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On Polynomial Collocation for Cauchy Singular Integral Equations with Fixed Singularities

Peter Junghanns[1], Andreas Rathsfeld[2]

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[1] Fakultät für Mathematik
TU Chemnitz
Reichenhainer Str. 39-41
D-09107 Chemnitz
Germany
E-Mail: peter.junghanns@mathematik.tu-chemnitz.de

[2] Weierstraß-Institut für Angewandte Analysis und Stochastik
Mohrenstr. 39
D-10117 Berlin
Germany
E-Mail: rathsfeld@wias-berlin.de

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Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
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World Wide Web: http://www.wias-berlin.de/

Abstract

In this paper we consider a polynomial collocation method for the numerical solution of Cauchy singular integral equations with fixed singularities over the interval, where the fixed singularities are supposed to be of Mellin convolution type. For the stability and convergence of this method in weighted $\mathbf{L}^2$ spaces, we derive necessary and sufficient conditions.

1 Introduction

Discretization schemes based on polynomial approximation are the most popular methods for the numerical solution of Cauchy singular integral equations (cf. e.g. [1, 2, 3, 8, 12, 11, 10, 15, 19] and [31, Chapter 9]). The reason is the well-known invariance property for polynomial spaces with respect to the integral operator if the last is multiplied by a correctly chosen weight function. Thus polynomial methods are spectral methods and exhibit optimal convergence properties.

On the other hand, the construction of the optimal weight functions, of the orthogonal polynomials, and of the corresponding collocation nodes is not so simple if the coefficients of the integral operators are not constant. Therefore, it is natural to use Chebyshev nodes even if the intrinsic weight function of the operator is different from the Chebyshev weight. Moreover, if additional fixed singularities occur, the invariance property holds only for the Cauchy singular part and not for the whole operator. Consequently, the usual approximation arguments do not apply, and, there is no motivation to choose complicated weights. Furthermore, iterative methods with integral equations, the coefficient functions of which change in every step of iteration, suggest to chose fixed collocation nodes independently of the coefficient functions (cf. [16]). Even in this case, in comparison to spline methods or trigonometric approaches, numerical experiments for several equations (cf. [29, 28, 34]) promise better approximation results for the polynomial collocation though the asymptotic orders of convergence are the same.

Polynomial methods have been considered for special integral equations e.g. in [24, 27, 28, 34]. These methods at least together with slight modifications are expected to converge for all invertible operators. Similarly, for properly chosen weight functions and the corresponding nodes, the invertibility of the Cauchy singular integral operator is the only condition needed to ensure the stability of the polynomial collocation. However, if the collocation nodes are chosen independently of the intrinsic weights, then there arise additional stability conditions expressed in form of the invertibility of related operators (cf. the special cases treated in [20, 18, 17]). Banach algebra techniques including local principles are the main tools to prove such results.

In the present paper we consider a Cauchy singular integral equation with additional fixed singularities and we propose a simple polynomial collocation method based on Chebyshev nodes of the second kind. We describe the integral equation in Sect. 2. The collocation method and the main result on the stability is formulated in Sect. 3. To prove the main result, in Sect. 4 we recall an abstract result on a general approximation method from [17]. This is based on the Banach algebra techniques due to Roch and Silbermann (cf. [33] and [31, Sect. 10.31–10.41]). In Sect. 5 we conclude the stability proof. Since a lot of results from [17] are employed, we recommend the reader to consult [17] before he turns to Sects. 4 and 5.

Finally, having solved the stability and convergence problems for singular integral equations with fixed singularities, the next essential task is to design algorithms for the assembling of the matrix of the corresponding collocation equations and for the efficient solution of the arising linear systems of equations. These issues will be stressed in future work.

