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Mean-square symplectic methods for Hamiltonian systems with multiplicative noise

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Abstract. Stochastic systems with multiplicative noise, phase flows of which have integral invariants, are considered. For such systems, numerical methods preserving the integral invariants are constructed using full implicit schemes of a new type for stochastic differential equations. In these full implicit schemes increments of Wiener processes are substituted by some truncated random variables. They are important for both theory and practice of numerical integration of stochastic differential equations. A special attention is paid to systems with separable Hamiltonians and to Hamiltonian systems with small noise. Liouvillian methods for stochastic systems preserving phase volume are also proposed. Some results of numerical experiments are presented.

1. Introduction

Consider the Cauchy problem for the system of stochastic differential equations (SDEs) in the sense of Stratonovich

$$dP = f(t,P,Q)dt + \sum_{r=1}^{m} \sigma_r(t,P,Q) \circ dw_r(t),\ P(t_0) = p, \tag{1.1}$$

$$dQ = g(t,P,Q)dt + \sum_{r=1}^{m} \gamma_r(t,P,Q) \circ dw_r(t),\ Q(t_0) = q,$$

where P, Q, f, g, σ_r, γ_r are n-dimensional column-vectors with the components P^i, Q^i, f^i, g^i, σ_r^i, γ_r^i, $i = 1,\dots,n$, and $w_r(t)$, $r = 1,\dots,m$, are independent standard Wiener processes.

We suppose that all the coefficients of considered systems are sufficiently smooth functions defined for $(t,p,q) \in [t_0, t_0+T] \times R^d$, $d = 2n$, and they provide the property of extendability of solutions to the interval $[t_0, t_0+T]$ (additional conditions in connection with considered methods consist in appropriate behavior of partial derivatives of the coefficients on infinity).

We denote by $X(t;t_0,x) = (P(t;t_0,p,q), Q(t;t_0,p,q))^\intercal = (P(t;t_0,p^1,\dots,p^n,q^1,\dots,q^n), Q(t;t_0,p^1,\dots,p^n,q^1,\dots,q^n))^\intercal$, $t_0 \le t \le t_0+T$, the solution of the problem (1.1). A more detailed notation is $X(t;t_0,x;\omega)$, where ω is an elementary event. It is known that $X(t;t_0,x;\omega)$ is a phase flow (diffeomorphism) for almost every ω. See its properties in, e.g. [1, 2].

If there are functions $H_r(t,p,q)$, $r = 0,\dots,m$, such that (see [1] and [3])

$$f^i = -\partial H_0/\partial q^i, \quad g^i = \partial H_0/\partial p^i, \tag{1.2}$$

$$\sigma_r^i = -\partial H_r/\partial q^i, \quad \gamma_r^i = \partial H_r/\partial p^i, \quad i = 1,\dots,n, \quad r = 1,\dots,m,$$

then the phase flow of (1.1) preserves symplectic structure:

$$dP \wedge dQ = dp \wedge dq, \tag{1.3}$$

i.e., the sum of the oriented areas of projections onto the coordinate planes $(p^1,q^1),\dots,(p^n,q^n)$ is an integral invariant [4].

Let P_k, Q_k, $k = 0,\dots,N$, $t_{k+1} - t_k = h_{k+1}$, $t_N = t_0 + T$, be a method for (1.1) based on the one-step approximation $\bar{P} = \bar{P}(t+h;t,p,q)$, $\bar{Q} = \bar{Q}(t+h;t,p,q)$. We say that the method preserves symplectic structure if

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq\,. \tag{1.4}$$

The present paper deals with symplectic integration of the Hamiltonian system with multiplicative noise (1.1), (1.2). It is a continuation of [3], where symplectic methods for Hamiltonian systems with additive noise were proposed. For symplectic integration of deterministic Hamiltonian systems see, e.g. [5, 6, 7, 8] and references therein.

As is known [5], in the case of deterministic general Hamiltonian systems symplectic Runge-Kutta (RK) methods are all implicit. Hence it is natural to expect that to construct symplectic methods for general Hamiltonian stochastic systems with multiplicative noise, full implicit methods are needed. The known implicit methods for stochastic systems with multiplicative noise (see [9, 10]) contain implicitness in deterministic terms only. In [11] an implicitness is introduced in stochastic terms as well. But the methods of [11] are of a very special form. In Section 2 a new class of full implicit methods of mean-square order 1/2 for general stochastic systems is proposed. In these implicit schemes increments of Wiener processes are substituted by some truncated random variables. They are important for both theory and practice of numerical integration of SDEs. In the commutative case a full implicit method of mean-square order 1 is also obtained. We use these full implicit methods in Section 3 to construct symplectic methods for general Hamiltonian systems with multiplicative noise. Sections 4 and 5 are devoted to special cases of separable Hamiltonians and systems with small noise respectively. In Section 6 some Liouvillian methods for stochastic systems preserving phase volume are constructed. Numerical tests are presented in the last section.

2. Full implicit methods

Construction of implicit methods for stochastic systems with additive noise does not cause any principal difficulties. However, all is much more intricate in the case of stochastic systems with multiplicative noise. The known implicit methods for such systems (see [9, 10]) contain implicitness restricted to deterministic terms, e.g., to the drift terms in the implicit Euler scheme. In [11], an implicitness is introduced in stochastic terms as well. But methods of [11] are of a very special form. In this section we construct a sufficiently large class of full implicit methods of mean-square order 1/2 for general stochastic systems.

2.1. The convergence theorem on mean-square methods from [9].

Let us recall some formulae of numerical methods for SDEs in the Ito sense

$$dX = a(t,X)dt + \sum_{r=1}^{m} b_r(t,X)dw_r(t), \; X(t_0) = X_0, \tag{2.1}$$

where X, $a(t,x^1,\dots,x^d)$, $b_r(t,x^1,\dots,x^d)$ are d-dimensional column-vectors with the components X^i, a^i, b_r^i, $i=1,\dots,d$, and $w_r(t)$, $r=1,\dots,m$, are independent standard Wiener processes.

Consider mean-square approximations of the solution to the system (2.1). A one-step mean-square approximation $\bar{X}_{t,x}(t+h)$, $t_0 \le t < t+h \le t_0+T$, is constructed depending on t, x, h, and $\{w_1(\vartheta)-w_1(t),\dots,w_m(\vartheta)-w_m(t);\; t\le\vartheta\le t+h\}$. Using the one-step approximation, we recurrently obtain the approximation X_k, $k=0,\dots,N$, $t_{k+1}-t_k = h_{k+1}$, $t_N = t_0+T$:

$$X_0 = X(t_0), \; X_{k+1} = \bar{X}_{t_k,X_k}(t_{k+1}).$$

For simplicity, we will take $t_{k+1} - t_k = h = T/N$. Note that X_0 may be a random variable which does not depend on the Wiener processes $w_r(t)$, $t \in [t_0, t_0 + T]$.

Suppose the functions $a(t,x)$ and $b_r(t,x)$ are defined and continuous for $t \in [t_0, t_0 + T]$, $x \in R^d$ and satisfy a uniform Lipschitz condition: for all $t \in [t_0, t_0 + T]$, $x, y \in R^d$ there is a constant $L > 0$ such that

$$|a(t,x) - a(t,y)| + \sum_{r=1}^{m} |b_r(t,x) - b_r(t,y)| \le L\,|x - y|\,. \tag{2.2}$$

Theorem 2.1. *(see [9]) Suppose the one-step approximation $\bar{X}_{t,x}(t+h)$ has order of accuracy p_1 for the expectation of the deviation and order of accuracy p_2 for the mean-square deviation; more precisely, for arbitrary $t_0 \le t \le t_0 + T - h$, $x \in R^d$ the following inequalities hold:*

$$\left| E\left(X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right)\right| \le K \cdot (1 + |x|^2)^{1/2} h^{p_1}, \tag{2.3}$$

$$\left[E\left|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right|^2\right]^{1/2} \le K \cdot (1 + |x|^2)^{1/2} h^{p_2}. \tag{2.4}$$

Also, let

$$p_2 \ge \frac{1}{2},\ p_1 \ge p_2 + \frac{1}{2}. \tag{2.5}$$

Then for any N and $k = 0, \ldots, N$ the following inequality holds:

$$\left[E\left|X_{t_0,X_0}(t_k) - \bar{X}_{t_0,X_0}(t_k)\right|^2\right]^{1/2} \le K \cdot (1 + E|X_0|^2)^{1/2} h^{p_2 - 1/2}, \tag{2.6}$$

i.e., the mean-square order of accuracy of the method constructed using the one-step approximation $\bar{X}_{t,x}(t+h)$ is $p = p_2 - 1/2$.

We note that all constants K mentioned above, as well as the ones that will appear in the sequel, depend in the final analysis on the system (2.1) and the approximations only and do not depend on X_0 and h.

The following evident lemma will be useful later on.

Lemma 2.1. *Let the one-step approximation $\bar{X}_{t,x}(t+h)$ satisfy the conditions of Theorem 2.1. Suppose that $\tilde{X}_{t,x}(t+h)$ is such that*

$$\left| E\left(\tilde{X}_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right)\right| = O(h^{p_1}), \tag{2.7}$$

$$\left[E\left|\tilde{X}_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\right|^2\right]^{1/2} = O(h^{p_2}) \tag{2.8}$$

with the same p_1 and p_2. Then the method based on the one-step approximation $\tilde{X}_{t,x}(t+h)$ has the same mean-square order of accuracy as the method based on $\bar{X}_{t,x}(t+h)$, i.e., its order is equal to $p = p_2 - 1/2$.

2.2. **The main idea and an example.** Let us start with an example. Consider the Ito scalar equation

$$dX = \sigma X\, dw(t). \tag{2.9}$$

The one-step approximation of the Euler method $\hat{X}$ is

(2.10) $$\hat{X} = x + \sigma x \Delta w(h).$$

We can represent this method in the form

$$\hat{X} = x + \sigma \hat{X} \Delta w + \sigma(x - \hat{X})\Delta w = x - \sigma^2 x (\Delta w)^2 + \sigma \hat{X} \Delta w.$$

As h is small, $(\Delta w)^2 \sim h$, and we obtain the following "natural" implicit method

(2.11) $$\tilde{X} = x - \sigma^2 x h + \sigma \tilde{X} \Delta w(h).$$

However, this method cannot be realized since $1 - \sigma \Delta w(h)$ can vanish for any small h. Further, for the formal value of $\tilde{X}$ from (2.11):

$$\tilde{X} = \frac{x(1 - \sigma^2 h)}{1 - \sigma \Delta w(h)},$$

we have $E|\tilde{X}| = \infty$. Clearly, method (2.11) is not suitable. The reason of this is unboundedness of the random variable $\Delta w(h)$ for any arbitrarily small h.

Our basic idea consists in replacement of $\Delta w(h) = \xi\sqrt{h}$, where ξ is $\mathcal{N}(0,1)$-distributed random variable, by another random variable $\zeta\sqrt{h} = \zeta_h\sqrt{h}$ such that $\zeta\sqrt{h}$ is bounded and the Euler-like method

(2.12) $$\check{X} = x + \sigma x \zeta \sqrt{h}$$

is of mean-square order $1/2$ as well. To achieve this, it is sufficient to require:

(2.13) $$E(\check{X} - \hat{X}) = O(h^{3/2}), \ E(\check{X} - \hat{X})^2 = O(h^2).$$

We take ζ as symmetric. Then $E(\check{X} - \hat{X}) = 0$. To satisfy the second equation in (2.13), the condition $E(\zeta_h - \xi)^2 = O(h)$ is sufficient.

We shall require a stronger inequality

(2.14) $$E(\zeta_h - \xi)^2 \le h^k, \ k \ge 1.$$

Let for $A_h > 0$

(2.15) $$\zeta_h = \begin{cases} \xi, \ |\xi| \le A_h, \\ A_h, \ \xi > A_h, \\ -A_h, \ \xi < -A_h. \end{cases}$$

Since

$$E(\zeta_h - \xi)^2 = \frac{2}{\sqrt{2\pi}} \int_{A_h}^{\infty} (x - A_h)^2 e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}} e^{-A_h^2/2} \int_{A_h}^{\infty} y^2 e^{-y^2/2} e^{-A_h y} dy < e^{-A_h^2/2},$$

(2.14) is fulfilled if $e^{-A_h^2/2} \le h^k$, i.e. $A_h^2 \ge 2k|\ln h|$. Thus, if

$$A_h = \sqrt{2k|\ln h|}, \ k \ge 1,$$

then the method based on the one-step approximation (2.12) has the mean-square order of convergence equal to $1/2$.

Lemma 2.2. *Let* $A_h = \sqrt{2k|\ln h|}, \ k \ge 1,$ *and* ζ_h *be defined by* (2.15). *Then the following inequality holds:*

(2.16) $$0 \le E(\xi^2 - \zeta_h^2) = 1 - E\zeta_h^2 \le (1 + 2\sqrt{2k|\ln h|})h^k.$$

Proof. We have

$$1 - E\zeta_h^2 = \frac{2}{\sqrt{2\pi}} \int_{A_h}^{\infty} (x^2 - A_h^2) e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}} \int_{A_h}^{\infty} \left[(x - A_h)^2 + 2A_h(x - A_h) \right] e^{-x^2/2} dx$$

$$\le e^{-A_h^2/2} + \frac{4A_h}{\sqrt{2\pi}} \int_{A_h}^{\infty} x e^{-x^2/2} dx = e^{-A_h^2/2} (1 + \frac{4A_h}{\sqrt{2\pi}}) \le (1 + 2A_h) e^{-A_h^2/2},$$

whence (2.16) follows. □

Now consider the following implicit method (for definiteness we put $k = 1$ and $A_h = \sqrt{2|\ln h|}$) :

$$\bar{X} = x - \sigma^2 x h + \sigma \bar{X} \zeta_h \sqrt{h}, \ \bar{X} = \frac{x(1 - \sigma^2 h)}{1 - \sigma \zeta_h \sqrt{h}}. \tag{2.17}$$

Since $|\zeta_h| \le \sqrt{2|\ln h|}$, this method is realizable for all h satisfying the inequality

$$2h|\ln h| < \frac{1}{\sigma^2}. \tag{2.18}$$

Proposition 2.1. *Method* (2.17) *is of mean-square order* 1/2.

