) + (\theta_t, \phi + \theta) \\ &= \int_\Omega \Big\{ - \mu_0[\nabla u + \alpha(u)\beta(\vartheta)\nabla w] \cdot (\theta \nabla \phi' + \phi' \nabla \theta) \\ &- \kappa_0 \nabla \theta \cdot \nabla(\phi + \theta) + (\beta(\theta) g + f) \; (\phi + \theta)\Big\} \; dx \\ &= \int_\Omega \Big\{ - \mu_0[\phi' \nabla u \cdot \nabla \theta + \alpha(u)\beta(\vartheta)\nabla w \cdot (\theta \nabla \phi' + \phi' \nabla \theta)] \\ &- \kappa_0 \nabla \theta \cdot \nabla(\phi + \theta) + \beta(\theta) g \; (\phi + \theta) + f\phi + \mu_0 \nabla u \cdot \nabla w \; \theta \Big\} \; dx \\ &\leq C(1 + \|\theta\|^2 + \|\nabla \phi'\|^2) \; - \frac{1}{2}\|\sqrt{\kappa_0}\nabla \theta\|^2.\end{aligned}$$

Hence Lemma 3.1(iii) and Gronwall's lemma imply (3.5). □

Now we are ready to prove our existence and uniqueness result.

Theorem 3.3. *Under the assumptions $(A_1) - (A_6)$ there exists a unique solution $\{u, \theta, w\}$ to Problem P.*

Proof. We define an operator $\mathcal{A} \in \Big((C(0,\tau; L^2))^2 \mapsto (C(0,\tau; L^2))^2\Big)$ by

$$\{U, \vartheta\} \mapsto \{u, \theta\} =: \mathcal{A}\{U, \vartheta\},$$

where $u(0) = u_0, \; \theta(0) = \theta_0$ and $\forall h \in L^2(0,\tau; H^1)$,

$$\int_0^\tau \Big\{(u_t, h) + \int_\Omega \mu_0[\nabla u + \alpha(u)\beta(\vartheta)\nabla w] \cdot \nabla h \; dx\Big\} \; dt = 0, \;\; w = \mathcal{P}(1 - 2U), \tag{3.6}$$

$$\int_0^\tau \Big\{(\theta_t, h) + \int_\Omega \Big[\kappa_0 \nabla \theta \cdot \nabla h - \mu_0[\nabla u + \alpha(u)\beta(\theta)\nabla w] \cdot \nabla(\phi'(u) + w) \; h\Big] \; dx\Big\} \; dt = 0. \tag{3.7}$$

The operator $\mathcal{A}$ is well defined by Lemma 3.1 and Lemma 3.2. Now we want to prove that $\mathcal{A}$ satisfies the contraction condition

$$\|\mathcal{A}\{U_1, \vartheta_1\} - \mathcal{A}\{U_2, \vartheta_2\}\|_{_\lambda} \leq \frac{1}{2}\|\{U_1, \vartheta_1\} - \{U_2, \vartheta_2\}\|_{_\lambda}, \tag{3.8}$$

where

$$\|\{U, \vartheta\}\|_{_\lambda} = \sup_{t \in [0,\tau]} \{e^{-\lambda t}(\|u(t)\|^2 + \|\vartheta(t)\|^2)\},$$

with a sufficiently large real number $\lambda > 0$. For this purpose let $\{U_i, \vartheta_i\},\ i = 1, 2$, be given. Set

$$\begin{aligned}
\{u_i, \theta_i\} &= \mathcal{A}\{U_i, \vartheta_i\},\ u = u_1 - u_2,\ U = U_1 - U_2,\ \ u_m = \frac{u_1 + u_2}{2}, \\
\phi_j &= \phi(u_j),\ j = 1, 2, m,\ \ \phi = \phi_1 - \phi_2,\ \ K = \frac{2}{\phi''_m} - \frac{1}{\phi''_1} - \frac{1}{\phi''_2} = \frac{u^2}{2}, \\
J_i &= -\mu_0[\nabla u_i + \alpha_i \beta_i \nabla w_i],\ \ \alpha_i = \alpha(u_i),\ \ w = w_1 - w_2, \\
\beta_i &= \beta(\vartheta_i),\ \beta = \beta_1 - \beta_2,\ \ \vartheta = \vartheta_1 - \vartheta_2,\ \ \theta = \theta_1 - \theta_2.
\end{aligned}$$

