The finite dimensional attractor for a 4th order system of Cahn-Hilliard type with a supercritical nonlinearity

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WIAS preprint No. 634 of January 25, 2001, ISSN 0946 - 8633

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2000 Mathematics Subject Classification: 35B40, 35B45

Key words and phrases: Cahn-Hilliard system, global attractors, fractal dimension

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Abstract. The paper is devoted to study the long-time behaviour of solutions of the following 4th order parabolic system in a bounded smooth domain $\Omega \subset\subset \mathbb{R}^n$:

$$b\partial_t u = -\Delta_x u \left(a\Delta_x u - \alpha\partial_t u - f(u) + \tilde{g}\right) , \tag{1}$$

where $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, a and b are given constant matrices such that $a + a^* > 0$, $b = b^* > 0$, $\alpha > 0$ is a positive number, and f and g are given functions. Note that the nonlinearity f is not assumed to be subordinated to the Laplacian. The existence of a finite dimensional global attractor for the system (1) is proved under some natural assumptions on the nonlinear term f.

Introduction

In the paper we study the longtime behaviour of solutions of the following 4th order parabolic system in a bounded smooth domain $\Omega \subset\subset \mathbb{R}^n$:

$$b\partial_t u = -\Delta_x \left(a\Delta_x u - \alpha\partial_t u - f(u) + \tilde{g}\right) , \tag{0.1}$$

where $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, a and b are given constant matrices such that $a + a^* > 0$ and $b = b^* > 0$, $\alpha > 0$ is a positive number, f and $\tilde{g}$ are given functions.

Systems of type (0.1) arise in mathematical study of phase transitions in multicomponent systems and are of great recent interest (see e.g. [4]–[6], [8], [19] and references therein). In particular, M.Gurtin (see [8]) proposed a model which takes into account microforces and described their influence in terms of $\partial_t u$. To derive the equation (0.1) he used a mass balance

$$\partial_t u = -\operatorname{div} h\,, \tag{0.1$'$}$$

where u is an order parameter (corresponding to a density of atoms) and h is the mass flux which is related to the chemical potential μ and to the order parameter by the formulae

$$h = -\nabla_x \mu \quad \text{and} \quad \mu = f(u) - \Delta_x u + \alpha\partial_t u\,. \tag{0.1$''$}$$

Inserting (0.1$''$) to (0.1$'$) one obtains an equation of type (0.1).

It is well known that in many cases the longtime behaviour of solutions of evolution PDE can be described in terms of an attractor of the semi-group generated by this equation (see [1], [9], [16] and references therein). Attractors for the system (0.1) under various assumptions on the nonlinear interaction function f have been constructed in [3], [11], [14], [15]. Note however, that to the best of our knowledge the finite dimensionality of the obtained attractor has been established only under the following growth restriction

$$|f(u)| \le C(1 + |u|^q), \quad q < q_{max} = \frac{n+2}{n-2}\,, \tag{0.2}$$

which guarantees the nonlinear terms in (0.1) to be subordinated to the linear ones in the corresponding energetic phase space. This assumption is used in order to obtain the differentiability with respect to the initial values, which is essential for the standard scheme of estimating the fractal dimension by using k-contraction maps (see [1], [9] or [16]).

In the present paper we mainly consider the case where the assumption (0.2) is not satisfied. In this case the regularity of solutions which can be deduced from the energetic arguments is not sufficient for obtaining the differentiability with respect to the initial values and consequently the standard scheme does not work. So additional arguments must be involved.

Additional regularity of the attractor ($\mathcal{A} \in H^4(\Omega)$) of the three dimensional ($n = 3$) potential Cahn–Hilliard system (0.1) without microforces ($\alpha = 0$, $a = a^*$, $f = \nabla_x F$) has been obtained in [11] under the assumption that the potential F is positive, sufficiently smooth and quasi-convex, i.e. $F(u) + K|u|^2$ is convex for an appropriate constant K (without the growth restriction (0.2)!). Note that this regularity is enough for proving the quasi-differentiability and applying the standard methods of investigating the attractor.

A finite dimensional attractor for the second order reaction-diffusion system ($b = 0$, $\alpha = 1$ in (0.1)) with a supercritical nonlinearity has been obtained in [18] for an arbitrary dimension n without proving the additional regularity of solutions. In that paper a new scheme of estimating the fractal dimension of invariant sets which does not require the corresponding map to be quasi-differentiable has been suggested.

In the present paper we extend this result to a more general class of 4th order Cahn–Hiliard systems (0.1) ($b \neq 0$) with microforces ($\alpha \neq 0$) and supercritical nonlinearity. We will assume that the nonlinear interaction function (which is not assumed to be potential) satisfies the following conditions:

$$(0.3) \qquad \begin{cases} 1.\ f \in C^1(\mathbb{R}^k, \mathbb{R}^k)\,, \\ 2.\ f(u).u \geq -C\,, \\ 3.\ f'(u) \geq -K\,. \end{cases}$$

For simplicity we complete the system (0.1) by the Dirichlet boundary conditions

$$(0.4) \qquad u\big|_{\partial\Omega} = \Delta_x u\big|_{\partial\Omega} = 0\,.$$

(The case of physically more relevant Neumann boundary conditions is discussed in Section 5).

It will be convenient for us to assume that $f(0) = 0$, $\alpha = 1$, and to rewrite the equation (0.1) in the following (formally) equivalent form:

$$(0.5) \qquad \begin{cases} \partial_t \left(1 + b(-\Delta_x)^{-1}\right) u - a\Delta_x u + f(u) = g, \quad x \in \Omega\,, \\ u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = 0\,, \end{cases}$$

where $(-\Delta_x)^{-1}$ is the inverse operator to the Laplacian with Dirichlet boundary conditions, the function $g = \tilde{g} - G$ and G is a solution of the following non-homogeneous Dirichlet boundary problem

$$(0.6) \qquad \Delta_x G = 0, \quad G\big|_{\partial\Omega} = \tilde{g}\big|_{\partial\Omega}\,.$$

It is assumed that the function g in (0.5) belongs to the space $L^2(\Omega)$

$$(0.7) \qquad g \in L^2(\Omega)$$

and the initial value u_0 is supposed to belong to the nonlinear set

$$(0.8) \qquad \mathbb{D} := \{v \in H^2(\Omega) \,:\, v\big|_{\partial\Omega} = 0, \;\; f(v) \in L^2(\Omega)\}$$

(i.e. $\mathbb{D}$ is the domain of definition of the nonlinear maximal monotone operator $v \to -a\Delta_x v + f(v) + Kv$ (in $L^2(\Omega)$) in notations of the monotone operator theory, see e.g. [2] or [7]) and the 'norm' in this set is defined by the following expression:

$$(0.9) \qquad \|v\|_{\mathbb{D}}^2 := \|v\|_{H^2(\Omega)}^2 + \|f(v)\|_{L^2(\Omega)}^2 .$$

A solution of the equation (0.5) is defined to be a function

$$(0.10) \qquad u \in C_w([0,T], \mathbb{D})$$

(i.e. $u \in C([0,T], H^2_w(\Omega))$, $u\big|_{\partial\Omega} = 0$, and $f(u) \in C([0,T], L^2_w(\Omega))$, where the symbol $'w'$ means the weak topology), which satisfies the equation (0.5) as a relation in $L^2(\Omega)$.

It is not difficult to verify that a solution of the equation (0.5) thus defined coincides with the variational solution of the initial problem (0.1), (0.4) if $g \in H^1(\Omega)$ and $g\big|_{\partial\Omega} = 0$ (see [16]). Thus, instead of studying the behaviour of solutions of the initial problem (0.1), (0.4) we will study below the longtime behaviour of (0.5).

The paper is organized as follows. The existence of a solution for the problem (0.5) and it's uniqueness is verified in Section 1. The extension of the semi-group $S_t : \mathbb{D} \to \mathbb{D}$ generated by the equation (0.5) in the spirit of the monotone operator theory in L^2 and the attractor $\mathcal{A}$ of the obtained semi-group are constructed in Section 2. Some regularity properties of the attractor are studied in Section 3. The proof of the fact that the attractor $\mathcal{A}$ has a finite fractal dimension is given in Section 4.

The case of Neumann boundary conditions

$$\partial_n u\big|_{\partial\Omega} = \partial_n \Delta_x u\big|_{\partial\Omega} = 0$$

is considered in Section 5.

Acknowledgements. The authors have greatly benefited from helpful comments of A.Miranville.

§1 Existence of solutions.

In this Section we deduce a number of a priori estimates for the problem (0.5) and prove that for every $u_0 \in \mathbb{D}$ this problem has a unique solution. The main result of this Section is the following Theorem.

Theorem 1.1. *Let the assumptions* (0.3) *and* (0.7) *be valid. Then for every* $u_0 \in \mathbb{D}$ *the problem* (0.5) *has a unique solution* u *in the class* (0.10) *and the following estimate holds:*

$$(1.1) \qquad \|u(t)\|_{\mathbb{D}}^2 \le C_1(\|u(0)\|_{\mathbb{D}}^2 + \|g\|_{L^2}^2) e^{2(K-\varepsilon)t}$$

with some $\varepsilon > 0$.

We give below only a formal proof of the estimate (1.1) which can be justified for instance using Galerkin's approximation method. To this end we need the following Lemmata.