2 The Integral Equation

We consider the following Cauchy singular integral equation with fixed singularities of Mellin convolution type.

$$\begin{aligned} a(x)u(x) + \frac{b(x)}{\mu(x)}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{\mu(y)u(y)}{y-x}\,\mathrm{d}y + \chi^-(x)\int_{-1}^{1}\mathbf{k}^-\!\left(\frac{1+x}{1+y}\right)\frac{u(y)}{1+y}\,\mathrm{d}y \qquad (2.1)\\ +\chi^+(x)\int_{-1}^{1}\mathbf{k}^+\!\left(\frac{1-x}{1-y}\right)\frac{u(y)}{1-y}\,\mathrm{d}y + \int_{-1}^{1}\mathbf{k}_0\,(x,y)\,u(y)\,\mathrm{d}y \;=\; f(x), \quad -1<x<1. \end{aligned}$$

In this equation the coefficient functions a, b are chosen from the space $PC[-1,1]$ of all piecewise continuous complex valued functions defined over $[-1,1]$. For definiteness, we suppose that the functions of $PC[-1,1]$ are continuous at the end points of the interval and at all but a finite number of interior points and that the function values are equal to the limits from the right at all interior points. The real valued weight function $\mu(x)$ is taken to be of Jacobi type $\mu(x) := v^{\mu_-,\mu_+}(x) := (1-x)^{\mu_-}(1+x)^{\mu_+}$, $-1<\mu_\pm<1$. By $\chi^-,\chi^+ \in C^\infty[-1,1]$ we denote cut-off functions, i.e. smooth non-negative functions such that $\chi^-(x) = 1$ for $-1 \le x \le -0.5$, that $\chi^-(x) = 0$ for $0 \le x \le 1$, and that $\chi^+(x) = \chi^-(-x)$. The Mellin kernels $\mathbf{k}^-$ and $\mathbf{k}^+$ are continuous functions over the half axis $\mathbb{R}_+ := (0,\infty)$ such that their Mellin symbols

$$\hat{\mathbf{k}}^\pm(z) \;:=\; \int_0^\infty y^{z-1}\mathbf{k}^\pm(y)\,\mathrm{d}y$$

are analytic in the strips $\{z\in\mathbb{C}:\ \alpha_\pm < \Re z < \beta_\pm\}$, $\alpha_\pm,\beta_\pm \in \mathbb{R}$ and satisfy

$$\sup_{z:\,\alpha_\pm<\Re z<\beta_\pm}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\hat{\mathbf{k}}^\pm(z)(1+|z|)^{1+k}\right| < \infty, \quad k = 0,1,2,\ldots\ . \qquad (2.2)$$

The kernel function $\mathbf{k}_0 :\ [-1,1]\times[-1,1] \longrightarrow \mathbb{C}$ is supposed to be continuous, the right-hand side f is taken from the weighted Lebesgue space $\mathbf{L}^2_\sigma$, and $u \in \mathbf{L}^2_\sigma$ is the unknown function. The inner product in the Hilbert space $\mathbf{L}^2_\sigma$ is given by

$$\langle u,v\rangle_\sigma \;:=\; \int_{-1}^{1} u(y)\overline{v(y)}\sigma(y)\,\mathrm{d}y, \quad \sigma(y) \;:=\; v^{\sigma_-,\sigma_+}(y) \;=\; (1-y)^{\sigma_+}(1+y)^{\sigma_-}\,, \qquad (2.3)$$
$$-1<\sigma_\pm<1.$$

Note that the class of equations (2.1) includes Cauchy singular integral equations (cf. e.g. [30, 14, 26]), boundary integral equations (cf. e.g. [4, 5]), and equations with fixed singularities (cf. e.g. [9]).