By (3.6) we get

$$\begin{aligned}
&\frac{d}{dt} \sum_{i=1,2} \int_\Omega (\phi_i - \phi_m)\ dx = \sum_{i=1,2} [(u_{i_t}, \phi'_i) - (u_{m_t}, \phi'_m)] \\
&\quad = \sum_{i=1,2} \int_\Omega J_i \cdot [\nabla \phi'_i - \phi''_m \nabla u_m]\ dx \\
&\quad = \sum_{i=1,2} \int_\Omega \frac{\phi''_m J_i}{2} \cdot \Big[\frac{2\nabla\phi'_i}{\phi''_m} - \frac{\nabla\phi'_1}{\phi''_1} - \frac{\nabla\phi'_2}{\phi''_2}\Big]\ dx \\
&\quad = \int_\Omega \frac{\phi''_m}{2} \Big\{K \sum_{i=1,2} J_i \cdot \nabla\phi'_i\ + [\alpha_2 J_1 - \alpha_1 J_2] \cdot \nabla\phi'\Big\}\ dx \\
&\quad = -\int_\Omega \frac{\phi''_m \mu_0}{2} \Big\{K \sum_{i=1,2} \alpha_i (\nabla\phi'_i + \beta_i \nabla w_i) \cdot \nabla\phi'_i \\
&\qquad\qquad + \alpha_1\alpha_2 [|\nabla\phi'|^2 + (\beta_1 \nabla w_1 - \beta_2 \nabla w_2) \nabla\phi']\Big\}\ dx \\
&\quad \le -\int_\Omega \frac{\phi''_m \mu_0}{2} \Big\{\frac{u^2}{4} \sum_{i=1,2} \alpha_i (|\nabla\phi'_i|^2 - (k_{1,\infty}|\Omega|)^2) \\
&\qquad\qquad + \alpha_1\alpha_2 [\frac{|\nabla\phi'|^2}{2} - |\nabla w|^2 - (k_{1,\infty}|\Omega|)^2 \beta^2]\Big\}\ dx.
\end{aligned}$$

Integrating this inequality, applying the convexity of ϕ, using Lemma 3.1(iii) and $|\beta(\vartheta_1) - \beta(\vartheta_2)| \le \frac{|\vartheta|}{b_0}$, we obtain

$$\|u(t)\|^2 + \int_0^t \int_\Omega \Big\{u^2 \sum_{i=1,2} |\nabla\phi'_i|^2 + |\nabla\phi'|^2\Big\}\ dx\ ds \le C \int_0^t \int_\Omega \Big\{u^2 + U^2 + \vartheta^2]\Big\}\ dx\ ds. \quad (3.9)$$

Further we get from (3.6), (3.7) and Lemma 3.1(iii)

$$\begin{aligned}
&\frac{d}{dt}\Big[(\phi, \theta) + \frac{1}{2}\|\theta\|^2\Big] = (u_{1_t}, \phi'_1\theta) - (u_{2_t}, \phi'_2\theta) + (\theta_t, \phi + \theta) \\
&= \int_\Omega \Big\{(J_1\phi'_1 - J_2\phi'_2) \cdot \nabla\theta + (J_1 \cdot \nabla\phi'_1 - J_2 \cdot \nabla\phi'_2)\ \theta \\
&-\kappa_0 \nabla\theta \cdot \nabla(\phi + \theta) + [\beta(\theta_1) g_1 - \beta(\theta_2) g_2 + f_1 - f_2]\ (\phi + \theta)\Big\}\ dx \\
&= \int_\Omega \Big\{(J_1\phi'_1 - J_2\phi'_2) \cdot \nabla\theta - \mu_0[\alpha_1\beta_1 \nabla w_1 \cdot \nabla\phi'_1 - \alpha_2\beta_2 \nabla w_2 \cdot \nabla\phi'_2] \\
&-\kappa_0 \nabla\theta \cdot \nabla(\phi + \theta) + [\beta(\theta_1) g_1 - \beta(\theta_2) g_2]\ (\phi + \theta) + (f_1 - f_2)\ \phi
\end{aligned}$$

$$+\mu_0[\nabla u_1 \cdot \nabla w_1 - \nabla u_2 \cdot \nabla w_2]\ \theta\Big\}\ dx$$
$$\le C\Big[\|u\|^2 + \|U\|^2 + \|\vartheta\|^2 + \|\theta\|^2 + \int_\Omega \Big\{u^2(|\nabla\phi_1'|^2 + |\nabla\phi_2'|^2) + \nabla|\phi'|^2\Big\}\ dx\Big].$$