Lemma 1.1. *Let u be a solution of the equation* (0.5)*. Then the following estimate holds*

$$\|u(t)\|^2_{H^1} \le C_1\left(\|u(0)\|^2_{H^1} + \|g\|^2_{L^2}\right) e^{2(K-\varepsilon)t}\,, \tag{1.2}$$

where K is the same as in (0.3) *and $\varepsilon > 0$ is small enough.*

Proof. Indeed, multiplying the equation (0.5) by $-\Delta_x u$ and integrating over Ω we obtain after the standard integration by parts that

$$\begin{aligned}\partial_t\left(\|\nabla_x u(t)\|^2_{L^2} + (bu(t),u(t))\right) + ((a+a^*)\Delta_x u(t),\Delta_x u(t)) +\\ + 2\left(f'(u(t))\nabla_x u(t),\nabla_x u(t)\right) + 2\left(g,\Delta_x u(t)\right) = 0\,,\end{aligned} \tag{1.3}$$

Using that $a+a^* > 0$, $f'(u) \ge -K$ and applying Friedrichs and Hölder's inequality we deduce from (1.3) that

$$\partial_t\left(\|\nabla_x u(t)\|^2_{L^2} + (bu(t),u(t)\right) + 2\varepsilon\|\nabla_x u(t)\|^2_{L^2} - 2K\|\nabla_x u(t)\|^2_{L^2} \le C_\varepsilon\|g\|^2_{L^2}\,, \tag{1.4}$$

where $\varepsilon > 0$ is a sufficiently small positive number. Applying Gronwall's inequality to the estimate (1.4) and using that b is positive, we obtain the assertion of the lemma.

Lemma 1.2. *Let u be a solution of the problem* (0.5)*. Then the following estimate is valid:*

$$\|\partial_t u(t)\|^2_{L^2} \le C_1(\|u(0)\|^2_{\mathbb{D}} + \|g\|^2_{L^2})e^{2(K-\varepsilon)t}\,, \tag{1.5}$$

where K is the same as in (0.3) *and $\varepsilon > 0$ is small enough.*

Proof. Differentiating the equation (0.5) with respect to t and denoting $\theta(t) = \partial_t u(t)$ we get

$$\begin{cases}\partial_t\left(1 + b(-\Delta_x)^{-1}\right)\theta(t) - a\Delta_x\theta(t) + f'(u(t))\theta(t) = 0\,,\\ \theta\big|_{t=0} = a\Delta_x u(0) - f(u(0)) + g\,,\quad \theta\big|_{\partial\Omega} = 0\,.\end{cases} \tag{1.6}$$

Multiplying this equation by $\theta(t)$, integrating over $x \in \Omega$ and arguing as in the proof of the previous lemma, we deduce that

$$\begin{aligned}\partial_t\left(\|\theta(t)\|^2_{L^2} + (b(-\Delta_x)^{-1/2}\theta,(-\Delta_x)^{-1/2}\theta)\right) +\\ + 2\varepsilon\|\theta(t)\|^2_{L^2} - 2K\|\theta(t)\|^2_{L^2} \le 0\,.\end{aligned} \tag{1.7}$$

Applying Gronwall's inequality to the estimate (1.7) we obtain the assertion of the lemma.

Lemma 1.3. *Let u be a solution of the problem* (0.5)*. Then the following estimate is valid:*

$$\|u(t)\|^2_{H^2} \le C_1(\|u(0)\|^2_{\mathbb{D}} + \|g\|^2_{L^2})e^{2(K-\varepsilon)t}\,, \tag{1.8}$$

where $\varepsilon > 0$ is small enough.

Proof. Let us rewrite the equation (0.5) in the form of an elliptic boundary problem

$$a\Delta_x u(t) - f(u(t)) = \partial_t\left(1 + b(-\Delta_x)^{-1}\right)u(t) - g \equiv h_u(t),\quad u(t)\big|_{\partial\Omega} = 0\,. \tag{1.9}$$

Multiplying (1.9) by $\Delta_x u(t)$ and integrating over $x \in \Omega$, we obtain, arguing as in the proof of Lemma 1.1, that

$$\|\Delta_x u(t)\|^2_{L^2} \le C_1 K \|\nabla_x u(t)\|^2_{L^2} + C_2(\|\partial_t u(t)\|^2_{L^2} + \|g\|^2_{L^2}) \,. \tag{1.10}$$

According to (L^2, H^2)-regularity of solutions of the Laplace equation, we have (see [17])

$$\|u(t)\|^2_{H^2} \le C\|\Delta_x u(t)\|^2_{L^2} \,. \tag{1.11}$$

Inserting the inequalities (1.2) and (1.5) into the right-hand side of (1.10) and using (1.11), we obtain (1.8) after simple calculations. Lemma 1.3 is proved.

Now we are in position to complete the proof of the estimate (1.1). Indeed, according to (0.9) we should estimate the H^2-norm of $u(t)$ and the L^2-norm of $f(u(t))$. The H^2-norm is already estimated in Lemma 1.3, so it remains to estimate $\|f(u(t))\|_{L^2}$. But it follows immediately from the equation (0.5) that

$$\|f(u(t))\|^2_{L^2} \le C\|u(t)\|^2_{H^2} + C\|\partial_t u(t)\|^2_{L^2} + C\|g\|^2_{L^2} \,. \tag{1.12}$$

Inserting the inequalities (1.5) and (1.8) into the right-hand side of (1.12) we obtain the estimate of the L^2-norm of $f(u(t))$. The estimate (1.1) is proved.

The existence of a solution $u \in C_w([0,T], \mathbb{D})$ for the problem (0.5) can be derived in a standard way using the a priori estimate (1.1) and the Galerkin's approximation method with a special choice of basis generated by eigenfunctions of the Laplacian (see for example [1], [16]). So it remains to prove the uniqueness.

Lemma 1.4. *Let $u_1, u_2 \in C_w([0,T], \mathbb{D})$ be two solutions of the equation* (0.5) *with the initial values $u_1(0)$ and $u_2(0)$ respectively. Then*

$$\|u_1(T) - u_2(T)\|^2_{L^2} + \int_T^{T+1} \|u_1(t) - u_2(t)\|^2_{H^1}\, dt \le \|u_1(0) - u_2(0)\|^2_{L^2} e^{2(K-\varepsilon)T} \tag{1.13}$$

for some positive $\varepsilon > 0$. Particularly, the problem (0.5) *has a unique solution for every $u_0 \in \mathbb{D}$.*

Proof. Let $w(t) = u_1(t) - u_2(t)$. Then the function w satisfies the equation

$$\begin{cases} \partial_t \left(1 + b(-\Delta_x)^{-1}\right) w(t) - a\Delta_x w(t) = f(u_2(t)) - f(u_1(t)) \,, \equiv h_{u_1,u_2}(t) \\ w\big|_{t=0} = u_1(0) - u_2(0) \,. \end{cases} \tag{1.14}$$

Note, that $h_{u_1,u_2} \in C_w([0,T], L^2)$. Moreover, since $f'(v) \ge -K$ then

$$(f(\xi_1) - f(\xi_2)).(\xi_1 - \xi_2) \ge -K|\xi_1 - \xi_2|^2 \tag{1.15}$$

for every $\xi_1, \xi_2 \in \mathbb{R}^k$. Thus,

$$(h_{u_1,u_2}(t), w(t)) \le K\|w(t)\|^2_{L^2} \,. \tag{1.16}$$

Multiplying now the equation (1.14) by $w(t)$, integrating over $x \in \Omega$ and using the estimate (1.16) we deduce that

$$\partial_t \left(\|w(t)\|^2_{L^2} + (b(-\Delta_x)^{-1/2} w(t), (-\Delta_x)^{-1/2} w(t)) \right) + \\ + \varepsilon\|w(t)\|^2_{H^1} - 2K\|w(t)\|^2_{L^2} \le 0 \,. \tag{1.17}$$

Applying Gronwall's inequality to (1.17) we obtain the estimate (1.13). Lemma 1.4 is proved. Theorem 1.1 is proved.

Remark 1.1. Note, that the dissipativity assumption $f(u).u \geq -C$ has not been used in the proof of Theorem 1.1, consequently this theorem remains valid without this assumption. However the dissipativity assumption will be essentially used in the next Section in order to prove the existence of an absorbing set for the semigroup, generated by the equation (0.5).

§2 The attractor.

In this Section we describe the longtime behavior of solutions of the autonomous equation (0.5) in terms of the attractor for the corresponding semigroup. Recall that, according to Theorem 1.1, the problem (0.5) generates a Lipschitz continuous semigroup $\{S_t,\, t \geq 0\}$ in $\mathbb{D}$:

$$S_t : \mathbb{D} \to \mathbb{D}, \quad S_t u_0 = u(t)\,. \tag{2.1}$$

Moreover, (1.1) implies that

$$\|u(t)\|^2_{\mathbb{D}} \leq C\left(\|u(0)\|^2_{\mathbb{D}} + \|g\|^2_{L^2}\right) e^{2(K-\varepsilon)t} \tag{2.2}$$

for a sufficiently small positive ε. But the right-hand side of (2.2) tends to $+\infty$ as $t \to \infty$. (The case $K-\varepsilon < 0$ is not considered because in this situation it is easy to prove that the attractor consists of a unique exponentially attracting equilibrium.) Hence, the estimate (2.2) does not guarantee that S_t will be bounded in $\mathbb{D}$ when $t \to \infty$. In fact under our assumptions we can prove that it will be bounded in L^2 or H^1 only and not in $\mathbb{D}$. To avoid this difficulty we extend by continuity the semigroup S_t, which is initially defined for only $u_0 \in \mathbb{D}$ to $\hat{S}_t : L^2 \to L^2$. Indeed, $\mathbb{D}$ is dense in L^2 and according to (1.13) S_t is uniformly continuous on $\mathbb{D}$ in L^2-metric for every fixed t. Consequently it can be extended in a unique way to a semigroup $\hat{S}_t$ on L^2 by

$$\hat{S}_t u_0 = L^2\text{–}\lim_{n\to\infty} S_t u_0^n, \quad u_0^n \in \mathbb{D}, \quad u_0 = L^2\text{–}\lim_{n\to\infty} u_0^n\,. \tag{2.3}$$

Moreover, the estimate (1.13) implies that

$$\|\hat{S}_t u_0^1 - \hat{S}_t u_0^2\|^2_{L^2} + \int_T^{T+1} \|\hat{S}_t u_0^1 - \hat{S}_t u_0^2\|^2_{H^1}\, dt \leq e^{2(K-\varepsilon)T}\|u_0^1 - u_0^2\|_{L^2} \tag{2.4}$$

for every $u_0^1, u_0^2 \in L^2$ and since $u_n(t) = S_t u_0^n \in C([0,T], L^2)$ if $u_0^n \in \mathbb{D}$ then $u(t) = \hat{S}_t u_0$ also belongs to $C([0,T], L^2)$ for every $u_0 \in L^2$.