We shortly write $Au = f$ for Equation (2.1) where $A := aI + b\mu^{-1}S\mu I + \chi^- K^- + \chi^+ K^+ + K_0$. In other words, aI is the operator of multiplication by a, the letter S stands for the Cauchy singular integral operator with kernel $1/(\pi\mathbf{i})\ 1/(y-x)$, the symbols $K^\pm$ for the Mellin convolution operators with kernels $\mathbf{k}^\pm([1\mp x]/[1\mp y])/[1\mp y]$, and K_0 for the compact integral operator with kernel $\mathbf{k}_0(x,t)$. We shall consider the equation $Au = f$ in the weighted space $\mathbf{L}^2_\sigma$. For our operator A to be bounded in $\mathbf{L}^2_\sigma$ we require (cf. (2.2))

$$|2\mu_\pm - \sigma_\pm| < 1, \tag{2.4}$$

$$\alpha_\pm < \frac{1}{2} + \frac{\sigma_\pm}{2} < \beta_\pm. \tag{2.5}$$

Moreover, for technical reasons (cf. the treatment of the discretized Mellin operators) we additionally require

$$\alpha_\pm < \frac{1}{2} + \frac{\sigma_\pm}{2} - \frac{1}{2} < \frac{1}{2} + \frac{\sigma_\pm}{2} + \frac{1}{2} < \beta_\pm, \tag{2.6}$$

$$\mu_\pm - \frac{\sigma_\pm}{2} \neq -\frac{1}{4}. \tag{2.7}$$

From [9, 23, 13] (cf. also [14] for the special case $K^- = 0 = K^+$), we infer the Fredholm property of the operator A on the left-hand side of Equation (2.1).

Theorem 2.1 *Suppose the conditions (2.4) and (2.5) are satisfied. Then we get*

i) The operator A is a bounded operator in $\mathbf{L}^2_\sigma$.

ii) The operator A is Fredholm if and only if

- *For any x in $(-1,1)$ and for $x' = -1, 1$, there holds $a(x\pm 0) + b(x\pm 0) \neq 0$ and $a(x\pm 0) - b(x \pm 0) \neq 0$ as well as $a(x') + b(x') \neq 0$ and $a(x') - b(x') \neq 0$.*
- *For any x in $(-1,1)$ such that a or b has a jump at x and for any ν with $0 \le \nu \le 1$, there holds*

$$\nu\frac{a(x+0)+b(x+0)}{a(x+0)-b(x+0)} + (1-\nu)\frac{a(x-0)+b(x-0)}{a(x-0)-b(x-0)} \neq 0.$$

- *For $x = \pm 1$ and for any z with $\Re z = 1/2 + \sigma_\pm/2$, there holds*

$$a(\pm 1) \mp b(\pm 1)(-\mathbf{i})\cot\Big(\pi[z - \mu_\pm]\Big) + \hat{\mathbf{k}}^\pm(z) \ \neq\ 0.$$

iii) If A is Fredholm, then its index is equal to minus the winding number of the closed continuous curve $\Gamma := \Gamma_- \cup \Gamma_1 \cup \Gamma_1' \cup \Gamma_2 \cup \Gamma_2' \cup \ldots \cup \Gamma_N \cup \Gamma_N' \cup \Gamma_{N+1} \cup \Gamma_+$ with the orientation given by the subsequent parametrizations. Here N is the number of discontinuity points x_i, $i = 1, \ldots, N$, of the coefficient functions a and b chosen *such that* $x_0 := -1 < x_1 < \ldots < x_N < x_{N+1} := 1$. *Using these* x_i, *the curves* Γ_i, $i = 1, \ldots, N+1$, *and* Γ_i', $i = 1, \ldots, N$, *are given by*

$$\begin{aligned}\Gamma_i &:= \left\{ \frac{a(y)+b(y)}{a(y)-b(y)} : \; x_{i-1} < y < x_i \right\}, \\ \Gamma_i' &:= \left\{ \nu \frac{a(x_i+0)+b(x_i+0)}{a(x_i+0)-b(x_i+0)} + (1-\nu)\frac{a(x_i-0)+b(x_i-0)}{a(x_i-0)-b(x_i-0)} : \; 0 \le \nu \le 1 \right\}.\end{aligned}$$

The curves $\Gamma_\pm$, *connecting the point* 1 *with one of the end points of* Γ_1 *and* Γ_{N+1}, *respectively, are given by the formula*

$$\left\{ \frac{a(\pm 1) \mp b(\pm 1)(-\mathbf{i})\cot\Big(\pi[z-\mu_\pm]\Big) + \hat{\mathbf{k}}^\pm(z)}{a(\pm 1) - b(\pm 1)} : \; z = \frac{1}{2} + \frac{\sigma_\pm}{2} - \mathbf{i}\nu, \; \nu \in \mathbb{R} \right\}.$$