Proof. Let us compare method (2.17) with the Euler method (2.10). We get

$$E\bar{X} = x(1 - \sigma^2 h) E \sum_{m=0}^{\infty} \sigma^m \zeta_h^m h^{m/2} = x(1 - \sigma^2 h) E \sum_{m=0}^{\infty} \sigma^{2m} \zeta_h^{2m} h^m.$$

It is obvious from here that the principal term in the expansion of $E(\bar{X} - \hat{X})$ is equal to $x\sigma^2 h(E\zeta_h^2 - 1)$. Due to Lemma 2.2, we obtain for all sufficiently small h :

$$|E(\bar{X} - \hat{X})| \le C|x|\sigma^2(1 + 2\sqrt{2|\ln h|})h^2, \tag{2.19}$$

where C is a positive constant.

Further

$$\begin{aligned} E(\bar{X} - \hat{X})^2 &= E(-\sigma^2 x h + \sigma \bar{X} \zeta_h \sqrt{h} - \sigma x \xi \sqrt{h})^2 \\ &\le 2\sigma^4 x^2 h^2 + 2E(\sigma \bar{X} \zeta_h \sqrt{h} - \sigma x \xi \sqrt{h})^2 \\ &= 2\sigma^4 x^2 h^2 + 2E(\sigma \cdot (x - \sigma^2 x h + \sigma \bar{X} \zeta_h \sqrt{h}) \zeta_h \sqrt{h} - \sigma x \xi \sqrt{h})^2 \\ &\le 2\sigma^4 x^2 h^2 + 2\sigma^2 x^2 h E(\zeta_h - \xi)^2 + C_1 x^2 h^2 \le C_2 x^2 h^2 \end{aligned} \tag{2.20}$$

for all sufficiently small h and some positive constants C_1 and C_2. The inequalities (2.19) and (2.20) imply the mean-square convergence of implicit method (2.17) with order 1/2. □

Introduction of implicitness in the stochastic term leads to appearance of the compensating term $-\sigma^2 x h$ in (2.17). This can be explained in the following way. Since $\bar{X}$ must be close to $x + \sigma x \zeta_h \sqrt{h}$, the expression $x + \sigma \bar{X} \zeta_h \sqrt{h}$ is close to $x + \sigma x \zeta_h \sqrt{h} + \sigma^2 x \zeta_h^2 h$. Consequently, making use of the compensating term results in $x + \sigma \bar{X} \zeta_h \sqrt{h} - \sigma^2 x h = x + \sigma x \zeta_h \sqrt{h} + \sigma^2 x (\zeta_h^2 - 1) h \approx x + \sigma x \zeta_h \sqrt{h}$, i.e., we get the correct result.

Now let us consider the expression $\sigma((1-\beta)x+\beta\bar{X})\zeta_h\sqrt{h}$ which introduces implicitness in the stochastic term with the parameter $0 \le \beta \le 1$. Clearly, the compensating term in this case is equal to $-\sigma^2\beta x h$. Thus, we derive the method:

$$\bar{X} = x - \sigma^2\beta x h + \sigma((1-\beta)x+\beta\bar{X})\zeta_h\sqrt{h},\ 0 \le \beta \le 1. \tag{2.21}$$

The following proposition can be proved analogously to Proposition 2.1.

Proposition 2.2. *The method* (2.21) *is of the mean-square order* 1/2 *as well as the methods:*

$$\bar{X} = x - \sigma^2\beta x\zeta_h^2 h + \sigma((1-\beta)x+\beta\bar{X})\zeta_h\sqrt{h},\ 0 \le \beta \le 1; \tag{2.22}$$

$$\bar{X} = x - \sigma^2\beta((1-\alpha)x+\alpha\bar{X})h + \sigma((1-\beta)x+\beta\bar{X})\zeta_h\sqrt{h},\ 0 \le \alpha,\beta \le 1. \tag{2.23}$$

2.3. **Convergence theorem.** Now we are in position to introduce full implicit methods for general systems of stochastic differential equations. For simplicity in writing we deal here with the scalar Ito SDE:

$$dX = a(t,X)dt + b(t,X)dw(t). \tag{2.24}$$

We suppose that $a(t,x),\ b(t,x),\ \dfrac{\partial b}{\partial x}(t,x)$ are continuous for $t_0 \le t \le T,\ x \in \mathbf{R}$, and there exists a positive constant L such that

$$|a(t,y)-a(t,x)| \le L|y-x|,\ |\frac{\partial b}{\partial x}(t,x)| \le L,\ t_0 \le t \le T,\ x,y \in \mathbf{R}. \tag{2.25}$$

Note that below the same letter L (or K, or C) is used for various constants.

Consider the following natural implicit one-step approximation

$$\bar{X} = x + a(t,\bar{X})h - b(t,x)\frac{\partial b}{\partial x}(t,x)h + b(t,\bar{X})\zeta_h\sqrt{h}, \tag{2.26}$$

where ζ_h is defined by (2.15) with $A_h = \sqrt{2|\ln h|}$ for definiteness.

Lemma 2.3. *There exist constants $K > 0$ and $h_0 > 0$ such that for any $h \le h_0,\ t_0 \le t \le T,\ x \in \mathbf{R}$ the equation* (2.26) *has a unique solution $\bar{X}$ which satisfies the inequality*

$$|\bar{X} - x| \le K(1+|x|)(|\zeta_h|\sqrt{h}+h). \tag{2.27}$$

The solution $\bar{X}$ of equation (2.26) *can be found by the method of simple iteration with x as the initial approximation.*

Proof. For any fixed $t,\ x$, and h, let us introduce the function

$$\varphi(z) = x + a(t,z)h - b(t,x)\frac{\partial b}{\partial x}(t,x)h + b(t,z)\zeta_h\sqrt{h}.$$

Then (2.26) can be written as

$$\bar{X} = \varphi(\bar{X}).$$

There is a positive constant C such that for any $z \in \mathbf{R}$

$$|\varphi(z) - x| \le |a(t,x)|h + |a(t,z)-a(t,x)|h + |b(t,x)||\zeta_h|\sqrt{h} + |b(t,z)-b(t,x)||\zeta_h|\sqrt{h}$$
$$+|b(t,x)\frac{\partial b}{\partial x}(t,x)|h \le C(1+|x|)(|\zeta_h|\sqrt{h}+h) + L|z-x|(|\zeta_h|\sqrt{h}+h).$$

Further, for any $z_1,\ z_2 \in \mathbf{R}$

$$|\varphi(z_2) - \varphi(z_1)| \le L|z_2 - z_1|(|\zeta_h|\sqrt{h} + h).$$

Clearly, there exist positive constants K and h_0 such that for any $h \le h_0,\ x \in \mathbf{R}$

$$L(|\zeta_h|\sqrt{h} + h) < 1$$

and if

$$|z - x| \le K(1 + |x|)(|\zeta_h|\sqrt{h} + h),$$

then

$$|\varphi(z) - x| \le K(1 + |x|)(|\zeta_h|\sqrt{h} + h).$$

Let us note that the constants K in the last two inequalities are the same. Now the lemma follows from the contraction mapping principle. □

In addition to (2.25) suppose that there exist continuous $\partial a/\partial t$, $\partial b/\partial t$, and $\partial^2 b/\partial x^2$ and

$$|\frac{\partial a}{\partial t}(t,x)| \le L(1+|x|),\ |\frac{\partial b}{\partial t}(t,x)| \le L(1+|x|),\ \ t_0 \le t \le T,\ x \in \mathbf{R}. \tag{2.28}$$

Theorem 2.2. *Assume* (2.25) *and* (2.28). *Let there exist* $\delta > 0$ *such that if* $|y - x| \le \delta(1 + |x|)$, *the inequality*

$$|b(t,x)\frac{\partial^2 b}{\partial x^2}(t,y)| \le L,\ t_0 \le t \le T, \tag{2.29}$$

holds.

Then the implicit method based on the one-step approximation (2.26) *converges in mean-square with order* 1/2.

Proof. Let $\hat{X}$ be the Euler approximation for (2.24):

$$\hat{X} = x + a(t,x)h + b(t,x)\Delta w(h).$$

Using the condition (2.25) only, we get

$$E|\bar{X} - \hat{X}|^2 \le E|a(t,\bar{X})h - a(t,x)h + b(t,\bar{X})\zeta_h\sqrt{h} - b(t,x)\Delta w(h) - b(t,x)\frac{\partial b}{\partial x}(t,x)h|^2$$
$$\le LE|a(t,\bar{X}) - a(t,x)|^2h^2 + LE|b(t,\bar{X}) - b(t,x)|^2\zeta_h^2 h$$
$$+Lb^2(t,x)E(\zeta_h - \xi)^2h + L|b(t,x)\frac{\partial b}{\partial x}(t,x)|^2h^2$$
$$\le LE|\bar{X} - x|^2h^2 + LE|\bar{X} - x|^2\zeta_h^2 h + L(1+|x|)^2E(\zeta_h - \xi)^2h + L(1+|x|)^2h^2.$$

Using Lemma 2.3, the fact that $E\zeta^4 < E\xi^4 = 3$, and (2.14), we obtain from here that

$$E|\bar{X} - \hat{X}|^2 \le L(1+|x|)^2h^2. \tag{2.30}$$

Let us proceed now to evaluation of $E(\bar{X} - \hat{X})$. We have

$$|E(\bar{X} - \hat{X})| \le |Ea(t,\bar{X}) - a(t,x)|h + |E(b(t,\bar{X}) - b(t,x))\zeta_h\sqrt{h} - b(t,x)\frac{\partial b}{\partial x}(t,x)h|. \tag{2.31}$$

Due to Lemma 2.3, $E|\bar{X} - x| \le K(1+|x|)(E|\zeta_h|\sqrt{h} + h)$. Then

$$|Ea(t,\bar{X}) - a(t,x)|h \le C(1+|x|)h^{3/2}. \tag{2.32}$$

We have

$$(2.33)\qquad (b(t,\bar{X})-b(t,x))\zeta_h\sqrt{h}-b(t,x)\frac{\partial b}{\partial x}(t,x)h$$
$$=\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot(\bar{X}-x)\zeta_h\sqrt{h}-b(t,x)\frac{\partial b}{\partial x}(t,x)h$$
$$=\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot(a(t,\bar{X})h+b(t,\bar{X})\zeta_h\sqrt{h}-b(t,x)\frac{\partial b}{\partial x}(t,x)h)\zeta_h\sqrt{h}$$
$$-b(t,x)\frac{\partial b}{\partial x}(t,x)h$$
$$=\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot(a(t,\bar{X})-b(t,x)\frac{\partial b}{\partial x}(t,x)h)\zeta_h h^{3/2}$$
$$+\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot b(t,\bar{X})\zeta_h^2 h-b(t,x)\frac{\partial b}{\partial x}(t,x)h,$$

where $0\le\theta\le 1$.

Since $|\bar{X}-x|\le\rho(1+|x|)$, where $\rho\to 0$ as $h\to 0$, we get $|\bar{X}|\le|x|+|\bar{X}-x|\le K(1+|x|)$ for all sufficiently small h. Therefore

$$(2.34)\qquad |E\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot a(t,\bar{X})\zeta_h h^{3/2}|\le KE|a(t,\bar{X})\zeta_h|h^{3/2}$$
$$\le KE(1+|\bar{X}|)|\zeta_h|h^{3/2}\le K(1+|x|)h^{3/2}.$$

Clearly,

$$|E\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot b(t,x)\frac{\partial b}{\partial x}(t,x)\zeta_h h^{3/2}|\le K(1+|x|)h^{3/2}.$$

Let us estimate the last two terms in (2.33). We obtain

$$\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot b(t,\bar{X})\zeta_h^2 h-b(t,x)\frac{\partial b}{\partial x}(t,x)h$$
$$=(\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))-\frac{\partial b}{\partial x}(t,x))b(t,\bar{X})\zeta_h^2 h$$
$$+\frac{\partial b}{\partial x}(t,x)(b(t,\bar{X})-b(t,x))\zeta_h^2 h+\frac{\partial b}{\partial x}(t,x)b(t,x)(\zeta_h^2-1)h$$
$$=\frac{\partial^2 b}{\partial x^2}(t,x+\theta_1(\bar{X}-x))\cdot\theta(\bar{X}-x)\cdot b(t,\bar{X})\zeta_h^2 h$$
$$+\frac{\partial b}{\partial x}(t,x)\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot(\bar{X}-x)\zeta_h^2 h+\frac{\partial b}{\partial x}(t,x)b(t,x)(\zeta_h^2-1)h,$$

where $0\le\theta,\ \theta_1\le 1$. Due to Lemma 2.3, we get $|x+\theta_1(\bar{X}-x)-\bar{X}|\le|\bar{X}-x|\le K(|\zeta_h|\sqrt{h}+h)(1+|x|)$. For all sufficiently small h we have $K(|\zeta_h|\sqrt{h}+h)<\delta$ and consequently due to (2.29)

$$(2.35)\qquad |\frac{\partial^2 b}{\partial x^2}(t,x+\theta_1(\bar{X}-x))\cdot b(t,\bar{X})|\le L.$$

Now using (2.35), the conditions (2.25), and Lemmas 2.2 and 2.3, we obtain for the last two terms in (2.33):

$$(2.36)\qquad |E\frac{\partial b}{\partial x}(t,x+\theta(\bar{X}-x))\cdot b(t,\bar{X})\zeta_h^2 h-b(t,x)\frac{\partial b}{\partial x}(t,x)h|\le K(1+|x|)h^{3/2}.$$

Thus, (2.31)-(2.36) give

$$|E(\bar{X}-\hat{X})| \le K(1+|x|)h^{3/2}.$$

Finally, applying Lemma 2.1 we prove this theorem. □

Remark 2.1. The condition (2.29) is satisfied if, for instance,

$$|b(t,x)| \le L,\ |\frac{\partial^2 b}{\partial x^2}(t,x)| \le L,\ t_0 \le t \le T,\ x \in \mathbf{R}, \tag{2.37}$$

or

$$|\frac{\partial^2 b}{\partial x^2}(t,x)| \le \frac{L}{1+|x|},\ t_0 \le t \le T,\ x \in \mathbf{R}, \tag{2.38}$$

holds.

Let us underline that the conditions of Theorem 2.2 are not necessary and the method is applicable more widely. This is true for other methods proposed in the paper as well.