Thus, we can naturally interpret the function $u(t) = \hat{S}_t u_0$ as a unique solution of the problem (0.5) for $u_0 \in L^2$ and study the longtime behavior of the semigroup $\hat{S}_t : L^2 \to L^2$. The following Theorem is of fundamental significance for these purposes.

Theorem 2.1. *Let $u_0 \in L^2$ and $u(t) = \hat{S}_t u_0$. Then $u \in C([0,T], L^2)$ for every $T \geq 0$ and*

$$\|u(T)\|^2_{L^2} + \int_T^{T+1} \|u(t)\|^2_{H^1}\, dt \leq C_1\|u(0)\|^2_{L^2} e^{-\varepsilon t} + C_2(1 + \|g\|^2_{L^2}) \tag{2.5}$$

for a sufficiently small positive $\varepsilon > 0$. Moreover, for every $t > 0$, $u(t) \in H^1$, $u \in C([t,T], H^1_w)$, and the following estimate is valid:

$$\|u(T)\|^2_{H^1} + \int_T^{T+1} \|u(t)\|^2_{H^2}\,dt \le C\frac{t+1}{t}\left(\|u(0)\|^2_{L^2}e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right). \tag{2.6}$$

Proof. According to (2.3), it is sufficient to deduce the estimates (2.5) and (2.6) only for $u_0 \in \mathbb{D}$. Let us prove firstly the estimate (2.5).

Multiplying the equation (0.5) by $u(t)$ and integrating over $x \in \Omega$, we obtain that

$$\begin{aligned} \partial_t\left(\|u(t)\|^2_{L^2} + (b(-\Delta_x)^{-1/2}u(t), (-\Delta_x)^{-1/2}u(t))\right) + \\ + 2\left((a+a^*)\nabla_x u(t), \nabla_x u(t)\right) = -2\left(f(u(t)), u(t)\right) + 2\left(g(t), u(t)\right). \end{aligned} \tag{2.7}$$

Using that $a + a^* > 0$ and $b = b^* > 0$, the dissipativity assumption (0.3) on $f(u)$, Hölder and Friedrichs inequalities we derive that

$$\begin{aligned} \partial_t\left(\|u(t)\|^2_{L^2} + (b(-\Delta_x)^{-1/2}u(t), (-\Delta_x)^{-1/2}u(t))\right) + \\ + \varepsilon\|u(t)\|^2_{L^2} + \varepsilon\|u(t)\|^2_{H^1} \le C\left(1 + \|g\|^2_{L^2}\right). \end{aligned} \tag{2.8}$$

Applying Gronwall's lemma to (2.8) we obtain the estimate (2.5).

Let us prove now the estimate (2.6). We give below only a formal deriving of it which can be justified by Galerkin's approximation method.

Multiplying the equation (0.5) by $t\Delta_x u(t)$ integrating over $x \in \Omega$, we obtain after integration by parts that

$$\begin{aligned} \partial_t\left(t\|\nabla_x u(t)\|^2_{L^2} + t(bu(t), u(t))\right) - \\ - \|u(t)\|^2_{H^1} - (bu(t), u(t)) + t\left((a+a^*)\Delta_x u(t), \Delta_x u(t)\right) = \\ = -2t\left(f'(u)\nabla_x u(t), \nabla_x u(t)\right) - 2t(g, \Delta_x u(t)). \end{aligned} \tag{2.9}$$

Using now the quasimonotonicity assumption (0.3) on $f(u)$ and that $a + a^* > 0$ and $b = b^* > 0$, we obtain as in the proof of the previous estimate that

$$\begin{aligned} \partial_t\left(t\|u(t)\|^2_{H^1} + t(bu(t), u(t))\right) + \varepsilon(t\|u(t)\|^2_{H^1}) + \varepsilon(t\|u(t)\|^2_{H^2}) \le \\ \le C\left((t+1)\|u(t)\|^2_{H^1} + t\|g\|^2_{L^2}\right) \end{aligned} \tag{2.10}$$

for a sufficiently small positive ε. Applying Gronwall's inequality to (2.10) and using the estimate (2.5) for the integral of $\|u(t)\|^2_{H^1}$, we obtain after simple calculations the estimate (2.6).

Thus, it remains to prove the continuity of $u(t)$ with respect to t. Indeed, the fact $u \in C([0,T], L^2)$ follows from (2.3) and from the continuity of solutions u_n for (0.5) with $u_n(0) \in \mathbb{D}$ proved in Theorem 1.1 ($u_n \in C([0,T], L^2)$).

The weak continuity in H^1 for $t > 0$ follows from $u \in C([0,T], L^2) \cap L^\infty([t,T], H^1)$ (see [12] for instance). Theorem 2.1 is proved.

Now we are in a position to construct a compact attractor for the semigroup $\hat{S}_t$ in L^2. Let us remind that a set $\mathcal{A} \subset L^2$ is called an attractor for $\hat{S}_t : L^2 \to L^2$, if

1. The set $\mathcal{A}$ is compact in L^2.
2. The set $\mathcal{A}$ is strictly invariant with respect to $\hat{S}_t$, i.e.

$$\hat{S}_t\mathcal{A} = \mathcal{A} \text{ for } t \geq 0\,. \tag{2.11}$$

3. $\mathcal{A}$ is an attracting set for $\hat{S}_t$ in L^2. The latter means that for every neighborhood $\mathcal{O}(\mathcal{A})$ of the set $\mathcal{A}$ in L^2 and for every bounded subset $B \subset L^2$ there exists $T = T(B, \mathcal{O})$ such that

$$\hat{S}_t B \subset \mathcal{O}(\mathcal{A}) \text{ for every } t \geq T\,. \tag{2.12}$$

(See [1], [9], [16] for details).

Theorem 2.2. *Let the assumptions* (0.3) *be valid and let* $g \in L^2$. *Then the semigroup* $\hat{S}_t$, *defined by* (2.3), *possesses a compact attractor* $\mathcal{A} \subset\subset L^2$ ($\mathcal{A} \subset H^1$) *which has the following structure*

$$\mathcal{A} = \Pi_0\mathcal{K}\,, \tag{2.13}$$

where $\mathcal{K}$ *denotes the set of all complete bounded trajectories of the semigroup* $\hat{S}_t$:

$$\mathcal{K} = \{\hat{u} \in C_b(\mathbb{R}, L^2) \ : \hat{S}_h u(t) = u(t+h) \ \text{ for } t \in \mathbb{R},\ h \geq 0, \|u(t)\|_{L^2} \leq C_u\} \tag{2.14}$$

and $\Pi_0 u \equiv u(0)$.

Proof. According to the abstract attractor existence theorem (e.g. see [1]) it is sufficient to verify that

1. The operators $\hat{S}_t : L^2 \to L^2$ are continuous for every fixed $t \geq 0$.
2. The semigroup $\hat{S}_t$ possesses a compact attracting set $\mathbb{K}$ in L^2.

The continuity is an immediate corollary of (2.4). So it remains only to verify the existence of the attracting set.

The estimate (2.6) implies that the H^1-ball

$$\mathbb{K} \equiv \{v \in H^1(\Omega) : \|v\|_{H^1} \leq R\}$$

will be the attracting (and even the absorbing) set for the semigroup $\hat{S}_t$ in L^2, if R is large enough. Since $H^1 \subset\subset L^2$ then $\mathbb{K}$ is compact in L^2 and consequently the semigroup $\hat{S}_t$ possesses the attractor $\mathcal{A} \subset \mathbb{K} \subset H^1$. Theorem 2.2 is proved.

§3 The regularity of solutions.

Let us remind that in Section 1 we have proved that the problem (0.5) has a unique solution $u(t) = S_t u_0$ for every $u_0 \in \mathbb{D}$. Then in Section 2 we have extended by continuity the semigroup S_t from $\mathbb{D}$ to $\hat{S}_t : L^2 \to L^2$ and proved that the semigroup thus obtained possesses the attractor $\mathcal{A}$ in L^2. This Section studies the following three problems which naturally arise after proving the above results:

1. In what sense the 'solution' $u(t) = \hat{S}_t u_0$ satisfies the equation (0.5) if u_0 only belongs to L^2 (but not from $\mathbb{D}$).
2. Whether the attractor $\mathcal{A}$ belongs to the space $\mathbb{D}$.

3. Under what assumptions on f the semigroup $\hat{S}_t$ possesses the following smoothing property:

$$\hat{S}_t : L^2 \to \mathbb{D} \quad \text{for every } t > 0\,. \tag{3.1}$$

Note also that these problems occur to be closely connected with the problem of the finite dimension of the attractor $\mathcal{A}$ which will be considered in the next Section.