Clearly, if operator A is Fredholm with index zero and if Equation (2.1) has only the trivial solution for the homogeneous right-hand side $f \equiv 0$, then A is invertible and Equation (2.1) is uniquely solvable for any right-hand side $f \in \mathbf{L}^2_\sigma$. The triviality of the solution to the homogeneous equation (triviality of the null space of A) is usually known in applications. In the case of a pure Cauchy singular operator A, i.e. if $K^- = 0$, $K^+ = 0$, $K_0 = 0$, $\mu_\pm = 0$, the triviality of the null space is known and we get

$$\begin{aligned}(-\mathbf{i})\cot\left(\pi\left(\frac{1}{2} + \frac{\sigma_\pm}{2} - \mathbf{i}\nu\right)\right) &= \frac{e^{\mathbf{i}2\pi(1/2+\sigma_\pm/2-\mathbf{i}\nu)}+1}{e^{\mathbf{i}2\pi(1/2+\sigma_\pm/2-\mathbf{i}\nu)}-1}, \\ \frac{a(\pm 1) \mp b(\pm 1)(-\mathbf{i})\cot\Big(\pi[z-\mu_\pm]\Big) + \hat{\mathbf{k}}^\pm(z)}{a(\pm 1) - b(\pm 1)} &= \frac{a(\pm 1) \mp b(\pm 1)(-\mathbf{i})\cot\Big(\pi z\Big)}{a(\pm 1) - b(\pm 1)} \\ &= \begin{cases} (1-\nu_0)\dfrac{a(+1)+b(+1)}{a(+1)-b(+1)} + \nu_0 \cdot 1, & \text{if sign is } + \\ (1-\nu_0)\cdot 1 + \nu_0 \dfrac{a(-1)+b(-1)}{a(-1)-b(-1)}, & \text{if sign is } - \end{cases} \\ \nu_0 &:= \frac{e^{\mathbf{i}2\pi(1/2+\sigma_\pm/2)}e^{-2\pi\nu}}{e^{\mathbf{i}2\pi(1/2+\sigma_\pm/2)}e^{-2\pi\nu}-1}.\end{aligned}$$

Obviously, the last expression of the symbol quotient depending on ν describes a circular arc connecting the point $[a(\pm 1)+b(\pm 1)]/[a(\pm 1)-b(\pm 1)]$ with 1 such that the straight line segment between $[a(\pm 1)+b(\pm 1)]/[a(\pm 1)-b(\pm 1)]$ and 1 is seen from the arc points under an angle of size $\pm 2\pi(1/2+\sigma_\pm/2)$. Thus the invertibility result of Gohberg and Krupnik is covered by the general theorem.

Example 2.1 *Assume, for simplicity,* $\sigma \equiv 1$. *Often (cf. [13]) the solution* u *of Equation (2.1) exhibits a singular behaviour at the end points* ± 1 *of the interval and, thus, is difficult to approximate. In this case a change of variable* $x = \Phi(\xi)$, $t = \Phi(\tau)$ *(cf. the transformation technique in [31], Chapter 11 and the references given in there) with a smooth parametrization* $\Phi : \; [-1,1] \longrightarrow [-1,1]$ *leads to*

$$\tilde{f}(\xi) \;=\; a\Big(\Phi(\xi)\Big)\tilde{u}(\xi) + \frac{b\Big(\Phi(\xi)\Big)}{\tilde{\mu}(\xi)}\frac{1}{\pi\mathbf{i}}\int_{-1}^{1}\frac{\tilde{\mu}(\tau)\tilde{u}(\tau)}{\tau-\xi}\,\mathrm{d}\tau + \int_{-1}^{1}\tilde{\mathbf{k}}_1(\xi,\tau)\,\tilde{u}(\tau)\,\mathrm{d}\tau,$$

