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Phase-field systems with vectorial order parameters including diffusional hysteresis effects

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Abstract

This paper is concerned with phase-field systems of Penrose-Fife type which model the dynamics of a phase transition with non-conserved vectorial order parameter. The main novelty of the model is that the evolution of the order parameter vector is governed by a system consisting of one partial differential equation and one partial differential inclusion, which in the simplest case may be viewed as a diffusive approximation of the so-called multi-dimensional stop operator, which is one of the fundamental hysteresis operators. Results concerning existence, uniqueness and continuous dependence on data are presented which can be viewed as generalizations of recent results by the authors to cases where a diffusive hysteresis occurs.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $1 \leq N \leq 3$, denote an open, bounded domain with smooth boundary Γ and unit outer normal field n, and let $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$, with some final time $T > 0$. We then consider the system of partial differential equations

$$\left(\theta + \frac{1}{2}\,|\chi|^2\right)_t - \Delta\left(-\frac{1}{\theta}\right) = f(x,t) \quad \text{in } Q\,, \tag{1.1}$$

$$w_t - \gamma\,\Delta w + \frac{\chi}{\theta} = 0 \quad \text{in } Q\,, \tag{1.2}$$

$$\chi_t - \mu\,\Delta\chi + \partial I_Z(\chi) + \sigma(\chi) \ni w_t \quad \text{in } Q, \tag{1.3}$$

subject to the boundary conditions

$$\frac{\partial}{\partial n}\left(-\frac{1}{\theta}\right) + n_0\left(-\frac{1}{\theta}\right) = h(x,t) \quad \text{on } \Sigma\,, \tag{1.4a}$$

$$\frac{\partial w}{\partial n} = \frac{\partial \chi}{\partial n} = 0 \quad \text{on } \Sigma \tag{1.4b}$$

and to the initial conditions

$$\theta(\cdot,0) = \theta_0\,, \quad w(\cdot,0) = w_0\,, \quad \chi(\cdot,0) = \chi_0\,, \quad \text{in } \Omega\,. \tag{1.5}$$

Here, the unknown θ is a scalar function on Q, $w := (w_1,\cdots,w_M)$ and $\chi := (\chi_1,\cdots,\chi_M)$ are vector functions on Q for a fixed $M \in \mathbb{N}$, and $\sigma : \mathbb{R}^M \to \mathbb{R}^M$ is a vector function. Besides, f and h are functions prescribed on Q and Σ, respectively, $n_0 > 0$ is a fixed constant, and γ and μ are real parameters. In what follows, we will always assume that $0 \leq \gamma \leq 1$ and $0 < \mu \leq 1$, even though any other bounded parameter intervals in $[0, +\infty)$ could be considered. Finally, Z is some nonempty, bounded, closed and convex subset of $\mathbb{R}^M$ such that $0 \in Z$.

The system (1.1)–(1.3) may be interpreted as a phase-field system modelling the dynamics of a phase transition occurring in the container Ω with non-conserved order parameter vector χ. In this connection, θ stands for the (positive) absolute temperature, and w is a quantity related to χ. In fact, if $\gamma = 0$ then w can be eliminated from the system, and (1.2), (1.3) reduce to the single inclusion

$$\chi_t - \mu \Delta \chi + \partial I_Z(\chi) + \sigma(\chi) + \frac{\chi}{\theta} \ni 0. \tag{1.6}$$

Note that the system (1.1), (1.6) is nothing but a phase-field model of the *Penrose-Fife type*, if χ is a scalar function, i. e. $M = 1$, and if $Z = [-1, 1]$. We refer the reader to [3, 4, 7, 16, 17] for its well-posedness and the asymptotic convergence as $\mu \searrow 0$. We also note that in the case $\mu = 0$, $\sigma \equiv 0$, equation (1.3) takes the form

$$\chi_t + \partial I_Z(\chi) \ni w_t, \tag{1.7}$$

and the input-output relation $w \mapsto \chi$ is nothing but the *stop operator with the characteristic set* Z, which is one of the basic examples for *hysteresis operators* (for monographs on hysteresis phenomena and their mathematical treatment, we refer the reader to [2, 9, 18]). Therefore, (1.3) constitutes a *diffusive approximation* to the stop operator, and we may interpret the system (1.2), (1.3) as a model for a phase evolution taking both diffusive and hysteresis effects into account. In that sense, the system (1.1)–(1.3) may be viewed as a first step to generalize the phase-field systems with hysteresis studied in the recent papers [10, 11, 12, 13, 14, 15] to the situation when the $w \mapsto \chi$-relation incorporates both hysteresis and diffusion.

In this paper, we study the system (1.1)–(1.5) in a more general setting. In fact, the functions $\frac{1}{2}|\chi|^2$ and $-\frac{1}{\theta}$ will be replaced by more general functions λ and α, respectively. It is the aim to show a well-posedness result and to study the asymptotic behaviour of the solutions in dependence of the two parameters γ and μ. We will be able to treat the case $\gamma \searrow 0$, while the dependence on μ turns out to be more difficult: we will not be able to handle the asymptotics as $\mu \searrow 0$, but only as $\mu \to \hat{\mu}$ for some $\hat{\mu} > 0$. Hence, the case (1.7) of the "pure" stop operator with $\gamma > 0$ will not be covered by our analysis.

The rest of the paper is organized as follows: In section 2, we give a detailed description of the considered problem, define our notion of a solution, and state the main results of the paper. Section 3 is concerned with the continuous dependence of solutions with respect to the initial and boundary data and to the function f. The main theorems stated in section 2 are then proved in the subsequent sections 3 to 5.

2 Statements of main results

Let us consider the following general assumptions:

(A1) α is a nondecreasing function from an open set $D(\alpha)$ into $\mathbb{R}$, which is locally Lipschitz continuous on $D(\alpha)$, and assume that α is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$; we fix a primitive $\hat{\alpha}$ of α, which is a proper lower semicontinuous and convex function on $\mathbb{R}$.

(A2) λ is a function of C^2 - class on $\mathbb{R}^M$; we denote by λ' the gradient operator of λ in $\mathbb{R}^M$, i.e. $\lambda'(\chi) = (\frac{\partial \lambda}{\partial \chi_1}(\chi), \cdots, \frac{\partial \lambda}{\partial \chi_M}(\chi))$ for $\chi := (\chi_1, \cdots, \chi_M)$.

(A3) σ is a vector field of C^1 - class in $\mathbb{R}^M$.

(A4) Z is a nonempty, bounded, closed and convex set in $\mathbb{R}^M$ such that $0 \in Z$; we denote by $I_Z(\cdot)$ the indicator function of Z on $\mathbb{R}^M$, namely

$$I_Z(\chi) := \begin{cases} 0 & \text{if } \chi \in Z \\ +\infty & \text{otherwise,} \end{cases}$$

and by $\partial I_Z(\cdot)$ its subdifferential in $\mathbb{R}^M$.

Now, our problem, referred to as $(P_{\gamma\mu})$, is of the following form:

$$(\theta + \lambda(\chi))_t - \Delta\alpha(\theta) = f(x,t) \quad \text{in } Q, \tag{2.1}$$

$$w_t - \gamma\Delta w - \alpha(\theta)\lambda'(\chi) = 0 \quad \text{in } Q, \tag{2.2}$$

$$\chi_t - \mu\Delta\chi + \partial I_Z(\chi) + \sigma(\chi) \ni w_t \quad \text{in } Q, \tag{2.3}$$

subject to the boundary conditions

$$\frac{\partial\,\alpha(\theta)}{\partial n} + n_0\,\alpha(\theta) = h(x,t) \quad \text{on } \Sigma, \tag{2.4}$$

$$\frac{\partial w}{\partial n} = \frac{\partial \chi}{\partial n} = 0 \quad \text{on } \Sigma, \tag{2.5}$$

and to the initial conditions

$$\theta(\cdot,0) = \theta_0, \quad w(\cdot,0) = w_0, \quad \chi(\cdot,0) = \chi_0, \quad \text{in } \Omega. \tag{2.6}$$

In order to describe our results, we use the following simple notations:

(1) $H := L^2(\Omega)$, equipped with the standard norm $|\cdot|_H$ and inner product $(\cdot,\cdot)_H$, and in any product space of H the same notations $|\cdot|_H$ and $(\cdot,\cdot)_H$ are often used to indicate the standard norm and inner product, respectively.

