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A Subdifferential Criterion for Calmness of Multifunctions

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Abstract

A criterion for the calmness of a class of multifunctions between finite-dimensional spaces is derived in terms of subdifferential concepts developed by Mordukhovich. The considered class comprises general constraint set mappings as they occur in optimization or mappings associated with a certain type of variational systems. The criterion for calmness is obtained as an appropriate reduction of Mordukhovich's well-known characterization of the stronger Aubin property (roughly spoken, one may pass to the boundaries of normal cones or subdifferentials when aiming at calmness).

1 Introduction

Frequently, the stability analysis of multifunctions $M : Y \rightrightarrows X$ between metric spaces X, Y, relies on the Aubin property, which is said to hold at some $(\bar{y}, \bar{x}) \in \mathrm{Gph}\, M$ (= graph of M), if there exist neighborhoods $\mathcal{V}, \mathcal{U}$ of $\bar{y}, \bar{x}$, respectively, as well as some $L > 0$ such that

$$d(x, M(y_2)) \leq L d(y_1, y_2) \quad \text{for all } y_1, y_2 \in \mathcal{V}, \text{ for all } x \in M(y_1) \cap \mathcal{U}.$$

This property is well known to be equivalent with the metric regularity of the inverse multifunction M^{-1} (cf. e.g. [8], Th. 9.43). In case of finite-dimensional spaces X, Y, it is possible to characterize equivalently the Aubin property of closed multifunctions by the following algebraic criterion (see [5]):

$$D^* M(\bar{y}, \bar{x})(0) = \{0\}, \tag{1}$$

where $D^* M$ refers to Mordukhovich's coderivative. A weaker concept of Lipschitz-like behaviour of multifunctions is calmness, which is satisfied at some $(\bar{y}, \bar{x}) \in \mathrm{Gph}\, M$, if there exist neighborhoods $\mathcal{V}, \mathcal{U}$ of $\bar{y}, \bar{x}$, respectively, as well as some $L > 0$ such that

$$d(x, M(\bar{y})) \leq L d(y, \bar{y}) \quad \text{for all } y \in \mathcal{V}, \text{ for all } x \in M(y) \cap \mathcal{U}.$$

The concept of calmness, applied to value functions of optimization problems, goes back to Clarke [1], who pointed out its relevance as a constraint qualification for obtaining nondegenerate Lagrange multipliers in optimization problems. To illustrate the analogous role that calmness of multifunctions plays in the same context, assume that M is a closed multifunction and that $\bar{x}$ is a local minimizer of some locally Lipschitzian function f on

$M(\bar{y})$. Then, there is some $K > 0$ such that $f(\bar{x}) \leq f(x) + Kd(x, M(\bar{y}))$ for all x in a neighborhood of $\bar{x}$ (see Prop. 2.4.3. in [2]). Now, if M is calm at $(\bar{y}, \bar{x})$, then the last inequality may be extended to $f(\bar{x}) \leq f(x) + KLd(y, \bar{y})$, which holds true for all $x \in M(y)$ with x close to $\bar{x}$ and y close to $\bar{y}$. This, however, is exactly the calmness condition which was shown in [2] (Prop. 6.4.4) to yield a nonsmooth (nondegenerate) multiplier rule for finite-dimensional optimization problems with Lipschitzian data.

For the derivation of multiplier rules, it is usual to indicate appropriate constraint qualifications which have a chance to be verified for the given data. Frequently, such constraint qualifications are associated with the Aubin property rather than calmness of the underlying constraint set mapping. This, however, may result in too strong conditions as is most easily seen from the convex example, in which $f(x) = x$ is minimized subject to $g(x) = |x| \leq 0$. Here, the Aubin property of the constraint set mapping (which is equivalent to Slater's condition) fails to hold for the minimizer due to $0 \in \partial g(0) = [-1, 1]$. On the other hand, calmness is fulfilled and, consequently, one has the multiplier rule $0 \in \partial f(0) + \lambda \partial g(0) = [1 - \lambda, 1 + \lambda]$ for some $\lambda \geq 0$. An appropriate constraint qualification in this example would be the condition $0 \notin \operatorname{bd} \partial g(0)$, where bd refers to the topological boundary. This can be considered as a weak Slater's condition, which is actually satisfied in the above example.

To put this idea into a more general context, we consider the following class of finite-dimensional multifunctions:

$$M(y) := \{x \in \omega | g(x) + y \in \Lambda\},$$

where $\omega \subseteq \mathbb{R}^p, \Lambda \subseteq \mathbb{R}^m$ are closed subsets and $g : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz. This class covers constraint sets of nonsmooth, finite-dimensional optimization but also some generalized equations, in particular nonlinear complementarity problems. Applying the criterion (1) for the Aubin property to this structure gives

$$\bigcup_{y^* \in N_\Lambda(g(\bar{x})) \setminus \{0\}} D^* g(\bar{x})(y^*) \cap (-N_\omega(\bar{x})) = \emptyset, \tag{2}$$

where N refers to Mordukhovich's normal cone. Now, the main result of this paper states that, under mild assumptions on ω and g, the weaker calmness property can be guaranteed under the weaker condition

$$\bigcup_{y^* \in N_\Lambda(g(\bar{x})) \setminus \{0\}} D^* g(\bar{x})(y^*) \cap (-\operatorname{bd} N_\omega(\bar{x})) = \emptyset.$$

Indeed, this criterion applies without any further assumptions on g given that the abstract constraint set ω (which typically has a simple structure) is convex or defined as an intersection or union of a finite number of smooth inequalities under the usual regularity condition. The result is no longer true for arbitrary closed sets ω, but at least for those being Clarke-regular it can be saved under the additional assumption that either g or ω is semismooth. If, moreover, g is Clarke-regular in the special case $\Lambda = \mathbb{R}^m_-$ (modelling a finite number of inequalities), one can even sharpen the above condition by passing to the boundary on the left-hand side as well. In this way, for instance the simple convex example mentioned above will be covered. Finally, the obtained criterion is applied to the case of nonlinear complementarity problems.

2 Basic concepts and notation

The following notation is employed throughout this paper: $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{B}_2$ is the respective unit ball. $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$, $\|\cdot\|^*$ is the corresponding dual norm, and $\mathbb{B}^*$ is the unit ball associated with $\|\cdot\|^*$. For a set ω, δ_ω and d^e_ω denote the indicator and the euclidean distance function, respectively. Finally, '$\sharp$' refers to the cardinality of sets.