$$-1 < \xi < 1, \qquad (2.8)$$

$$\begin{aligned}
\tilde{f}(\xi) &:= f(\Phi(\xi))\sqrt{\Phi'(\xi)}, \quad \tilde{u}(\tau) := u(\Phi(\tau))\sqrt{\Phi'(\tau)},\\
\tilde{\mu}(\xi) &:= (1+\xi)^{\tilde{\mu}_-}(1-\xi)^{\tilde{\mu}_-},\\
\tilde{\mathbf{k}}_1(\xi,\tau) &:= \frac{b\Big(\Phi(\xi)\Big)}{\pi\mathbf{i}}\left[\frac{\mu\Big(\Phi(\tau)\Big)}{\mu\Big(\Phi(\xi)\Big)}\frac{\sqrt{\Phi'(\xi)\cdot\Phi'(\tau)}}{\Phi(\tau)-\Phi(\xi)} - \frac{\tilde{\mu}(\tau)}{\tilde{\mu}(\xi)}\frac{1}{\tau-\xi}\right] +\\
&\quad \Bigg\{\chi^-\Big(\Phi(\xi)\Big)\,\mathbf{k}^-\left(\frac{1+\Phi(\xi)}{1+\Phi(\tau)}\right)\frac{1}{1+\Phi(\tau)} +\\
&\quad \chi^+\Big(\Phi(\xi)\Big)\,\mathbf{k}^+\left(\frac{1-\Phi(\xi)}{1-\Phi(\tau)}\right)\frac{1}{1-\Phi(\tau)} + \mathbf{k}_0\Big(\Phi(\xi),\Phi(\tau)\Big)\Bigg\}\sqrt{\Phi'(\xi)\cdot\Phi'(\tau)}.
\end{aligned}$$

If a certain number of derivatives of Φ vanishes at $\tau = \pm 1$, then $\tilde{u}$ is much smoother than u at ± 1. Similarly to the old kernel, the new kernel function is again a kernel with fixed singularities of Mellin convolution type at ± 1. Since the mapping $u \mapsto \tilde{u}$ is a unitary operator in $\mathbf{L}^2[-1,1]$, the solvability properties in $\mathbf{L}^2[-1,1]$ do not change. To improve the singular behaviour of the solution at ± 1, one can choose e.g. $\Phi(\tau) := \mp 2^{1-m}(1\mp\tau)^m \pm 1$, $m > 1$. For an improvement at both end points, the parametrization can be chosen to be the sigmoidal transform $\Phi(\tau) := 1 - 2(1-\tau)^m/[(1+\tau)^m + (1-\tau)^m]$. A more sophisticated transformation can be found in [21]. Note that this transformation technique is useful also to enlarge the strip of analyticity and to enforce condition (2.6).

Example 2.2 *Suppose the kernel function $\mathbf{k}_2(x,y) = p(1+x,1+y)/q(1+x,1+y)$ is the quotient of two homogeneous polynomials p and q such that the degree of p is the degree of q minus one and such that $q(1+x,1+y)$ does not vanish over $[-1,1]\times[-1,1]$. Then $\mathbf{k}_2$ is of the Mellin convolution type $\mathbf{k}_2(x,y) = \mathbf{k}^-([1+x]/[1+y])/[1+y]$ where $\mathbf{k}^-(z) := p(z,1)/q(z,1)$. Using the decomposition of rational functions into partial fractions, we can split this kernel into the sum of elementary kernels. In particular, to the Mellin convolution kernel*

$$\mathbf{k}_2(x,y) = \frac{1}{\pi\mathbf{i}}\frac{(1+x)^k(1+y)^{j-k}}{[(1+y)e^{i\omega} - (1+x)]^{j+1}}$$

defined for the parameters $0 < \omega < 2\pi$, $j,k \in \mathbb{N}$, $0 \le k \le j$, there corresponds the Mellin symbol function

$$\hat{\mathbf{k}}^-(z) = -\mathbf{i}(-1)^j e^{-\mathbf{i}(j+1)\omega}\begin{pmatrix} z+k-1\\ j\end{pmatrix}\frac{e^{\mathbf{i}(\omega-\pi)(z+k)}}{\sin\pi(z+k)}.$$

Similarly, kernels of the form $\mathbf{k}_3(x,y) = p(1-x,1-y)/q(1-x,1-y)$ can be treated as kernels with fixed singularity at 1. See also the treatment of such operators and their applications in [4, 5, 9].