2.4. **General construction.** Let

$$dX^i = a^i(t,X)dt + \sum_{r=1}^{m} b_r^i(t,X)dw_r(t),\ i=1,...,d. \tag{2.39}$$

Introduce the one-step approximation:

$$\bar{X}^i = x^i + \sum_{k=1}^{l} \lambda_k^i a^i(t+\nu_k^i h, (1-\alpha_{k1}^i)x^1 + \alpha_{k1}^i \bar{X}^1, ..., (1-\alpha_{kd}^i)x^d + \alpha_{kd}^i \bar{X}^d)h \tag{2.40}$$

$$+\sum_{r=1}^{m}\sum_{k=1}^{l} \mu_{rk}^i b_r^i(t+\nu_{rk}^i h, (1-\beta_{rk1}^i)x^1 + \beta_{rk1}^i \bar{X}^1, ..., (1-\beta_{rkd}^i)x^d + \beta_{rkd}^i \bar{X}^d)\zeta_{rh}\sqrt{h} + A^i,$$

where $0 \le \nu,\alpha,\beta \le 1$, $\lambda,\mu \ge 0$, $\sum_{k=1}^{l}\lambda_k^i = 1$, $\sum_{k=1}^{l}\mu_{rk}^i = 1$, $i=1,...,d$, l is a positive integer, and A^i are some expressions to be found. Substituting the Euler-like approximation

$$\hat{X}^j = x^j + a^j(t,x)h + \sum_{s=1}^{m} b_s^j(t,x)\zeta_{sh}\sqrt{h}$$

instead of $\bar{X}^j$, $j=1,...,d$, in b_r^i, we obtain

$$b_r^i(t+\nu_{rk}^i h, (1-\beta_{rk1}^i)x^1 + \beta_{rk1}^i \bar{X}^1, ..., (1-\beta_{rkd}^i)x^d + \beta_{rkd}^i \bar{X}^d)$$

$$\approx b_r^i(t,x) + \sum_{j=1}^{d} \frac{\partial b_r^i}{\partial x^j}(t,x)\beta_{rkj}^i \sum_{s=1}^{m} b_s^j(t,x)\zeta_{sh}\sqrt{h}.$$

It is clear from here that either

$$A^i = -\sum_{r=1}^{m}\sum_{k=1}^{l}\mu_{rk}^i \sum_{j=1}^{d} \frac{\partial b_r^i}{\partial x^j}(t,x)\beta_{rkj}^i \sum_{s=1}^{m} b_s^j(t,x)\zeta_{sh}\sqrt{h}\zeta_{rh}\sqrt{h} \tag{2.41}$$

or

$$A^i = -\sum_{r=1}^{m}\sum_{k=1}^{l}\mu_{rk}^i \sum_{j=1}^{d} \frac{\partial b_r^i}{\partial x^j}(t,x)\beta_{rkj}^i b_r^j(t,x)h \tag{2.42}$$

can be put in (2.40).

Substituting one of these expressions in (2.40), we obtain a multi-parametric family of implicit methods. It is also possible to introduce implicitness in A^i by changing t, x as it was done in the terms connecting with a^i. Moreover, the family can be extended if some a^i or b_r^i are represented as a sum of terms. In this case for different terms the coefficients $\lambda,\ \nu,\ \alpha,\ \mu,\ \beta$ can differ.

It can be proved that under appropriate conditions of smoothness and boundedness on the coefficients of (2.39) the method based on the one-step approximation (2.40) with A^i as in (2.41) or (2.42) is of mean-square order 1/2. The proof is analogous to the proof of Theorem 2.2.

Here and below we will not precisely indicate conditions on the coefficients a and b_r letting that appropriate conditions on the coefficients hold. These conditions can be restored using the general theory [9] and Theorem 2.2.

Let us give an example of full implicit methods:

$$\bar{X} = x + a(t,\bar{X})h - \sum_{r=1}^{m}\sum_{j=1}^{d} \frac{\partial b_r}{\partial x^j}(t,\bar{X}) b_r^j(t,\bar{X})h + \sum_{r=1}^{m} b_r(t,\bar{X})\zeta_{rh}\sqrt{h}.$$

Further, in the case of SDEs in the sense of Stratonovich

$$dX = a(t,X)dt + \sum_{r=1}^{m} b_r(t,X) \circ dw_r(t) \tag{2.43}$$

we construct the derivative-free full-implicit method (midpoint method):

$$X_{k+1} = X_k + a(t_k + \frac{h}{2}, \frac{X_k + X_{k+1}}{2})h + \sum_{r=1}^{m} b_r(t_k, \frac{X_k + X_{k+1}}{2})\,(\zeta_{rh})_k\sqrt{h}. \tag{2.44}$$

For $b_r^i = 0$, this method coincides with the well-known deterministic midpoint scheme, which has the second order of convergence.

In the general case the method (2.44) is of mean-square order 1/2. In the commutative case, i.e., when $\Lambda_i b_r = \Lambda_r b_i$ (here the operator $\Lambda_r := (b_r, \partial/\partial x)$) or in the case of a system with one noise (i.e., $m = 1$) the midpoint method (2.44) has the first mean-square order of convergence which is stated in the next theorem.

Theorem 2.3. *Suppose that the commutative conditions $\Lambda_i b_r = \Lambda_r b_i,\ i, r = 1, \ldots, m$, are fulfilled. Let ζ_{rh} be defined by* (2.15) *with $A_h = \sqrt{4|\ln h|}$. Then the method* (2.44) *for the system* (2.43) *has the first mean-square order of convergence.*

Proof. Let $\tilde{X}$ be the following approximation of solution to (2.43):

$$\tilde{X} = x + a(t + \frac{h}{2}, x)h + \sum_{r=1}^{m} b_r(t,x)\Delta w_r(h) + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} \Lambda_i b_r(t,x)\Delta w_i(h)\Delta w_r(h)$$
$$+ \frac{1}{2}\sum_{r=1}^{m} \Lambda_r b_r(t,x)\left(\Delta w_r(h)\right)^2.$$

It is known [9] that the method based on this one-step approximation has the first mean-square order of convergence under the commutative conditions of this theorem.

Denote by $\bar{X}$ the one-step approximation of the midpoint method (2.44):

$$\bar{X} = x + a(t + \frac{h}{2}, \frac{x+\bar{X}}{2})h + \sum_{r=1}^{m} b_r(t, \frac{x+\bar{X}}{2})\zeta_{rh}\sqrt{h}.$$

Expanding the right-hand side of $\bar{X}$ about x and using the assumption $\Lambda_i b_r = \Lambda_r b_i$, we obtain

$$\bar{X} = x + a(t + \frac{h}{2}, x)h + \sum_{r=1}^{m} b_r(t,x)\zeta_{rh}\sqrt{h} + h\sum_{i=1}^{m-1}\sum_{r=i+1}^{m} \Lambda_i b_r(t,x)\zeta_{ih}\zeta_{rh}$$
$$+ \frac{h}{2}\sum_{r=1}^{m} \Lambda_r b_r(t,x)\left(\zeta_{rh}\right)^2 + \rho.$$

On the same way as in Theorem 2.2 it is possible to show that

$$|E\rho| = O(h^2),\ E\rho^2 = O(h^3).$$

We have

$$R = \bar{X} - \tilde{X} = \sum_{r=1}^{m} b_r(t,x)\left(\zeta_{rh}\sqrt{h} - \Delta w_r(h)\right) + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} \Lambda_i b_r(t,x)\left(\zeta_{ih}\zeta_{rh}h - \Delta w_i(h)\Delta w_r(h)\right)$$
$$+ \sum_{r=1}^{m} \Lambda_r b_r(t,x)\left[h\left(\zeta_{rh}\right)^2 - \left(\Delta w_r(h)\right)^2\right] + \rho.$$

Using Lemma 2.2, we obtain

$$|ER| = O(h^2).$$

Now analyze ER^2 :

$$ER^2 \le L(1+|x|^2)\left[\sum_{r=1}^{m} E\left(\zeta_{rh}\sqrt{h} - \Delta w_r(h)\right)^2 + \sum_{i=1}^{m-1}\sum_{r=i+1}^{m} E\left(\zeta_{ih}\zeta_{rh}h - \Delta w_i(h)\Delta w_r(h)\right)^2\right.$$
$$\left. + \sum_{r=1}^{m} E\left(h\left(\zeta_{rh}\right)^2 - \left(\Delta w_r(h)\right)^2\right)^2\right] + O(h^3).$$

The first and the second terms in the square brackets are $O(h^3)$ due to (2.14). From the inequality $E(\zeta_{rh} - \xi_r)^4 \le 3h^2$ which is proved analogously to (2.14), it is easy to see that the third term is also $O(h^3)$. So, $ER^2 = O(h^3)$. Finally applying Lemma 2.1, we prove the theorem. □

3. Symplectic methods for general Hamiltonian system

Here, using the results of the previous section, we construct symplectic methods for general Hamiltonian system with multiplicative noise (1.1), (1.2). Its Ito form reads

$$\text{(3.1)} \qquad dP^i = f^i dt + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n} \frac{\partial \sigma_r^i}{\partial p^j}\sigma_r^j dt + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n} \frac{\partial \sigma_r^i}{\partial q^j}\gamma_r^j dt + \sum_{r=1}^{m} \sigma_r^i dw_r(t)$$

$$dQ^i = g^i dt + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial\gamma_r^i}{\partial p^j}\sigma_r^j dt + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial\gamma_r^i}{\partial q^j}\gamma_r^j dt + \sum_{r=1}^{m}\gamma_r^i dw_r(t).$$

Introduce the following implicit method:

$$(3.2)\qquad P_{k+1} = P_k + fh - \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}(\frac{\partial\sigma_r}{\partial p^j}\sigma_r^j - \frac{\partial\sigma_r}{\partial q^j}\gamma_r^j)h + \sum_{r=1}^{m}\sigma_r\cdot(\zeta_{rh})_k\sqrt{h}$$

$$Q_{k+1} = Q_k + gh - \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}(\frac{\partial\gamma_r}{\partial p^j}\sigma_r^j - \frac{\partial\gamma_r}{\partial q^j}\gamma_r^j)h + \sum_{r=1}^{m}\gamma_r\cdot(\zeta_{rh})_k\sqrt{h},$$

where all the functions have t, P_{k+1}, Q_k as their arguments.

Theorem 3.1. *The implicit method* (3.2) *for the system* (3.1) *is symplectic and of the mean-square order* 1/2.

Proof. The method (3.2) belongs to the family (2.40) and consequently the assertion about its order of convergence follows from the previous section. Let us prove symplecticness of the method. It is convenient to write the one-step approximation corresponding to (3.2) in the form

$$(3.3)\quad \bar{P}^i = p^i - \frac{\partial H_0}{\partial q^i}h - \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial^2 H_r}{\partial q^i\partial p^j}\frac{\partial H_r}{\partial q^j}h - \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial^2 H_r}{\partial q^i\partial q^j}\frac{\partial H_r}{\partial p^j}h - \sum_{r=1}^{m}\frac{\partial H_r}{\partial q^i}\zeta_{rh}\sqrt{h}$$

$$\bar{Q}^i = q^i + \frac{\partial H_0}{\partial p^i}h + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial^2 H_r}{\partial p^i\partial p^j}\frac{\partial H_r}{\partial q^j}h + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial^2 H_r}{\partial p^i\partial q^j}\frac{\partial H_r}{\partial p^j}h + \sum_{r=1}^{m}\frac{\partial H_r}{\partial p^i}\zeta_{rh}\sqrt{h},$$

where all the functions have $t, \bar{P}, q$ as their arguments.

Introduce the function $F(t,p,q)$ ($h,\ \zeta_{rh}$ are fixed here):

$$F(t,p,q) = H_0(t,p,q)h + \frac{1}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial H_r}{\partial q^j}(t,p,q)\frac{\partial H_r}{\partial p^j}(t,p,q)h + \sum_{r=1}^{m}H_r(t,p,q)\zeta_{rh}\sqrt{h}.$$

Then (3.3) can be written as

$$(3.4)\qquad \bar{P}^i = p^i - \frac{\partial F}{\partial q^i}(t,\bar{P},q)$$

$$\bar{Q}^i = q^i + \frac{\partial F}{\partial p^i}(t,\bar{P},q).$$

We have (the arguments everywhere are $t,\ \bar{P},\ q$):

$$\sum_{i=1}^{n} d\bar{P}^i\wedge d\bar{Q}^i = \sum_{i=1}^{n} d\bar{P}^i\wedge(dq^i + \sum_{j=1}^{n}F''_{p^ip^j}d\bar{P}^j + \sum_{j=1}^{n}F''_{p^iq^j}dq^j)$$

$$= \sum_{i=1}^{n} d\bar{P}^i\wedge dq^i + \sum_{i=1}^{n}\sum_{j=1}^{n}F''_{p^ip^j}d\bar{P}^i\wedge d\bar{P}^j + \sum_{i=1}^{n}\sum_{j=1}^{n}F''_{p^iq^j}d\bar{P}^i\wedge dq^j.$$

Since $d\bar{P}^i \wedge d\bar{P}^j = -d\bar{P}^j \wedge d\bar{P}^i$, we get

$$\sum_{i=1}^n d\bar{P}^i \wedge d\bar{Q}^i = \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n \sum_{j=1}^n F''_{p^i q^j} d\bar{P}^i \wedge dq^j \tag{3.5}$$

$$= \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n \sum_{j=1}^n F''_{q^i p^j} d\bar{P}^j \wedge dq^i.$$

Further

$$d\bar{P}^i = dp^i - \sum_{j=1}^n F''_{q^i p^j} d\bar{P}^j - \sum_{j=1}^n F''_{q^i q^j} dq^j.$$

Substituting $\sum_{j=1}^n F''_{q^i p^j} d\bar{P}^j$ from here in (3.5), we obtain

$$\sum_{i=1}^n d\bar{P}^i \wedge d\bar{Q}^i = \sum_{i=1}^n d\bar{P}^i \wedge dq^i + \sum_{i=1}^n (dp^i - d\bar{P}^i - \sum_{j=1}^n F''_{q^i q^j} dq^j) \wedge dq^i$$

$$= \sum_{i=1}^n dp^i \wedge dq^i - \sum_{i=1}^n \sum_{j=1}^n F''_{q^i q^j} dq^j \wedge dq^i = \sum_{i=1}^n dp^i \wedge dq^i.$$

□

A more general symplectic method for the Hamiltonian system (1.1), (1.2) has the form

$$P_{k+1} = P_k + f(t_k + \beta h, \alpha P_{k+1} + (1-\alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k)h \tag{3.6}$$

$$+ (\frac{1}{2} - \alpha) \sum_{r=1}^m \sum_{j=1}^n (\frac{\partial \sigma_r}{\partial p^j} \sigma_r^j - \frac{\partial \sigma_r}{\partial q^j} \gamma_r^j) h + \sum_{r=1}^m \sigma_r \cdot (\zeta_{rh})_k \sqrt{h}$$

$$Q_{k+1} = Q_k + g(t_k + \beta h, \alpha P_{k+1} + (1-\alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k)h$$

$$+ (\frac{1}{2} - \alpha) \sum_{r=1}^m \sum_{j=1}^n (\frac{\partial \gamma_r}{\partial p^j} \sigma_r^j - \frac{\partial \gamma_r}{\partial q^j} \gamma_r^j) h + \sum_{r=1}^m \gamma_r \cdot (\zeta_{rh})_k \sqrt{h},$$

where σ_r, γ_r, $r = 1, \ldots, m$, and their derivatives are calculated at $(t_k, \alpha P_{k+1} + (1 - \alpha)P_k, (1-\alpha)Q_{k+1} + \alpha Q_k)$, and α, $\beta \in [0,1]$ are parameters.