(2) $V := H^1(\Omega)$, equipped with the norm

$$|v|_V := \left\{ \int_\Omega |\nabla v|^2 \, dx + n_0 \int_\Gamma |v|^2 \, d\Gamma \right\}^{\frac{1}{2}}, \quad \forall \, v \in V,$$

and its dual space is denoted by V^* with dual norm $|\cdot|_{V^*}$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V, and by F the duality mapping from V onto V^*; by definition, F is given by the formula

$$\langle Fv, u \rangle = \int_\Omega \nabla v \cdot \nabla u \, dx \, + \, n_0 \int_\Gamma v \, u \, d\Gamma, \quad \forall \, v \, , \, u \in V.$$

(3) We denote by Δ_0 the Laplace operator in H with homogeneous Neumann boundary condition, i.e. we have, by definition, $v = \Delta_0 w$ if and only if $w \in H^2(\Omega)$, $v \in H$ and $v = \Delta w$ a.e. in Ω, with $\frac{\partial w}{\partial n} = 0$ a.e. on Γ. In the product space H^M, we denote simply by $\Delta_0 w$ the vector $(\Delta_0 w_1, \cdots, \Delta_0 w_M)$, for $w := (w_1, \cdots, w_M)$.

In what follows, we denote by $|\cdot|$ both the absolute value of reals and the Euclidean norm of vectors in $\mathbb{R}^M$, and also by $|\Omega|$ the Lebesgue measure of Ω in $\mathbb{R}^N$, $N = 1, 2, 3$. With the above notations, we now give a weak formulation for problem $(P_{\gamma\mu})$.

Definition 2.1 Suppose that data $f \in L^2(0, T; H)$, $h \in L^2(0, T; L^2(\Gamma))$, $\theta_0 \in H$, and w_0, $\chi_0 \in H^M$ are given. We then call a triple $\{e, w, \chi\}$ with $e := \theta + \lambda(\chi)$ a *(weak) solution* to $(P_{\gamma\mu})$ for real parameters $\mu > 0$ and $\gamma \geq 0$, if the following conditions are satisfied:

(a) $e \in W^{1,2}(0, T; V^*) \cap L^2(0, T; H)$, $\alpha(\theta) \in L^2(0, T; V)$, and w, $\chi \in W^{1,2}(0, T; H^M)$. Moreover, $\chi \in L^2(0, T; H^2(\Omega)^M)$ and $w \in L^2(0, T; H^2(\Omega)^M)$ if $\gamma > 0$.

(b) Equation (2.1) and the boundary condition (2.4) are satisfied in the sense that

$$e'(t) \, + \, F\alpha(\theta(t)) = f^*(t) \quad \text{in } V^*, \quad \text{for a.e. } t \in (0, T), \tag{2.8}$$

where the prime denotes the time derivative $\frac{d}{dt}$, and where $f^* \in L^2(0, T; V^*)$ is defined by

$$\langle f^*(t), z \rangle = (f(t), z)_H \, + \int_\Gamma h(\cdot, t) \, z \, d\Gamma \quad \forall \, z \in V, \quad \text{for a.e. } t \in (0, T). \tag{2.9}$$

(c) Equations (2.2), (2.3) and the boundary conditions (2.5) are satisfied in the sense that

$$w'(t) - \gamma \, \Delta_0 w(t) - \alpha(\theta(t)) \, \lambda'(\chi(t)) = 0 \text{ in } H^M, \quad \text{for a.e. } t \in (0, T), \tag{2.10}$$

$$\chi'(t) \, - \, \mu \, \Delta_0 \chi(t) \, + \, \partial I_Z(\chi(t)) \, + \, \sigma(\chi) \, \ni \, w'(t) \quad \text{in } H^M, \quad \text{for a.e. } t \in (0, T). \tag{2.11}$$

If $\gamma = 0$, then the term $\gamma \, \Delta_0 w$ is neglected in (2.10).

(d) The initial condition (2.6) is satisfied in the sense that

$$e(0) = e_0 := \theta_0 + \lambda(\chi_0) \quad \text{in } V^*, \quad w(0) = w_0 \quad \text{in } H, \quad \chi(0) = \chi_0 \quad \text{in } H.$$

We can now state the main results of this paper. Concerning existence, we have the following result:

Theorem 2.2 *In addition to the conditions* (A1) *to* (A4)*, suppose that one of the following (a),(b) and (c) holds:*

(a) $\alpha \le 0$ *on* $D(\alpha)$*, and* λ *is a convex function on* $\mathbb{R}^M$ *such that*

$$\tilde{\chi} \cdot \lambda'(\chi) \ge 0, \qquad \forall \chi \in Z,\ \forall \tilde{\chi} \in \partial I_Z(\chi). \tag{2.12}$$

(b) $D(\alpha) = \mathbb{R}$*, and* α *is Lipschitz continuous on* $\mathbb{R}$*.*

(c) $\gamma = 0$*.*

Further suppose that $f \in L^2(0,T;H)$, $\theta_0 \in H$ *with* $\hat{\alpha}(\theta_0) \in L^1(\Omega)$, $w_0 \in V^M$, *as well as* $\chi_0 \in V^M$ *with* $\chi_0(x) \in Z$ *for a. e.* $x \in \Omega$. *Also, for the boundary datum* $h \in L^2(0,T;L^2(\Gamma))$ *assume that* $\frac{h}{n_0} = \alpha(\tilde{h})$ *a. e. on* Σ *for some* $\tilde{h} \in L^2(0,T;L^2(\Gamma))$. *Then, for* $\gamma \in [0,1]$ *and* $\mu \in (0,1]$, *the problem* $(P_{\gamma\mu})$ *has at least one solution* $\{e,w,\chi\}$ *which satisfies the further regularity properties* $e \in L^\infty(0,T;H)$, *and* $\chi \in L^\infty(0,T;V^M)$. *Moreover, if* $\gamma > 0$ *then* $w \in L^\infty(0,T;V^M)$.

The second theorem is concerned with the convergence of the problems $(P_{\gamma\mu})$ with respect to the parameters γ and μ.

Theorem 2.3 *Assume that condition (a) or (b) is satisfied and that* f, h, θ_0, w_0 *and* χ_0 *are as in Theorem 2.2. Let* $\{\gamma_n\}$ *and* $\{\mu_n\}$ *be two sequences of strictly positive numbers such that* $\gamma_n \to 0$ *and* $\mu_n \to \mu$ *as* $n \to +\infty$, *for a positive number* μ. *Besides, let* $\{e_n, w_n, \chi_n\}$ *be solutions to* $(P_{\gamma_n\mu_n})$. *Then* $\{e_n, w_n, \chi_n\}$ *converges to the unique solution* $\{e,w,\chi\}$ *to problem* $(P_{0\mu})$ *in the sense that*

$$e_n \to e \ \text{ strongly in } C([0,T];V^*), \quad e_n' \to e' \ \text{ weakly in } L^2(0,T;V^*), \tag{2.13}$$

$$\alpha(\theta_n) \to \alpha(\theta) \ \text{ weakly in } L^2(0,T;V), \quad w_n \to w \ \text{ weakly in } W^{1,2}(0,T;H^M), \tag{2.14}$$

$$\chi_n \to \chi \ \text{ weakly}^* \text{ in } L^\infty(0,T;V^M), \quad \chi_n' \to \chi' \ \text{ weakly in } L^2(0,T;H^M), \tag{2.15}$$

where $\theta_n := e_n - \lambda(\chi_n)$ *and* $\theta := e - \lambda(\chi)$.

The typical example such as mentioned in the introduction satisfies condition (a) of Theorem 2.2. The proofs of the above theorems will be given in sections 4 and 5.