3 A Collocation Method with Polynomial Trial Functions

We consider a new Jacobi weight function $\vartheta := v^{\vartheta_-,\vartheta_+}$ setting $\vartheta_\pm := 1/4 - \sigma_\pm/2$, and, for $n \in \mathbb{N}$, we introduce the trial space $\vartheta\mathbb{P}_n$ as the space of complex valued polynomials of degree less than n multiplied by the weight ϑ. The collocation points are chosen to be the Chebyshev nodes of the second kind, i.e. they are defined by the formula $x^\varphi_{kn} := \cos(\pi k/(n+1))$, $k = 1, 2, \dots, n$. Now the collocation method seeks an approximation $u_n \in \vartheta\mathbb{P}_n$ for the exact solution of $Au = f$ by solving

$$(Au_n)(x^\varphi_{kn}) = f(x^\varphi_{kn}), \; k = 1, 2, \dots, n. \tag{3.1}$$

This system can be written as an operator equation $A_n u_n := M_n f$, where $A_n := M_n A|_{\vartheta\mathbb{P}_n}$ and where M_n denotes the interpolation defined by $M_n f \in \vartheta\mathbb{P}_n$ and $M_n f(x^\varphi_{kn}) = f(x^\varphi_{kn})$, $k = 1, \dots, n$. Introducing the Chebyshev polynomial of the second kind $U_n(x) := \sqrt{2/\pi}\,\sin((n+1)\arccos(x))/\sin(\arccos(x))$ and choosing the Lagrange basis

$$\tilde{\ell}^\varphi_{kn}(x) := \frac{\vartheta(x)}{\vartheta(x^\varphi_{kn})} \frac{U_n(x)}{(x - x^\varphi_{kn})U_n'(x^\varphi_{kn})}, \tag{3.2}$$

the interpolation projection M_n is defined by $M_n f := \sum_1^n f(x^\varphi_{kn})\tilde{\ell}^\varphi_{kn}$.

Now the collocation is convergent if u_n tends to u for any right-hand side f. However, since we even do not have $M_n f \longrightarrow f$ for any $f \in \mathbf{L}^2_\sigma$, we change the notion of convergence. We say the collocation method is convergent if, for any $f \in \mathbf{L}^2_\sigma$ and for any $f_n \in \vartheta\mathbb{P}_n$ with $f_n \longrightarrow f$, the solutions u_n of $A_n u_n = f_n$ exist uniquely at least for sufficiently large n and if these solutions tend to the exact solution u of $Au = f$ in the norm of $\mathbf{L}^2_\sigma$. In particular, we can choose $f_n := L_n f$ where L_n is the orthogonal projection of $\mathbf{L}^2_\sigma$ onto $\vartheta\mathbb{P}_n$. Next, we endow the trial space $\vartheta\mathbb{P}_n$ with the norm induced from the space $\mathbf{L}^2_\sigma$. We call the collocation method stable if the approximate operators A_n are invertible at least for n sufficiently large and if the norms $\|A_n\|$ and $\|[A_n]^{-1}\|$ are bounded uniformly with respect to n. Note that this notion is equivalent to the boundedness of the condition numbers of the matrices of A_n with respect to the scaled basis $\{\tilde{\ell}^\varphi_{kn}/\omega_{kn}\}$ with $\omega_{kn} := \sqrt{\pi/(n+1)}\sqrt{\varphi(x^\varphi_{kn})\sigma(x^\varphi_{kn})}$, $\varphi := v^{1/2,1/2}$. Hence, stability is important for the solution of the linear system of equations corresponding to (3.1). On the other hand, if the approximate operators $A_n P_n$ tend strongly to $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$, then it is well known (cf. e.g. [31], Chapter 1) that the stability is equivalent to the convergence of the collocation method.