This together with (3.9) yields

$$\begin{aligned}\|u(t)\|^2 + \|\theta(t)\|^2 &\le c\int_0^t e^{c(t-s)}[\|U(s)\|^2 + \|\vartheta(s)\|^2]\ ds \\ &\le c \sup_{s\in[0,t]}[e^{-\lambda s}(\|U(s)\|^2 + \|\vartheta(s)\|^2)]\int_0^t e^{[ct+(\lambda-c)s]}\ ds \\ &\le \|\{U,\vartheta\}\|_\lambda\ \frac{ce^{ct}}{\lambda-c}[e^{(\lambda-c)t}-1] \le \frac{ce^{\lambda t}}{\lambda-c}\|\{U,\vartheta\}\|_\lambda.\end{aligned}$$

Hence for $\lambda = 3c$ we obtain (3.8). Thus by Banach's fixed point theorem $\mathcal{A}$ has a unique fixed point $\{u,\theta\}$. Now it is easy to check that $\{u,\theta,w\}$ with $w = \mathcal{P}(1-2u)$ is the unique solution to *Problem P*. □

4. Global behaviour

In this section we study the behaviour of the solution to *Problem P* as time t tends to infinity. The key for proving global a priori estimates are conservation of mass and energy E (comp. (2.4) resp. (2.5)) and increasing of entropy S (2.3) along the solution.

Lemma 4.1. *Let $\{u,w,\theta\}$ be the solution of Problem P and set*

$$E(t) = E(u(t),T(t)),\quad S(t) = S(u(t),T(t)),\quad T = B^{-1}(\theta).$$

Then for $t\in[0,\tau]$ following relations hold:

(i) $\quad \overline{u(t)} = \overline{u_0},$

(ii) $\quad E(t) = E(0),$

(iii) $\quad \dfrac{dS}{dt} = D \ge 0,\quad D := \displaystyle\int_\Omega \Big\{\frac{|J_u|^2}{\mu} + \kappa|\nabla\log T|^2\Big\}\ dx,$

(iv) $\quad \displaystyle\int_0^\tau D\ dr \le c_0 := E(0) - S(0) + (B(1)+2)|\Omega| + k_{0,1}.$

Proof. Choosing $h=1$ in (2.18) we get (i).
(ii) Choosing $h = \phi' + w$ in (2.18) and $h=1$ in (2.19), we obtain

$$\frac{dE}{dt} = (\theta_t,1) + (u_t,\phi'+w) = 0.$$

(iii) By (2.3), (A_2), (2.22), (2.18),(2.19) and (A_4) we find

$$\begin{aligned}\frac{dS}{dt} &= -(u_t,\phi'(u)) + (\theta_t,\frac{1}{T}) = \int_\Omega\Big\{-J_u\cdot\nabla\phi' - \kappa\nabla T\cdot\nabla\frac{1}{T} - J_u\cdot\frac{\nabla(\phi'+w)}{T}\Big\}\,dx \\ &= \int_\Omega\Big\{-\frac{J_u}{T}\cdot[(1+T)\nabla\phi' + \nabla w] + \kappa|\nabla\log T|^2\Big\}\ dx \\ &= \int_\Omega\Big\{-\frac{J_u(1+T)\phi''(u)}{T}\cdot[\nabla u + \alpha(u)\beta(\theta)\nabla w] + \kappa|\nabla\log T|^2\Big\}\ dx\ =\ D.\end{aligned}$$

(iv) Using $(ii), (iii)$ and the elementary estimate

$$\phi(u) = u\log u + (1-u)\log(1-u) \geq u(1-\frac{1}{u}) + (1-u)(1-\frac{1}{1-u}) \geq -1, \quad (4.1)$$