3 Continuous dependence of solutions on the data

In this section, we prove the continuous dependence of solutions to $(P_{\gamma\mu})$ with respect to the initial and boundary data and to the function f (which implies the uniqueness of the solutions) in any of the following three special cases:

(Case 1) $N = 1$, and there is a constant $K_0 > 0$ satisfying

$$(\alpha(\theta_1) - \alpha(\theta_2))\,(\theta_1 - \theta_2) \;\geq\; \frac{K_0\,|\alpha(\theta_1) - \alpha(\theta_2)|^2}{|\alpha(\theta_1)\,\alpha(\theta_2)| + 1} \quad \forall\,\theta_i \in D(\alpha),\; i = 1, 2\,. \tag{3.1}$$

(Case 2) $D(\alpha) = \mathbb{R}$, and α is Lipschitz continuous on $\mathbb{R}$, say, there is a constant $K_0 > 0$ satisfying

$$K_0\,|\alpha(\theta_1) - \alpha(\theta_2)| \;\leq\; |\theta_1 - \theta_2| \quad \forall\,\theta_i \in \mathbb{R}\,,\; i = 1, 2\,. \tag{3.2}$$

(Case 3) It holds $\gamma = 0$.

Theorem 3.1 *Let $\{e_i, w_i, \chi_i\}$ be two solutions to $(P_{\gamma\mu})$, for $\gamma \in [0,1]$, $\mu \in (0,1]$, corresponding to the initial data $\{e_{0i}, w_{0i}, \chi_{0i}\}$, to the boundary data h_i, and to the source terms f_i, for $i = 1, 2$. We then have the following results:*

(i) *Assume that* (Case 1) *is given. Then it holds, for all $s \in [0,T]$,*

$$\begin{aligned}
&|e_1(s) - e_2(s)|_{V^*}^2 + C_1\,|w_1(s) - w_2(s)|_H^2 + |\chi_1(s) - \chi_2(s)|_H^2 \\
&+ C_2 \left\{ K_0 \int_0^s \int_\Omega \frac{|\alpha(\theta_1) - \alpha(\theta_2)|^2}{|\alpha(\theta_1)\,\alpha(\theta_2)| + 1}\,dx\,dt \right. \\
&\left. + \int_0^s (|\nabla(w_1 - w_2)|_H^2 + |\nabla(\chi_1 - \chi_2)|_H^2)(t)\,dt \right\} \\
\leq\; & \exp\left\{ C_3 \int_0^s \Big(|\alpha(\theta_1(t))|_V^2 + |\alpha(\theta_2(t))|_V^2 + 1 \Big)\,dt \right\} \\
&\times \left\{ |e_{01} - e_{02}|_{V^*}^2 + C_1 |w_{01} - w_{02}|_H^2 + |\chi_{01} - \chi_{02}|_H^2 + C_4 \int_0^s |f_1^*(t) - f_2^*(t)|_{V^*}^2\,dt \right\},
\end{aligned} \tag{3.3}$$

where $f_i^* \in L^2(0,T;V^*)$ is determined by h_i and f_i as in (2.9)*, and where C_k, $1 \leq k \leq 4$, are positive constants depending on $\gamma \in [0,1]$, $\mu \in (0,1]$, σ, and λ.*

(ii) *Assume that* (Case 2) *is given. Then it holds, for all $s \in [0,T]$,*

$$\begin{aligned}
&|e_1(s) - e_2(s)|_{V^*}^2 + C_1\,|w_1(s) - w_2(s)|_H^2 + |\chi_1(s) - \chi_2(s)|_H^2 \\
&+ C_2 \left\{ \int_0^s (K_0\,|\alpha(\theta_1) - \alpha(\theta_2)|_H^2 + |\nabla(w_1 - w_2)|_H^2 + |\nabla(\chi_1 - \chi_2)|_H^2)(t)\,dt \right\} \\
\leq\; & \exp\left\{ C_3 \int_0^s \Big(|\alpha(\theta_1(t))|_V^2 + |\alpha(\theta_2(t))|_V^2 + 1 \Big)\,dt \right\} \\
&\times \left\{ |e_{01} - e_{02}|_{V^*}^2 + C_1 |w_{01} - w_{02}|_H^2 + |\chi_{01} - \chi_{02}|_H^2 + C_4 \int_0^s |f_1^* - f_2^*|_{V^*}^2(t)\,dt \right\},
\end{aligned} \tag{3.4}$$

where f_i^* , $i = 1, 2$, *and* C_k , $1 \le k \le 4$, *are defined as in* (i).

(iii) *Assume* (Case 3) *is given. Then it holds, for all* $s \in [0, T]$,

$$
\begin{aligned}
&|e_1(s) - e_2(s)|_{V^*}^2 + |\chi_1(s) - \chi_2(s)|_H^2 + C_2 \int_0^s |\nabla(\chi_1 - \chi_2)|_H^2(t)\, dt \\
\le\ & \exp\left\{ C_3 \int_0^s \left(|\alpha(\theta_1(t))|_V^2 + |\alpha(\theta_2(t))|_V^2 + 1 \right) dt \right\} \\
&\times \left\{ |e_{01} - e_{02}|_{V^*}^2 + |\chi_{01} - \chi_{02}|_H^2 + C_4 \int_0^s |f_1^* - f_2^*|_{V^*}^2(t)\, dt \right\} , \qquad (3.5)
\end{aligned}
$$

where f_i^* , $i = 1, 2$, *and* C_k , $1 \le k \le 4$, *are defined as in* (i). *Moreover, the functions* w_i *are determined by*

$$w_i(s) = w_{0i} + \int_0^s \alpha(\theta_i(t))\, \lambda'(\chi_i(t))\, dt ,$$

for all $s \in [0, T]$ *and* $i = 1, 2$.

Proof. In what follows, we will suppress the argument t for the sake of brevity whenever this is appropriate and does not lead to confusion. First, we take the difference of the equalities (2.8) and (2.10), and of the inequalities (2.11), respectively, for two solutions $\{e_i, w_i, \chi_i\}$ to obtain, with the abbreviating notations $\alpha_i := \alpha(\theta_i)$, $\lambda_i := \lambda(\chi_i)$, $\lambda_i' := \lambda'(\chi_i)$, and $\sigma_i := \sigma(\chi_i)$, $i = 1, 2$,

$$(e_1 - e_2)' + F(\alpha_1 - \alpha_2) = f_1^* - f_2^* \quad \text{in } V^*, \text{ a.e. in } (0, T), \qquad (3.6)$$

$$(w_1 - w_2)' - \gamma\, \Delta_0(w_1 - w_2) - (\alpha_1 \lambda_1' - \alpha_2 \lambda_2') = 0 \quad \text{in } H^M, \text{ a.e. in } (0, T), \quad (3.7)$$

$$(\chi_1 - \chi_2)' - \mu\, \Delta_0(\chi_1 - \chi_2) + (\tilde{\chi}_1 - \tilde{\chi}_2) + \sigma_1 - \sigma_2 = (w_1 - w_2)' \quad \text{in } H^M, \text{ a.e. in } (0, T), \qquad (3.8)$$

where $\tilde{\chi}_i \in \partial I_Z(\chi_i)$ a.e. in Q for $i = 1, 2$.

We now perform the following computations:

(i) Take the inner product in V^* between both sides of (3.6) and $e_1 - e_2$.

(ii) Take the inner product in H^M between both sides of (3.7) and $\chi_1 - \chi_2$.

(iii) Take the inner product in H^M between both sides of (3.8) and $\chi_1 - \chi_2$.

(iv) Take the inner product in H^M between both sides of (3.7) and $w_1 - w_2$.