Next, we recall some basic concepts from nonsmooth analysis needed in this paper. For a closed subset $A \subseteq \mathbb{R}^k$, the contingent and Clarke's tangent cone, respectively, to A at some point $\bar{x} \in A$ are defined by

$$\begin{aligned} K_A(\bar{x}) &= \left\{d \in \mathbb{R}^k \mid \exists t_n \downarrow 0,\ d_n \to d:\ \bar{x} + t_n d_n \in A\right\}. \\ T^c_A(\bar{x}) &= \left\{d \in \mathbb{R}^k \mid \forall t_n \downarrow 0,\ x_n \to \bar{x}\ (x_n \in A)\, \exists d_n \to d : x_n + t_n d_n \in A\right\}. \end{aligned}$$

The respective normal cones are obtained as

$$\hat{N}_A(\bar{x})\,(N^c_A(\bar{x})) = \left\{d^* \in \mathbb{R}^k \mid \langle d^*, d\rangle \le 0\ \forall d \in K_A(\bar{x})\,(T^c_A(\bar{x}))\right\}.$$

In contrast, the Mordukhovich normal cone is defined as a generally nonconvex object via

$$N_A(\bar{x}) = \left\{d^* \in \mathbb{R}^k \mid \exists d^*_n \to d^*, x_n \to \bar{x}\ (x_n \in A) : d^*_n \in \hat{N}_A(x_n)\right\}.$$

A is called (Clarke-) regular at $\bar{x}$, if $K_A(\bar{x}) = T^c_A(\bar{x})$ or, equivalently, $N_A(\bar{x}) = N^c_A(\bar{x}) = \hat{N}_A(\bar{x})$. By $\operatorname{epi} f = \left\{(x,\alpha) \in \mathbb{R}^{k+1} | f(x) \le \alpha\right\}$, denote the epigraph of a lower semicontinuous function $f : \mathbb{R}^k \to \mathbb{R}$. Now, the normal cones induce subdifferentials of f via

$$\partial f(\bar{x})\,(\partial^c f(\bar{x})) = \left\{x^* \in \mathbb{R}^k | (x^*, -1) \in N_{\operatorname{epi} f}(\bar{x})\,(N^c_{\operatorname{epi} f}(\bar{x}))\right\},$$

where ∂ and ∂^c refer to Mordukhovich's and Clarke's subdifferentials respectively. A more general construction is Mordukhovich's coderivative $D^*Z(\bar{x},\bar{y}) : \mathbb{R}^l \rightrightarrows \mathbb{R}^k$ of some multifunction $Z : \mathbb{R}^k \to \mathbb{R}^l$ at some point $(\bar{x},\bar{y}) \in \operatorname{cl} \operatorname{Gph} Z$:

$$D^*Z(\bar{x},\bar{y})(y^*) = \left\{x^* \in \mathbb{R}^k | (x^*, -y^*) \in N_{\operatorname{Gph} Z}(\bar{x},\bar{y})\right\}.$$

For single-valued, locally Lipschitzian mappings $g = (g_1, \ldots, g_l) : \mathbb{R}^k \to \mathbb{R}^l$, the basic relation between coderivative and subdifferential of its components is $D^*g(\bar{x})(y^*) = \partial\left(\sum_{i=1}^l y^*_i g_i\right)(\bar{x})$. For a detailed treatment of the objects mentioned here, we refer to [8], [2] and [6].

For technical reasons, we shall make use of the concept of semismooth functions introduced by Mifflin in [3]:

Definition 2.1 *A function $F : \mathbb{R}^k \to \mathbb{R}$ is called semismooth at $\bar{x} \in \mathbb{R}^k$ if it is locally Lipschitz at $\bar{x}$ and the following property holds true: for each $d \in \mathbb{R}^k$ and for any sequences $t_n \downarrow 0$, $d_n \to d$, $\alpha_n \in \partial^c F(\bar{x} + t_n d_n)$, the limit $\lim_{n\to\infty} \langle \alpha_n, d \rangle$ exists.*

The following statement was shown in [3] (Lemma 2):

Lemma 2.2 *If $F : \mathbb{R}^k \to \mathbb{R}$ is semismooth at $\bar{x} \in \mathbb{R}^k$, then the directional derivative $F'(\bar{x}; d)$ exists for all $d \in \mathbb{R}^k$ and equals the limit $\lim_{n\to\infty} \langle \alpha_n, d \rangle$, where α_n is any of the sequences from Definition 2.1.*

Via the euclidean distance function d^e, the concept of semismoothness may be carried over to sets.

Definition 2.3 *A set $A \subseteq \mathbb{R}^k$ is called semismooth at $\bar{x} \in \operatorname{cl} A$ if for any sequence $x_n \to \bar{x}$ with $x_n \in A$ and $\|x_n - \bar{x}\|^{-1}(x_n - \bar{x}) \to d$ it holds that $\langle \alpha_n, d \rangle \to 0$ for all selections $\alpha_n \in \partial^c d^e_A(x_n)$.*

Proposition 2.4 *If $A \subseteq \mathbb{R}^k$ is closed and d^e_A is semismooth at $\bar{x} \in A$, then A is semismooth at $\bar{x}$.*

Proof. Let x_n, α_n be arbitrary sequences in Definition 2.3. Taking $t_n := \|x_n - \bar{x}\|$ and $d_n := t_n^{-1}(x_n - \bar{x})$ in Definition 2.1, we derive from Lemma 2.2 the existence of the directional derivative $d^{e\prime}_A(\bar{x}; d)$ as well as

$$\begin{aligned}\langle \alpha_n, d \rangle \to d^{e\prime}_A(\bar{x}; d) &= \lim_{t\downarrow 0, d'\to d} t^{-1}(d^e_A(\bar{x} + td') - d^e_A(\bar{x})) \\ &= \lim_{n\to\infty} t_n^{-1}(d^e_A(x_n) - d^e_A(\bar{x})) = 0,\end{aligned}$$

where the representation of the directional derivative relies on d^e_A being Lipschitz. ■

3 Characterization of calmness

We start with the main result of this paper.

Theorem 3.1 *Consider the multifunction $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ given by*

$$M(y) := \{x \in \omega | g(x) + y \in \Lambda\}, \tag{3}$$

where $\omega \subseteq \mathbb{R}^p$ and $\Lambda \subseteq \mathbb{R}^m$ are closed subsets and $g : \mathbb{R}^p \to \mathbb{R}^m$ is locally Lipschitz at some $\bar{x}$ with $(0, \bar{x}) \in \operatorname{Gph} M$. Assume that

1. ω is regular at $\bar{x}$;

2. One of the following two conditions holds true:

(a) There is some norm $\|\cdot\|_+$ on $\mathbb{R}^m$ such that the value function $P(x) := \min_{z\in\Lambda} \|z - g(x)\|_+$ is semismooth at $\bar{x}$;
or

(b) ω is semismooth at $\bar{x}$.

3. the constraint qualification

$$\bigcup_{y^*\in N_\Lambda(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(y^*) \cap (-\operatorname{bd} N_\omega(\bar{x})) = \emptyset. \tag{4}$$

holds true.

Then, M is calm at $(0, \bar{x})$.