Theorem 3.2. *The implicit method* (3.6) *for the system* (1.1), (1.2) *is symplectic and of the mean-square order* 1/2.

Proof. As in the previous theorem, we need to prove symplecticness of the method only. Introduce the function

$$G(t,p,q) = H_0(t + \beta h, p, q)h + (\frac{1}{2} - \alpha) \sum_{r=1}^m \sum_{j=1}^n \sigma_r^j(t,p,q)\gamma_r^j(t,p,q)h + \sum_{r=1}^m H_r(t,p,q)\zeta_{rh}\sqrt{h}.$$

It is not difficult to verify that the one-step approximation corresponding to (3.6) can be written in the form:

$$\bar{P}^i = p^i - \frac{\partial G}{\partial q^i}(t, \alpha\bar{P} + (1-\alpha)p, (1-\alpha)\bar{Q} + \alpha q) \tag{3.7}$$

$$\bar{Q}^i = q^i + \frac{\partial G}{\partial p^i}(t, \alpha\bar{P} + (1-\alpha)p, (1-\alpha)\bar{Q} + \alpha q).$$

Let $\alpha \neq 0$. Then (3.7) is equivalent to

$$\tilde{P}^i = p^i - \frac{\partial(\alpha G)}{\partial q^i}(t, \tilde{P}, \tilde{q})$$

$$\bar{Q}^i = \tilde{q}^i + \frac{\partial(\alpha G)}{\partial p^i}(t, \tilde{P}, \tilde{q}),$$

where $\tilde{P} = \alpha \bar{P} + (1-\alpha)p$, $\tilde{q} = (1-\alpha)\bar{Q} + \alpha q$. It follows from the proof of Theorem 3.1 that $\sum_{i=1}^n d\tilde{P}^i \wedge d\bar{Q}^i = \sum_{i=1}^n dp^i \wedge d\tilde{q}^i$, whence $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$. The case $\alpha = 0$ is proved as Theorem 3.1. □

The method (3.2) is a particular case of (3.6) when $\alpha = 1$, $\beta = 0$. If $\alpha = \beta = 1/2$ the method (3.6) becomes the midpoint method (cf. (2.44)):

(3.8)
$$P_{k+1} = P_k + f(t_k + \frac{h}{2}, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2})h + \sum_{r=1}^m \sigma_r(t_k, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2})\,(\zeta_{rh})_k\sqrt{h}$$

$$Q_{k+1} = Q_k + g(t_k + \frac{h}{2}, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2})h + \sum_{r=1}^m \gamma_r(t_k, \frac{P_k + P_{k+1}}{2}, \frac{Q_k + Q_{k+1}}{2})\,(\zeta_{rh})_k\sqrt{h}.$$

Remark 3.1. In the commutative case, i.e., when $\Lambda_i b_r = \Lambda_r b_i$ or in the case of a system with one noise (i.e., $m = 1$) the symplectic method (3.8) for (1.1), (1.2) has the first mean-square order of convergence.

Remark 3.2. In the case of separable Hamiltonians at noise, i.e., when $H_r(t,p,q) = U_r(t,q) + V_r(t,p)$, $r = 1, \ldots m$, we can obtain symplectic methods for (1.1), (1.2) which are explicit in stochastic terms and do not need truncated random variables. For instance, (3.2) acquires the form

(3.9)
$$P_{k+1} = P_k + f(t_k, P_{k+1}, Q_k)h + \frac{h}{2}\sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r}{\partial q^j}(t_k, Q_k) \cdot \gamma_r^j(P_{k+1}) + \sum_{r=1}^m \sigma_r(t_k, Q_k)\Delta_k w_r,$$

$$Q_{k+1} = Q_k + g(t_k, P_{k+1}, Q_k)h - \frac{h}{2}\sum_{r=1}^m \sum_{j=1}^n \frac{\partial \gamma_r}{\partial p^j}(P_{k+1}) \cdot \sigma_r^j(t_k, Q_k) + \sum_{r=1}^m \gamma_r(t_k, P_{k+1})\Delta_k w_r.$$

Remark 3.3. It is possible to construct full explicit symplectic methods for the following partitioned system:

(3.10)
$$dP = f(t,Q)dt + \sum_{r=1}^m \sigma_r(t,Q) \circ dw_r(t),\ P(t_0) = p,$$

$$dQ = g(P)dt + \sum_{r=1}^m \gamma_r(t) dw_r(t),\ Q(t_0) = q,$$

with

$$f^i = -\partial U_0/\partial q^i,\ g^i = \partial V_0/\partial p^i,\ \sigma_r^i = -\partial U_r/\partial q^i,\ r = 1,\dots m,\ i = 1,\dots,n.$$

For instance, the explicit partitioned Runge-Kutta method (cf. (4.6) – (4.7))

$$\mathcal{Q}_1 = Q_k + \alpha h g(P_k), \tag{3.11}$$

$$\mathcal{P}_1 = P_k + hf(t_k + \alpha h, \mathcal{Q}_1) + \frac{h}{2}\sum_{r=1}^{m}\sum_{j=1}^{n}\frac{\partial \sigma_r}{\partial q^j}(t_k, \mathcal{Q}_1)\cdot\gamma_r^j(t_k),$$

$$\mathcal{Q}_2 = \mathcal{Q}_1 + (1-\alpha)hg(\mathcal{P}_1),$$

$$P_{k+1} = \mathcal{P}_1 + \sum_{r=1}^{m}\sigma_r(t_k, \mathcal{Q}_2)\Delta_k w_r, \tag{3.12}$$

$$Q_{k+1} = \mathcal{Q}_2 + \sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r,\ k = 0,\dots,N-1,$$

with the parameter $0 \le \alpha \le 1$ is symplectic and of the mean-square order 1/2.

A particular case of the system (3.10) is considered in the next section, where explicit symplectic methods of a higher order are proposed.

4. Explicit symplectic methods in the case of separable Hamiltonians

Consider a special case of the Hamiltonian system (1.1), (1.2) such that

$$H_0(t,p,q) = V_0(p) + U_0(t,q),\ H_r(t,p,q) = U_r(t,q),\ r = 1,\dots m. \tag{4.1}$$

In this case we get the following system in the sense of Stratonovich

$$dP = f(t,Q)dt + \sum_{r=1}^{m}\sigma_r(t,Q)\circ dw_r(t),\ P(t_0) = p, \tag{4.2}$$

$$dQ = g(P)dt,\ Q(t_0) = q,$$

with

$$f^i = -\partial U_0/\partial q^i,\ g^i = \partial V_0/\partial p^i,\ \sigma_r^i = -\partial U_r/\partial q^i,\ r = 1,\dots m,\ i = 1,\dots,n. \tag{4.3}$$

We note that it is not difficult to consider a slightly more general separable Hamiltonian $H_0(t,p,q) = V_0(t,p) + U_0(t,q)$ but we restrict ourselves to H_0 from (4.1).

It is obvious that the system (4.2) has the same form in the sense of Ito.

For $V_0(p) = \frac{1}{2}(M^{-1}p,p)$ with M a constant, symmetric, invertible matrix, the system (4.2) takes the form

$$dP = f(t,Q)dt + \sum_{r=1}^{m}\sigma_r(t,Q)dw_r(t),\ P(t_0) = p, \tag{4.4}$$

$$dQ = M^{-1}Pdt,\ Q(t_0) = q.$$

This system can be written as a second-order differential equation with multiplicative noise

$$\frac{d^2Q}{dt^2} = M^{-1}f(t,Q) + M^{-1}\sum_{r=1}^{m}\sigma_r(t,Q)\dot{w}_r(t). \tag{4.5}$$

Due to specific features of the system (4.2), (4.3) we have succeeded in construction of explicit partitioned Runge-Kutta (PRK) methods of a higher order.

4.1. **First-order methods.** A PRK method for (4.2) has the form (cf. (3.11)-(3.12)):

$$\mathcal{Q}_1 = Q_k + \alpha h g(P_k),\ \ \mathcal{P}_1 = P_k + hf(t_k+\alpha h, \mathcal{Q}_1), \tag{4.6}$$
$$\mathcal{Q}_2 = \mathcal{Q}_1 + (1-\alpha)hg(\mathcal{P}_1),$$

$$P_{k+1} = \mathcal{P}_1 + \sum_{r=1}^{m}\sigma_r(t_k,\mathcal{Q}_2)\Delta_k w_r,\ \ Q_{k+1} = \mathcal{Q}_2,\ k=0,\dots,N-1, \tag{4.7}$$

where $0 \le \alpha \le 1$ is a parameter.

Theorem 4.1. *The explicit method* (4.6) – (4.7) *for the system* (4.2) *with* (4.3) *is symplectic and of the first mean-square order.*

Proof. In the case of the system (4.2) the operators Λ_r take the form $\Lambda_r = (\sigma_r, \partial/\partial p)$. Since σ_r do not depend on p, we get $\Lambda_i\sigma_j = 0$. It is known [9] that in such a case the Euler method has the first mean-square order of accuracy. Comparing the method (4.6)–(4.7) with the Euler method and using Lemma 2.1, it is not difficult to get that the method (4.6)–(4.7) is of the first mean-square order as well.

Due to (4.3), $\partial\sigma_r^i/\partial q^j = \partial\sigma_r^j/\partial q^i$. Using this, we obtain $dP_{k+1}\wedge dQ_{k+1} = d\mathcal{P}_1\wedge d\mathcal{Q}_2$. It is easy to prove that $d\mathcal{P}_1\wedge d\mathcal{Q}_2 = d\mathcal{P}_1\wedge d\mathcal{Q}_1 = dP_k\wedge dQ_k$. Therefore the method (4.6)–(4.7) is symplectic. □

Remark 4.1. By swapping the roles of p and q, we can propose the following symplectic method of the first mean-square order for the system (4.2) with (4.3):

$$\mathcal{P} = P_k + \alpha h f(t_k, Q_k),\ \ \mathcal{Q} = Q_k + hg(\mathcal{P}) \tag{4.8}$$

$$P_{k+1} = \mathcal{P} + (1-\alpha)hf(t_{k+1},\mathcal{Q}) + \sum_{r=1}^{m}\sigma_r(t_k,\mathcal{Q})\Delta_k w_r,\ \ Q_{k+1} = \mathcal{Q},\ k=0,\dots,N-1. \tag{4.9}$$

4.2. **Methods of order 3/2.** Consider the relations

$$\mathcal{P}_i = p + h\sum_{j=1}^{s}\alpha_{ij}f(t+c_jh,\mathcal{Q}_j) + \sum_{j=1}^{s}\sum_{r=1}^{m}\sigma_r(t+d_jh,\mathcal{Q}_j)\left(\lambda_{ij}\varphi_r + \mu_{ij}\psi_r\right), \tag{4.10}$$
$$\mathcal{Q}_i = q + h\sum_{j=1}^{s}\hat{\alpha}_{ij}g(\mathcal{P}_j),\ \ i=1,\dots,s,$$

$$\bar{P} = p + h\sum_{i=1}^{s}\beta_i f(t+c_ih,\mathcal{Q}_i) + \sum_{i=1}^{s}\sum_{r=1}^{m}\sigma_r(t+d_ih,\mathcal{Q}_i)\left(\nu_i\varphi_r + \varkappa_i\psi_r\right), \tag{4.11}$$
$$\bar{Q} = q + h\sum_{i=1}^{s}\hat{\beta}_i g(\mathcal{P}_i),$$

where φ_r, ψ_r do not depend on p and q, the parameters α_{ij}, $\hat{\alpha}_{ij}$, β_i, $\hat{\beta}_i$, λ_{ij}, μ_{ij}, ν_i, $\varkappa_i$ satisfy the conditions

$$\beta_i\hat{\alpha}_{ij} + \hat{\beta}_j\alpha_{ji} - \beta_i\hat{\beta}_j = 0, \tag{4.12}$$

$$\nu_i\hat{\alpha}_{ij} + \hat{\beta}_j\lambda_{ji} - \nu_i\hat{\beta}_j = 0, \quad \varkappa_i\hat{\alpha}_{ij} + \hat{\beta}_j\mu_{ji} - \varkappa_i\hat{\beta}_j = 0, \quad i, j = 1, \dots, s,$$

and c_i, d_i are arbitrary parameters.

If $\sigma_r \equiv 0$ the relations (4.10)–(4.11) coincide with a general form of s-stage PRK methods for deterministic differential equations (see, e.g., [5, p. 34]). It is known [8, 5] that the symplectic condition holds for $\bar{P}$, $\bar{Q}$ from (4.10)–(4.11) with (4.12) in the case of $\sigma_r \equiv 0$. By a generalization of the proof of Theorem 6.2 from [5], we prove the following lemma (another generalization is given in [3]).

Lemma 4.1. *The relations* (4.10) – (4.11) *with conditions* (4.12) *preserve symplectic structure, i.e.,* $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$.