From (i), we obtain that a.e. in $(0, T)$

$$\frac{1}{2} \frac{d}{dt} |e_1 - e_2|_{V^*}^2 + (\alpha_1 - \alpha_2, \theta_1 - \theta_2)_H + (\alpha_1 - \alpha_2, \lambda_1 - \lambda_2)_H = (f_1^* - f_2^*, e_1 - e_2)_{V^*}. \quad (3.9)$$

Next, prior to performing (ii), we note that

$$|\lambda(\chi_1) - \lambda(\chi_2) - \lambda'(\chi_1) \cdot (\chi_1 - \chi_2)| \le L(\lambda') |\chi_1 - \chi_2|^2,$$

where $L(\lambda')$ denotes the (finite) Lipschitz constant of λ' on Z. Therefore it follows (cf. [6]) that

$$(\alpha_1\lambda_1'-\alpha_2\lambda_2')\cdot(\chi_1-\chi_2) \le (\alpha_1-\alpha_2)(\lambda_1-\lambda_2)+L(\lambda')\,(|\alpha_1|+|\alpha_2|)\,|\chi_1-\chi_2|^2. \qquad (3.10)$$

Now, on account of (3.10), the second calculation (ii) yields

$$(w_1'-w_2',\chi_1-\chi_2)_H + \gamma\,(\nabla(w_1-w_2),\nabla(\chi_1-\chi_2))_H$$
$$\le (\alpha_1-\alpha_2,\lambda_1-\lambda_2)_H + L(\lambda')\int_\Omega (|\alpha_1|+|\alpha_2|)\,|\chi_1-\chi_2|^2\,dx \qquad (3.11)$$

a. e. on $(0,T)$. The computation (iii) yields, with the (finite) Lipschitz constant $L(\sigma)$ of σ on Z, that a. e. in $(0,T)$ it holds

$$\frac{1}{2}\frac{d}{dt}|\chi_1-\chi_2|_H^2+\mu\,|\nabla(\chi_1-\chi_2)|_H^2 \le L(\sigma)\,|\chi_1-\chi_2|_H^2+(w_1'-w_2',\chi_1-\chi_2)_H. \quad (3.12)$$

Finally, we have by (iv) that a. e. in $(0,T)$

$$\frac{1}{2}\frac{d}{dt}|w_1-w_2|_H^2 + \gamma\,|\nabla(w_1-w_2)|_H^2$$
$$\le M(\lambda')\int_\Omega|\alpha_1-\alpha_2|\,|w_1-w_2|\,dx + L(\lambda')\int_\Omega|\alpha_1|\,|\chi_1-\chi_2|\,|w_1-w_2|\,dx\,, \quad (3.13)$$

where $M(\lambda') := \sup_{\chi\in Z}|\lambda'(\chi)|$.

We now have to estimate each of the cases (Case k), $k=1,2,3$, individually.

(Case 1): Assume that $\gamma>0$ (the case $\gamma=0$ is treated below in (Case 3)). Using (3.1), we derive from (3.9) that

$$\frac{1}{2}\frac{d}{dt}|e_1-e_2|_{V^*}^2 + K_0\int_\Omega\frac{|\alpha_1-\alpha_2|^2}{|\alpha_1\,\alpha_2|+1}\,dx + (\alpha_1-\alpha_2,\lambda_1-\lambda_2)_H$$
$$\le \frac{1}{2}|f_1^*-f_2^*|_{V^*}^2+\frac{1}{2}|e_1-e_2|_{V^*}^2 \quad \text{a.e. on } (0,T)\,. \qquad (3.14)$$

Since $N=1$, $L^\infty(\Omega)$ is compactly embedded in V, so that there is some $c_0>0$ satisfying

$$|z|_{L^\infty(\Omega)} \le c_0\,|z|_V \quad \forall\, z\in V\,.$$

Hence, the second term in the right-hand side of (3.11) is dominated by the expression

$$L(\lambda')\,c_0\,(|\alpha_1|_V^2+|\alpha_2|_V^2+1)\,|\chi_1-\chi_2|_H^2. \qquad (3.15)$$

Similarly, employing Young's inequality, we find that the first term on the right-hand side of (3.13) is dominated by

$$\varepsilon\,M(\lambda')\,K_0\int_\Omega\frac{|\alpha_1-\alpha_2|^2}{|\alpha_1\,\alpha_2|+1}\,dx + c_\varepsilon\,M(\lambda')\,(|\alpha_1|_V^2+|\alpha_2|_V^2+1)\,|w_1-w_2|_H^2\,, \quad (3.16)$$

while the second term can be estimated by

$$\frac{1}{2}\, c_0\, L(\lambda')\, |\alpha_1|_V \,(|\chi_1 - \chi_2|_H^2 \,+\, |w_1 - w_2|_H^2), \tag{3.17}$$

where ε is an arbitrary positive number, and c_ε is a positive constant depending only on ε. Now, adding (3.11), (3.12), (3.13) multiplied by $C_1 := \frac{\gamma+1}{2\mu^2}$, and (3.14), and using (3.15)–(3.17) with sufficiently small ε, we obtain an inequality of the form

$$\begin{aligned}
&\frac{d}{dt}\left\{|e_1 - e_2|_{V^*}^2 \,+\, C_1\,|w_1 - w_2|_H^2 \,+\, |\chi_1 - \chi_2|_H^2\right\}\\
&+C_2\left\{K_0 \int_\Omega \frac{|\alpha_1 - \alpha_2|^2}{|\alpha_1\,\alpha_2| \,+\, 1}\, dx \,+\, |\nabla(w_1 - w_2)|_H^2 \,+\, |\nabla(\chi_1 - \chi_2)|_H^2\right\}\\
\le\ & C_3\,(|\alpha_1|_V^2 \,+\, |\alpha_2|_V^2 \,+\, 1)\left\{|e_1 - e_2|_{V^*}^2 + C_1\,|w_1 - w_2|_H^2 + |\chi_1 - \chi_2|_H^2\right\}\\
&+C_4\,|f_1^* - f_2^*|_{V^*}^2\,,
\end{aligned} \tag{3.18}$$

a.e. on $(0,T)$, where C_2, C_3, C_4 can be chosen to be positive constants depending only on γ, μ, λ, and σ. Using Gronwall's lemma, we can conclude (3.3) from (3.18).

<u>(Case 2):</u> As before, we assume that $\gamma > 0$. We use the following inequality, which for $N \le 3$ is easily derived from the standard interpolation inequality:

$$\int_\Omega |z|\,|u|\,|v|\,dx \,\le\, \varepsilon\,(|\nabla u|_H^2 + |\nabla v|_H^2) + c_\varepsilon\,|z|_V^2\,(|u|_H^2 + |v|_H^2) \quad \forall\, z\,,\, u\,,\, v \in V, \tag{3.19}$$

where ε is an arbitrary positive number, and where c_ε is a positive constant depending only on ε. If α satisfies (3.2) then inequality (3.14) with $K_0 \int_\Omega \frac{|\alpha_1 - \alpha_2|^2}{|\alpha_1\,\alpha_2| + 1}\,dx$ replaced by the expression $K_0\,|\alpha_1 - \alpha_2|_H^2$ holds. Also, using (3.19), we see that the second term on the right-hand side of (3.11) is a.e. in $(0,T)$ dominated by the expression

$$\varepsilon\, L(\lambda')\,|\nabla(\chi_1 - \chi_2)|_H^2 \,+\, c_\varepsilon\, L(\lambda')(|\alpha_1|_V^2 \,+\, |\alpha_2|_V^2)\,|\chi_1 - \chi_2|_H^2\,. \tag{3.20}$$

Besides, the first and second terms on the right-hand side of (3.13) are respectively dominated by the expressions

$$\varepsilon\, M(\lambda')\,|\alpha_1 - \alpha_2|_H^2 \,+\, c_\varepsilon\, M(\lambda')\,|w_1 - w_2|_H^2\,, \tag{3.21}$$

$$\varepsilon\, L(\lambda')\,(|\nabla(w_1 - w_2)|_H^2 + |\nabla(\chi_1 - \chi_2)|_H^2) \,+\, c_\varepsilon\, L(\lambda')\,|\alpha_1|_V^2\,(|w_1 - w_2|_H^2 + |\chi_1 - \chi_2|_H^2)\,. \tag{3.22}$$

Now, just as in (Case 1), taking (3.20) to (3.22) into account, we see that (3.18), with the expression $K_0 \int_\Omega \frac{|\alpha_1 - \alpha_2|^2}{|\alpha_1\,\alpha_2| + 1}\,dx$ replaced by $K_0\,|\alpha_1 - \alpha_2|_H^2$, holds. Consequently, (3.4) is satisfied.