we get

$$\begin{aligned} S(0) \quad &+ \int_0^t Dds = S(0) + \int_0^t dS = S(t) = \int_\Omega \Big\{\psi'(T) - \phi(u)\Big\}\, dx \\ &= \int_\Omega \Big\{\int_1^T \frac{b(r)}{r}dr - \phi(u)\Big\}\, dx \leq \int_\Omega \Big\{\int_1^{T\geq 1} \frac{b(r)}{r}dr - \phi(u)\Big\}\, dx \\ &\leq \int_\Omega \Big\{\int_1^{T\geq 1} b(r)dr - \phi(u)\Big\}\, dx \leq \int_\Omega \Big\{\int_0^T b(r)dr - \phi(u)\Big\}\, dx \\ &= \int_\Omega \Big\{B(1) + T\psi'(T) - \psi(T) - \phi(u)\Big\}\, dx \\ &= E(t) + \int_\Omega \Big\{B(1) - 2\phi(u) - u\mathcal{P}(1-u)\Big\}\, dx \\ &= E(0) + \int_\Omega \Big\{B(1) - 2\phi(u) - u\mathcal{P}(1-u)\Big\}\, dx \leq c_0 + S(0). \end{aligned}$$

□

This lemma furnishes following global estimates:

Lemma 4.2. *Let $\{u, \theta, w\}$ be the solution of Problem P and*

$$T = B^{-1}(\theta), \quad v = (1+T)\phi'(u) + w.$$

Then

$$\begin{aligned} (i) \qquad & \sup_{t\in[0,\infty)} \Big\{\|\theta(t)\|_1,\ b_0\|T(t)\|_1\Big\} \leq c_1 := E(0) + (B(1)+1)|\Omega| + k_{0,1}, \\ (ii) \qquad & \sup_{t\in[0,\infty)} \Big\{\|\psi'(T(t))\|_1,\ b_0\|\log T(t)\|_1\Big\} \leq c_0 + c_1, \\ (iii) \qquad & \int_0^\infty \int_\Omega \Big\{\frac{1}{\phi''(u)}\Big|\nabla\Big(\frac{v}{1+T}\Big)\Big|^2 + |\nabla \log T|^2\Big\}\, dx\ dt \leq C. \end{aligned}$$

Proof. (i) By $T(t) \geq 0$, Lemma 4.1(ii), $\|u(t)\|_\infty \leq 1$ and (4.1) we find

$$\begin{aligned} b_0\|T(t)\|_1 \quad &\leq \quad \|\theta(t)\|_1 = \int_\Omega \int_0^{T(t)} b(r)\ dr\ dx = \int_\Omega \Big\{B(1) + \int_1^{T(t)} b(r)\ dr\Big\}\, dx \\ &= \int_\Omega \Big\{B(1) + [T\psi' - \psi](t)\Big\}\, dx \\ &= E(t) + \int_\Omega \Big\{B(1) - [\phi + \mathcal{P}(1-u)](t)\Big\}\, dx \\ &= E(0) + B(1)|\Omega| - \int_\Omega [\phi + \mathcal{P}(1-u)](t)\ dx \leq c_1. \end{aligned}$$

(ii) We denote by $\psi'(T)_\pm = \frac{1}{2}(|\psi'(T)| \pm \psi'(T)) \geq 0$ the positive resp. negative part of $\psi'(T)$, such that

$$\psi'(T) = \psi'(T)_+ - \psi'(T)_-. \quad (4.2)$$

Then the foregoing estimate implies

$$\begin{aligned} \|\psi'(T)_+\|_1 &= \int_{\{T(x)\ge 1\}} \Big\{ \int_1^{T(x)} \frac{b(r)}{r}\ dr \Big\}\ dx \\ &\le \int_{\{T(x)\ge 1\}} \Big\{ \int_1^{T(x)} b(r)\ dr \Big\}\ dx \le \int_\Omega \Big\{ \int_1^{T(x)} b(r)dr + B(1) \Big\}\ dx \le c_1. \end{aligned}$$

On the other hand, using (4.1) and (4.2), we infer from Lemma 4.1(iii, iv)

$$\begin{aligned} \|\psi'(T)_-\|_1 &= \int_\Omega (\psi'(T)_+ - \psi'(T))\ dx \le \int_\Omega (\psi'(T)_+ - \psi'(T) + \phi(u) + 1)\ dx \\ &= \int_\Omega (\psi'(T)_+ + 1)\ dx - S(t) = \int_\Omega (\psi'(T)_+ + 1)\ dx - \int_0^t dS - S(0) \\ &= \|\psi'(T)_+\|_1 + |\Omega| - \int_0^t D\ dr - S(0) \le c_1 + |\Omega| - S(0) = c_0. \end{aligned}$$