Proof. Denote for a while: $f_i = f(t + c_i h, \mathcal{Q}_i)$, $g_i = g(\mathcal{P}_i)$, $\sigma_{ri} = \sigma_r(t + d_i h, \mathcal{Q}_i)$. We get

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq + h\sum_{j=1}^{s}\hat{\beta}_j dp \wedge dg_j + h\sum_{i=1}^{s}\beta_i df_i \wedge dq + h^2\sum_{i=1}^{s}\sum_{j=1}^{s}\beta_i\hat{\beta}_j df_i \wedge dg_j \tag{4.13}$$

$$+\sum_{i=1}^{s}\sum_{r=1}^{m}\left(\nu_i\varphi_r + \varkappa_i\psi_r\right) d\sigma_{ri} \wedge dq + h\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{r=1}^{m}\left(\nu_i\varphi_r + \varkappa_i\psi_r\right)\hat{\beta}_j d\sigma_{ri} \wedge dg_j.$$

Then we express $dp \wedge dg_i$ from

$$d\mathcal{P}_j \wedge dg_j = dp \wedge dg_j + h\sum_{i=1}^{s}\alpha_{ji} df_i \wedge dg_j + \sum_{i=1}^{s}\sum_{r=1}^{m}\left(\lambda_{ji}\varphi_r + \mu_{ji}\psi_r\right) d\sigma_{ri} \wedge dg_j$$

and substitute it in (4.13). Analogously, we act with $df_i \wedge dq$ and $d\sigma_{ri} \wedge dq$ finding them from the similar expressions for $df_i \wedge d\mathcal{Q}_i$ and $d\sigma_{ri} \wedge d\mathcal{Q}_i$. As a result, using (4.12), we obtain

$$d\bar{P} \wedge d\bar{Q} = dp \wedge dq + h\sum_{i=1}^{s}\hat{\beta}_i d\mathcal{P}_i \wedge dg_i + h\sum_{i=1}^{s}\beta_i df_i \wedge d\mathcal{Q}_i$$
$$+\sum_{i=1}^{s}\sum_{r=1}^{m}\left(\nu_i\varphi_r + \varkappa_i\psi_r\right) d\sigma_{ri} \wedge d\mathcal{Q}_i.$$

Taking into account skew-symmetry of the wedge product and that the vector-functions f, g, σ_r are gradients, f, σ_r do not depend on p, and g does not depend on q, it is not difficult to see that each of the terms $d\mathcal{P}_i \wedge dg_i$, $df_i \wedge d\mathcal{Q}_i$, $d\sigma_{ri} \wedge d\mathcal{Q}_i$ vanishes. Therefore $d\bar{P} \wedge d\bar{Q} = dp \wedge dq$. □

Introduce the 2-stage explicit PRK method for the system (4.2), (4.3):

$$\mathcal{Q}_1 = Q_k,\ \ \mathcal{P}_1 = P_k + \frac{h}{4} f(t_k, \mathcal{Q}_1) + \frac{1}{2}\sum_{r=1}^{m} \sigma_r(t_k, \mathcal{Q}_1)\left(3(J_{r0})_k - \Delta_k w_r\right), \tag{4.14}$$

$$\mathcal{Q}_2 = \mathcal{Q}_1 + \frac{2}{3} h g(\mathcal{P}_1),$$

$$\mathcal{P}_2 = \mathcal{P}_1 + \frac{3}{4} h f(t_k + \frac{2}{3}h, \mathcal{Q}_2) + \frac{3}{2}\sum_{r=1}^{m} \sigma_r(t_k + \frac{2}{3}h, \mathcal{Q}_2)\left(-(J_{r0})_k + \Delta_k w_r\right),$$

$$P_{k+1} = \mathcal{P}_2,\quad Q_{k+1} = \mathcal{Q}_2 + \frac{h}{3} g(\mathcal{P}_2),\ \ k = 0,\ldots,N-1, \tag{4.15}$$

where

$$J_{r0} := \frac{1}{h}\int_t^{t+h} \left(w_r(\vartheta) - w_r(t)\right) d\vartheta. \tag{4.16}$$

Theorem 4.2. *The explicit PRK method* (4.14) – (4.15) *for system* (4.2), (4.3) *preserves symplectic structure and has the mean-square order* 3/2.

Proof. The method (4.14)-(4.15) has the form of (4.10)-(4.11) and its parameters satisfy the conditions (4.12). Then, Lemma 4.1 implies that this method preserves symplectic structure.

Let us now prove mean-square order of convergence of (4.14)-(4.15). To this end, introduce the one-step approximation for (4.2):

$$\tilde{P} = p + \sum_{r=1}^{m} \sigma_r \Delta w_r + hf + \sum_{r=1}^{m}\left[\frac{\partial \sigma_r}{\partial t} + \sum_{i=1}^{n} g^i \frac{\partial \sigma_r}{\partial q^i}\right] I_{0r} + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + \sum_{i=1}^{n} g^i \frac{\partial f}{\partial q^i}\right], \tag{4.17}$$

$$\tilde{Q} = q + hg + \sum_{r=1}^{m}\sum_{i=1}^{n} \sigma_r^i \frac{\partial g}{\partial p^i} I_{r0} + \frac{h^2}{2}\left[\sum_{i=1}^{n} f^i \frac{\partial g}{\partial p^i} + \frac{1}{2}\sum_{r=1}^{m}\sum_{i,j=1}^{n} \sigma_r^i \sigma_r^j \frac{\partial^2 g}{\partial p^i \partial p^j}\right],$$

where

$$I_{0r} = \int_t^{t+h} (\vartheta - t)\, dw_r(\vartheta),\ \ I_{r0} = \int_t^{t+h} \left(w_r(\vartheta) - w_r(t)\right) d\vartheta = hJ_{r0}, \tag{4.18}$$

and all the coefficients are calculated at (t, p, q). We note that

$$\left(\Delta w_r - J_{r0}\right) h = I_{0r}.$$

Using the general theory of numerical integration of SDEs [9], it is not difficult to show that the method based on (4.17) is of the mean-square order 3/2. Our nearest aim is to prove that the one-step approximation $\bar{P}$, $\bar{Q}$ corresponding to the method (4.14)-(4.15) is such that

$$\left| E\begin{bmatrix} \bar{P} - \tilde{P} \\ \bar{Q} - \tilde{Q} \end{bmatrix} \right| = O(h^3),\ \ \left(E\begin{bmatrix} \bar{P} - \tilde{P} \\ \bar{Q} - \tilde{Q} \end{bmatrix}^2 \right)^{1/2} = O(h^2). \tag{4.19}$$

Expanding the right-hand sides of the approximation $\bar{P}$, $\bar{Q}$ about (t,p,q), we obtain

$$\bar{P} = p + hf + \frac{h^2}{2}\frac{\partial f}{\partial t} + \frac{3}{4}h\sum_{i=1}^{n}\Delta\mathcal{Q}_2^i\frac{\partial f}{\partial q^i} + \sum_{r=1}^{m}\sigma_r\Delta w_r + \frac{3}{2}\sum_{r=1}^{m}\sum_{i=1}^{n}\Delta\mathcal{Q}_2^i\frac{\partial\sigma_r}{\partial q^i}(\Delta w_r - J_{r0}) + h\sum_{r=1}^{m}\frac{\partial\sigma_r}{\partial t}(\Delta w_r - J_{r0}) + \rho_1, \tag{4.20}$$

$$\bar{Q} = q + hg + \frac{h}{3}\sum_{i=1}^{n}\left(2\Delta\mathcal{P}_1^i + \Delta\mathcal{P}_2^i\right)\frac{\partial g}{\partial p^i} + \frac{h}{6}\sum_{i,j=1}^{n}\left(2\Delta\mathcal{P}_1^i\Delta\mathcal{P}_1^j + \Delta\mathcal{P}_2^i\Delta\mathcal{P}_2^j\right)\frac{\partial^2 g}{\partial p^i\partial p^j} + \rho_2,$$

$$\Delta\mathcal{P}_1 := \mathcal{P}_1 - p = \frac{h}{4}f + \frac{1}{2}\sum_{r=1}^{m}\sigma_r\left(3J_{r0} - \Delta w_r\right),$$

$$\Delta\mathcal{Q}_2 := \mathcal{Q}_2 - q = \frac{2}{3}hg + \frac{2}{3}h\sum_{i=1}^{n}\Delta\mathcal{P}_1^i\frac{\partial g}{\partial p^i} + \frac{h}{3}\sum_{i,j=1}^{n}\Delta\mathcal{P}_1^i\Delta\mathcal{P}_1^j\frac{\partial^2 g}{\partial p^i\partial p^j} + r_1,$$

$$\Delta\mathcal{P}_2 := \mathcal{P}_2 - p = hf + \sum_{r=1}^{m}\sigma_r\Delta w_r + r_2,$$

where all the coefficients are calculated at (t,p,q).

Due to properties of the Wiener process and Ito integrals, we have

$$E\Delta w_i = 0,\ E\Delta w_i\Delta w_j = \delta_{ij}h,\ E\Delta w_i\Delta w_j\Delta w_k = 0,\ E\left(\Delta w_i\right)^4 = 3h^2, \tag{4.21}$$

$$EJ_{i0} = 0,\quad EJ_{i0}J_{j0} = \delta_{ij}\frac{h}{3},\ EJ_{i0}J_{j0}J_{k0} = 0,\ E\left(J_{i0}\right)^4 = \frac{h^2}{3},$$

$$E\Delta w_iJ_{j0} = \delta_{ij}\frac{h}{2},\ E\Delta w_i\Delta w_jJ_{k0} = 0,\ E\Delta w_iJ_{j0}J_{k0} = 0.$$

Then, under appropriate conditions on smoothness and boundedness of the coefficients of (4.2), we get

$$|E\rho_i| = O(h^3),\quad E\left(\rho_i\right)^2 = O(h^5),\quad i = 1,2, \tag{4.22}$$

$$|Er_1| = O(h^3),\quad E\left(r_1\right)^2 = O(h^5),\quad |Er_2| = O(h^2),\quad E\left(r_2\right)^2 = O(h^3).$$

In addition to (4.21) we note that

$$E\left(\Delta w_r - J_{r0}\right)\left(3J_{r0} - \Delta w_r\right) = 0,\ E\left(3J_{r0} - \Delta w_r\right)^2 = h. \tag{4.23}$$

Using (4.21)-(4.23), we obtain form (4.20):

$$\bar{P} = p + \sum_{r=1}^{m}\sigma_r\Delta w_r + hf + \sum_{r=1}^{m}\left[\frac{\partial\sigma_r}{\partial t} + \sum_{i=1}^{n}g^i\frac{\partial\sigma_r}{\partial q^i}\right]I_{0r} + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + \sum_{i=1}^{n}g^i\frac{\partial f}{\partial q^i}\right] + R_1,$$

$$\bar{Q} = q + hg + \sum_{r=1}^{m}\sum_{i=1}^{n}\sigma_r^i\frac{\partial g}{\partial p^i}I_{r0} + \frac{h^2}{2}\left[\sum_{i=1}^{n}f^i\frac{\partial g}{\partial p^i} + \frac{1}{2}\sum_{r=1}^{m}\sum_{i,j=1}^{n}\sigma_r^i\sigma_r^j\frac{\partial^2 g}{\partial p^i\partial p^j}\right] + R_2$$

with R_i, $i = 1,2$, such that

$$|ER_i| = O(h^3),\quad E\left(R_i\right)^2 = O(h^4),\quad i = 1,2.$$

This implies (4.19). It follows from (4.19) and Lemma 2.1 that the method (4.14)-(4.15) is of the mean-square order 3/2. □

Remark 4.2. The random variables $\Delta_k w_r(h)$, $(J_{r0})_k$ have a Gaussian joint distribution, and they can be simulated at each step by $2m$ independent $\mathcal{N}(0,1)$-distributed random variables ξ_{rk} and η_{rk}, $r = 0, \dots, m$:

$$\Delta_k w_r(h) = \xi_{rk}\sqrt{h},\ (J_{r0})_k = \left(\xi_{rk}/2 + \eta_{rk}/\sqrt{12}\right)\sqrt{h}\,.$$

As a result, the method (4.14)-(4.15) takes the constructive form.

Remark 4.3. It is very unusual that the direct expansion of (4.14)-(4.15) does not contain the habitual term $\dfrac{h^2}{4}\sum_{r=1}^{m}\sum_{i,j=1}^{n}\dfrac{\partial^2 g}{\partial p^i \partial p^j}\sigma_r^i\sigma_r^j$. The similar term in the expansion contains some combinations of Δw_r and J_{r0} instead of h. This is necessary for a method conserving symplectic structure. At the same time this new reception allows to construct new Runge-Kutta methods for general (not only Hamiltonian) stochastic systems with additive noise (see a similar remark in [3]).

Remark 4.4. In the case of $\sigma_r = 0$, $r = 1, \dots, m$, the method (4.14)-(4.15) coincides with the well-known deterministic symplectic PRK method of the second order. Attracting other explicit deterministic second-order PRK methods from [5, 8], it is possible to construct other explicit symplectic methods of the order 3/2 for the system (4.2), (4.3).

5. Symplectic methods for Hamiltonian systems with small multiplicative noise

Here, using ideas of [12], we propose specific methods adapted to the Hamiltonian system with *small* multiplicative noise:

$$dP = f(t,P,Q)dt + \varepsilon\sum_{r=1}^{m}\sigma_r(t,P,Q)\circ dw_r(t),\ P(t_0) = p, \tag{5.1}$$

$$dQ = g(t,P,Q)dt + \varepsilon\sum_{r=1}^{m}\gamma_r(t,P,Q)\circ dw_r(t),\ Q(t_0) = q,$$

$$f^i = -\partial H_0/\partial q^i,\ g^i = \partial H_0/\partial p^i, \tag{5.2}$$

$$\sigma_r^i = -\partial H_r/\partial q^i,\quad \gamma_r^i = \partial H_r/\partial p^i,\quad r = 1,\dots,m,\quad i = 1,\dots,n,$$

where $\varepsilon > 0$ is a small parameter.

The errors of mean-square methods adapted to systems with small noise are estimated in terms of products $h^i\varepsilon^j$, where h is the step-size of discretization and ε is a small parameter at noise [12]. Usually, global error has the form $O(h^j + \varepsilon^k h^l)$ with $j > l,\ k > 0$. Thanks to the fact that the accuracy order of such methods is equal to a comparatively small l, they are not too complicated, while due to the large j and the small factor ε^k at h^l, their errors are fairly low. This allows us to construct effective mean-square methods.

5.1. **Systems with Hamiltonians of the general form.** First we note that in application to the system with small noise (5.1)-(5.2) the method (3.6) is of the mean-square order $O(h + \varepsilon^2 h^{1/2})$ and the midpoint method (3.8) is of the order $O(h^2 + \varepsilon h + \varepsilon^2 h^{1/2})$ (cf. [12]). In the commutative case or in the case of one noise the error of method (3.8) is estimated as $O(h^2 + \varepsilon h)$.