Proof. Assume by contradiction that M is not calm at $(0, \bar{x})$. By definition, there exist sequences $x_n \to \bar{x}$, $x_n \in M(y_n)$, $y_n \to 0$, such that $d(x_n, M(0)) > nd(0, y_n)$, where the distance on the right-hand side is generated by $\|\cdot\|_+$. Hence,

$$d(x_n, M(0)) > nd(0, M^{-1}(x_n)). \tag{5}$$

In particular, $x_n \in M(y_n)$ implies that $x_n \in \omega$. Clearly, $d(x_n, M(0)) > 0$ for all n due to (5). Further, for the function defined in assumption 2, we have

$$P(x) = d(g(x), \Lambda) = d(0, -g(x) + \Lambda) = d(0, M^{-1}(x)) \quad \forall x \in \omega. \tag{6}$$

We observe that $P(x_n) > 0$ since otherwise $x_n \in M(0)$ in contrast to the statement above. Consequently, each x_n is an ε-minimizer of the function $P + \delta_\omega$ with $\varepsilon := P(x_n)$, (recall that $\bar{x} \in M(0)$, hence $\bar{x} \in \omega$ and $\inf(P + \delta_\omega) = P(\bar{x}) = 0$). Since $P + \delta_\omega$ is a proper, lower semicontinuous function, the application of Ekeland's variational principle (with ε as above and $\lambda := (n/2)P(x_n)$) yields for each $n \in \mathbb{N}$ the existence of a point $\tilde{x}_n$ such that

$$\begin{aligned} P(\tilde{x}_n) + \delta_\omega(\tilde{x}_n) &\leq P(x_n) + \delta_\omega(x_n) & (7)\\ \|\tilde{x}_n - x_n\| &\leq (n/2)P(x_n) & (8)\\ \tilde{x}_n &\in \arg\min\{P(x) + (2/n)\|\tilde{x}_n - x\| \,|x \in \omega\}. & (9) \end{aligned}$$

Note that (7) implies $P(\tilde{x}_n) + \delta_\omega(\tilde{x}_n) \leq P(x_n)$ due to $x_n \in \omega$, hence $\delta_\omega(\tilde{x}_n) = 0$. As a consequence, $\tilde{x}_n \in \omega$ and the formulation of (9) is justified. From (8), (6) and (5), we infer that $\tilde{x}_n \to \bar{x}$ and $P(\tilde{x}_n) > 0$. Indeed, $P(\tilde{x}_n) = 0$ would imply the contradiction to (5)

$$d(x_n, M(0)) \leq \|\tilde{x}_n - x_n\| \leq (n/2)P(x_n) = (n/2)d(0, M^{-1}(x_n)).$$

Applying the necessary optimality conditions to (9), we deduce that

$$0 \in \partial P(\tilde{x}_n) + N_\omega(\tilde{x}_n) + (2/n)\mathbb{B}^*.$$

Hence, there exist sequences $\eta_n \in \partial P(\tilde{x}_n)$ and $\beta_n \in -N_\omega(\tilde{x}_n)$ such that $\|\eta_n - \beta_n\|^* \leq 2/n$ for all $n \in \mathbb{N}$. Since P is Lipschitz near $\bar{x}$, the sequence $\{\eta_n\}$ is bounded. Consequently, due to the last relation, $\{\beta_n\}$ must be bounded too. By extracting appropriate subsequences, one arrives at

$$\eta_{n'} \to \eta \in \partial P(\bar{x}) \quad \text{and} \quad \beta_{n'} \to \eta \in -N_\omega(\bar{x}) \tag{10}$$

by virtue of the multifunctions $\partial P(\cdot)$ and $N_\omega(\cdot)$ having closed graph.

Next, we verify the following relation

$$\eta \in \{D^* g(\bar{x})(y^*) | y^* \in N_\Lambda(g(\bar{x})) \setminus \{0\}\}. \tag{11}$$

To this aim, denote $\sigma(x,z) := \|g(x) - z\|_+$ and $A(x) := \{z \in \Lambda | P(x) = \sigma(x,z)\}$. Since $\Lambda \neq \emptyset$ (due to $g(\bar{x}) \in \Lambda$), one has that $A(x) \neq \emptyset$ for each $x \in \mathbb{R}^p$. Furthermore, well-known results from parametric optimization (e.g. [8], Cor. 7.42) imply that $\operatorname{Gph} A$ is closed and A is uniformly bounded around each $x \in \mathbb{R}^p$. This, along with the fact that σ is locally Lipschitz, allows to apply Theorem 4.1 in [4] to the function P. One gets the inclusion

$$\partial P(\tilde{x}_n) \subseteq \bigcup_{y^* \in \mathbb{R}^m, z \in A(\tilde{x}_n)} \{x_1^* + x_2^* \in \mathbb{R}^p | x_1^* \in D^* Q(\tilde{x}_n, z)(y^*), (x_2^*, y^*) \in \partial\sigma(\tilde{x}_n, z)\},$$

where $Q : \mathbb{R}^p \rightrightarrows \mathbb{R}^m$ denotes the constant multifunction $Q(x) := \Lambda \; \forall x \in \mathbb{R}^p$. Clearly $\operatorname{Gph} Q = \mathbb{R}^p \times \Lambda$, $N_{\operatorname{Gph} Q}(\tilde{x}_n, z) = \{0\} \times N_\Lambda(z)$, and the definition of the coderivative implies that

$$D^* Q(\tilde{x}_n, z)(y^*) = \begin{cases} 0 & \text{if } y^* \in -N_\Lambda(z) \\ \emptyset & \text{else} \end{cases}.$$

Consequently,

$$\partial P(\tilde{x}_n) \subseteq \bigcup_{y^* \in -N_\Lambda(z), z \in A(\tilde{x}_n)} \{x^* \in \mathbb{R}^p | (x^*, y^*) \in \partial\sigma(\tilde{x}_n, z)\}. \tag{12}$$

Putting $F(x,z) := g(x) - z$, we have $\sigma = \|\cdot\|_+ \circ F$, and the chain rule for Lipschitz mappings in [6] (Cor. 5.8) yields that

$$\partial\sigma(\tilde{x}_n, z) \subseteq \bigcup_{s \in \partial\|\cdot\|_+(g(\tilde{x}_n) - z)} \partial \langle s, F \rangle (\tilde{x}_n, z). \tag{13}$$

Since $\langle s, F \rangle (x,z) = s^T g(x) - s^T z$, the sum rule in [6] (Cor. 4.6) provides that

$$\partial \langle s, F \rangle (\tilde{x}_n, z) = [\partial \langle s, g \rangle (\tilde{x}_n) \times \{0\}] + [\{0\} \times \{-s\}] = \partial \langle s, g \rangle (\tilde{x}_n) \times \{-s\}.$$

Furthermore, since $P(\tilde{x}_n) > 0$ implies $F(\tilde{x}_n, z) \neq 0$ for all $z \in A(\tilde{x}_n)$, we derive from convex analysis that

$$\partial \|\cdot\|_+ (g(\tilde{x}_n) - z) = \{s \in \mathbb{S}^* | \langle s, g(\tilde{x}_n) - z \rangle = \|g(\tilde{x}_n) - z\|_+\} \quad (z \in A(\tilde{x}_n)),$$

where $\mathbb{S}^*$ denotes the unit sphere in $\mathbb{R}^m$ equipped with the dual norm of $\|\cdot\|_+$. Combining the previous relations with (13), one arrives at

$$\partial\sigma(\tilde{x}_n, z) \subseteq \{D^*g(\tilde{x}_n)(s) \times \{-s\} | s \in \mathbb{S}^*\} \quad (z \in A(\tilde{x}_n)),$$

where we used the relation $D^*g(\tilde{x}_n)(s) = \partial \langle s, g \rangle (\tilde{x}_n)$ which is valid for Lipschitzian mappings, cf. [6] (Prop. 4.6). Inserting the last inclusion into (12) gives

$$\partial P(\tilde{x}_n) \subseteq \bigcup_{z \in A(\tilde{x}_n)} \{D^*g(\tilde{x}_n)(s) | s \in \mathbb{S}^* \cap N_\Lambda(z)\},$$

which holds for all $n \in \mathbb{N}$ since n was arbitrarily fixed. Therefore, along with the sequence $\eta_{n'}$ defined in (10), we have sequences $s_{n'}$ and $z_{n'}$ such that

$$\eta_{n'} \in D^*g(\tilde{x}_{n'})(s_{n'}); \quad s_{n'} \in \mathbb{S}^* \cap N_\Lambda(z_{n'}); \quad z_{n'} \in A(\tilde{x}_{n'}).$$