Thus (4.2) yields

$$\|\psi'(T)\|_1 = \|\psi'(T)_+ - \psi'(T)_-\|_1 \le \|\psi'(T)_+\|_1 + \|\psi'(T)_-\|_1 \le c_1 + c_0$$

and consequently

$$\begin{aligned} b_0\|\log T(t)\|_1 &= b_0 \int_\Omega |\log T| dx = b_0 \int_\Omega \Big| \int_1^T \frac{dr}{r} \Big| dx \le \int_\Omega \Big| \int_1^T \frac{b(r)}{r} dr \Big| dx \\ &= \|\psi'(T(t))\|_1. \end{aligned}$$

(iii) Letting $\tau \to \infty$ we see from Lemma 4.1$(iii), (iv)$ that

$$\int_0^\infty \int_\Omega \Big\{ \frac{|J_u|^2}{\mu} + \kappa |\nabla \log T|^2 \Big\}\ dx\ dt \le c_0. \tag{4.3}$$

By (2.12) we have

$$J_u = -\mu \Big[\nabla \frac{v}{T} - (\phi' + w) \nabla \frac{1}{T} \Big] = -\frac{\mu_0}{\phi''} \Big[\nabla \Big(\frac{v}{1+T} \Big) + \frac{wT\nabla \log T}{(1+T)^2} \Big].$$

Since $\|w(t)\|_\infty \le k_{0,\infty}|\Omega|$, $\phi'' \ge 4$ and $\kappa \ge \kappa_0\ b_0$, this implies

$$\frac{1}{\phi''} \Big| \nabla \Big(\frac{v}{1+T} \Big) \Big|^2 + |\nabla \log T|^2 \le C \Big(\frac{|J_u|^2}{\mu} + \kappa |\nabla \log T|^2 \Big).$$

Hence (4.3) yields (ii). □

Now we can state our main result concerning the asymptotic behaviour of the solution.

Theorem 4.3. *Let* $\{u, \theta, w\}$ *be the solution to Problem* P *guaranteed by Theorem* 3.3 *and*

$$T = B^{-1}(\theta), \quad v = (1+T)\phi'(u) + w.$$

Then a sequence $t_k \to \infty,\ k = 1, 2, \ldots,$ *functions* $u^*,\ w^*$ *and constants* $\theta^*,\ v^*,\ T^*,$ $\beta^* = \beta(\theta^*)$ *exist, such that* $u_k = u(t_k),\ \theta_k = \theta_k(t_k),\ w_k = w(t_k),\ v_k = v(t_k),\ T_k = T(t_k),\ \alpha_k = \alpha(u_k),\ \beta_k = \beta(\theta_k),\ \phi'_k = \phi'(u_k)$ *satisfy:*

(i) $\quad u_k \to u^*$ *and* $\log T_k \to \log T^*$ *strongly in* L^2, *weakly in* H^1 *and a. e. in* Ω,

(ii) $\quad \log T_k \to \log T^*$ *strongly in* H^1,

(iii) $\quad \theta_k \to \theta^*$ *and* $\psi'(T_k) \to \psi'(T^*)$ *weak* * *in* L^1 *and a. e. in* Ω,

(iv) $\quad w_k \to w^*$ *strongly in* H^1 *and a. e. in* Ω,

(v) $\quad v_k \to v^*$ *a. e. in* Ω,

(vi) $\quad \overline{u^*} = \overline{u_0}, \quad E(u^*, T^*) = E(u_0, T_0), \quad S(u^*, T^*) = \lim_{t\to\infty} S(u(t), T(t)),$

(vii) $\quad w^* = \mathcal{P}(1 - 2u^*), \quad u^* = \dfrac{1}{1 + e^{\beta^*(w^* - v^*)}},$

$(viii)$ $\quad (u^*, T^*)$ *is (possibly local) solution of the constrained maximum problem*

$$S(u, T) \to max, \quad \overline{u} = \overline{u_0}, \quad E(u, T) = E(u_0, T_0).$$

Proof. (i) By Lemma 4.2(iii) there exists a sequence $t_j \in [j, j+1]$, $j = 1, 2 \ldots$, such that

$$\int_\Omega \left\{ \alpha_j \left| \nabla(\beta_j v_j) \right|^2 + |\nabla \log T_j|^2 \right\} dx \to 0. \tag{4.4}$$