Let us now obtain a symplectic method of order $O(h^4 + \ldots)$. To this end, introduce the full implicit method

$$\mathcal{P}_1 = P_k + h\frac{\varkappa}{2}f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1), \tag{5.3}$$

$$\mathcal{Q}_1 = Q_k + h\frac{\varkappa}{2}g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1),$$

$$\mathcal{P}_2 = P_k + h[\varkappa f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-2\varkappa}{2}f(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2)],$$

$$\mathcal{Q}_2 = Q_k + h[\varkappa g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + \frac{1-2\varkappa}{2}g(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2)],$$

$$\mathcal{P}_3 = P_k + h[\varkappa f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)f(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \frac{\varkappa}{2}f(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)],$$

$$\mathcal{Q}_3 = Q_k + h[\varkappa g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)g(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \frac{\varkappa}{2}g(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)],$$

$$\mathcal{P}_4 = P_k + h[\varkappa f(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)f(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \varkappa f(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)],$$

$$\mathcal{Q}_4 = Q_k + h[\varkappa g(t_k + \frac{\varkappa}{2}h, \mathcal{P}_1, \mathcal{Q}_1) + (1-2\varkappa)g(t_k + \frac{h}{2}, \mathcal{P}_2, \mathcal{Q}_2) + \varkappa g(t_k + \frac{2-\varkappa}{2}h, \mathcal{P}_3, \mathcal{Q}_3)],$$

$$P_{k+1} = \mathcal{P}_4 + \varepsilon \sum_{r=1}^{m} \sigma_r(t_k, \frac{\mathcal{P}_4 + P_{k+1}}{2}, \frac{\mathcal{Q}_4 + Q_{k+1}}{2})\,(\zeta_{rh})_k\,, \tag{5.4}$$

$$Q_{k+1} = \mathcal{Q}_4 + \varepsilon \sum_{r=1}^{m} \gamma_r(t_k, \frac{\mathcal{P}_4 + P_{k+1}}{2}, \frac{\mathcal{Q}_4 + Q_{k+1}}{2})\,(\zeta_{rh})_k\,,$$

where ζ_{rh} is defined in (2.15) with $A_h = \sqrt{2|\ln h|}$ and the number $\varkappa$ is equal to

$$\varkappa = \frac{1}{3}(2 + 2^{1/3} + 2^{-1/3}). \tag{5.5}$$

Let us note that the method (5.3)-(5.5) is reduced under $\sigma_r \equiv 0$, $\gamma_r \equiv 0$, $r = 1, \ldots, m$, to the well-known fourth-order symplectic RK method for deterministic Hamiltonian systems (see, e.g., [5, p. 101]).

Theorem 5.1. *The implicit method* (5.3) – (5.5) *for system* (5.1) – (5.2) *is symplectic and its mean-square error is estimated as* $O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2})$.

Proof. The fact that the error of (5.3)-(5.5) is estimated as $O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2})$ follows from a standard routine error analysis and from the mean-square theorem of [12]. Further, taking into account (5.2), we obtain that $dP_{k+1} \wedge dQ_{k+1} = d\mathcal{P}_4 \wedge d\mathcal{Q}_4$ (for proving this fact it suffices to put in (3.8) $f = g = 0$). Since $\mathcal{P}_4$, $\mathcal{Q}_4$ correspond to the symplectic deterministic method [5, p. 101], we have $d\mathcal{P}_4 \wedge d\mathcal{Q}_4 = dp \wedge dq$. Thus, the method (5.3)-(5.5) is symplectic. □

Remark 5.1. By other deterministic fourth-order symplectic methods (see, e.g. [8, 5]), other symplectic methods with the error $O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2})$ for the system (5.1) – (5.2) can be constructed. It is possible to get a symplectic method of the order $O(h^4 + \varepsilon^2 h^{1/2})$ as well.

5.2. **Systems with separable Hamiltonians.** Consider the special case of system (5.1)-(5.2) (cf. (4.2)-(4.3))

$$dP = f(t,Q)dt + \varepsilon \sum_{r=1}^{m} \sigma_r(t,Q) \circ dw_r(t), \; P(t_0) = p, \tag{5.6}$$

$$dQ = g(P)dt, \; Q(t_0) = q,$$

where $f^i = -\partial U_0/\partial q^i, \; g^i = \partial V_0/\partial p^i$, and $\sigma_r^i = -\partial U_r/\partial q^i, \; r = 1, \ldots m, \; i = 1, \ldots, n.$

An important particular case of (5.6) is a second-order differential equation with small multiplicative noise (cf. (4.5)).

The method (4.14)-(4.15) applied to (5.6) is of the order $O(h^2 + \varepsilon^2 h^{3/2})$.

On the basis of the fourth-order deterministic PRK method from [5, p. 109], we construct the following method for the system (5.6):

$$\mathcal{Q}_1 = Q_k, \tag{5.7}$$

$$\mathcal{P}_1 = P_k + h\frac{\varkappa}{2} f(t_k, \mathcal{Q}_1) + \varepsilon \sum_{r=1}^{m} \sigma_r(t_k, \mathcal{Q}_1)((1-\alpha)(J_{r0})_k + \frac{\alpha}{2}\Delta_k w_r),$$

$$\mathcal{Q}_2 = \mathcal{Q}_1 + h\varkappa g(\mathcal{P}_1),$$

$$\mathcal{P}_2 = \mathcal{P}_1 + h\frac{1-\varkappa}{2} f(t_k + \varkappa h, \mathcal{Q}_2) + \varepsilon \sum_{r=1}^{m} \sigma_r(t_k + \varkappa h, \mathcal{Q}_2)(\alpha(J_{r0})_k - \frac{\alpha}{2}\Delta_k w_r),$$

$$\mathcal{Q}_3 = \mathcal{Q}_2 + h(1-2\varkappa) g(\mathcal{P}_2),$$

$$\mathcal{P}_3 = \mathcal{P}_2 + h\frac{1-\varkappa}{2} f(t_k + (1-\varkappa)h, \mathcal{Q}_3) + \varepsilon \sum_{r=1}^{m} \sigma_r(t_k + (1-\varkappa)h, \mathcal{Q}_3)(\alpha(J_{r0})_k - \frac{\alpha}{2}\Delta_k w_r),$$

$$\mathcal{Q}_4 = \mathcal{Q}_3 + h\varkappa g(\mathcal{P}_3),$$

(5.8)

$$P_{k+1} = \mathcal{P}_3 + h\frac{\varkappa}{2} f(t_k + h, \mathcal{Q}_4) + \varepsilon \sum_{r=1}^{m} \sigma_r(t_k + h, \mathcal{Q}_4)(-(1+\alpha)(J_{r0})_k + (1+\frac{\alpha}{2})\Delta_k w_r),$$

$$Q_{k+1} = \mathcal{Q}_4, \quad k = 0, \ldots, N-1,$$

where

$$\varkappa = (2 + 2^{1/3} + 2^{-1/3})/3 \quad \text{and} \quad \alpha = \pm 1/\sqrt{\varkappa}, \tag{5.9}$$

and J_{r0} is as in (4.16).

Theorem 5.2. *The explicit method* (5.7) – (5.9) *for the system* (5.6) *is symplectic and its mean-square error is estimated as* $O(h^4 + \varepsilon h^2 + \varepsilon^2 h^{3/2})$.

Proof. The method (5.7)-(5.9) has the form of (4.10)-(4.11) and its parameters satisfy the conditions (4.12). Then Lemma 4.1 implies that this method preserves symplectic structure. Alternatively, this fact can be proved directly by using the evident chain of equalities:

$$dP_{k+1} \wedge dQ_{k+1} = d\mathcal{P}_3 \wedge d\mathcal{Q}_4 = d\mathcal{P}_3 \wedge d\mathcal{Q}_3 = d\mathcal{P}_2 \wedge d\mathcal{Q}_3 = d\mathcal{P}_2 \wedge d\mathcal{Q}_2$$
$$= d\mathcal{P}_1 \wedge d\mathcal{Q}_2 = d\mathcal{P}_1 \wedge d\mathcal{Q}_1 = dP_k \wedge dQ_k \, .$$

Using ideas of the proof of Theorem 4.2 and the mean-square theorem of [12], we establish that the method (5.7)-(5.9) is of order $O(h^4 + \varepsilon h^2 + \varepsilon^2 h^{3/2})$ (of course, the corresponding calculations require much routine work). □

6. Liouvillian methods for stochastic systems preserving phase volume

In the previous sections we constructed some Hamiltonian methods for stochastic Hamiltonian systems. These systems (as well as the methods) preserve the symplectic structure and, consequently, preserve the phase volume. In this section we deal with a more general class of systems which preserve the phase volume but may not preserve the symplectic structure.

Let us start with the deterministic d-dimensional system

$$\frac{dX}{dt} = a(t, X), \quad X(t_0) = x, \tag{6.1}$$

the phase flow $X(t; t_0, x)$ of which preserves the phase volume. Note that the dimension d may be odd.

Let $D_0 \in R^d$ be a domain with finite volume. The transformation $X(t; t_0, x)$ maps D_0 into the domain D_t. The volume V_t of the domain D_t is equal to

$$V_t = \int_{D_t} dX^1 \dots dX^d = \int_{D_0} \left| \frac{D(X^1, \dots, X^d)}{D(x^1, \dots, x^d)} \right| dx^1 \dots dx^d.$$

Then, the volume-preserving condition consists in the equality

$$\left| \frac{D(X^1(t), \dots, X^d(t))}{D(x^1, \dots, x^d)} \right| = 1 \tag{6.2}$$

or, equivalently, it consists in preservation of the d-form $dX^1 \wedge dX^2 \wedge \dots \wedge dX^d$.

According to the Liouville theorem (see, e.g., [4]), the phase flow of (6.1) preserves phase volume if and only if

$$\frac{\partial a^1(t,x)}{\partial x^1} + \dots + \frac{\partial a^d(t,x)}{\partial x^d} = div\, a = 0. \tag{6.3}$$

Numerical methods preserving the phase volume are called Liouvillian [13, 14]. Due to our best knowledge, there are no constructive Liouvillian methods for the deterministic system (6.1), (6.3) of a general form (see [13, 14, 15, 16] and references therein). Some constructive Liouvillian methods for particular cases of (6.1), (6.3) can be found in [13, 14, 15, 16]. It was shown in [14] that certain methods known to be symplectic are also phase volume preserving. However, it was also demonstrated that in general the relation between these two properties is rather delicate: neither of them implies the other.

Consider the Cauchy problem for the d-dimensional system of SDEs in the sense of Ito:

$$dX = a(t, X)dt + \sum_{r=1}^{m} b_r(t, X) dw_r(t), \; X(t_0) = x, \tag{6.4}$$

the phase flow $X(t; t_0, x; \omega)$ of which preserves phase volume, i.e., for which the condition (6.2) holds.

It is known (see [17, 18] and also [3]) that the phase flow of (6.4) preserves phase volume if and only if

$$div\,(a-\frac{1}{2}\sum_{r=1}^{m}\frac{\partial b_r}{\partial x}b_r)=0,\quad div\,b_r=0,\ r=1,\dots,m. \tag{6.5}$$

Let X_k, $k=0,\dots,N$, $t_{k+1}-t_k=h_{k+1}$, $t_N=t_0+T$:

$$X_0=X(t_0),\ X_{k+1}=\bar{X}_{t_k,X_k}(t_{k+1}),$$

be a mean-square method for (6.4) based on the one-step approximation $\bar{X}_{t,x}(t+h)=\bar{X}(t+h;t,x)$. It is clear that a method preserves phase volume if its one-step approximation satisfies the equality

$$\left|\frac{D(\bar{X}^1,\dots,\bar{X}^d)}{D(x^1,\dots,x^d)}\right|=1 \tag{6.6}$$

or equivalently

$$d\bar{X}^1\wedge\dots\wedge d\bar{X}^d=dx^1\wedge\dots\wedge dx^d. \tag{6.7}$$

Taking into account that there are no constructive Liouvillian methods for a general deterministic Liouvillian system, we restrict ourselves here to some particular cases of the stochastic system (6.4), (6.5).

6.1. **Liouvillian methods for partitioned systems with multiplicative noise.** Consider the particular case of (6.4):

$$\begin{aligned} dX&=f(t,Y)dt+\sum_{r=1}^{m}\sigma_r(t,Y)dw_r(t),\ X(t_0)=x,\\ dY&=g(t,X)dt+\sum_{r=1}^{m}\gamma_r(t)dw_r(t),\ Y(t_0)=y, \end{aligned} \tag{6.8}$$

where X, f, σ_r are l-dimensional column vectors and Y, g, γ_r are n-dimensional column vectors.

It is not difficult to check that the coefficients of (6.8) satisfy (6.5), i.e., the phase flow of system (6.8) preserves phase volume. Note that if $l=n$ and there are U_r, $r=0,\dots m$, and V_0 such that $f^i=-\partial U_0/\partial y^i$, $g^i=\partial V_0/\partial x^i$, and $\sigma_r=-\partial U_r/\partial y^i$, $r=1,\dots m$, $i=1,\dots,l$, then the system (6.8) possesses the symplectic property (cf. (3.10), we pay attention that the system (3.10) is in the sense of Stratonovich).

Introduce the PRK method for (6.8) (cf. (3.11)-(3.12)):

$$\begin{aligned} \mathsf{Y}_1&=Y_k+\alpha hg(t_k,X_k),\\ \mathsf{X}_1&=X_k+hf(t_k+\alpha h,\mathsf{Y}_1),\\ \mathsf{Y}_2&=\mathsf{Y}_1+(1-\alpha)hg(t_{k+1},\mathsf{X}_1), \end{aligned} \tag{6.9}$$

$$\begin{aligned} X_{k+1}&=\mathsf{X}_1+\sum_{r=1}^{m}\sigma_r(t_k,\mathsf{Y}_2)\Delta_k w_r,\\ Y_{k+1}&=\mathsf{Y}_2+\sum_{r=1}^{m}\gamma_r(t_k)\Delta_k w_r,\ k=0,\dots,N-1, \end{aligned} \tag{6.10}$$

with the parameter $0 \le \alpha \le 1$.

If $\sigma_r = \gamma_r = 0$, $r = 1, \dots, m$, this method coincides with the deterministic Liouvillian method [13, 14, 16].

Theorem 6.1. *The method* (6.9)-(6.10) *for system* (6.8) *is Liouvillian and of the mean-square order* 1/2.