Since $\tilde{x}_{n'} \to \bar{x}$ and A is uniformly bounded around $\bar{x}$, we may extract subsequences such that $s_{n''} \to \bar{s} \in \mathbb{S}^*$ and $z_{n''} \to \bar{z}$. By closedness of Gph A (see remark above), it follows that $\bar{z} \in A(\bar{x}) = \{g(\bar{x})\}$ (due to $g(\bar{x}) \in \Lambda$). Furthermore, $(\eta_{n''}, -s_{n''}) \in N_{\mathrm{Gph}\, g}(\tilde{x}_{n''}, g(\tilde{x}_{n''}))$ according to the definition of the coderivative. Since the graph of the normal cone mapping $N_{\mathrm{Gph}\, g}$ is closed, we infer that $\bar{s} \in N_\Lambda(g(\bar{x}))$ and, by (10),

$$(\eta_{n''}, -s_{n''}) \to (\eta, -\bar{s}) \in N_{\mathrm{Gph}\, g}(\bar{x}, g(\bar{x})).$$

Consequently, $\eta \in D^*g(\bar{x})(\bar{s})$ with $\bar{s} \in \mathbb{S}^* \cap N_\Lambda(g(\bar{x}))$, which eventually implies (11).

Now, (5), (6) and (8) yield $\|x_n - \bar{x}\| > nP(x_n) \geq (n/2)P(x_n) \geq \|\tilde{x}_n - x_n\|$. Taking into account the already obtained relations $P(\bar{x}) = 0 < P(\tilde{x}_n) \leq P(x_n)$ (see (7)), one arrives at

$$0 < \frac{P(\tilde{x}_n) - P(\bar{x})}{\|\tilde{x}_n - \bar{x}\|} \leq \frac{P(x_n)}{\|x_n - \bar{x}\| - \|\tilde{x}_n - x_n\|} < \frac{2}{n}. \tag{14}$$

From the sequence $\tilde{x}_{n'}$ (corresponding to $\eta_{n'}$ in (10)) we extract a subsequence $\tilde{x}_{n^*}$ (recall that $\tilde{x}_{n'} \neq \bar{x}$) such that

$$\lim_{n^* \to \infty} \frac{\tilde{x}_{n^*} - \bar{x}}{\|\tilde{x}_{n^*} - \bar{x}\|} = \xi \quad \text{for some} \quad \xi \in \mathbb{R}^p \quad \text{with} \quad \|\xi\| = 1$$

Clearly, $\xi \in K_\omega(\bar{x}) = T^c_\omega(\bar{x})$ by assumption 1. Since $\eta_{n^*} \in \partial P(\tilde{x}_{n^*})$ and $\beta_{n^*} \in -N_\omega(\tilde{x}_{n^*})$ (see derivation on top of (10)), the trivial representation

$$\tilde{x}_{n^*} = \bar{x} + \lambda_{n^*}\xi_{n^*} \quad \text{with } \lambda_{n^*} := \|\tilde{x}_{n^*} - \bar{x}\| \downarrow 0 \quad \text{and } \xi_{n^*} := \frac{\tilde{x}_{n^*} - \bar{x}}{\|\tilde{x}_{n^*} - \bar{x}\|} \to \xi,$$

provides that $\eta_{n^*} \in \partial P(\bar{x} + \lambda_{n^*}\xi_{n^*})$ and $\beta_{n^*} \in -N_\omega(\bar{x} + \lambda_{n^*}\xi_{n^*})$. Under assumption 2(a), Lemma 2.2 gives

$$\langle \eta, \xi \rangle = \lim_{n^* \to \infty} \langle \eta_{n^*}, \xi \rangle = P'(\bar{x}; \xi). \tag{15}$$

With P being locally Lipschitz and using (14), its directional derivative may be represented as

$$P'(\bar{x}; \xi) = \lim_{t \downarrow 0, \xi' \to \xi} \frac{P(\bar{x} + t\xi') - P(\bar{x})}{t} = \lim_{n^* \to \infty} \frac{P(\bar{x} + \lambda_{n^*}\xi_{n^*}) - P(\bar{x})}{\lambda_{n^*}} = 0, \tag{16}$$

whence $\langle \eta, \xi \rangle = 0$. Let us verify the same relation under assumption 2(b): It is evident in case $\eta = 0$, so let $\eta \neq 0$. Due to (10) it follows that $\beta_{n^*} \neq 0$ for n^* large enough. Next, we refer to the identity $N_\omega(\tilde{x}_{n^*}) \cap \mathbb{B}_2 = \partial d^e_\omega(\tilde{x}_{n^*})$ (see Ex. 8.53 in [8]). Consequently, with $\tilde{\beta}_{n^*} := -\beta_{n^*} / \|\beta_{n^*}\|_2$, we obtain $\tilde{\beta}_{n^*} \in \partial d^e_\omega(\tilde{x}_{n^*}) \subseteq \partial^c d^e_\omega(\tilde{x}_{n^*})$. Assumption 2(b) then yields

$$\|\eta\|^{-1} \langle \eta, \xi \rangle = -\lim_{n^* \to \infty} \left\langle \tilde{\beta}_{n^*}, \xi \right\rangle = 0,$$

whence again $\langle \eta, \xi \rangle = 0$.

Using that $\langle \eta, \xi \rangle = 0$ under any of the two assumptions 2(a) or 2(b), one gets for arbitrarily small $\varepsilon > 0$ that $\langle \eta - \varepsilon\xi, \xi \rangle = -\varepsilon < 0$. Since $\xi \in T^c_\omega(\bar{x})$, this implies that $\eta - \varepsilon\xi \notin -N^c_\omega(\bar{x})$. On the other hand, $\eta \in -N^c_\omega(\bar{x})$ according to (10). Consequently, $\eta \in \operatorname{bd} -N^c_\omega(\bar{x}) = -\operatorname{bd} N^c_\omega(\bar{x})$, which together with (11) provides a contradiction to (4). ■

Note that, as a result of the regularity assumption 1 in Theorem 3.1, one may replace N^c_ω by N_ω in the constraint qualification (4). The obtained result may be illustrated in one dimension as follows:

Example 3.2 *In Theorem 3.1, let $\Lambda := \mathbb{R}_-$, $g(x) := x$ and $\omega := \mathbb{R}_+$. Then, the multifunction M in (3) is easily verified to be calm at the point $(0,0)$ of its graph. Clearly, assumptions 1 and 2 (actually both, 2(a) and 2(b)) are satisfied. Furthermore, the constraint qualification (4) reduces to the condition $\nabla g(0) (= 1) \notin -\operatorname{bd} N^c_\omega(0) (= \{0\})$, which is certainly satisfied. On the other hand, we have $\nabla g(0) \in -N_\omega(0) (= -N^c_\omega(0) = \mathbb{R}_+)$, so that the criterion (2), designed for the stronger Aubin property, fails to apply.*

The following example illustrates that the regularity of ω in assumption 1 of Theorem 3.1 and Corollary 4.1 cannot be dispensed with ingeneral, so the constraint qualification (4) is not sufficient for calmness in case of arbitrary closed sets ω:

Example 3.3 *In the context of Theorem 3.1, define $\Lambda := \mathbb{R}_-$, $g(x) := x^2$ and $\omega := \{n^{-1/2} | n \in \mathbb{N}\} \cup \{0\}$. Then, ω is closed but fails to be regular at $\bar{x} := 0$. Obviously,*

$M(0) = \{0\}$, hence $(0, \bar{x}) \in \operatorname{Gph} M$. Furthermore, assumption 2(a) of Theorem 3.1 is satisfied since

$$P(x) = \min_{z \le 0} |z - g(x)| = \max\{g(x), 0\}$$

is semismooth as the maximum of two semismooth functions ([3], Th. 5). Finally, one easily verifies that $T^c_\omega(\bar{x}) = \{0\}$, hence $N^c_\omega(\bar{x}) = \mathbb{R}$. As a consequence, $-\operatorname{bd} N^c_\omega(\bar{x}) = \emptyset$ so that assumption 3 is trivially fulfilled. On the other hand, M is immediately seen not to be calm at $(0, \bar{x})$ (take sequences $x_n := n^{-1/2}$ and $y_n := -n^{-1}$ for establishing a contradiction to the definition of calmness).

Let us recall that regularity and semismoothness of a set ω are completely independent properties (assumptions 1 and 2(b) of Theorem 3.1):

Example 3.4 *Let $\omega := \operatorname{epi}(-|x|) \subseteq \mathbb{R}^2$. Then,*

$$d^e_\omega(x, y) = \max\{0, \min\{(x - y)/\sqrt{2}, -(x + y)/\sqrt{2}\}\}$$

is semismooth as a min-max-composition of semismooth functions (cf. [3]). Invoking Proposition 2.4, we see that ω is a semismooth set which clearly fails to be regular at $(0, 0)$. Conversely, define

$$\omega := \bigcup_{n \in \mathbb{N}} \{(x, y) \in \mathbb{R}^2 | x \le -n^{-1},\ 0 \le y \le n^{-2}\}.$$

Calculating $K_\omega(0, 0) = T^c_\omega(0, 0) = \mathbb{R}_-$, we verify that ω is regular at $(\bar{x}, \bar{y}) = (0, 0)$. On the other hand, taking the sequence $(x_n, y_n) := (-n^{-1}, n^{-2})$, we get $(x_n, y_n) \in \omega$ and

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\|^{-1} ((x_n, y_n) - (\bar{x}, \bar{y})) \to d := (-1, 0).$$

Since $(1, 0) \in \partial^c d^e_\omega(x_n, y_n)$, it follows with $\alpha_n \equiv (1, 0)$ the contradiction $\langle \alpha_n, d \rangle = -1$ to Definition 2.3. Hence, ω is not semismooth at $(\bar{x}, \bar{y})$.

In the rest of this section we are going to identify structures of the abstract constraint set ω which render superfluous all technical assumptions of Theorem 3.1 such that the constraint qualification (4) becomes the only condition of calmness for M. First, we indicate a situation where ω satisfies assumptions 1 and 2(b) of Theorem 3.1. To this aim, let a set A be described by the following system of inequalities:

$$A = \{x \in \mathbb{R}^k | f_i(x) \le 0,\ i = 1, \dots, l\}. \tag{17}$$

Further, for $x \in A$ denote by $I(x)$ the standard set of active inequalities

$$I(x) := \{i \in \{1, 2, \dots, l\} \mid f_i(x) = 0\}.$$

Lemma 3.5 *Let A be given as in (17). Assume that the f_i are continuously differentiable and that the set of gradients*

$$\{\nabla f_i(x) | i \in I(x)\} \tag{18}$$

is positively linearly independent at each $x \in A$. Then, A is regular and semismooth at each of its points.

Proof. For the regularity part see [2] (Cor. 2, p.56). The same reference confirms that

$$N_A^c(x) = \left\{ \sum_{i=1}^{l} \lambda_i \nabla f_i(x) | \lambda_i \geq 0,\ \lambda_i = 0 \text{ for } i \notin I(x) \right\} \quad \text{for all } x \in A.$$

As for the semismoothness of A at some arbitrary $\bar{x} \in A$, let x_n, α_n and d be arbitrarily given as in Definition 2.3. The index set

$$I := \{i \in \{1, \ldots, l\} \,|\, \text{there is a subsequence } \{x_{n'}\} \text{ with } i \in I(x_{n'})\}$$

satisfies $I \subseteq I(\bar{x})$ (by continuity of the f_i's) as well as

$$\langle \nabla f_i(\bar{x}), d \rangle = \lim_{n' \to \infty} \left\langle \nabla f_i(\bar{x}), \frac{x_{n'} - \bar{x}}{\|x_{n'} - \bar{x}\|} \right\rangle = 0 \text{ for all } i \in I \tag{19}$$

(by differentiability of the f_i's and by $f_i(\bar{x}) = f_i(x_{n'}) = 0$ for $i \in I$). Since $x_n \in A$ and $\alpha_n \in \partial^c d_A^e(x_n) \subseteq N_A^c(x_n) \cap \mathbb{B}_2$ (cf. [2], Prop. 2.4.2. and Th. 2.5.6), there exist $\lambda_i^{(n)} \geq 0$ such that

$$\alpha_n = \sum_{i=1}^{l} \lambda_i^{(n)} \nabla f_i(x_n) \quad \text{and } \lambda_i^{(n)} = 0 \text{ for } i \in \{1, \ldots, l\} \setminus I(x_n).$$

By definition, one has $I(x_n) \subseteq I$ for n large enough, hence

$$\langle \alpha_n, d \rangle = \sum_{i \in I} \lambda_i^{(n)} \langle \nabla f_i(x_n), d \rangle \,. \tag{20}$$

Now, assumption (18) is well known to be equivalent with the existence of some $\xi \in \mathbb{R}^k$ such that $\langle \nabla f_i(\bar{x}), \xi \rangle > 0$ for $i \in I(\bar{x})$. Hence, for some $\varepsilon > 0$ and for n large enough, one has $\langle \nabla f_i(x_n), \xi \rangle \geq \varepsilon \quad (i \in I(\bar{x}))$. This implies

$$\langle \alpha_n, \xi \rangle = \sum_{i \in I} \lambda_i^{(n)} \langle \nabla f_i(x_n), \xi \rangle$$

and, using $\|\alpha_n\| \leq 1$, we deduce that

$$0 \leq \lambda_i^{(n)} \leq \frac{\langle \alpha_n, \xi \rangle}{\langle \nabla f_i(x_n), \xi \rangle} \leq \frac{\|\xi\|}{\varepsilon} \quad (i \in I).$$