From this, $\|\nabla w_j\|_\infty \le k_{1,\infty}|\Omega|$, $\|\alpha_j\|_{L^\infty} \le \frac{1}{4}$ and

$$\left[\nabla u + \alpha(u)\beta(\theta)\nabla w\right]_j = \left[\alpha(u)[\nabla(\beta(\theta)\ v) + wT\beta^2(\theta)\nabla \log T]\right]_j$$

we obtain $\|\nabla u_j\|^2 \le C$. Hence $\|u_j\|_\infty \le 1$, Lemma 4.2(ii), (4.4) and Poincare's inequality imply

$$\|u(t_j)\|_{H^1} + \|\log T(t_j)\|_{H^1} \le C.$$

Thus, because of the compactness of the embedding of H^1 into L^2, there exist subsequences $(u_k) \subset (u_j)$ and $(T_k) \subset (T_j)$ satisfying (i).
(ii) follows from (i) and (4.4).
(iii) follows from (i), Lemma 4.2(i), (ii) and the weak* compactness of bounded sets in L^1.
(iv) is a consequence of (i) and assumption (A_3).
(v) Let $g_k = \arctan\left[\exp\frac{-\beta_k v_k}{2}\right]$. By $0 \le g_k(x) \le 1$, we can suppose that $g_k \rightharpoonup g^*$ in L^2. Using that $\|\beta_k w_k\|_\infty \le k_{0,\infty}|\Omega|$ and (4.4), we find

$$\lim_{k\to\infty} \int_\Omega |\nabla g_k|^2\ dx = \frac{1}{4} \lim_{k\to\infty} \int_\Omega \frac{e^{-\beta_k v_k}|\nabla(\beta_k v_k)|^2}{(1 + e^{-\beta_k v_k})^2}\ dx$$
$$\le \frac{1}{4} \lim_{k\to\infty} \int_\Omega \frac{e^{-\beta_k w_k}\alpha_k|\nabla(\beta_k v_k)|^2}{min[1, e^{-2\beta_k w_k}]}\ dx \le C \lim_{k\to\infty} \int_\Omega \alpha_k|\nabla(\beta_k\ v_k)|^2\ dx \to 0\ ,$$

and by Fatou's lemma

$$\int_\Omega |\nabla g^*|^2\ dx = 0,\ \text{i. e. },\ g^* = const.\ \text{in}\ \Omega.$$

Now $(i), (ii)$ and (iv) imply

$$\begin{aligned} g^* = \lim_{k\to\infty} g_k &= \lim_{k\to\infty} \arctan\Big[\sqrt{\frac{1-u_k}{u_k}}\; e^{\frac{\beta_k w_k}{2}}\Big] \\ &= \arctan\Big[\sqrt{\frac{1-u^*}{u^*}}\; e^{\frac{\beta^* w^*}{2}}\Big] \quad \text{a. e. in } \Omega. \end{aligned}$$

That is

$$u^* = \frac{1}{1+(\tan g^*)^2 e^{\beta^* w^*}} \; . \tag{4.5}$$

Since $\|w^*\|_\infty \le k_{0,\infty}|\Omega| =: c$, we have

$$\frac{1}{1+(\tan g^*)^2 e^{\beta^* c}} \le u^*(x) \le \frac{1}{1+(\tan g^*)^2 e^{-\beta^* c}} \; . \tag{4.6}$$

From this and $\overline{u^*} = \overline{u_0} < 1$ we infer $0 < g^* < \infty$. Thus by (4.6) an $\epsilon > 0$ exists such that $\epsilon < u^*(x) < 1-\epsilon \; \forall x \in \Omega$. So we get

$$v_k = (1+T_k)\log\frac{u_k}{1-u_k} + w_k \to (1+T^*)\log\frac{u^*}{1-u^*} + w^* =: v^* \quad \text{a. e. in } \Omega. \tag{4.7}$$

By (4.5) this means

$$v^* = -\frac{2\log(\tan g^*)}{\beta^*} = const.$$

(vi) By (i) and Lemma 4.1(i) we have $\overline{u^*} = \overline{u_0}$. From (2.4), (A_2), Lemma 4.1 and (ii) resp. (2.3) and Lemma 4.1$(iii), (iv)$ we have

$$\begin{aligned} E(u_0,T_0) = \lim_{k\to\infty} E(u_k,T_k) &= \lim_{k\to\infty}\int_\Omega [\phi(u_k) + \theta_k - B(1) + u_k\mathcal{P}(1-u_k)]\,dx \\ \text{resp. } \lim_{t\to\infty} S(u(t),T(t)) &= \lim_{k\to\infty}\int_\Omega [\psi'(u_k) - \phi(u_k)]\,dx. \end{aligned}$$