<u>(Case 3):</u> Sum up (3.9), (3.11) with $\gamma = 0$, and (3.12), and use (3.19) in order to estimate the second terms on the right-hand sides of (3.11) and (3.13). As before,

we then obtain

$$\frac{d}{dt}\left\{|e_1 - e_2|^2_{V^*} + |\chi_1 - \chi_2|^2_H\right\} + C_2\,|\nabla(\chi_1 - \chi_2)|^2_H$$
$$\leq\ C_3\,(|\alpha_1|^2_V + |\alpha_2|^2_V + 1)\left\{|e_1 - e_2|^2_{V^*} + |\chi_1 - \chi_2|^2_H\right\} + C_4\,|f^*_1 - f^*_2|^2_{V^*}$$

a.e. on $(0,T)$, whence the required inequality (3.5) follows. □

4 Approximate solutions and their uniform estimates

In this section, we consider an approximate problem for problem $(P_{\gamma\mu})$ with positive γ, μ. To this end, let α_δ be a Lipschitz continuous, globally bounded and nondecreasing function on $\mathbb{R}$ with parameter $\delta \in (0,1]$ such that α_δ converges to α on $\mathbb{R}\times\mathbb{R}$ in the sense of graphs as $\delta \to 0$. In this paper, choosing a strictly decreasing family $\{r_\delta\}$ and a strictly increasing family $\{s_\delta\}$ in $\mathbb{R}$ with respect to δ satisfying

$$r_\delta \downarrow \inf D(\alpha)\,, \quad s_\delta \uparrow \sup D(\alpha)\,, \quad \text{as } \delta \to 0\,,$$

we take as α_δ the function

$$\alpha_\delta(r) := \begin{cases} \alpha(r_\delta) & \text{for } r \le r_\delta\,, \\ \alpha(r) & \text{for } r_\delta < r < s_\delta\,, \\ \alpha(s_\delta) & \text{for } r \ge s_\delta\,. \end{cases}$$

Clearly, the range $R(\alpha_\delta)$ of α_δ is bounded. In this case, a primitive $\hat{\alpha}_\delta$ of α_δ can be chosen so that $\hat{\alpha}_\delta \to \hat{\alpha}$ uniformly on each compact subset of $D(\alpha)$ as $\delta \to 0$. Moreover, for the initial and boundary data θ_0 and $\tilde{h}$ smooth approximations $\theta_{0\delta}$ and $\tilde{h}_\delta$ are chosen such that, as $\delta \to 0$,

$$\theta_{0\delta} \to \theta_0 \text{ in } H\,, \quad \hat{\alpha}_\delta(\theta_{0\delta}) \to \hat{\alpha}(\theta_0) \text{ in } L^1(\Omega)\,,$$

as well as

$$\tilde{h}_\delta \to \tilde{h} \text{ in } L^2(0,T;L^2(\Gamma)), \quad h_\delta := n_0\,\alpha_\delta(\tilde{h}_\delta) \to h \text{ in } L^2(0,T;L^2(\Gamma))\,.$$

Also, let $f^*_\delta \in L^2(0,T;V^*)$ be the function determined from f and h_δ just as f^* in (2.9) of Definition 2.1. We now refer to $(P^\delta_{\gamma\mu})$ as the problem $(P_{\gamma\mu})$ with θ_0, f^*, α, replaced, respectively, by $\theta_{0\delta}$, f^*_δ, α_δ, in Definition 2.1. We have the following result.

Proposition 4.1 *Let $\gamma > 0$, $\mu > 0$, $\delta > 0$. Then problem $(P^\delta_{\gamma\mu})$ has a unique solution $\{e_{\gamma\mu\delta}, w_{\gamma\mu\delta}, \chi_{\gamma\mu\delta}\}$ such that $e_{\gamma\mu\delta} \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H)$, $\alpha_\delta(\theta_{\gamma\mu\delta}) \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ with $\theta_{\gamma\mu\delta} = e_{\gamma\mu\delta} - \lambda(\chi_{\gamma\mu\delta})$, as well as $w_{\gamma\mu\delta}$, $\chi_{\gamma\mu\delta} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$.*

Proof. The construction of a solution is based on the standard fixed point argument for continuous operators in compact convex sets. To this end, we consider the following three Cauchy problems

$$w' - \gamma\,\Delta_0 w = \bar u\,\lambda'(\bar\chi) \ \text{ in } \ H^M, \ \text{ a.e. in } (0,T), \ \ w(0) = w_0, \tag{4.1}$$

$$\chi' - \mu\,\Delta_0\chi + \partial I_Z(\chi) + \sigma(\chi) \ni w' \ \text{ in } \ H^M, \ \text{ a.e. in } (0,T), \ \ \chi(0) = \chi_0, \tag{4.2}$$

$$\theta' + F\alpha_\delta(\theta) = f_\delta^* - \lambda(\chi)' \ \text{ in } \ V^*, \ \text{ a.e. in } (0,T), \ \ \theta(0) = \theta_0, \tag{4.3}$$

for each pair of functions $(\bar u, \bar\chi) \in X$, where

$$X := \left\{ (\bar u, \bar\chi);\ \begin{array}{l} \bar u \in L^2(0,T;H),\ \bar u \in R(\alpha_\delta) \ \text{ a.e. in } Q \\ \bar\chi \in L^2(0,T;H^M),\ \bar\chi \in Z \ \text{ a.e. in } Q \end{array} \right\}.$$

It is well-known that for each $(\bar u, \bar\chi) \in X$ the problem (4.1) admits a unique solution w in $W^{1,2}(0,T;H^M) \cap L^\infty(0,T;V^M)$ satisfying the bound

$$|w|_{W^{1,2}(0,T;H^M)} + |w|_{L^\infty(0,T;V^M)} \le A_0\,(1 + |w_0|_V), \tag{4.4}$$

where $A_0 > 0$ is independent of the choice of $(\bar u, \bar\chi) \in X$ since

$$\sup_{(\bar u,\bar\chi)\in X} |\bar u \lambda'(\bar\chi)|_{L^2(0,T;H^M)} \le T^{\frac12}\,|\Omega|^{\frac12} \sup_{r\in\mathbb{R},\,p\in Z} \{|\alpha_\delta(r)|\,|\lambda'(p)|\} < +\infty. \tag{4.5}$$

In fact, (4.4) is easily obtained from testing (4.1) by w'. Next, for this function w problem (4.2) has a unique solution χ in $W^{1,2}(0,T;H^M) \cap L^\infty(0,T;V^M)$ satisfying the bound

$$|\chi|_{W^{1,2}(0,T;H^M)} + |\chi|_{L^\infty(0,T;V^M)} \le A_1\,(1 + |\chi_0|_V + |w'|_{L^2(0,T;H^M)}), \tag{4.6}$$

where $A_1 > 0$ is independent of w'; in fact, (4.6) follows from the inequality obtained by multiplying (4.2) by χ'. Finally, owing to the result in [5; Theorem 1.5], the problem (4.3) has for this function χ a unique solution θ belonging to $W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H^M)$ such that $\alpha_\delta(\theta) \in L^2(0,T;V)$, and satisfying the bound

$$|\theta|_{W^{1,2}(0,T;V^*)} + |\theta|_{L^\infty(0,T;H)} + |\alpha_\delta(\theta)|_{W^{1,2}(0,T;H)} + |\alpha_\delta(\theta)|_{L^\infty(0,T;V)}$$

$$\le A_2 \left\{ 1 + |\theta_{0\delta}|_V + |\chi'|_{L^2(0,T;H^M)} + |f|_{L^2(0,T;H)} + |\tilde h_\delta|_{W^{1,2}(0,T;L^2(\Gamma))} \right\}, \tag{4.7}$$

where $A_2 > 0$ is independent of χ'; indeed, the bound (4.7) is obtained by multiplying (4.3) by θ, θ', and $\frac{d}{dt}(\alpha_\delta(\theta))$, and by using the condition that $\alpha_\delta(\tilde h_\delta) = h_\delta/n_0$ on Σ.