This boundedness property allows to pass to the limit $n \to \infty$ in (20) upon taking into account (19). One gets the desired relation $\langle \alpha_n, d\rangle \to 0$ in order to verify semismoothness of A at $\bar{x}$. ■

As a consequence, the result of Theorem 3.1 simplifies as follows for common structures of the abstract constraints:

Corollary 3.6 *In the setting of Theorem 3.1, let ω be convex or described by a finite number of smooth inequalities as in (17) which satisfy the regularity condition (18). Then, the constraint qualification (4) implies calmness of M at $(0, \bar{x})$.*

Proof. In both cases, ω is a regular and semismooth set. For the second case, this was shown in Lemma 3.5. For convex ω, regularity is clear, while semismoothness follows from semismoothness of the convex distance function d^e_ω ([3], Prop. 3) via Proposition 2.4. ■

Another relevant instance of abstract sets ω which allow direct application of the criterion (4) without further technical assumptions is given by unions of smooth inequalities (in contrast to intersections as in (17)). At the same time, this structure reflects a situation where our criterion (4) coincides with condition (2) ensuring the Aubin property. Note that in general, as pointed out by Example 3.2, both conditions differ significantly.

Definition 3.7 *We call $A \subseteq \mathbb{R}^k$ a 'disjunctive set' if there exists a continuously differentiable mapping $f : \mathbb{R}^k \to \mathbb{R}^l$ for some $l \in \mathbb{N}$ such that*

$$\begin{aligned} A &= \left\{x \in \mathbb{R}^k | \exists i \in \{1, \dots l\} : f_i(x) \le 0\right\} \quad \textit{and} \\ \operatorname{rank}\{\nabla f_i(x) | i \in J(x)\} &= \sharp J(x)\, \forall x \in A, \end{aligned}$$

where $J(x) = \{i \in \{1, \dots l\}\, | f_i(x) = \min_{j \in \{1,\dots l\}} f_j(x) = 0\}$ denotes a modified set of active indices.

Similar to conventional sets of active indices, the continuity of f implies that $J(x) \subseteq J(\bar{x})$ for x close to $\bar{x}$ $(x, \bar{x} \in A)$.

Proposition 3.8 *Let $A \subseteq \mathbb{R}^k$ be a disjunctive set. Then*

$$N_A(\bar{x}) = \mathbb{R}_+ \cdot \bigcup_{i \in J(\bar{x})} \{\nabla f_i(\bar{x})\} \tag{21}$$

for all $\bar{x} \in A$.

Proof. The assertion is obvious for the cases $\sharp J(\bar{x}) \le 1$, since then either $J(\bar{x}) = \emptyset$ (hence $\bar{x} \in \operatorname{int} A$ and $N_A(\bar{x}) = \{0\}$) or $J(\bar{x}) = \{i^*\}$ for some $i^* \in \{1, \dots l\}$ (then, locally around $\bar{x}$, the set A is described by the single continuously differentiable inequality $f_{i^*}(x) \le 0$ with, according to Def. 3.7, $\nabla f_{i^*}(\bar{x}) \neq 0$; hence $N_A(\bar{x}) = \mathbb{R}_+ \cdot \nabla f_{i^*}(\bar{x})$). Now, let $\sharp J(\bar{x}) \geq 2$ and, without loss of generality, assume that $\{1, 2\} \subseteq J(\bar{x})$, hence $f_1(\bar{x}) = f_2(\bar{x}) = 0$. Since $\nabla f_1(\bar{x})$ and $\nabla f_2(\bar{x})$ are linearly independent according to Definition 3.7, there exists some ξ with $\langle \nabla f_2(\bar{x}), \xi \rangle \leq 0$ and $\langle \nabla f_1(\bar{x}), \xi \rangle > 0$. Clearly, for the polar to the contingent cone, introduced in Section 2, it holds that $\hat{N}_A(\bar{x}) \subseteq \mathbb{R}_+ \cdot \nabla f_1(\bar{x})$ due to $\{x | f_1(x) \leq 0\} \subseteq A$. On the other hand, the first of the preceding inequalities ensures that ξ belongs to the contingent cone of the set $\{x | f_2(x) \leq 0\} \subseteq A$ at $\bar{x}$, hence ξ belongs to $K_A(\bar{x})$. Then, by the second of the preceding inequalities, $\nabla f_1(\bar{x}) \notin \hat{N}_A(\bar{x})$. Summarizing, we arrive at $\hat{N}_A(\bar{x}) = \{0\}$.

Now, let $x \in A$ be close to but different from $\bar{x}$. In each of the cases $\sharp J(x) = 0, 1$, the cones $\hat{N}_A(x)$, $N_A(x)$ coincide, and one has $\hat{N}_A(x) = \{0\}$ or $\hat{N}_A(x) = \mathbb{R}_+ \cdot \nabla f_{i^*}(x)$, respectively, for some $i^* \in J(\bar{x})$ according to the remarks above related to $\bar{x}$ rather than x. If, instead, $\sharp J(x) \geq 2$, then $\hat{N}_A(x) = \{0\}$ (again according to the remarks above related to $\bar{x}$ rather than x). Recalling, how N_A is generated from $\hat{N}_A$ (see Section 2), this altogether yields the inclusion '$\subseteq$' in (21). For the reverse inclusion, it suffices to show that $\nabla f_i(\bar{x}) \in N_A(\bar{x})$ for all $i \in J(\bar{x})$. This, however, follows again from the full rank condition in Definition 3.7 which ensures, for each $i \in J(\bar{x})$, the existence of some sequence $x_n \to \bar{x}$ such that $f_i(x_n) = 0$ and $f_j(x_n) > 0$ for $j \in J(\bar{x}) \setminus \{i\}$. Then, $J(x_n) = \{i\}$ and (see above) $\hat{N}_A(x) = \mathbb{R}_+ \cdot \nabla f_i(x_n)$, so $0 \neq \nabla f_i(x_n) \in \hat{N}_A(x)$. Passing to the limit $n \to \infty$, one obtains the desired relation $\nabla f_i(\bar{x}) \in N_A(\bar{x})$ by continuous differentiability of f. ∎

Corollary 3.9 *Let $A \subseteq \mathbb{R}^k$ be a disjunctive set and $k > 1$. Then $N_A(\bar{x}) = \operatorname{bd} N_A(\bar{x})$ for all $\bar{x} \in A$.*

The last corollary confirms the coincidence of Mordukhovich's and our criterion ((2) and (4), respectively) for disjunctive sets ω, so there is no chance to distinguish algebraically between calmness and Aubin property in this situation. As a simple example, take $\overline{x} = 0$ and $\omega = \text{epi}(-|x|)$, which is a disjunctive set with $f_1(x) = x$, $f_2(x) = -x$. Then, formally, Theorem 3.1 cannot be applied due to violation of assumption 1. Nevertheless, (4) can be invoked by virtue of its coincidence with (2) according to Corollary 3.9.