We start here with the most simple case where the nonlinear term f satisfies the following *growth* restriction:

$$|f(u)| \le C(1 + |u|^p) \quad \text{where } p \le p_{max} \equiv 1 + \frac{4}{n-4}\,, \tag{3.2}$$

if $n > 4$ and p is arbitrary if $n = 4$ (for $n \le 3$ we need not *any* growth restriction!). In this case one can easily verify (using Sobolev embedding theorem) that $f(v) \in L^2$ if $v \in H^2$. Thus,

$$\mathbb{D} = H^2(\Omega) \cap \{v\big|_{\partial\Omega} = 0\} \tag{3.3}$$

and therefore the nonlinearity $f(u)$ is subordinated to the linear term $\Delta_x u$.

Theorem 3.1. *Let the assumption* (3.2) *holds. Then the semigroup* $\hat{S}_t$ *possesses the smoothing property in the form of* (3.1) *and consequently for every* $u_0 \in L^2$ $u(t) = \hat{S}_t u_0$ *satisfies* (0.5) *in the sense of distributions. Moreover,*

$$\|u(1)\|^2_{\mathbb{D}} \le Q(\|u_0\|^2_{L^2} + \|g\|^2_{L^2}) \tag{3.4}$$

for a certain monotonous function Q *depending on* f *and therefore*

$$\mathcal{A} \subset \mathbb{D}\,. \tag{3.5}$$

Proof. Indeed, according to (2.6) $u \in L^2([s,T], H^2)$ for every $s > 0$. Hence due to Fubini's theorem $u(t) \in H^2$ for almost all $t \in \mathbb{R}_+$. Then, according to (3.3), $u(t) \in \mathbb{D}$ for almost all $t \in \mathbb{R}_+$. But Theorem 1.1 implies that $\hat{S}_t : \mathbb{D} \to \mathbb{D}$, therefore $u(t) \in \mathbb{D}$ for every $t > 0$. Let us prove now the estimate (3.4).

Indeed, according to (2.6),

$$\int_{1/2}^1 \|u(t)\|^2_{H^2} dt \le C(\|u(0)\|^2_{L^2} + 1 + \|g\|^2_{L^2})\,.$$

The latter means that there exists a point $t_0 \in [1/2, 1]$, such that

$$\|u(t_0)\|^2_{H^2} \le 2C(\|u(0)\|^2_{L^2} + 1 + \|g\|^2_{L^2}) \tag{3.6}$$

and hence, according to (3.2) and the embedding theorem

$$\|u(t_0)\|^2_{\mathbb{D}} \equiv \|u(t_0)\|^2_{H^2} + \|f(u(t_0))\|^2_{L^2} \le Q(\|u(t_0)\|^2_{H^2}) \tag{3.7}$$

for a certain monotonous function Q. The estimate (3.4) follows now from the inequality (2.2) with $u_0 = u(t_0)$ applied in the point $t = 1 - t_0$ and from the estimates (3.6) and (3.7).

Thus it remains to prove the embedding (3.5). But this fact is an immediate corollary of the estimates (2.5) and (3.4). Theorem 3.1 is proved.

Remark 3.1. Let $n \le 3$. Then Theorem 3.1 and the embedding theorem imply that under the assumptions of Section 2

$$\hat{S}_t : L^2(\Omega) \to C(\Omega) \;\text{ for } t > 0 \text{ and } \;\mathcal{A} \subset C(\Omega)\,. \tag{3.8}$$

Assume now that $n \ge 4$, (3.2) holds with $p < p_0$ and the right-hand side $g \in L^r(\Omega)$ for a some $r > \frac{n}{2}$. Then using the L^q-regularity theory for the heat equations (see [10]) one can derive that the assertions (3.8) remain valid in this case as well. Moreover the space C in (3.8) can be replaced by $H^{2,r} \subset\subset C$.

Note also that the growth condition (3.2) is essentially less restrictive than (0.4).

Let us consider now the case when the nonlinearity f is not subordinated to the linear part $\Delta_x u$.

Theorem 3.2. *Let the assumptions of Theorem 2.2 holds and let $u(t) = \hat{S}_t u_0$ with $u_0 \in L^2$. Then the function $f(u(t)).u(t)$ belongs to $L^1([0,T], L^1(\Omega))$ and satisfies the estimate*

$$\int_T^{T+1} \|f(u(t)).u(t)\|_{L^1} dt \le C\|u_0\|^2_{L^2} e^{-\varepsilon T} + C(1 + \|g\|^2_{L^2})\,, \tag{3.9}$$

for every $T \ge 0$.

Proof. Indeed, let $u(t) = L^2\text{–}\lim_{n\to\infty} u_n(t)$ where u_n be the solution of (0.5) with the initial condition $u_n(0) \in \mathbb{D}$. Multiply the equation (0.5) (with u replaced by u_n) by $u_n(t)$ and integrate over $(t,x) \in [T, T+1] \times \Omega$. We will have after evident transformations that

$$\begin{aligned} \int_T^{T+1} (f(u_n(t)), u_n(t))\, dt = 1/2 \left(\|u_n(T)\|^2_{L^2} - \|u_n(T+1)\|^2_{L^2}\right) + \\ 1/2 \Big((b(-\Delta_x)^{-1/2} u_n(T), (-\Delta_x)^{-1/2} u_n(T)) - \\ - (b(-\Delta_x)^{-1/2} u_n(T+1), (-\Delta_x)^{-1/2} u_n(T+1)) \Big) \\ - \int_T^{T+1} (a\nabla_x u_n(t), \nabla_x u_n(t)) dt + \int_T^{T+1} (g, u_n(t))\, dt\,. \end{aligned} \tag{3.10}$$

Inserting the estimate (2.5) into the right-hand side of (3.10) and using the fact that $f(u).u \ge -C$ we obtain

$$\int_T^{T+1} \|f(u_n(t)).u_n(t)\|_{L^1} dt \le C\|u_n(0)\|^2_{L^2} e^{-\varepsilon T} + C(1 + \|g\|^2_{L^2})\,. \tag{3.11}$$

Note that without loss of generality we may assume that $u_n \to u$ a.e. in $[T, T+1]\times\Omega$ and consequently $|f(u_n).u_n| \to |f(u).u|$ a.e. Passing now to the limit $n \to \infty$ in (3.11), we derive the estimate (3.9). Theorem 3.2 is proved.

Corollary 3.1. *Let the assumptions of Theorem 2.2 hold and let the function $f(u)$ satisfy the inequality*

$$|f(v)| \le C\left(|f(v).v| + 1 + |v|^2\right) \quad \text{for every } v \in \mathbb{R}^k\,. \tag{3.12}$$

Then 1. $f(u) \in L^1([0,T],L^1)$, 2. $\partial_t u \in L^1([s,T],L^1)$ for every $s>0$ and $u(t) = \hat{S}_t u_0$ satisfies (0.5) *in the sense of distributions.*

In general situations (where the estimate (3.12) is not assumed to be fulfilled) the function $u(t) = \hat{S}_t u_0$ can be interpreted as a unique solution of a variational inequality (see e.g. [2]) which corresponds to the system (0.5). In order to derive this inequality we assume that $u \in C_w([0,T],\mathbb{D})$ is a solution of (0.5) and $v \in C_w([0,T],\mathbb{D}) \cap C^1_w([0,T],L^2)$ be an arbitrary test function. Multiply now the equation (0.5) by $u(t)-v(t)$ and intergate over $[0,T]\times\Omega$. Then, integrating by parts and using the inequalities

$$(f(u)-f(v),u-v) \ge K\|u-v\|^2_{L^2}, \quad -(a\Delta_x u - \Delta_x v, u-v) \ge 0\,,$$

we derive the following inequality:

(3.13)

$$1/2\left((1+b(-\Delta_x)^{-1})u(t),u(t)) - 2v(t)\right)\Big|_{t=0}^{t=T} + \int_0^T ((1+b(-\Delta_x)^{-1})u(t),\partial_t v(t))\,dt \le$$
$$\le \int_0^T (a\Delta_x v(t) - f(v(t)) + g(t), u(t)-v(t))\,dt + K\int_0^T \|u(t)-v(t)\|^2_{L^2}\,dt\,.$$

Approximating now a 'solution' $u(t) = \hat{S}_t u_0$ by $u_n(t) := S_t u_0^n$, $u_0^n \in \mathbb{D}$, and $u_0^n \to u_0$ in L^2 and passing to the limit $n\to\infty$ in the inequalities (3.13) for the solutions u_n, we derive that $u(t) \in C([0,T],L^2(\Omega))$ also satisfies (3.13). The following Theorem shows that this property characterises the 'solution' $u(t)$.

Theorem 3.3. *Let the above assumptions hold and let $u \in C([0,T_1],L^2(\Omega))$ satisfy the inequality* (3.13) *for every $T \in [0,T_1]$ and every test function*

$$v \in C_w([0,T_1],\mathbb{D}) \cap C^1_w([0,T_1],L^2(\Omega))\,,$$

then $u(t) = \hat{S}_t u_0$.

Proof. Indeed, let $v_0^n \in \mathbb{D}$, $v_0^n \to u_0 := u(0)$ in $L^2(\Omega)$ and $v_n(t) := S_t v_0^n$. Then by definition $v_n(t) \to v(t)$ in $C([0,T_1],L^2(\Omega))$, where $v(t) := \hat{S}_t u_0$. Let us prove that $u(t) \equiv v(t)$. Indeed, taking $v_n(t)$ as a test function in (3.13), we derive using the equation (0.5) for v_n and integration by parts that

$$\left((1+b(-\Delta_x)^{-1})(u(t)-v_n(t)),u(t)-v_n(t)\right)\Big|_{t=0}^{t=T} \le 2K\int_0^T \|u(t)-v_n(t)\|^2_{L^2}\,dt\,.$$

Passing to the limit $n\to\infty$ in this inequality and using that $u(0)=v(0)$ and that the matrix b is positive we obtain that

$$\|u(T)-v(T)\|^2_{L^2} \le 2K\int_0^T \|u(t)-v(t)\|^2_{L^2}\,dt\,.$$

Gronwall's inequality implies now that $u(t) \equiv v(t)$. Theorem 3.3 is proved.