Now, let $S : X \to X$ denote the operator that assigns to each $(\bar u, \bar\chi) \in X$ the pair of functions (u, χ) with $u := \alpha_\delta(\theta)$ which satisfies (4.1) to (4.3). Moreover, put

$$\begin{aligned} A_3 &:= A_1\,[1 + |\chi_0|_V + A_0\,(1 + |w_0|_V)], \\ A_4 &:= A_2 \left[1 + |\theta_{0\delta}|_V + A_3 + |f|_{L^2(0,T;H)} + |\tilde h_\delta|_{W^{1,2}(0,T;L^2(\Gamma))} \right], \end{aligned}$$

and

$$X_0 \ := \ \left\{ (\bar u, \bar\chi) \in X; \ \begin{array}{l} |\bar u|_{W^{1,2}(0,T;H)} + |\bar u|_{L^\infty(0,T;V)} \le A_4 \, , \quad \bar u \in R(\alpha_\delta) \ \text{ a.e. in } \ Q \\ |\bar\chi|_{W^{1,2}(0,T;H^M)} + |\bar\chi|_{L^\infty(0,T;V^M)} \le A_3 \, , \quad \bar\chi \in Z \ \text{ a.e. in } \ Q \end{array} \right\} .$$

Obviously, X_0 is a nonempty, compact and convex subset of $L^2(0,T;H) \times L^2(0,T;H^M)$. Also, it follows from (4.4) to (4.7) that $S(X_0) \subset X_0$.

Next, we prove that S is continuous in X_0 with respect to the topology of $L^2(0,T;H) \times L^2(0,T;H^M)$. To this end, assume that $(\bar u_n, \bar\chi_n) \in X$, $\bar u_n \to \bar u$ in $L^2(0,T;H)$, and $\bar\chi_n \to \bar\chi$ in $L^2(0,T;H^M)$, and denote by w_n, χ_n, and θ_n, respectively, the solution to (4.1) with $(\bar u, \bar\chi)$ replaced by $(\bar u_n, \bar\chi_n)$, the solution to (4.2), and the solution to (4.3), respectively. Then, by (4.7), and owing to the well-known general results concerning the continuous depencence of the solutions to the evolution equations (4.1), (4.2), (4.3) (cf. [1, 5]), we can conclude that

$$\begin{array}{l} w_n \to w \quad \text{strongly in } W^{1,2}(0,T;H) \, , \quad \chi_n \to \chi \quad \text{strongly in } L^2(0,T;H^M) \, , \\ \chi_n' \to \chi' \quad \text{weakly in } L^2(0,T;H^M) \, , \quad \theta_n \to \theta \quad \text{weakly in } L^2(0,T;H) \, , \\ \theta_n' \to \theta' \quad \text{weakly in } L^2(0,T;V^*) \, , \quad \alpha_\delta(\theta_n) \to \alpha_\delta(\theta) \quad \text{strongly in } L^2(0,T;H) \, , \end{array}$$

and the limits w, χ, θ are solutions to (4.1), (4.2), (4.3), respectively. This shows that $S(\bar u, \bar\chi) = (\alpha_\delta(\theta), \chi)$, which proves the continuity of S in X_0 with respect to the topology of $L^2(0,T;H) \times L^2(0,T;H^M)$.

It now follows from Schauder's fixed point theorem that S has at least one fixed point (u, χ) in X_0, which in turn gives a triple $\{e, w, \chi\}$ such that $u = \alpha_\delta(\theta)$, $e := \theta + \lambda(\chi)$, and w is the solution of (4.1) with $\bar u \, \lambda'(\bar\chi) = u \, \lambda'(\chi)$. Consequently, this triple is a solution to $(P^\delta_{\gamma\mu})$. The uniqueness of a solution of $(P^\delta_{\gamma\mu})$ is a consequence of (Case 2) of Theorem 3.1. □

In the remainder of this section, we will derive some uniform estimates for the approximate solutions $\{e_{\gamma\mu\delta}, w_{\gamma\mu\delta}, \chi_{\gamma\mu\delta}\}$ and $\theta_{\gamma\mu\delta} := e_{\gamma\mu\delta} - \lambda(\chi_{\gamma\mu\delta})$ constructed in Proposition 4.1. For simplicity, fixing the parameters $\gamma \in (0,1]$ and $\mu \in (0,1]$, we denote them by $\{e_\delta, w_\delta, \chi_\delta\}$, where $\theta_\delta := e_\delta - \lambda(\chi_\delta)$, for each $\delta \in (0,1]$. We then have

$$\theta_\delta' \, + \, \lambda(\chi_\delta)' \, + \, F\alpha_\delta(\theta_\delta) = f_\delta^* \ \text{ in } \ V^* , \ \text{ a.e. in } \ (0,T) \, , \tag{4.8}$$

$$w_\delta' \, - \, \gamma \, \Delta_0 w_\delta = \alpha_\delta(\theta_\delta) \, \lambda'(\chi_\delta) \ \text{ in } H^M , \ \text{ a.e. in } \ (0,T) \, , \tag{4.9}$$

$$\chi_\delta' \, - \, \mu \, \Delta_0 \chi_\delta \, + \, \partial I_Z(\chi_\delta) \, + \, \sigma(\chi_\delta) \ni w_\delta' \ \text{ in } H^M , \ \text{ a.e. on } \ (0,T) \, , \tag{4.10}$$

with initial conditions $\theta_\delta(0) = \theta_{0\delta}$, $w_\delta(0) = w_0$, and $\chi_\delta(0) = \chi_0$.

Proposition 4.2 *Assume that the same conditions as in Theorem 2.2 are satisfied. Moreover, for the parameters γ and μ, assume that $0 < \gamma \le L_0 \, \mu$ for some constant $L_0 > 0$. Then there exist constants $a_k > 0$, $1 \le k \le 6$, independent of all the parameters γ, μ, $\delta \in (0,1]$, such that the function*

$$E_\delta(t) := \int_\Omega \hat\alpha_\delta(\theta_\delta(t)) \, dx \, + \, a_1 \, \gamma \, |\nabla w_\delta(t)|_H^2 \, + \, a_2 \, \mu \, |\nabla \chi_\delta(t)|_H^2 \, + \, a_3 \, |\chi_\delta(t)|_H^2 \, + \, |\theta_\delta(t)|_H^2 \, ,$$

$0 \le t \le T$, *satisfies the inequality*

$$\frac{d}{dt}\,E_\delta \,+\, a_4\,\Big[|\alpha_\delta(\theta_\delta)|_V^2 \,+\, |w_\delta'|_H^2 \,+\, |\chi_\delta'|_H^2\Big]$$

$$\le\, a_5\,E_\delta \,+\, a_6\,\Big[|f|_H^2 \,+\, |h_\delta|^2_{L^2(\Gamma)} \,+\, |\tilde h_\delta|^2_{L^2(\Gamma)} \,+\, 1\Big] \quad \textit{a.e. in } (0,T)\,. \tag{4.11}$$

Proof. We test the functions $\alpha_\delta := \alpha_\delta(\theta_\delta)$ to (4.8) in $V^*\times V$, and χ_δ to (4.9) and (4.10) in $H^M\times H^M$. It then follows with the help of Young's inequality that

$$\frac{d}{dt}\int_\Omega \hat\alpha_\delta(\theta_\delta)\,dx \,+\, (\lambda(\chi_\delta)',\alpha_\delta)_H \,+\, \frac12\,|\alpha_\delta|_V^2 \;\le\; \frac12\,|f_\delta^*|^2_{V^*} \;\text{ a.e. in } (0,T)\,, \tag{4.12}$$

$$(w_\delta',\chi_\delta)_H \,\le\, \varepsilon|\alpha_\delta|_V^2 + B_1(\varepsilon) \,+\, \frac{\gamma}{2}\,|\nabla w_\delta|_H^2 \,+\, \frac{\gamma}{2}\,|\nabla\chi_\delta|_H^2 \;\text{ a.e. in } (0,T)\,, \tag{4.13}$$

$$\frac12\,\frac{d}{dt}\,|\chi_\delta|_H^2 \,+\, \mu\,|\nabla\chi_\delta|_H^2 \;\le\; (w_\delta',\chi_\delta)_H \,+\, B_2 \;\text{ a.e. in } (0,T)\,, \tag{4.14}$$

with an arbitrary small positive number ε and a positive constant $B_1(\varepsilon)$ depending only on ε, where $B_2 := |\Omega|\,\sup_{q\in Z}\{|\sigma(q)|^2\,|q|^2\}$, and where in (4.14) the monotonicity of the subdifferential ∂I_Z (cf. (A4)) has been used.