4 Application to optimization and nonlinear complementarity problems

With particular choices of Λ, we can specialize the results of Section 3 for various constraint mappings arising in applications. The simplest case corresponds to standard mathematical programs with inequality constraints, where $\Lambda = \mathbb{R}^m_-$ and, consequently,

$$N_\Lambda(g(\bar{x})) = \{y^* \in \mathbb{R}^m | y_i^* \geq 0 \text{ for } i \in I(\bar{x}) \text{ and } y_i^* = 0 \text{ otherwise}\}, \tag{22}$$

with $I(\bar{x}) := \{i \in \{1, \ldots, m\} | g_i(\bar{x}) = 0\}$ being the set of active indices.

Corollary 4.1 *Consider the multifunction M in (3) with $\Lambda = \mathbb{R}^m_-$ at a point $(0, \bar{x}) \in \operatorname{Gph} M$. Assume that*

1. *ω is regular at $\bar{x}$.*
2. *One of the following two conditions holds true:*
   (a) *All components g_i of g are semismooth at $\bar{x}$.*
   or
   (b) *ω is semismooth at $\bar{x}$.*
3. *The following constraint qualification holds true:*

$$\sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) \cap (-\operatorname{bd} N^c_\omega(\bar{x})) = \emptyset \quad \text{for all } \lambda_i \geq 0 \text{ with } \sum_{i\in I(\bar{x})} \lambda_i = 1 \tag{23}$$

Then, M is calm at $(0, \bar{x})$.

Proof. Choosing $\|\cdot\|_+$ as the l_∞- norm, the specific structure $\Lambda = \mathbb{R}^m_-$ considered here provides

$$P(x) = \max_{1,\cdots,m} [g_i(x)]_+ = \max\left\{g^1(x), \dots, g^m(x), 0\right\}. \tag{24}$$

Now, the last expression is a composition of the semismooth function $\max\{\cdot, \dots, \cdot\}$ with a mapping having semismooth components according to assumption 2. Applying Theorem 5 in [3], one derives the semismoothness of P at $\bar{x}$. Obviously, it suffices now to check assumption 3 of Theorem 3.1. Assuming its violation, there would exist some η with

$$\eta \in \bigcup_{v^*\in N_\Lambda(g(\bar{x}))\setminus\{0\}} D^*g(\bar{x})(v^*) \cap (-\operatorname{bd} N^c_\omega(\bar{x})).$$

In particular, according to (22), there exists some $v^* \in \mathbb{R}^m_+ \setminus \{0\}$ such that $v^*_i = 0$ for $i \notin I(\bar{x})$ and

$$\eta \in D^*g(\bar{x})(v^*) = \partial\left(\sum_{i\in I(\bar{x})} v^*_i g_i\right)(\bar{x}) \subseteq \sum_{i\in I(\bar{x})} v^*_i \partial g_i(\bar{x}),$$

where we used that $\partial(\lambda f) = \lambda\partial(f)$ for $\lambda \geq 0$ and the sum rule $\partial(f_1 + f_2) \subseteq \partial(f_1) + \partial(f_2)$ for locally Lipschitzian functions. Since $v^* \neq 0$ and $-\operatorname{bd} N^c_\omega(\bar{x})$ is a cone, we have $c^{-1}\eta \in -\operatorname{bd} N^c_\omega(\bar{x})$ for $c := \sum_{i\in I(\bar{x})} v^*_i > 0$ as well as

$$c^{-1}\eta \in \sum_{i\in I(\bar{x})} (c^{-1}v^*_i) g_i(\bar{x}).$$

This contradicts assumption 3 above, hence assumption 3 of Theorem 3.1 has to be satisfied. ■

It is clear that the technical assumptions 1 and 2 of Corollary 4.1 can be circumvented in special situations in the same way as described in Section 3 with respect to Theorem 3.1. Hence, constraint qualification (23) is automatically sufficient for calmness of M in the following cases:

- ω is convex or described by a finite number of smooth inequalities as in (17) which satisfy the regularity condition (18);
- ω is a disjunctive set as in Definition 3.7;
- ω is a regular set and g is convex or continuously differentiable or a maximum or minimum over a continuously and compactly indexed family of continuously differentiable functions (in all of which cases g becomes semismooth, see [3]).

The constraint qualification (23) can be weakened under the additional assumptions of regularity for g:

Theorem 4.2 *Consider the multifunction M in (3) with $\Lambda = \mathbb{R}^m_-$ at a point $(0, \bar{x}) \in \operatorname{Gph} M$. Assume that*

1. *ω is closed and regular at $\bar{x}$.*
2. *All components g_i are regular and semismooth at $\bar{x}$.*
3. *The following constraint qualification holds true:*

$$\left(\operatorname{bd} \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x})\,(\bar{x})\right) \cap \left(-\operatorname{bd} N^c_\omega(\bar{x})\right) = \emptyset \quad \textit{for all } \lambda \in \mathbb{R}^m_+ \textit{ with } \sum_{i \in I(\bar{x})} \lambda_i = 1.$$

Then, M is calm at $(0, \bar{x})$.

Proof. We define P as in the proof of Corollary 4.1 and introduce the function

$$Q(x) = \max\left\{g^1(x), \ldots, g^m(x)\right\},$$

which is regular according to assumption 2. If $Q(\bar{x}) < 0$, i.e., $g(\bar{x}) \in \operatorname{int} \Lambda$, then the continuity of g entails calmness of Mat $(0, \bar{x})$ in a trivial way. Hence, let $Q(\bar{x}) = P(\bar{x}) = 0$. Then, due to the regularity in assumption 2, one has (see [2], Th. 2.8.2)

$$\partial^c Q(\bar{x}) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i \partial^c g_i(\bar{x}) | \lambda_i \geq 0\,(i \in I(\bar{x})), \sum_{i \in I(\bar{x})} \lambda_i = 1 \right\}. \tag{25}$$

Suppose now that M fails to be calm at $(0,\bar{x})$. Repeating the proof of Theorem 3.1, one may deduce the existence of some $\xi \in T^c_\omega(\bar{x})$ and some $\eta \in -\operatorname{bd} N^c_\omega(\bar{x})$ satisfying $\langle \eta, \xi \rangle = P'(\bar{x};\xi)$(see (15)). Furthermore, the relation $P(\tilde{x}_n) > 0$ from the proof of Theorem 3.1 implies that $Q(\tilde{x}_n) = P(\tilde{x}_n)$. In view of (16), the relation above now turns into

$$\langle \eta, \xi \rangle = Q'(\bar{x};\xi) = \max\left\{\langle \eta', \xi \rangle \,|\eta' \in \partial^c Q(\bar{x})\right\},$$

where the second equality relies on the fact that, again by assumption 2, the conventional directional derivative of Q coincides with its directional derivative in the sense of Clarke. This, in turn, is the support function of Clarke's subdifferential (see [2]). As a consequence, $\eta \in \operatorname{bd} \partial^c Q(\bar{x})$, which contradicts assumption 3 according to (25). ■

In the trivial case of a single inequality $g(x) \le 0$ (without abstract constraints), the constraint qualification in Theorem 4.2 turns into the condition $0 \notin \operatorname{bd} \partial g(\bar{x})$. Of course, in the smooth case, this amounts to the condition $\nabla g(\bar{x}) \neq 0$ which is sufficient even for the stronger Aubin property of the constraint set mapping. A substantial gain over the criterion $0 \notin \partial g(\bar{x})$ (sufficient for the Aubin property) therefore occurs in a nonsmooth setting, for instance in the simple convex example discussed in the Introduction. Combining the last remarks with those following Corollary 3.9 and with Example 3.2, we have identified several significant circumstances - independently for the set ω and for the function g - under which the criteria (4) and (2) differ or coincide.