Remark 3.2. Approximating the test function $v(t)$ in (3.13) by piecewise constant with respect to t ones, one can establish that instead of the inequality (3.13) it is sufficient to require that for every $v_0 \in \mathbb{D}$ and every $0 \le \tau \le T \le T_1$

$$(3.14)\quad 1/2\left((1+b(-\Delta_x)^{-1})u(t), u(t)-2v_0\right)\Big|_{t=\tau}^{t=T} \le$$
$$\le \int_\tau^T (a\Delta_x v_0 - f(v_0) + g, u(t) - v_0)\, dt + K\int_\tau^T \|u(t)-v_0\|^2_{L^2}\, dt$$

(which is similar to the standard variational inequalities for monotone operator theory, see [2]).

Now we are going to study the smoothing properties of (0.5) for the case where the main part of the nonlinearity f has a gradient structure.

Theorem 3.4. *Let the assumptions of Theorem 2.2 be valid and let the function f have the structure*

$$(3.15)\qquad f(v) = f_1(v) + f_2(v)\,,$$

where the function f_1 also satisfies (0.3) *and $f_1(v) = \nabla_v F(v)$, and the function f_2 be subordinated to f_1 in the following sense*

$$(3.16)\qquad |f_2(v)|^2 \le C_1 F(v) + C_2\left(1+|v|^2\right)\,.$$

Then the semigroup $\hat{S}_t$, defined by (2.3)*, maps L^2 to $\mathbb{D}$ for every $t>0$. Moreover,*

$$(3.17)\qquad \|u(t)\|^2_{\mathbb{D}} \le C\frac{1+t^2}{t^2}\left(\|u(0)\|^2_{L^2}e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right)$$

and therefore $\mathcal{A} \subset \mathbb{D}$.

The proof of this theorem is based on a number of lemmata.

Lemma 3.1. *Under the assumptions of Theorem 3.3 the following estimate is valid:*

$$(3.18)\qquad -C\left(1+\ln(|v|+1)\right) \le F(v) \le C\left(|f(v).v| + 1 + |v|^2\right)\ \textit{for every } v \in \mathbb{R}^k$$

and consequently

$$(3.19)\qquad \int_T^{T+1} \|f_2(u(t))\|^2_{L^2} + \|F(u(t))\|_{L^1}\, dt \le C\|u(0)\|^2_{L^2}e^{-\varepsilon t} + C(1+\|g\|^2_{L^2})\,.$$

The proof of this lemma is given in [18].

Lemma 3.2. *Let the assumptions of Theorem 3.3 hold. Then for $T>0$*

$$(3.20)\qquad \int_T^{T+1} \|\partial_t u(t)\|^2_{L^2}\, dt \le C\frac{T+1}{T}\left(\|u(0)\|^2_{L^2}e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right)\,.$$

Proof. Let us multiply the equation (0.5) by $t\partial_t u(t)$ and integrate over $t \in [0, 2]$:

$$\int_0^2 t\left(\|\partial_t u(t)\|^2_{L^2} + (b(-\Delta_x)^{-1/2}u(t), (-\Delta_x)^{-1/2}u(t))\right)\,dt = \tag{3.21}$$

$$= \int_0^2 (a\Delta_x u(t), t\partial_t u(t))\,dt - 2F(u(2)) +$$

$$+ \int_0^2 F(u(t))\,dt - \int_0^2 t(f_2(u(t)), \partial_t u(t))\,dt + \int_0^2 t(g, \partial_t u)\,dt\,.$$

Applying the Hölder inequality together with (2.5) and (3.19) to the right-hand side of (3.21), we deduce that

$$\int_0^2 t\|\partial_t u(t)\|^2_{L^2}\,dt \le C\left(\int_0^2 t\|\Delta_x u(t)\|^2_{L^2}\,dt + 1 + \|g\|^2_{L^2} + \|u_0\|^2_{L^2}\right).$$

Arguing as in the proof of estimate (2.6) one can easily derive that

$$\int_0^2 t\|\Delta_x u(t)\|^2_{L^2}\,dt \le C\left(\|u_0\|^2_{L^2} + 1 + \|g\|^2_{L^2}\right) \tag{3.22}$$

and therefore

$$\int_0^2 t\|\partial_t u(t)\|^2_{L^2}\,dt \le C_1\left(1 + \|g\|^2_{L^2} + \|u_0\|^2_{L^2}\right). \tag{3.23}$$

Note that the estimate (3.23) implies (3.20). Indeed, for $T \le 1$ we derive from (3.23) that

$$T\int_T^{T+1} \|\partial_t u(t)\|^2_{L^2}\,dt \le C_1\left(1 + \|g\|^2_{L^2} + \|u_0\|^2_{L^2}\right).$$

And if $T \ge 1$, then according to (3.23) and (2.5)

$$\int_T^{T+1} \|\partial_t u(t)\|^2_{L^2}\,dt \le C(\|u(T-1)\|^2_{L^2} + 1 + \|g\|^2_{L^2}) \le C_1\left(\|u(0)\|^2_{L^2}e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right)$$

Lemma 3.2 is proved.

Lemma 3.3. *Let the assumptions of Theorem 3.3 hold. Then for* $t > 0$

$$\|\partial_t u(t)\|^2_{L^2} \le C\frac{1+t^2}{t^2}\left(\|u(0)\|^2_{L^2}e^{-\varepsilon t} + 1 + \|g\|^2_{L^2}\right). \tag{3.24}$$

Proof. Let us differentiate the equation (0.5) with respect to t and denote $\theta(t) = \partial_t u(t)$. We will obtain the equation

$$\partial_t\left(1 + b(-\Delta_x)^{-1}\right)\theta(t) = a\Delta_x\theta(t) - f'(u(t))\theta(t)\,. \tag{3.25}$$

Multiplying the equation (3.25) by $t^2\partial_t\theta$ and using the monotonicity assumption on f, we derive that

$$\partial_t\left(t^2\|\theta(t)\|^2_{L^2} + t^2(b(-\Delta_x)^{-1/2}\theta(t), (-\Delta_x)^{-1/2}\theta(t))\right) - 2t\|\theta(t)\|^2_{L^2} \le \tag{3.26}$$

$$\le -t^2((a+a^*)\partial_t\theta, \partial_t\theta) + 2t(b(-\Delta_x)^{-1/2}\theta(t), (-\Delta_x)^{-1/2}\theta(t)) + 2Kt^2\|\theta(t)\|^2_{L^2}$$

and therefore

$$\partial_t \left(t^2 \|\theta(t)\|^2_{L^2} + (b(-\Delta_x)^{-1/2}\theta(t), (-\Delta_x)^{-1/2}\theta(t)) \right) + \\ + \varepsilon \left(t^2 \|\theta(t)\|^2_{L^2} \right) \le Ct(t+1)\|\partial_t u(t)\|^2_{L^2}\,. \tag{3.27}$$

Applying Gronwall's inequality to the estimate (3.27) and using the estimate (3.23) for $\partial_t u(t)$ in the right-hand side of (3.27), we obtain the assertion of the lemma.

Now we are in a position to complete the proof of the Theorem. Indeed, the estimate (3.24) inserted in (1.10) gives us that

$$\|u(t)\|^2_{H^2} \le C\frac{1+t^2}{t^2}\left(\|u(0)\|^2_{L^2} e^{-\varepsilon t} + 1 + \|g\|^2_{L^2} \right)\,.$$

Inserting this estimate into (1.12) we derive the analogous estimate for the norm of $f(u(t))$. Theorem 3.4 is proved.

Remark 3.3. The model example of the nonlinearity $f(u)$ for which the assumptions of previous Theorem hold is the following:

$$f_1(u) = (a_1u_1|u_1|^{p_1}, \cdots, a_ku_k|u_k|^{p_k}) \quad , \quad f_2(u) = Lu\,, \tag{3.28}$$

where $a_i > 0$, $p_i > 0$ and L is and arbitrary linear operator ($L \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$).

§4 The dimension of the attractor

In this Section we prove that under some additional assumptions on the nonlinear term $f(u)$ the attractor $\mathcal{A}$ of the equation (0.5) has a finite fractal dimension. Note that the usual way of estimating the fractal dimension of invariant sets involving the Liapunov exponents and k-contraction maps (see for instance [16]) requires the semigroup to be quasidifferentiable with respect to the initial data on the attractor. But in our case where $f(u)$ is not subordinated to the linear part $\Delta_x u$ (in the sense of (3.3)) we were able to prove only that $\mathcal{A} \subset \mathbb{D}$ (under the assumptions of previous Section) which is not sufficient to obtain the differentiability. To avoid this difficulty we use below a another scheme of estimating the dimension of invariant sets introduced in [18] which works without the differentiability assumptions.

First of all we remind here the definition and the simplest properties of the fractal dimension (see [16] for further details).