Proof. Let us check that the one-step approximation $\bar{X}$, $\bar{Y}$ corresponding to (6.9)-(6.10) satisfies (6.7). Using properties of exterior products, we obtain

$$d\bar{X}^1 \wedge \dots \wedge d\bar{X}^l \wedge d\bar{Y}^1 \dots \wedge d\bar{Y}^n = (d\mathsf{X}_1^1 + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^1}{\partial y^j} d\mathsf{Y}_2^j) \wedge \dots \tag{6.11}$$

$$\wedge (d\mathsf{X}_1^{l-1} + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^{l-1}}{\partial y^j} d\mathsf{Y}_2^j) \wedge (d\mathsf{X}_1^l + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^l}{\partial y^j} d\mathsf{Y}_2^j) \wedge d\mathsf{Y}_2^1 \wedge \dots \wedge d\mathsf{Y}_2^n$$

$$= (d\mathsf{X}_1^1 + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^1}{\partial y^j} d\mathsf{Y}_2^j) \wedge \dots \wedge (d\mathsf{X}_1^{l-1} + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^{l-1}}{\partial y^j} d\mathsf{Y}_2^j)$$

$$\wedge (d\mathsf{X}_1^l \wedge d\mathsf{Y}_2^1 \wedge \dots \wedge d\mathsf{Y}_2^n + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^l}{\partial y^j} d\mathsf{Y}_2^j \wedge d\mathsf{Y}_2^1 \wedge \dots \wedge d\mathsf{Y}_2^n)$$

$$= (d\mathsf{X}_1^1 + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^1}{\partial y^j} d\mathsf{Y}_2^j) \wedge \dots \wedge (d\mathsf{X}_1^{l-1} + \sum_{r=1}^m \sum_{j=1}^n \frac{\partial \sigma_r^{l-1}}{\partial y^j} d\mathsf{Y}_2^j) \wedge d\mathsf{X}_1^l \wedge d\mathsf{Y}_2^1 \wedge \dots \wedge d\mathsf{Y}_2^n$$

$$= \dots = d\mathsf{X}_1^1 \wedge \dots \wedge d\mathsf{X}_1^l \wedge d\mathsf{Y}_2^1 \wedge d\mathsf{Y}_2^2 \wedge \dots \wedge d\mathsf{Y}_2^n .$$

Since (6.9) corresponds to the deterministic Liouvillian method, it follows from (6.11) that the method (6.9)-(6.10) is Liouvillian.

To prove the mean-square order of (6.9)-(6.10), we compare it with the Euler method and apply Lemma 2.1 as usual. □

Now put $\gamma_r = 0$, $r = 1, \dots, m$, in (6.8) (cf. (4.2)):

$$dX = f(t, Y)dt + \sum_{r=1}^m \sigma_r(t, Y)dw_r(t), \ X(t_0) = x, \tag{6.12}$$

$$dY = g(t, X)dt, \ Y(t_0) = y.$$

The Liouvillian method (6.9)-(6.10) in application to (6.12) is of the first mean-square order (cf. Theorem 4.1).

Introduce the PRK method for (6.12):

$$\mathsf{Y}_1 = Y_k, \quad \mathsf{X}_1 = X_k + \frac{h}{4} f(t_k, \mathsf{Y}_1) + \frac{1}{2}\sum_{r=1}^{m} \sigma_r(t_k, \mathsf{Y}_1)\left(3(J_{r0})_k - \Delta_k w_r\right), \tag{6.13}$$

$$\mathsf{Y}_2 = \mathsf{Y}_1 + \frac{2}{3} h g(t_k + \frac{h}{4}, \mathsf{X}_1),$$

$$\mathsf{X}_2 = \mathsf{X}_1 + \frac{3}{4} h f(t_k + \frac{2}{3}h, \mathsf{Y}_2) + \frac{3}{2}\sum_{r=1}^{m} \sigma_r(t_k + \frac{2}{3}h, \mathsf{Y}_2)\left(-(J_{r0})_k + \Delta_k w_r\right),$$

$$X_{k+1} = \mathsf{X}_2, \quad Y_{k+1} = \mathsf{Y}_2 + \frac{h}{3} g(t_{k+1}, \mathsf{X}_2), \quad k = 0, \dots, N-1. \tag{6.14}$$

This method applied to (4.2) gives the symplectic method (4.14)-(4.15).

Theorem 6.2. *The method* (6.13)-(6.14) *for the system* (6.12) *is Liouvillian and of the mean-square order* 3/2.

Proof. By the arguments similar to ones used to obtain (6.11) in Theorem 6.1, we prove that the one-step approximation corresponding to (6.13)-(6.14) satisfies the volume-preserving condition (6.7). For a proof of the mean-square order see Theorem 4.2. □

6.2. **Liouvillian methods for a volume-preserving system with additive noise.** The d-dimensional system with additive noise

$$dX = a(t, X)dt + \sum_{r=1}^{m} b_r(t) dw_r(t), \; X(t_0) = x, \tag{6.15}$$

possesses the volume-preserving property if and only if the condition (6.3) holds.

Theorem 6.3. *Let* $\bar{X} = X + A(t, X, \bar{X}; h)$ *be a one-step approximation corresponding to the first-order Liouvillian method for the deterministic system* (6.1), (6.3). *Then the method for the stochastic system* (6.15), (6.3):

$$X_{k+1} = X_k + A(t_k, X_k, X_{k+1}; h) + \sum_{r=1}^{m} b_r(t_k)\Delta_k w_r \tag{6.16}$$

is Liouvillian and of the first mean-square order.

Proof. We have for the one-step approximation $\bar{X}$ corresponding to (6.16): $d\bar{X}^i = dx^i + dA^i$, $i = 1, \dots, d$. Since these expressions coincide with the ones for the deterministic Liouvillian method, the approximation $\bar{X}$ satisfies (6.7) and the method is Liouvillian. The mean-square order of (6.16) easily follows from the general theory [9]. □

Due to this theorem, construction of first-order Liouvillian methods for Liouvillian systems with additive noise reduces to construction of such methods for deterministic Liouvillian systems. For instance, consider the following Liouvillian system

(6.17)

$$dX^i = a^i(t, X^1, \dots, X^{i-1}, X^{i+1}, \dots, X^d)dt + \sum_{r=1}^{m} b_r^i(t) dw_r(t), \; X(t_0) = x, \; i = 1, \dots, d.$$

In [16] an explicit first-order Liouvillian method for the deterministic system (6.1) with $a(t,x)$ as in (6.17) was proposed. Using it, we obtain

$$X_{k+1}^i = X_k^i + ha^i(t_k, X_{k+1}^1, \dots, X_{k+1}^{i-1}, X_k^{i+1}, \dots, X_k^d) + \sum_{r=1}^m b_r^i(t_k)\Delta_k w_r, \tag{6.18}$$
$$i = 1, \dots, d, \; k = 0, \dots, N-1.$$

Corollary 6.1. *The method* (6.18) *for* (6.17) *is Liouvillian and of the first mean-square order.*

Note that the Liouvillian method (6.9)-(6.10) for the system (6.8) with $\sigma_r(t,y) = \sigma_r(t)$, $r = 1, \dots, m$, (the partitioned system with additive noise) is of the first mean-square order. Further, for the partitioned system (6.8) with $\sigma_r(t,y) = \sigma_r(t)$, $r = 1, \dots, m$, a parametric family of 2-stage explicit Liouvillian PRK methods of mean-square 3/2 is derived. The form of these methods coincide with the symplectic method (5.11)-(5.14) from [3]. Let us also note that for the particular case of system (6.12) with $\sigma_r(t,y) = \sigma_r(t)$ and $g(t,x) = M^{-1}x$ where M is a constant, symmetric, invertible matrix, we succeed in construction of a Liouvillian method of the third mean-square order. The form of this method coincides with the third-order symplectic method (6.24)-(6.25) from [3].

7. Numerical tests

We test symplectic methods proposed in the previous sections on systems of linear stochastic equations. It turns out that it is possible to construct specific symplectic methods for linear systems and we start this section with consideration of such methods.

7.1. Explicit symplectic methods for a general second-order system of linear Ito SDEs.

Consider the two-dimensional linear system

$$\begin{aligned} dX^1 &= (a_{11}X^1 + a_{12}X^2)dt + (b_{11}X^1 + b_{12}X^2)dw(t) \\ dX^2 &= (a_{21}X^1 + a_{22}X^2)dt + (b_{21}X^1 + b_{22}X^2)dw(t), \end{aligned} \tag{7.1}$$

with conditions providing the preservation of phase area:

$$b_{11} + b_{22} = 0, \; a_{11} + a_{22} - (b_{11}^2 + b_{12}b_{21}) = 0.$$

Of course, implicit methods of Section 3 can be applied to this system. Here we derive explicit area-preserving methods for (7.1) using ideas of the method of fractional steps. Linearity of the right-hand sides of (7.1) allows us to present them as a sum of simple terms such that it is easy to construct a phase-area preserving method for each of the terms. A superposition of these partial methods gives a phase-area preserving method for (7.1). On this way we obtain the explicit method based on the following one-step approximation, $\bar{X} = (\bar{X}^1, \bar{X}^2)^\top$:

$$\bar{X} = S_4 S_3 S_2 S_1 x, \tag{7.2}$$

where

$$S_1 = \begin{bmatrix} 1 & b_{12}\Delta w \\ b_{21}\Delta w & 1 + b_{12}b_{21}(\Delta w)^2 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 1 + b_{11}\Delta w + \frac{1}{2}b_{11}^2(\Delta w)^2 & -\frac{1}{2}b_{11}^2(\Delta w)^2 \\ -\frac{1}{2}b_{11}^2(\Delta w)^2 & 1 - b_{11}\Delta w + \frac{1}{2}b_{11}^2(\Delta w)^2 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 1 & c_{12}h \\ c_{21}h & 1 + c_{12}c_{21}h^2 \end{bmatrix}, \; S_4 = \begin{bmatrix} 1 + c_{11}h + \frac{1}{2}c_{11}^2h^2 & -\frac{1}{2}c_{11}^2h^2 \\ -\frac{1}{2}c_{11}^2h^2 & 1 - c_{11}h + \frac{1}{2}c_{11}^2h^2 \end{bmatrix},$$

and

$$c_{11} = a_{11} - \frac{1}{2}b_{11}^2, \; c_{12} = a_{12} - b_{11}b_{12} + \frac{1}{2}b_{11}^2, \; c_{21} = a_{21} - b_{11}b_{21} + \frac{1}{2}b_{11}^2.$$

The method (7.2) preserves phase area (one can see that the determinants of S_i, $i = 1, \ldots, 4$, are equal to 1). It is also not difficult to prove that this method is of the mean-square order 1/2.

By swapping the roles of x^1 and x^2 in (7.2), we get another method. Moreover, methods obtained by any rearrangement of matrices S_1, S_2, S_3, S_4 preserve phase area and have the mean-square order 1/2 as well. It is easy to construct similar area-preserving methods for the two-dimensional system, the right-hand sides of which are the same as in (7.1) except they have the drift coefficients $a_{ij}(x^j)$, $j \neq i$, instead of $a_{ij}x^j$. They can also be generalized to linear systems with multiplicative noise of an arbitrary dimension both in the symplectic version for the Hamiltonian systems and in the phase-volume preserving version for the Liouvillian systems.

7.2. **Example 1.** The system of SDEs in the sense of Stratonovich (Kubo oscillator)

$$dX^1 = -aX^2dt - \sigma X^2 \circ dw(t), \;\; X^1(0) = x^1, \qquad (7.3)$$
$$dX^2 = aX^1dt + \sigma X^1 \circ dw(t), \;\; X^2(0) = x^2,$$

is often used for testing numerical methods (see, e.g., [19]). Here a and σ are constants and $w(t)$ is a one-dimensional standard Wiener process.

The phase flow of this system preserves symplectic structure. Moreover, the quantity $\mathcal{H}(x^1, x^2) = (x^1)^2 + (x^2)^2$ is conservative for this system, i.e.

$$\mathcal{H}(X^1(t), X^2(t)) = \mathcal{H}(x^1, x^2) \;\text{ for } t \geq 0.$$

This means that a phase trajectory of (7.3) belongs to the circle with center at the origin and with the radius $\sqrt{\mathcal{H}(x^1, x^2)}$.

We test here four methods. In application to (7.3) the symplectic PRK method (3.9) takes the form:

$$X_{k+1}^1 = X_k^1 - aX_k^2h - \frac{\sigma^2}{2}X_{k+1}^1h - \sigma X_k^2\Delta_k w, \qquad (7.4)$$
$$X_{k+1}^2 = X_k^2 + aX_{k+1}^1h + \frac{\sigma^2}{2}X_k^2h + \sigma X_{k+1}^1\Delta_k w.$$

This method is implicit in the deterministic part only.

The midpoint method (3.8) applied to the system with one noise (7.3) reads

$$X_{k+1}^1 = X_k^1 - a\frac{X_k^2 + X_{k+1}^2}{2}h - \sigma\frac{X_k^2 + X_{k+1}^2}{2}\left(\zeta_h\right)_k\sqrt{h}, \qquad (7.5)$$
$$X_{k+1}^2 = X_k^2 + a\frac{X_k^1 + X_{k+1}^1}{2}h + \sigma\frac{X_k^1 + X_{k+1}^1}{2}\left(\zeta_h\right)_k\sqrt{h}.$$

This is a full implicit method. Note that due to specific features of the system (7.3), the formula (7.5) is valid (solvable) not only in the case of the truncated random variable ζ_h but also if we put $\Delta_k w$ instead of $\left(\zeta_h\right)_k\sqrt{h}$.

The explicit method (7.2) (we pay attention that (7.2) is given for the Ito system) is written in the case of (7.3) as

$$\begin{pmatrix} X^1_{k+1} \\ X^2_{k+1} \end{pmatrix} = \begin{pmatrix} 1-\dfrac{\sigma^2}{2}h+\dfrac{\sigma^4}{8}h^2 & -\dfrac{\sigma^4}{8}h^2 \\ -\dfrac{\sigma^4}{8}h^2 & 1+\dfrac{\sigma^2}{2}h+\dfrac{\sigma^4}{8}h^2 \end{pmatrix} \begin{pmatrix} 1 & -ah \\ ah & 1-a^2h^2 \end{pmatrix} \times \begin{pmatrix} 1 & -\sigma\Delta_k w \\ \sigma\Delta_k w & 1-\sigma^2(\Delta_k w)^2 \end{pmatrix} \begin{pmatrix} X^1_k \\ X^2_k \end{pmatrix}. \tag{7.6}$$

The method (7.5) has the first mean-square method of convergence. The methods (7.4) and (7.6) are of mean-square order 1/2 as well as the Euler method:

$$X^1_{k+1} = X^1_k - aX^2_k h - \frac{\sigma^2}{2}X^1_k h - \sigma X^2_k \Delta_k w, \tag{7.7}$$
$$X^2_{k+1} = X^2_k + aX^1_k h - \frac{\sigma^2}{2}X^2_k h + \sigma X^1_k \Delta_k w,$$

which, of course, is not symplectic.

Figure 1 gives a sample phase trajectory of (7.3) simulated by the symplectic methods (7.4), (7.5), and (7.6) and by the Euler method (7.7). The initial condition is $x^1 = 1$, $x^2 = 0$. Then, the corresponding exact phase trajectory belongs to the circle with center at the origin and with the unit radius.