Thus the remaining relations in (vi) are consequences of (i), (iii) and Lebesgue's dominated convergence theorem.
(vii) By (i), (iv) we can take the limit $k \to \infty$ in (2.20) to prove the first relation in (iv). The second one follows from (4.7).
$(viii)$ Following the arguments of the introduction leading to (2.7), (2.8), we see that the equations (vii) coincide with the Euler-Lagrange equations associated to the constrained maximum problem for the entropy functional S, i. e., (u^*, T^*) is (possibly local) maximizer of S. □

Theorem 4.4. *Suppose that*

$$(A_7) \qquad \gamma := \frac{k_{0,2}\,(b_0 + c_\phi)}{2b_0} < 1 \quad \text{or} \quad E(u_0,T_0) > B(\gamma-1)\,|\Omega| + k_{0,1}.$$

Then the equilibrium state $\{u^*, v^*, w^*, T^*\}$ given by Theorem 4.3 is unique. Moreover, the assertions of Theorem 4.3 hold with $t_k \to \infty$ replaced by $t \to \infty$.

Proof. Let $\{u_i, v_i, w_i, T_i\}$, $i = 1, 2$, be equilibrium states such that

$$w_i = \mathcal{P}(1-2u_i), \quad v_i = (1+T_i)\phi'(u_i) + w_i = const., \quad 0 \le T_i = const.,$$
$$\overline{u_i} = \overline{u_0}, \quad E(u_i, T_i) = E(u_0, T_0), \quad S(u_i, T_i) = \lim_{t\to\infty} S(ut), T(t)) =: S_\infty.$$

From the energy equality we can estimate T_i from below as follows:

$$\begin{aligned}
|\Omega| B(T_i) &= |\Omega| \int_0^{T_i} b(r)\ dr = |\Omega| B(1) + \int_\Omega \int_1^{T_i} b(r)\ dr\ dx \\
&= |\Omega| B(1) + \int_\Omega [T_i \psi'(T_i) - \psi(T_i)]\ dx \\
&= E(u_0, T_0) - \int_\Omega [\phi(u_i) - u_i \mathcal{P}(1-u_i)]\ dx \ge E(u_0, T_0) - k_{0,1}\ , \text{ i. e.,} \\
T_i &\ge B^{-1}\Big[\max\big[0, \frac{E(u_0,T_0) - k_{0,1}}{|\Omega|}\big]\Big].
\end{aligned}$$

Further, by (2.16), we have

$$\begin{aligned}
&|\phi(u_1) - \phi(u_2)|^2 \le (c_\phi \varrho)^2, \\
&\varrho^2(u_1, u_2) := (u_1 - u_2)(\phi'(u_1) - \phi'(u_2)) \ge 4\|u_1 - u_2\|^2,
\end{aligned}$$

and thus by $S(u_i, T_i) = S_\infty$

$$\begin{aligned}
b_0 \Big|\log\frac{T_1}{T_2}\Big| &= b_0 \Big|\int_{T_2}^{T_1} \frac{dr}{r}\Big| \le \Big|\int_{T_2}^{T_1} \frac{b(r)dr}{r}\Big| = \Big|\psi'(T_1) - \psi'(T_2)\Big| \\
&= \frac{1}{|\Omega|}\Big|\int_\Omega [\phi(u_1) - \phi(u_2)]\ dx\Big| \le \frac{c_\phi \|\varrho\|}{|\Omega|^{\frac{1}{2}}}\ .
\end{aligned}$$