Definition 4.1. Let X be a metric space and let M be a precompact set in X. Then, according to Hausdorff's criterium the set M can be covered by a finite number of ε-balls in X for every $\varepsilon > 0$. Denote by $N_\varepsilon(M, X)$ the minimal number of ε-balls in X which cover M. Then the Kolmogorov entropy of the set M in X is defined to be the following number

$$\mathcal{H}_\varepsilon(M, X) \equiv \log_2 N_\varepsilon(M, X) \tag{4.1}$$

and the fractal (entropy, box-counting) dimension of M can be defined in the following way

$$d_F(M) = d_F(M, X) = \limsup_{\varepsilon \to 0} \frac{\mathcal{H}_\varepsilon(M, X)}{\log_2 \frac{1}{\varepsilon}}\,. \tag{4.2}$$

The following properties of the fractal dimension can be easily deduced from it's definition:

Proposition 4.1. *1. Let M be a compact k dimensional Lipschitz manifold in X. Then $d_F(M,X) = k$.*

2. Let X and Y be metric spaces $M \subset X$ and $L : X \to Y$. Assume that the map L is globally Lipschitz continuous on M. Then

$$d_F(L(M), Y) \le d_F(M, X)\,. \tag{4.3}$$

Particularly, the fractal dimension preserves under Lipschitz continuous homeomorphisms.

The following Theorem is of fundamental significance in our study the dimension of attractors.

Theorem 4.1. *Let H_1 and H be Banach spaces, H_1 be compactly embedded into H and let $K \subset\subset H$. Assume that there exists a map $L : K \to K$, such that $L(K) = K$ and the following 'smoothing' property is valid*

$$\|L(k_1) - L(k_2)\|_{H_1} \le C\|k_1 - k_2\|_H \tag{4.4}$$

for every $k_1, k_2 \in K$. Then the fractal dimension of K in H is finite and can be estimated in the following way:

$$d_F(K, H) \le \mathcal{H}_{1/4C}(B(1, 0, H_1), H)\,, \tag{4.5}$$

where C is the same as in (4.4) and $B(1, 0, H_1)$ means the unit ball in the space H_1.

The proof of this Theorem is given in [18].

Now we are ready to formulate the main result of this Section.

Theorem 4.2. *Let the assumptions of Theorem 2.2 hold and let $\mathcal{A}$ be the attractor of the equation (0.5). Assume that for a sufficiently small $\delta > 0$ the following regularity assumption is valid*

$$\|f'(\alpha u_0 + (1-\alpha)u_1)\|_{L^{2-\delta}(\Omega)} \le C \tag{4.6}$$

uniformly with respect to $u_0, u_1 \in \mathcal{A}$ and $\alpha \in [0, 1]$. Then the fractal dimension of the attractor $\mathcal{A}$ is finite.

$$d_F(\mathcal{A}, L^2(\Omega)) < \infty\,. \tag{4.7}$$

We are going to apply Theorem 4.1. In order to do so we need some estimates for the difference $v(t) = u_1(t) - u_2(t)$ between two solutions u_1 and u_2 belonging to the attractor.

Lemma 4.1. *Let the assumptions of the theorem hold and let $\varepsilon > 0$ and $\delta > 0$ satisfies the condition*

$$0 < k(\varepsilon, \delta) \equiv \frac{4\varepsilon + \delta - \varepsilon\delta}{1 - (\varepsilon + \delta)} \le \frac{4}{n-2}\,.$$

Then the following estimate is valid:

$$\|\partial_t v\|_{L^{1+\varepsilon}([1,2], L^{1+\varepsilon}(\Omega))} + \|v\|_{L^2([1,2], H^1(\Omega))} \le C\|v(0)\|_{L^2}\,. \tag{4.8}$$

Proof. Recall, that the function $v(t)$ satisfies the equation

$$\partial_t \left(1 + b(-\Delta_x)^{-1}\right) v(t) = a\Delta_x v - l(t)v \ , v\big|_{\partial\Omega} = 0 \tag{4.9}$$

with $l(t) = \int_0^1 f'(su_1(t) + (1-s)u_2(t))\, ds$. Since $u_1(t), u_2(t) \in \mathcal{A}$, the assumption (4.6) implies that

$$\|l(t)\|_{L^{2-\delta}} \le C_1 \,. \tag{4.10}$$

Let us estimate the $L^{1+\varepsilon}$-norm of the function $h_v(t) = l(t)v(t)$ using Hölder inequality, the estimate (4.10), and Sobolev's embedding theorem $H^1 \subset L^p$ if $p \le 2 + \frac{4}{n-2}$:

$$\|h_v(t)\|_{L^{1+\varepsilon}} \le \|l(t)\|_{L^{2-\delta}} \|v(t)\|_{L^{2+k(\varepsilon,\delta)}} \le C_2 \|v(t)\|_{H^1} \,. \tag{4.11}$$

It follows from the estimates (2.4) and (4.14) that

$$\|h_v\|_{L^{1+\varepsilon}([0,2],L^{1+\varepsilon})} \le C_3 \|v(0)\|_{L^2} \,. \tag{4.12}$$

Let us rewrite (4.9) as a linear non-homogeneous problem in Ω

$$\partial_t \left(1 + b(-\Delta_x)^{-1}\right) v = a\Delta_x v - h_v(t) \,. \tag{4.13}$$

Then according to the $L^{1+\varepsilon}$-regularity theorem for the linear equation (4.16) (this theorem for $b > 0$ can be easily deduced from the one for a standard parabolic equation using the compact perturbations arguments) and using the smoothing property for the corresponding homogeneous problem (see for instance [10]), we derive that

$$\begin{aligned} \|\partial_t v\|_{L^{1+\varepsilon}([1,2],L^{1+\varepsilon})} + \|\Delta_x v\|_{L^{1+\varepsilon}([1,2],L^{1+\varepsilon})} \le \\ \le C \left(\|v(0)\|_{L^{1+\varepsilon}} + \|h_v\|_{L^{1+\varepsilon}([0,2],L^{1+\varepsilon})}\right) \le C_4 \|v(0)\|_{L^2} \,. \end{aligned} \tag{4.14}$$

The estimate (4.14) together with (2.4) completes the proof of Lemma 4.1.

Lemma 4.2. *Let the assumptions of the previous lemma hold. Then*

$$\|v(1)\|^2_{L^2} \le C \int_0^1 \|v(t)\|^2_{L^2}\, dt \,. \tag{4.15}$$

Proof. Indeed, multiplying the equation (4.12) by $tv(t)$ and integrating over $x \in \Omega$, we obtain using the fact that $l(t) \ge -K$

$$\begin{aligned} \partial_t \left(t\|v(t)\|^2_{L^2} + t(b(-\Delta_x)^{-1/2} v(t), (-\Delta_x)^{-1/2} v(t))\right) - \\ - 2K \left(t\|v(t)\|^2_{L^2}\right) \le C\|v(t)\|^2_{L^2} \,. \end{aligned} \tag{4.16}$$

Applying Gronwall's inequality to the estimate (4.16), we obtain the assertion of the lemma.

Thus, combining the results of Lemmata 4.1 and 4.2, we derive that

$$\|\partial_t v\|_{L^{1+\varepsilon}([2,3],L^{1+\varepsilon}(\Omega))} + \|v\|_{L^2([2,3],H^1(\Omega))} \le C\|v\|_{L^2([0,1],L^2)} \,. \tag{4.17}$$

Now we are in the position to complete the proof of the theorem. To this end we introduce the space

$$\mathcal{W} = \{u \in L^2([0,1], H^1) : \partial_t u \in L^{1+\varepsilon}([0,1], L^{1+\varepsilon})\}. \tag{4.18}$$

It is known (see [13]) that the space $\mathcal{W}$ is compactly embedded into $L^2([0,1], L^2)$.

Let us consider the restriction $\mathcal{K}\big|_{[0,1]}$ of the kernel $\mathcal{K}$, defined by (2.14) and the map

$$L : \mathcal{K}\big|_{[0,1]} \to \mathcal{K}\big|_{[0,1]}, \ (Lu)(t) = \hat{S}_2 u(t). \tag{4.19}$$

Since the attractor is strictly invariant with respect to $\hat{S}_t$, we have

$$L\left(\mathcal{K}\big|_{[0,1]}\right) = \mathcal{K}\big|_{[0,1]}$$

and due to (4.17),

$$\|L(u_1) - L(u_2)\|_{\mathcal{W}} \le C\|u_1 - u_2\|_{L^2([0,1],L^2)}.$$

Consequently, according to Theorem 4.1,

$$d_F\left(\mathcal{K}\big|_{[0,1]}, L^2([0,1], L^2(\Omega))\right) < \infty. \tag{4.20}$$

The finite dimensionality of $\mathcal{A}$ in $L^2(\Omega)$ is an immediate corollary of (4.20), (4.15) and the second assertion of Proposition 4.1. Theorem 4.2 is proved.

Thus, we have proved that the attractor is finite dimensional under the regularity assumption (4.6). But it is still not clear how to verify this condition in applications. The following corollary gives an answer on this question.

Corollary 4.1. *Let the attractor $\mathcal{A}$ be bounded in $\mathbb{D}$ (for instance let the assumptions of Theorem 3.3 be valid). Let us assume also that there exists a convex function $\Psi : \mathbb{R}^k \to \mathbb{R}_+$, such that*

$$K_2\Psi(v) - C_2 \le \|f'(v)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} \le K_1\Psi(v) + C_1, \ \forall v \in \mathbb{R}^k, \tag{4.21}$$

where $K_i > 0$. Moreover, it is assumed that the derivative f' satisfies the estimate

$$\|f'(v)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} \le C(|f(v)|^{1+\beta} + 1) \tag{4.22}$$

for a sufficiently small $\beta > 0$ and every $v \in \mathbb{R}^k$. Then the assumption (4.6) is satisfied and consequently the attractor $\mathcal{A}$ has a finite fractal dimension.