Next, test w_δ' and χ_δ' to (4.9) and (4.10), respectively, in $H^M\times H^M$, to get by Young's inequality

$$|w_\delta'|_H^2 \,+\, \frac{\gamma}{2}\,\frac{d}{dt}\,|\nabla w_\delta|_H^2 \;=\; (\alpha_\delta\lambda'(\chi_\delta),w_\delta')_H \;\text{ a.e. in } (0,T)\,, \tag{4.15}$$

as well as

$$\frac14\,|\chi_\delta'|_H^2 \,+\, \frac{\mu}{2}\,\frac{d}{dt}\,|\nabla\chi_\delta|_H^2 \;\le\; \frac12\,|w_\delta'|_H^2 \,+\, B_2 \;\text{ a.e. in } (0,T). \tag{4.16}$$

Also, test $\alpha_\delta\lambda'(\chi_\delta)$ to (4.10) in $H^M\times H^M$ to get

$$\begin{aligned}&(\lambda(\chi_\delta)',\alpha_\delta)_H - (\alpha_\delta\lambda'(\chi_\delta),w_\delta')_H\\ =\;& -\mu(\nabla\chi_\delta,\nabla(\alpha_\delta\lambda'(\chi_\delta)))_H - (\tilde\chi_\delta,\alpha_\delta\lambda'(\chi_\delta))_H - (\sigma(\chi_\delta),\alpha_\delta\lambda'(\chi_\delta))_H,\end{aligned} \tag{4.17}$$

where $\tilde\chi_\delta$ is a function in $L^2(0,T;H^M)$ satisfying $\tilde\chi_\delta\in\partial I_Z(\chi_\delta)$ a.e. on $(0,T)$.

Moreover, by testing formally θ_δ to (4.8) in $H\times H$, we see that

$$\begin{aligned}&\frac12\,\frac{d}{dt}|\theta_\delta|_H^2 + (\lambda(\chi_\delta)',\theta_\delta)_H \,+\, (\nabla\alpha_\delta,\nabla\theta_\delta)_H\\ +\;& n_0\int_\Gamma(\alpha_\delta-\alpha_\delta(\tilde h_\delta))\,\theta_\delta\,d\Gamma = (f,\theta_\delta)_H \qquad \text{a.e. in } (0,T)\,.\end{aligned} \tag{4.18}$$

Besides, by the monotonicity of the function $\alpha_\delta(\cdot)$, and owing to the conditions on $\tilde h_\delta$, we have

$$(\nabla\alpha_\delta,\nabla\theta_\delta)_H \ge 0, \quad n_0\int_\Gamma(\alpha_\delta-\alpha_\delta(\tilde h_\delta))\,\theta_\delta\,d\Gamma \,\ge\, n_0\int_\Gamma(\alpha_\delta-\alpha_\delta(\tilde h_\delta))\,\tilde h_\delta\,d\Gamma\,. \tag{4.19}$$

Therefore, it follows from (4.18) and (4.19) that

$$\frac{d}{dt}|\theta_\delta|_H^2 \le \frac{1}{8}|\alpha_\delta|_V^2 + \varepsilon|\chi_\delta'|_H^2 + B_3(\varepsilon)|\theta_\delta|_H^2 + B_4\left(|f|_H^2 + |h_\delta|_{L^2(\Gamma)}^2 + |\tilde h_\delta|_{L^2(\Gamma)}^2 + 1\right) \text{ a.e. in } (0,T), \tag{4.20}$$

where $B_3(\varepsilon)$ is a positive constant depending only on any small positive number ε and $B_4 > 0$ is a positive constant independent of all the parameters γ, μ, δ, ε. Note here that (4.18) to (4.20) are just formal computations because of the lack of regularity properties of θ_δ, but (4.20) can be rigorously verified via an appropriate further regularization of problem (4.10).

Now, consider the case when condition (a) is satisfied. In this case, note from the non-positiveness of α and (2.12) that $(\tilde\chi_\delta, \alpha_\delta\lambda'(\chi_\delta))_H \le 0$. Hence, by (4.17) we have

$$(\lambda(\chi_\delta)', \alpha_\delta)_H - (\alpha_\delta\lambda'(\chi_\delta), w_\delta')_H \ge -\varepsilon|\alpha_\delta|_V^2 - B_5(\varepsilon)(\mu^2|\nabla\chi_\delta|_H^2 + 1), \tag{4.21}$$

where ε is an arbitrary positive number and $B_5(\varepsilon)$ is a positive constant depending only on ε. Now, add the inequalities (4.12)–(4.16), (4.20) and use (4.21). We then find that a. e. in $(0,T)$ it holds

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega \hat\alpha_\delta(\theta_\delta)\,dx + \frac{\gamma}{2}|\nabla w_\delta|_H^2 + \frac{\mu}{2}|\nabla\chi_\delta|_H^2 + \frac{1}{2}|\chi_\delta|_H^2 + |\theta_\delta|_H^2\right\}\\
&\qquad + \left(\frac{3}{8} - 2\varepsilon\right)|\alpha_\delta|_V^2 + \frac{1}{2}|w_\delta'|_H^2 + \left(\frac{1}{4} - \varepsilon\right)|\chi_\delta'|_H^2\\
\le\ & \left(\frac{L_0\mu}{2} + B_5(\varepsilon)\mu^2\right)|\nabla\chi_\delta|_H^2 + \frac{\gamma}{2}|\nabla w_\delta|_H^2 + B_3(\varepsilon)|\theta_\delta|_H^2 + \frac{1}{2}|f_\delta^*|_{V^*}^2\\
&+ B_4\left(|f|_H^2 + |h_\delta|_{L^2(\Gamma))}^2 + |\tilde h_\delta|_{L^2(\Gamma)}^2 + 1\right) + B_1(\varepsilon) + 2B_2 + B_5(\varepsilon).
\end{aligned} \tag{4.22}$$

Therefore, we can easily derive an inequality of the form (4.11) from (4.22) by a suitable choice of the a_k, $1 \le k \le 6$, with sufficiently small $\varepsilon > 0$.

Secondly, consider the case when condition (b) is satisfied. In this case, by the Lipschitz continuity of α we see that

$$|(\lambda(\chi_\delta)', \alpha_\delta)_H| \le \varepsilon|\chi_\delta'|_H^2 + B_6(\varepsilon)(|\theta_\delta|_H^2 + 1)$$

and

$$|(\alpha_\delta\lambda'(\chi_\delta), w_\delta')_H| \le \varepsilon|w_\delta'|_H^2 + B_6(\varepsilon)(|\theta_\delta|_H^2 + 1),$$

where ε is an arbitrary positive number and $B_6(\varepsilon)$ is a positive constant depending only on ε. Summing up (4.12)–(4.16) and (4.20), and using the above two inequalities, we obtain an inequality similar to the form (4.22).

Finally, when condition (c) is satisfied, we have

$$(w_\delta', \chi_\delta')_H = (\lambda(\chi_\delta)', \alpha_\delta)_H \tag{4.23}$$

and

$$\frac{1}{2}|\chi_\delta'|_H^2 + \frac{\mu}{2}|\nabla\chi_\delta|_H^2 \le (w_\delta', \chi_\delta')_H + \frac{1}{2}B_2, \tag{4.24}$$

a.e. on $(0,T)$, which are derived by multiplying (4.9) and (4.10) by χ'_δ. Summing up (4.12)–(4.14), (4.20), (4.23) and (4.24), we again obtain an inequality of the form (4.22).