We now give an example that highlights the necessity of the additional regularity assumption on g in Theorem 4.2:

Example 4.3 *In the context of Theorem 4.2, let $m = 2$ and define $g_1(x) := x^2$, $g_2(x) := -|x|$, $\omega := \mathbb{R}$, $\bar{x} := 0$. Then, $M(0) = \{0\}$, hence $(0,\bar{x}) \in \operatorname{Gph} M$. Obviously, assumptions 1 and the semismoothness part of assumption 2 are satisfied (convex and concave functions are semismooth, see Prop. 3 in [3]). Furthermore, we have for all $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = 1$ that*

$$\begin{aligned}\left(\operatorname{bd} \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x})\right) \cap (-\operatorname{bd} N^c_\omega(\bar{x})) &= \\ (\operatorname{bd}\{\lambda_1 \cdot \{0\} + \lambda_2 \cdot [-1,1]\}) \cap (-\operatorname{bd}\{0\}) &= (\operatorname{bd}[-1,1]) \cap \{0\} = \emptyset.\end{aligned}$$

This entails assumption 3. Hence, all assumptions of Theorem 4.2 with the exception of the regularity of g_2 are satisfied. Now, with the same sequences as in the end of Example 3.3, it is easily checked, that M fails to be calm at $\bar{x}$.

Consider now a situation associated with a parametric nonlinear complementarity problem (NCP):

For a given $p \in \mathbb{R}^k$, find $x \in \mathbb{R}^n_+$ such that

$$F(p,x) \ge 0, \quad \langle x, F(p,x) \rangle = 0, \quad (p,x) \in \omega, \tag{26}$$

where $F:\mathbb{R}^k\times\mathbb{R}^n\to\mathbb{R}^n$ is assumed to be continuously differentiable and $\omega\subseteq\mathbb{R}^k\times\mathbb{R}^n$ is closed. Putting $\Phi(p,x):=(x,-F(p,x))$, the non-abstract part of (26) can be equivalently written in the form $\Phi(p,x)\in\operatorname{Gph}N_{\mathbb{R}^n_+}$, in which case N reduces to the classical normal cone of convex analysis. We define a multifunction $M:\mathbb{R}^{2n}\rightrightarrows\mathbb{R}^k\times\mathbb{R}^n$ by

$$M(y)=\left\{(p,x)\in\omega|\Phi(p,x)+y\in\operatorname{Gph}N_{\mathbb{R}^n_+}\right\}.$$

Theorem 4.4 *Let $(0,\bar{p},\bar{x})\in\operatorname{Gph}M$ and assume that*

1. *ω is closed and regular at $(\bar{p},\bar{x})$.*
2. *the constraint qualification*

$$\left.\begin{array}{l}\left(-\left[\nabla_p F(\bar{p},\bar{x})\right]^T z, w-\left[\nabla_x F(\bar{p},\bar{x})\right]^T z\right)\in-\operatorname{bd}N^c_\omega(\bar{p},\bar{x})\\ \text{for some}\quad (w,z)\in N_{\operatorname{Gph}N_{\mathbb{R}^n_+}}(\bar{x},-F(\bar{p},\bar{x}))\end{array}\right\}\Longrightarrow w=0,\ z=0$$

is satisfied.

Then, M is calm at $(0,\bar{p},\bar{x})$.

Proof. Our aim is to apply Theorem 3.1 with

$$m:=2n,\ p:=k+n,\ g:=\Phi,\ \Lambda:=\operatorname{Gph}N_{\mathbb{R}^n_+}.$$

Endowing the space $\mathbb{R}^n\times\mathbb{R}^n$ with the norm

$$\|(v_1,v_2)\|_+:=\sqrt{\sum_{i=1}^n\left(\max\left\{|v_1^i|,|v_2^i|\right\}\right)^2},$$

the following point-to-set distance has been calculated in [7] (Prop. 5.1):

$$\operatorname{dist}(0,-\Phi(p,x)+\operatorname{Gph}N_{\mathbb{R}^n_+})=\|\min\{x,F(p,x)\}\|_2,$$

where the minimum has to be understood componentwise. The left-hand side, however, is exactly the value function P of assumption 2(a) in Theorem 3.1. Since concave functions (like 'min') and convex functions (like $\|\cdot\|_2$) are semismooth, P itself is semismooth as a composition.

Finally, observing that

$$D^*\Phi(\bar{p},\bar{x})((w,z)=\begin{bmatrix}0 & -\left[\nabla_p F(\bar{p},\bar{x})\right]^T\\ \mathbf{I}_n & -\left[\nabla_x F(\bar{p},\bar{x})\right]^T\end{bmatrix}\begin{bmatrix}w\\ z\end{bmatrix},$$

we verify that assumption 2 above entails assumption 3 in Theorem 3.1. Summarizing, assumptions 1, 2(a) and 3 of that Theorem are satisfied. ■

5 Conclusion

In a number of perturbed equilibrium problems, including the above NCP, the map M attains the form

$$M(y) = \{(p,x) \in \omega \mid \Phi(p,x) + y \in \operatorname{Gph} Q\}, \tag{27}$$

where $\Phi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^{2n}$ is continuously differentiable and $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ is a multifunction with the closed graph. In this situation the presented theory can be applied, provided we endow $\mathbb{R}^{2n}$ with a suitable norm $\|\cdot\|_+$ such that the value function

$$P(p,x) := \min_{z \in \operatorname{Gph} Q} \|z - \Phi(p,x)\|_+$$

satisfies the requirements of Theorem 3.1. The choice of this norm depends naturally on the structure of the (possibly complicated) nonconvex set $\operatorname{Gph} Q$.

Consider now the optimization problem

$$\begin{array}{ll} \text{minimze} & \Theta(x) \\ \text{subject to} & \\ & x \in M(0) \cap \omega, \end{array} \tag{28}$$

where $\Theta : \mathbb{R}^n \to \mathbb{R}$ is a lipschitzian objective, and assume that $\hat{x}$ is its local solution. By virtue of [9], Lemma 3.1, under the assumptions of Theorem 3.1 there exists a real $R > 0$ and neighborhood $\mathcal{U}$ of $\hat{x}$ such that $\hat{x}$ solves the penalized problem

$$\begin{array}{ll} \text{minimize} & \Theta(x) + R\,P(x) \\ \text{subject to} & \\ & x \in \omega \cap \mathcal{U}. \end{array} \tag{29}$$

Function P can thus be used as a penalty for the numerical solution of (28). Moreover, on the basis of (29) one can derive necessary optimality conditions for (28) so that the assumptions of Theorem 3.1 create a (rather nonrestrictive) constraint qualification.

References

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