Indeed, since the function Ψ is convex, we have

$$\begin{aligned} \|f'(\alpha v_1 - (1-\alpha)v_2)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} &\le K_1\alpha\Psi(v_1) + K_1(1-\alpha)\Psi(v_2) + C_2 \le \\ &\le \frac{K_1}{K_2}\left(\alpha\|f'(v_1)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} + (1-\alpha)\|f'(v_2)\|_{\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)} + C\right) \end{aligned} \tag{4.23}$$

for every $v_1, v_2 \in \mathbb{R}^k$ and $\alpha \in [0,1]$. Thus, (4.6) is fulfilled if

$$\|f'(u_0)\|_{L^{2-\delta}(\Omega)} \le C \ \text{ for every } u_0 \in \mathcal{A}. \tag{4.24}$$

In order to verify the assumption (4.24) we use the estimate (4.22). Indeed, according to (4.22) and due to the fact that $\mathcal{A}$ is bounded in $\mathbb{D}$

$$\|f'(u_0)\|^{2-\delta}_{L^{2-\delta}} \le C(\|f(v)\|^2_{L^2} + 1) \le C(\|u\|^2_{\mathbb{D}} + 1) \le C_1$$

for $\delta = 2 - \frac{2}{1+\beta}$. Corollary 4.1 is proved.

Remark 4.1. Since the solutions of the equation $y' = y^{1+\beta}$ blow up in finite time, (4.22) is not a growth restriction but only some kind of regularity assumption.

§5 The case of Newmann boundary conditions.

In this Section we briefly consider the 4-order parabolic system of the Cahn-Hilliard type with *Neumann* boundary conditions:

$$
(5.1)\qquad \begin{cases} b\partial_t u = -\Delta_x\left(a\Delta_x u - \partial_t u - f(u) + \tilde{g}\right) , \\ \partial_n u\big|_{\partial\Omega} = \partial_n \Delta_x u\big|_{\partial\Omega} = 0 \, , \\ u\big|_{t=0} = u_0 \, . \end{cases}
$$

It is assumed as before that $u = (u^1, \cdots, u^k)$ is a vector-valued unknown function, $b = b^* > 0$ and a, $a + a^* > 0$ are given. As in the case of Dirichlet boundary conditions the solution u of the equation (5.1) is defined to be a function $u \in C_w([0,T], D(A)) \cup C^1_w([0,T], L^2(\Omega))$, where

$$
D(A) := \{u \in W^{2,2}(\Omega) : \quad \partial_n u\big|_{\partial\Omega} = 0, \quad f(u) \in L^2(\Omega)\}
$$

and the equality (5.1) should be understood in the variational sence (see e.g. [16]).

The main difference to the case of Dirichlet boundary conditions considered before is the fact that the Laplacian Δ_x with Newmann boundary conditions has zero eigenvalue, moreover, the multiplicity of zero eigenvalue is equal to k in the case of systems. This leads to appearing of k concervation laws for the initial system (5.1) and to some additional conditions to the external force $\tilde{g}$. Indeed, integrating the equation (5.1) over $x \in \Omega$ we obtain after the standard integration by parts that

$$
(5.2)\qquad b\partial_t \int_{x\in\Omega} u(t)\, dx = \int_{\partial\Omega} \partial_n \tilde{g}\, dS \, .
$$

The equality (5.2) shows that one can expect the bounedness of solution $u(t)$ (even in $L^1(\Omega)$-norm) only if the right-hand side of (5.2) equals to zero, i.e. we should impose the following restrictions on $\tilde{g}$:

$$
(5.3)\qquad \int_{\partial\Omega} \partial_n \tilde{g}^i(x)\, dS = 0, \quad i = 1, \cdots, k \;\; (\tilde{g} = (\tilde{g}^1, \cdots, \tilde{g}^k)) \, .
$$

It will be assumed everywhere below that the condition (5.3) is satisfied. Therefore, the relation (5.2) gives us k conservation laws (b is invertible!):

$$
(5.4)\qquad \left\langle u^i(t)\right\rangle = \left\langle u^i(0)\right\rangle := m^i, \quad i = 1, \cdots, k, \quad \langle f\rangle := \frac{1}{|\Omega|}\int_{x\in\Omega} f(x)\, dx \, .
$$

Thus, it seems resonable to consider the restrictions of the dynamical system, generated by (5.1) to invariant surfaces

$$
(5.5)\qquad \mathcal{T}_m := \{u \in D(A) \, : \left\langle u^i\right\rangle = m^i, \quad i = 1, \cdots, k\}, \quad m := (m^1, \cdots, m^k)
$$

and to study the dynamics on these surfaces.

Our task now is to rewrite the equation (5.1) in the form of (0.5) (as we have done for the case of Dirichlet boundary condtions). Note however that the Laplace operator with Neumann boundary conditions is not invertible and a priori it is not clear how to do so.

Define now the inverse operator $L := (-\Delta_x)^{-1}$ by the following formula: $w = Lv$ is a unique solution of the following equation

$$\Delta_x w = v, \quad \partial_n w\big|_{\partial\Omega} = 0, \quad \langle w\rangle = 0\,, \tag{5.6}$$

which is defined on $L^2(\Omega) \cap \{\langle v\rangle = 0\}$ (it is well known that this definition is correct). Let us apply the operator L to both sides of the equation (5.1) (it is possible to do because $\langle b\partial_t u\rangle = 0$ due to the conservation laws). Then we derive after the standard computations that the solution u should satisfy the following equation:

$$\begin{cases} (1+bL)\partial_t u = a\Delta_x u - f(u) + \langle f(u)\rangle + g, \\ \partial_n u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0\,, \end{cases} \tag{5.7}$$

where the function $g = \tilde g - \langle \tilde g\rangle - G$ and G is a solution of the following non-homogeneous Neumann boundary problem

$$\Delta_x G = 0, \quad \partial_n G\big|_{\partial\Omega} = \partial_n \tilde g\big|_{\partial\Omega}, \quad \langle G\rangle = 0\,, \tag{5.8}$$

which is uniquely solvable due to (5.3).

The obtained equation (5.7) is a compact perturbation (by the term $bL\partial_t u$) of the mass-preserving Allen–Cahn equation. Thus, in contrast to the case of Dirichlet boundary conditions we will have now the additional non-local term $\langle f(u)\rangle$ in the right-hand side of (5.7). Note however that this term is also compact (and even one dimensional) perturbation and the methods applied above to the case of Dirichlet boundary conditions should work (after minor changing) in this situation as well. The main aim of this Section is to verify that it is really so.

It is assumed that the non-linear term $f(u)$ satisfied the assumptions (0.3) (as before) and also the following additional condition: for every $\mu > 0$ there is a constant C_μ, such that

$$|f(u)| \le \mu |f(u).u| + C_\mu, \quad \forall u \in \mathbb{R}^k\,. \tag{5.9}$$

Note, that the assumption (5.9) is always true in the scalar case ($k = 1$) and looks not very restrictive even in the case of systems ($k \ge 2$). We need this additional assumptions in order to obtain the appropriate estimates for the non-local term $\langle f(u)\rangle$.

We assume also that the external force $g \in L^2(\Omega)$ and has a zero mean value $\langle g\rangle = 0$ (which is necessary in order to obtain the conservation laws (5.4)).

The analogue of Theorem 1.1 for this case is the following one.

Theorem 5.1. *Let the above assumptions hold. Then for every $m \in \mathbb{R}^k$ and every $u_0 \in \mathcal{T}_m$ the problem (5.7) possesses a unique solution which satisfies the following estimate:*

$$\|u(t)\|_{D(A)} \le C_m \|u(0)\|^2_{D(A)} e^{2Kt} + C_m(1 + \|g\|^2_{0,2})\,, \tag{5.10}$$

where the constant C_m depends only on m and is independent of g and u_0. Moreover, if $u_0 \in \mathcal{T}_m$, then $u(t) \in \mathcal{T}_m$ for every $t \ge 0$.

Proof. Indeed, in order to verify the invariance of $\mathcal{T}_m$ it is sufficient to integrate the equation (5.7) over $x \in \Omega$ and to take into account the assumption $\langle g\rangle = 0$ and the fact that, by definition of the operator L, $\langle L\partial_t u\rangle = 0$.

The proof of the estimate (5.10) is completely analogous to the proof of Theorem 1.1 because $\langle \Delta_x u\rangle = 0$ and $\langle \partial_t u\rangle = 0$. Consequently the non-local term $\langle f(u)\rangle$ will disappear in the relations (1.3), (1.7) and (1.10) and therefore repeating word by word the proofs of Lemmata 1.1–1.3 we derive the estimates (1.2), (1.5) and (1.8) for the case of equation (5.7).

The only problem is to obtain the estimate for the non-local term $\langle f(u)\rangle$ which is necessary in order to obtain the estimate $\|f(u)\|_{0,2}$ expressing $f(u)$ from the equation (5.7) (as in (1.12)). Thus, we restrict ourselves to give only the proof of such estimate.

Lemma 5.1. *Let the above assumptions hold. Then*

$$|\langle f(u(t))\rangle| \le C_m\|u(0)\|^2_{D(A)}e^{2Kt} + C_m(\|g\|^2_{0,2}+1)\,, \tag{5.11}$$

where $\alpha > 0$ is an appropriate positive constant.