This ends the proof of the proposition. □

Corollary 4.3 *There is a constant $N_0 > 0$, independent of all the parameters γ, μ, $\delta \in (0,1]$, such that*

$$
\begin{aligned}
&|\theta_\delta|^2_{W^{1,2}(0,T;V^*)} + |\theta_\delta|^2_{L^\infty(0,T;H)} + |\alpha_\delta(\theta_\delta)|^2_{L^2(0,T;V)} + \mu\,|\nabla\chi_\delta|^2_{L^\infty(0,T;H^M)} \\
&+ |\chi'_\delta|^2_{L^2(0,T;H^M)} + \gamma\,|\nabla w_\delta|^2_{L^\infty(0,T;H^M)} + |w'_\delta|^2_{L^2(0,T;H^M)} \\
&\le N_0 \Big[|f|^2_{L^2(0,T;H)} + |h_\delta|^2_{L^2(0,T;L^2(\Gamma))} + |\tilde h_\delta|^2_{L^2(0,T;L^2(\Gamma))} \\
&\quad + |\hat\alpha_\delta(\theta_{0\delta})|_{L^1(\Omega)} + |\theta_{0\delta}|^2_H + |\chi_0|^2_V + |w_0|^2_V + 1 \Big].
\end{aligned}
\tag{4.25}
$$

Proof. Applying Gronwall's Lemma to inequality (4.11) in Proposition 4.2, we immediately obtain a uniform estimate of the form (4.25), except the one for $|\theta_\delta|_{W^{1,2}(0,T;V^*)}$. The estimate for $|\theta_\delta|_{W^{1,2}(0,T;V^*)}$ follows from the relation $\theta'_\delta = -F\alpha_\delta - \frac{d}{dt}\lambda(\chi_\delta) + f^*_\delta$. □

5 Proofs of the existence results

We first prove Theorem 2.2.

Proof of Theorem 2.2. Consider the case when $\mu > 0$ and $\delta > 0$, and let $\{e_\delta, w_\delta, \chi_\delta\}$ with $\theta_\delta := e_\delta - \lambda(\chi_\delta)$ be the family of approximate solutions for $(P_{\gamma\mu})$ constructed in the previous section. Then, according to Corollary 4.3, invoking some standard compactness results, we can claim that there are a sequence $\{\delta_n\}$ with $\delta_n \to 0$ as $n \to +\infty$, as well as functions $\theta \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H)$, $\alpha^* \in L^2(0,T;V)$, $w \in W^{1,2}(0,T;H^M) \cap L^\infty(0,T;V^M)$, and $\chi \in W^{1,2}(0,T;H^M) \cap L^\infty(0,T;V^M)$, such that

$$\theta_n := \theta_{\delta_n} \to \theta \text{ strongly in } C([0,T];V^*) \text{ and weakly}^* \text{ in } L^\infty(0,T;H), \tag{5.1}$$

$$\alpha_n := \alpha_{\delta_n}(\theta_n) \to \alpha^* \text{ weakly in } L^2(0,T;V), \tag{5.2}$$

$$
\begin{aligned}
w_n := w_{\delta_n} \to w, \quad \chi_n := \chi_{\delta_n} \to \chi \quad , \quad &\text{both strongly in } C([0,T];H^M) \\
&\text{and weakly in } L^2(0,T;V^M),
\end{aligned}
\tag{5.3}
$$

$$w'_n \to w', \quad \chi'_n \to \chi', \text{ both weakly in } L^2(0,T;H^M). \tag{5.4}$$

Hence, $\chi \in Z$ a. e. in Q, and

$$\lambda(\chi_n) \to \lambda(\chi) \text{ both strongly in } C([0,T];H) \text{ and weakly in } W^{1,2}(0,T;H), \quad (5.5)$$

as well as

$$\lambda'(\chi_n) \to \lambda'(\chi), \quad \sigma(\chi_n) \to \sigma(\chi), \quad \text{both strongly in } C([0,T];H^M). \quad (5.6)$$

Now, take $\delta = \delta_n$ in (4.8) to (4.10), and pass to the limit as $n \to +\infty$. Then, by the convergences (5.1) to (5.6), we have

$$\theta' + \lambda(\chi)' + F\alpha^* = f^* \text{ in } V^*, \text{ a. e. in } (0,T), \quad (5.7)$$

$$w' - \gamma\,\Delta_0 w = \alpha^*\,\lambda'(\chi) \text{ in } H^M, \text{ a. e. in } (0,T), \quad (5.8)$$

$$\chi' - \mu\,\Delta_0\chi + \partial I_Z(\chi) + \sigma(\chi) \ni w' \text{ in } H^M, \text{ a. e. in } (0,T). \quad (5.9)$$

Therefore, in order to complete our proof it suffices to show that $\alpha^* = \alpha(\theta)$. This is shown as follows. By (5.1) and (5.2), we have that

$$\int_0^T \langle F\alpha_n, F^{-1}(\theta_n - \theta)\rangle\, dt \to 0 \text{ as } n \to +\infty,$$

so that

$$\lim_{n\to+\infty} \int_0^T (\alpha_n, \theta_n)_H\, dt = \int_0^T (\alpha^*, \theta)_H\, dt. \quad (5.10)$$

Since $\alpha_n \to \alpha$ in H in the sense of graphs, it follows from (5.10) that $\alpha^* = \alpha(\theta)$ a. e. in Q. Thus, $\{e, w, \chi\}$ with $e := \theta + \lambda(\chi)$ is a solution to our system $(P_{\gamma\mu})$.

In the case $\gamma = 0$ and $\mu > 0$ the proof is only a slight modification of the above, so we may omit the details. □

Proof of Theorem 2.3. Let $\{\gamma_n\}$ and $\{\mu_n\}$ be as in the statement of Theorem 2.3, and denote by $\{e_n, w_n, \chi_n\}$, with $\theta_n := e_n - \lambda(\chi_n)$, the sequence of solutions of $(P_{\gamma_n\mu_n})$. Then, from (4.25) in Corollary 4.1, we may assume, by taking subsequences if necessary, that the following convergences hold for some functions $\theta \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H)$, $\alpha^* \in L^2(0,T;V)$, $w \in W^{1,2}(0,T;H^M)$ and $\chi \in W^{1,2}(0,T;H^M) \cap L^\infty(0,T;V^M)$, with $\chi \in Z$ a. e. in Q:

$$\theta_n \to \theta \text{ both strongly in } C([0,T];V^*) \text{ and weakly}^* \text{ in } L^\infty(0,T;H), \quad (5.11)$$

$$\alpha_n := \alpha(\theta_n) \to \alpha^* \text{ weakly in } L^2(0,T;V), \quad (5.12)$$

$$w_n \to w \text{ weakly in } W^{1,2}(0,T;H^M), \quad (5.13)$$

$$\chi_n \to \chi \text{ strongly in } C([0,T];H^M) \text{ and weakly in } W^{1,2}(0,T;H^M) \cap L^2(0,T;V^M). \quad (5.14)$$

Just as in the proof of Theorem 2.1, it follows from the convergences (5.11) to (5.14) that $\alpha^* = \alpha(\theta)$ a. e. in Q, and $\{e, w, \chi\}$, with $e := \theta + \lambda(\chi)$, is a solution to $(P_{0\mu})$, since the sequence $\{\gamma_n \Delta_0 w_n \ (= w_n' - \alpha(\theta_n)\,\lambda(\chi_n))\}$ is bounded in $L^2(0,T;H^M)$ and converges weakly to 0 in $L^2(0,T;H^M)$. Moreover, by the uniqueness result for $(P_{0\mu})$ (cf. (Case 3) in Theorem 3.1) we can conclude that the above convergences hold for the entire sequences $\{\gamma_n\}$ and $\{\mu_n\}$, and (2.12) to (2.14) hold. This ends the proof of the theorem. □

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