Putting these inequalities together and using mass equality and that $v_i, T_i = const.$, we obtain with $u = u_1 - u_2$

$$\begin{aligned}
\|\varrho\|^2 &= \Big(u, \frac{v_1 - w_1}{1+T_1} - \frac{v_2 - w_2}{1+T_2}\Big) = \Big(u, \frac{w_2}{1+T_2} - \frac{w_1}{1+T_1}\Big) \\
&= \frac{1}{2}\Big(u, \Big[\frac{1}{1+T_1} + \frac{1}{1+T_2}\Big](w_2 - w_1) + \Big[\frac{1}{1+T_1} - \frac{1}{1+T_2}\Big](w_1 + w_2)\Big) \\
&\le \frac{k_{0,2}\|u\|}{1+\min[T_1, T_2]}\Big[2\|u\| + \frac{|T_1 - T_2||\Omega|^{\frac{1}{2}}}{1+\max[T_1, T_2]}\Big] \\
&\le \frac{k_{0,2}\|u\|}{1+\min[T_1, T_2]}\Big[2\|u\| + \Big|\log\frac{T_1}{T_2}\Big||\Omega|^{\frac{1}{2}}\Big] \\
&\le \frac{\gamma}{1 + B^{-1}\Big[\min\big[0, \frac{E(u_0,T_0) - k_{0,1}}{|\Omega|}\big]\Big]}\ \|\varrho\|^2.
\end{aligned}$$

Because of assumption (A_7) this estimate implies $\varrho(u_1, u_2) = 0$ and consequently

$$\{u_1, v_1, w_1, T_1\} = \{u_2, v_2, w_2, T_2\} = \{u^*, v^*, w^*, T^*\}.$$

Finally, this and the global a priori estimates and compactness arguments applied in the proof of Theorem 4.3 ensure that

$$\lim_{t\to\infty}\{u(t), v(t), w(t), T(t)\} = \{u^*, v^*, w^*, T^*\}$$

in the sense of Theorem 4.3, $(i) - (v)$. □

References

[1] R.A. Adams, *Sobolev spaces*, Pure Appl. Math., Academic Press, New York - London, 1975.

[2] G. Albinus, H. Gajewski, R. Hünlich, *Thermodynamic design of energy models of semiconductor devices*, Preprint No. **573**, WIAS (2000), to appear.

[3] N. Alikakos, *An Application of the invariance principle to reaction–diffusion equations*, J. Differential Equations **333** (1979), 201–225.

[4] H.W. Alt, I. Pawlow, *A mathematical model of dynamics of non–isothermal phase separation*, Phys. D **59** (1992), 389–416.

[5] H.W. Alt, I. Pawlow, *Existence of solutions for non–isothermal phase separation*, Adv. Math. Sci. Appl. **1** (1992), 319-409.

[6] J.W. Cahn, J.E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys. **28** (1958), 258–267.

[7] C.–K. Chen, P.C. Fife, *Nonlocal models of phase transitions in solids*, Adv. Math. Sci. Appl. **10**, (2000), 821–849.

[8] C. Ebmeyer, J. Frehse, *Mixed boundary value problems for nonlinear elliptic equations in multidimensional non-smooth domains*, Math. Nachr. **203**, (1999), 47–74.

[9] C.M. Elliot, H. Garcke, *On the Cahn–Hilliard equation with degenerate mobility*, SIAM J. Math. Anal. **27**, (1996), 404–423.

[10] H. Garcke, *On the Cahn–Hilliard equation with non–constant mobility*, FBP News **4**, (1994), 16–17.

[11] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie–Verlag, Berlin, 1974.

[12] H. Gajewski, I.V. Skrypnik, *To the uniqueness problem for nonlinear parabolic equations*, Preprint No. **658**, WIAS (2001), to appear.

[13] H. Gajewski, K. Zacharias, *On a nonlocal phase separation model*, Preprint No. 656, WIAS (2001), to appear.

[14] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions I. Macroscopic limits*, J. Statist. Phys. **87**, (1997), 37–61.

[15] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions II. Interface motion*, SIAM J. Appl. Math. **58**, (1998), 1707–1729.

[16] A. Kufner, O. John, S. Fučík, *Function spaces*, Academia, Prague, 1977.

[17] O.A. Ladyzhenskaja, V.A. Solonnikov, N.N. Uraltseva, *Linear and quasilinear equations of parabolic type*, Math. Monographs, A.M.S., Providence, vol. **23**, 1974.

[18] E.H. Lieb, M. Loss, *Analysis*, A.M.S., Providence, 1997.

[19] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier–Villars, Paris, 1969.

[20] O. Penrose, P.C. Fife, *Thermodynamically consistent models of phase–field type for the kinetics of phase transitions*, Phys. D **43** (1990), 44–62.

[21] W. Shen, S. Zheng, *On the Coupled Cahn–Hilliard Equations*, Comm. Partial Differential Equations **18** (1993), 701–727.