We see that the Euler method is not appropriate for simulation of the oscillator (7.3) on long time intervals while the symplectic methods preserve conservative properties of the Kubo oscillator.

These experiments also demonstrate that the midpoint method is much more accurate than the other methods applied. It is not difficult to check that $\mathcal{H}(x^1, x^2)$ is conserved by the midpoint method (7.5) but it is not conserved by the other symplectic methods: PRK method (7.4) and method (7.6). This is similar to the deterministic case. Indeed, it is known [7, 5] that symplectic deterministic RK methods (e.g., the midpoint scheme) conserve all quadratic functions that are conserved by the Hamiltonian system being integrated, while deterministic PRK methods do not possess this property.

7.3. **Example 2.** Consider the system of Ito equations

$$dX^1 = bX^2 dt \tag{7.8}$$
$$dX^2 = aX^1 dt + \sigma X^1 dw(t),$$

where a, b, and σ are some constants. Note that if $b = 1$ and $a < 0$, (7.8) is a linear oscillator with multiplicative noise.

This system is of the form (4.2). In application to (7.8) the Euler method reads

$$X^1_{k+1} = X^1_k + hbX^2_k \tag{7.9}$$
$$X^2_{k+1} = X^2_k + X^1_k \cdot (ha + \sigma\Delta_k w).$$

and the explicit PRK method (4.6)-(4.7) with $\alpha = 1$ has the form (we note that X^1, X^2 here correspond to Q, P in (4.2)):

$$X^1_{k+1} = X^1_k + hbX^2_k \tag{7.10}$$
$$X^2_{k+1} = X^2_k + X^1_{k+1} \cdot (ha + \sigma\Delta_k w).$$

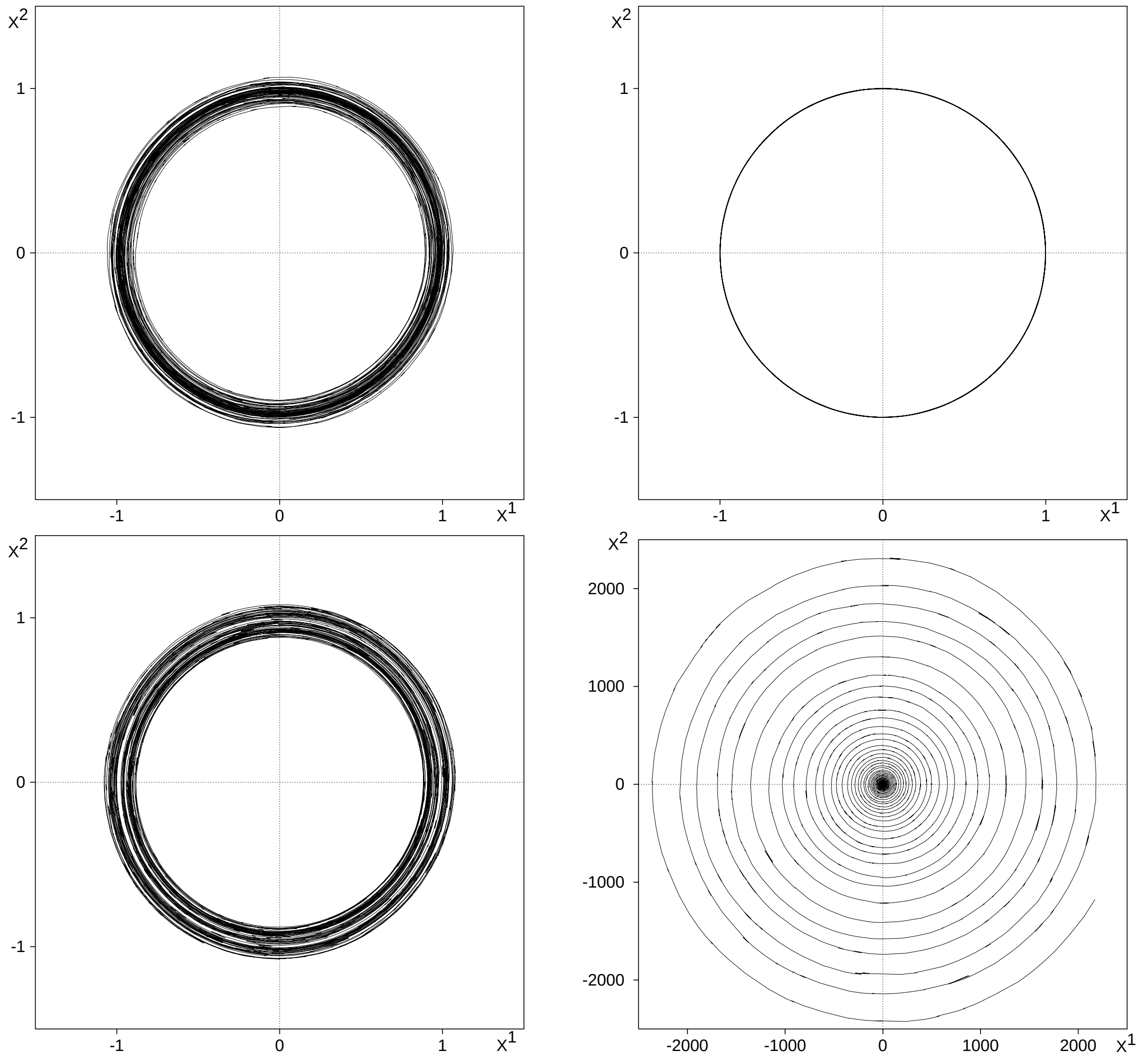

FIGURE 1. A sample phase trajectory of (7.3) with $X^1(0) = 1$, $X^2(0) = 0$ obtained by the symplectic method (7.4) (top left), the midpoint method (7.5) (top right), the explicit method (7.6) (bottom left) and by the Euler method (7.7) (bottom right) for $a = 2$, $\sigma = 0.3$, $h = 0.02$ on the time interval $t \leq 200$.

Both methods are of the first mean-square order.

Figure 2 presents evolution of domains in the phase plane of system (7.8). The initial domain is the circle with center at $(1, 0)$ and with the radius 0.1. In our experiments we take $a = -1$, $b = 1$, $\sigma = 0.2$, and $h = 0.02$. For these a and b, the period of free oscillations of (7.8) is equal to 2π.

The left part of Figure 2 corresponds to the symplectic method (7.10) and the right one to the Euler method (7.9). The each part contains three series of images of the initial domain. The fist series has 6 images, including the initial one, and presents the evolution on the time interval $[0, 5]$. So, all these images belong to the first period of the oscillator (7.8). The images are plotted once per 50 time steps and the last image in the first series corresponds to $t = 5$. The second and third series (each of 6 images again) for

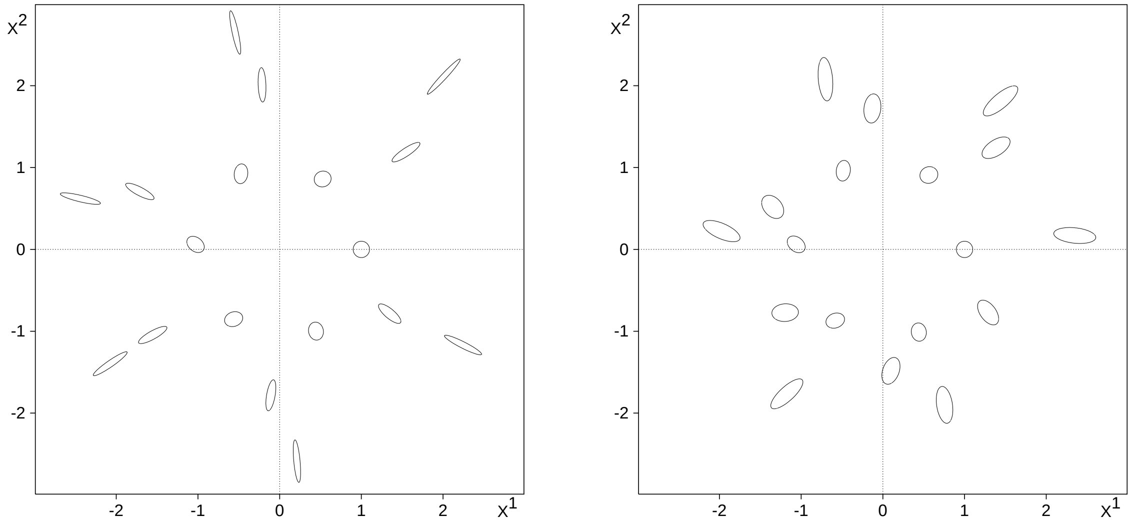

FIGURE 2. The evolution of domains in the phase plane of system (7.8) for $a = -1$, $b = 1$, $\sigma = 0.2$, and $h = 0.02$. Images of the initial circle are obtained at various time moments by the mapping in the case of symplectic method (7.10) (left) and by the mapping in the case of the Euler method (7.9) (right).

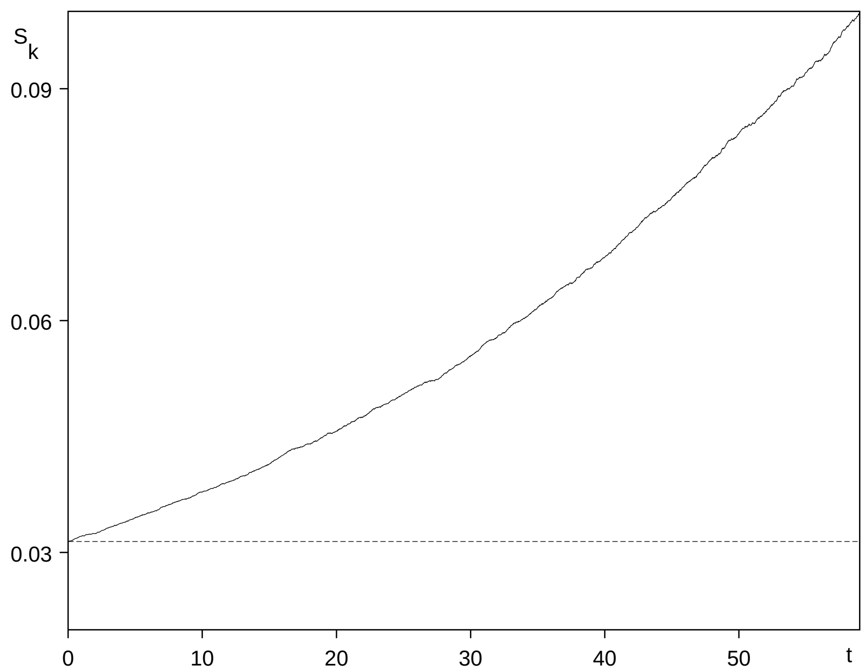

FIGURE 3. A typical behavior of the phase area S_k in the case of the Euler method (7.9) (solid line). Dashed line corresponds to preservation of the phase area by the system (7.8) and the symplectic method (7.10). The parameters are as in Fig. 2.

the symplectic method (left figure) are given on the 8^{th} and 13^{th} periods of oscillations respectively while these series for the Euler method (right figure) correspond to 5^{th} and 7^{th} periods. This difference is caused by the fact that the amplitude of oscillations simulated

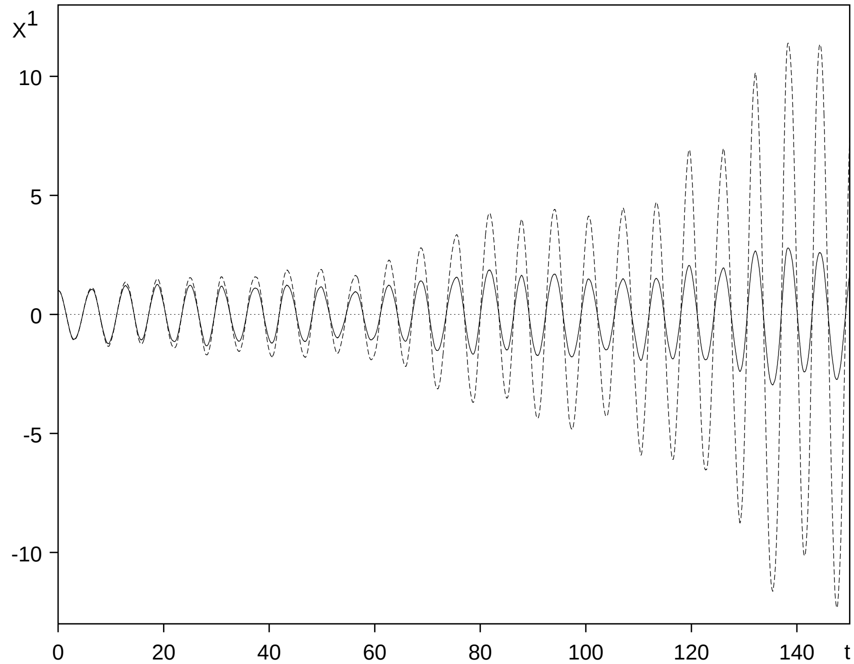

FIGURE 4. A sample trajectory of (7.8) with the parameters as in Fig. 2. Solid line – the symplectic method (7.10), dashed line – the Euler method (7.9).

by the Euler method grows essentially faster than the amplitude in simulations by the symplectic method. We note that the symplectic method with the taken h and time interval is quite accurate (see below). We also observe preservation of domains areas in the case of symplectic method (7.10) and growth of the areas in the case of the Euler method (7.9).

The phase area S_k in the case of the Euler method (7.9) changes at a one-step as $S_{k+1} = S_k \cdot (1 - hb(ha + \sigma\Delta_k w))$. A typical behavior of S_k is given on Fig. 3. It is easy to get that the mean $ES_k \approx S_0 \exp(-abht_k)$ and for $h^3 t_k \ll 1$ the standard deviation $(E(S_k - ES_k)^2)^{1/2} \approx S_0 \exp(-abht_k)(\exp(b^2\sigma^2 h^2 t_k) - 1)^{1/2}$.

A sample trajectory of (7.8) simulated by the symplectic method (7.10) and the Euler method (7.9) is plotted on Fig. 4. The trajectory obtained by the symplectic method with $h = 0.02$ (solid line) visually coincides with the one obtained with a smaller step, e.g. with $h = 0.002$ using the same sample path for the Wiener process, i.e., this trajectory visually coincides with the exact solution of (7.8). This figure clearly demonstrates that the Euler method is unacceptable for simulation of the solution to (7.8) on a long time interval while the symplectic method (7.10) produces quite accurate results despite both methods have the same mean-square order of accuracy.

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