Proof. Indeed, taking the inner product in $\mathbb{R}^k$ of the equation (5.7) with the function $u(t)$, integrating over $x \in \Omega$ and taking into account that $\langle u(t)\rangle = m$, we derive the relation

$$\begin{aligned}((1+bL)\partial_t u(t), u(t)) = -(a\nabla_x u(t), \nabla_x u(t)) - \\ -(f(u(t)), u(t)) + (g, u(t)) + m.\langle f(u(t))\rangle\,.\end{aligned} \tag{5.12}$$

Using the estimates (1.2) and (1.8) one can easily derive from (5.12) that

$$\begin{aligned}\int_\Omega |f(u(t)).u(t)|\,dx \le |m|\cdot|\Omega|\int_\Omega |f(u(t))|\,dx + \\ + C_m\|u_0\|^2_{D(A)}e^{2Kt} + C_m(1+\|g\|^2_{0,2})\,.\end{aligned} \tag{5.13}$$

Applying the assumption (5.9) in order to estimate the first term in the right-hand side of (5.13) and taking $\mu < 1/(2|m|\cdot|\Omega|)$, we derive that

$$\int_\Omega |f(u(t).u(t)|\,dx \le C'_m\|u_0\|^2_{D(A)}e^{2Kt} + C'_m(1+\|g\|^2_{0,2})\,. \tag{5.14}$$

The estimate (5.14) together with the assumption (5.9) imply the estimate (5.11). Lemma 5.1 is proved.

Having the estimate (5.11) together with (1.5) and (1.8), we may express the value of $f(u)$ and obtain as before the estimate for $\|f(u)\|_{0,2}$:

$$\|f(u(t))\|_{0,2} \le C_m\|u_0\|^2_{D(A)}e^{2Kt} + C_m(1+\|g\|^2_{0,2})\,. \tag{5.15}$$

The estimates (1.8) and (5.15) imply (5.10). The existence of a solution can be easily proved basing on the estimate (5.10) by the Galerkin method. The uniqueness of a solution follows from Lemma 1.4, which can be reproved word by word for the case of Neumann boundary conditions (since $\langle u_1 - u_2\rangle = 0$ the non-local term will disappear again). Theorem 5.1 is proved.

Thus, the semi-group $S_t^{(m)} : \mathcal{T}_m \to \mathcal{T}_m$, generated by the equation (5.7)

$$u(t) := S_t^{(m)}u_0, \quad u_0 \in \mathcal{T}_m \tag{5.16}$$

is well defined for every $m \in \mathbb{R}^k$.

As in the case of Dirichlet boundary conditions we may extend in a unique way the semi-group $S_t^{(m)} : \mathcal{T}_m \to \mathcal{T}_m$ to the semi-group $\hat{S}_t^{(m)} : \hat{\mathcal{T}}_m \to \hat{\mathcal{T}}_m$ (by the formulae (2.3)), where

$$\hat{\mathcal{T}}_m := [\mathcal{T}_m]_{L^2(\Omega)} = L^2(\Omega) \cap \{\langle u_0 \rangle = m\}\,. \tag{5.17}$$

(As in the case of Dirichlet boundary conditions the weak solutions $\hat{u}(t) := \hat{S}_t^{(m)} u_0$ of (5.7) for $u_0 \in \hat{\mathcal{T}}_m$ can be characterized in terms of the appropriate variational inequality (see (3.13) and (3.14)).

The analogues of Theorem 2.1 and 2.2 for the case of Neumann boundary conditions will be the following one.

Theorem 5.2. *Let the above assumptions hold. Then for every $u_0 \in \hat{\mathcal{T}}_m$ a weak solution $\hat{u}(t)$ of the equation (5.7) belongs to $C([0,T], L^2(\Omega)) \cap C_w([\tau,T], W^{1,2}(\Omega))$ for every $\tau > 0$ and the estimates (2.5) and (2.6) hold with constants C, C_1, C_2 depending on m. Moreover, for every $m \in \mathbb{R}^k$ the semi-group $\hat{S}_t^{(m)} : \hat{\mathcal{T}}_m \to \hat{\mathcal{T}}_m$ possesses a global attractor $\mathcal{A}^{(m)} \in \hat{\mathcal{T}}_m \cap W^{1,2}(\Omega)$.*

Proof. The result of this theorem can be obtained analogously to Theorem 2.1 and 2.2. That is why we restrict ourselves to derive only the dissipative estimate (2.5).

Let $u(t) = v(t) + m$. Then $\langle v(t) \rangle = 0$ and the left-hand side of (5.12) can be rewritten in the following way:

$$\begin{aligned} &((1+bL)\partial_t u(t), u(t)) = ((1+bL)\partial_t v(t), v(t)+m) = \\ &\quad = ((1+bL)\partial_t v(t), v(t)) = 1/2\partial_t[\|v(t)\|_{0,2}^2 + (bL^{1/2}v(t), L^{1/2}v(t))]\,. \end{aligned} \tag{5.18}$$

Using the facts that $a + a^* > 0$, $\langle v \rangle = 0$ and the operator L is bounded, we can estimate the first term in the right-hand side of (5.12) in the following way:

$$\begin{aligned} (a\nabla_x u, \nabla_x u) &= (a\nabla_x v, \nabla_x v) \ge 2\alpha \|\nabla_x v\|_{0,2}^2 \ge \alpha\|\nabla_x v\|_{0,2}^2 + \beta\|v\|_{0,2}^2 \ge \\ &\ge \alpha\|\nabla_x v\|_{0,2}^2 + \beta'\left(\|v\|_{0,2}^2 + (bL^{1/2}v, L^{1/2}v)\right) \end{aligned} \tag{5.19}$$

for some positive constants $\alpha, \beta, \beta' > 0$.

The non-linear terms in the right-hand side of (5.12) can be estimated in a standard way using the second assumption of (0.3) and the assumption (5.9) (see also the proof of Lemma 5.1)

$$-(f(u(t)).u(t)) + m.\langle f(u(t))\rangle \ge -C'_m\,. \tag{5.20}$$

Inserting the estimates (5.18)–(5.20) in the relation (5.12), we derive that

$$\begin{aligned} &\partial_t \left(\|v(t)\|_{0,2}^2 + (bL^{1/2}v(t), L^{1/2}v(t))\right) + \\ &+ \beta'\left(\|v(t)\|_{0,2}^2 + (bL^{1/2}v(t), L^{1/2}v(t))\right) + \alpha\|\nabla_x v(t)\|_{0,2}^2 \le C_m(1 + \|g\|_{0,2}^2)\,. \end{aligned} \tag{5.21}$$

Applying Gronwall's inequality to the relation (5.21) and taking into account the facts that $b = b^* > 0$ and $L = L^* > 0$, we derive the estimate (2.5). The smoothing property (2.6) can be established completely analogous. Having the estimates (2.5) and (2.6) and repeating word by word the proof of Theorem 2.2, we obtain the existence of the attractors $\mathcal{A}^{(m)}$. Theorem 5.2 is proved.

It is not difficult to see that the regularity results of Section 3 remains true for the case of Newmann boundary conditions as well.

Theorem 5.3. *Let the assumptions of Theorem 5.1 hold and let in addition the condition* (3.2) *be satisfied. Then for every* $t > 0$ *the semigroup* $\hat{S}_t^{(m)} : \hat{\mathcal{T}}_m \to D(A)$ *and the estimate* (3.4) *is valid. Consequently, for every* $m \in \mathbb{R}^k$ *the attractor* $\mathcal{A}^{(m)}$ *belongs to* $D(A)$ *and is bounded in it.*

The proof of this theorem is analogous to Theorem 3.1 and so we omit it here.

In the case where the non-linearity is not subbordinated to the linear terms we have the analogue of Theorem 3.4.

Theorem 5.4. *Let the assumptions of Theorem 5.1 hold and let in addition the non-linear term satisfy the conditions* (3.15) *and* (3.16). *Then*

$$\|\hat{u}(t)\|_{D(A)} \le C_m \frac{1+t^2}{t^2} \left(\|u_0\|^2_{0,2} e^{-\mu t} + 1 + \|g\|^2_{0,2} \right) , \tag{5.22}$$

and consequently the attractor $\mathcal{A}^{(m)}$ *belongs to* $D(A)$ *and is bounded in it.*

The proof of this Theorem is the same as in the case of Dirichlet boundary conditions.

In conclusion of this Section we give the analogue of Theorem 4.2 for the case of Neumann boundary conditions which gives us the finite dimensionality of the attractors $\mathcal{A}^{(m)}$.

Theorem 5.5. *Let the assumptions of Theorem 5.1 hold and let in addition* (4.6) *be true (for instanse let the conditions of Corollary 4.1 be satisfied). Then the attractors* $\mathcal{A}^{(m)}$ *have finite fractal dimension in* $L^2(\Omega)$*:*

$$\dim_F(\mathcal{A}^{(m)}, L^2(\Omega)) \le C_m < \infty \, . \tag{5.23}$$

The assertion of this Theorem can be verified in the same way as in the case of Dirichlet boundary conditions (see Theorem 4.2).

Example 5.1. Let us consider the scalar case $k = 1$ and the polynomial non-linearity

$$f(u) = u^{2l+1} + \sum_{i=1}^{2l} a_i u^i, \quad a_i \in \mathbb{R}, \tag{5.24}$$

where $l \in \mathbb{N}$ is a certain integer number. Then all assumptions of Theorem 5.5 are evidently satisfied and consequently for every $l \in \mathbb{N}$ and for every dimension n the equation (5.1) possesses the attractors $\mathcal{A}^{(m)}$, $m \in \mathbb{R}$, and their dimension in $L^2(\Omega)$ is finite. Moreover, it is worth to emphasize that in the case $n = 3$ due to (3.8) the solution $u(t) \in C(\Omega)$ for $t > 0$. If this fact is established one can easily derive by standard arguments the global existence of *classical* solutions for the equation (5.1) if the initial data u_0 and the external force g are smooth enough.

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