To formulate our main result, we need some technical notation. We define $\varrho = \vartheta^{-1}\varphi = \sqrt{\sigma\varphi}$, i.e. $\varrho := v^{\varrho_-,\varrho_+}$ with $\varrho_\pm := -\vartheta_\pm + 1/2 = \sigma_\pm/2 + 1/4$. Furthermore, we introduce operators $A_\pm$ which will turn out to be limits of the discretized operators A_n. We define the operators $A_\pm := B_\pm + C_\pm$ acting in the space ℓ^2 of square summable infinite sequences by

$$\begin{aligned} B_\pm &:= \left((j+1)^{1/2+\sigma_\pm} b^\pm_{(j+1),(k+1)} (k+1)^{-1/2-\sigma_\pm}\right)_{j,k=0}^\infty, \\ b^\pm_{j,k} &:= a(\pm 1)\delta_{j,k} \pm b(\pm 1) \frac{1}{\pi \mathbf{i}} \frac{k^{2\varrho_\pm}}{j^{2\varrho_\pm}} \frac{2k}{j^2 - k^2} \tilde{\delta}_{j,k} \end{aligned}$$

$$\tilde{\delta}_{j,k} \;:=\; \begin{cases} 2 & \text{if } k \equiv j+1 \bmod 2 \\ 0 & \text{if } k \equiv j \bmod 2 \end{cases} .$$

The operator $C_\pm = ((j+1)^{1/2+\sigma_\pm} c^\pm_{(j+1),(k+1)} (k+1)^{-1/2-\sigma_\pm})_{j,k=0}^\infty \in \mathcal{L}(\ell^2)$ is defined as follows. For fixed $\varrho_\pm$ and $\mu_\pm$ and for $\zeta_\pm := \mu_\pm - \varrho_\pm$, we set

$$\begin{aligned} C^{\mu_\pm} &:= \Big((j+1)^{1/2+\sigma_\pm} c^{\mu_\pm}_{(j+1),(k+1)} (k+1)^{-1/2-\sigma_\pm}\Big)_{j,k=0}^\infty , \\ c^{\mu_\pm}_{j,k} &:= b^\pm_{(k+1)} \delta_{k,j} \pm \frac{1}{\pi \mathbf{i}} \frac{k^{2\mu_\pm}}{j^{2\mu_\pm}} \frac{2k(1-\delta_{k,j})}{j^2-k^2} \mp \frac{1}{\pi \mathbf{i}} \frac{k^{2\varrho_\pm}}{j^{2\varrho_\pm}} \frac{2k(1-\delta_{k,j})}{j^2-k^2} \mp \\ &\quad \frac{1}{\pi \mathbf{i}} \frac{k^{2\mu_\pm}}{j^{2\mu_\pm}} \frac{2k(1-\delta_{k,j})}{j^2-k^2} \frac{(-1)^{k+1}}{\sqrt{2\pi}} d_k^{\zeta_\pm} \pm d_j^{\zeta_\pm} \frac{1}{\pi \mathbf{i}} \frac{k^{2\varrho_\pm}}{j^{2\varrho_\pm}} \frac{2k(1-\delta_{k,j})}{j^2-k^2} \frac{(-1)^{k+1}}{\sqrt{2\pi}} , \qquad (3.3) \\ b^\pm_k &:= \frac{4(-1)^{k+1} k}{\mathbf{i}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1}{[(k\pi)^2 - s^2]^2} \, s \sin s \, \mathrm{d}s , \qquad (3.4) \\ d_k^{\zeta_\pm} &:= 2\sqrt{\frac{2}{\pi}} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{s}{k}\right)^{2\zeta_\pm} - 1}{(k\pi)^2 - s^2} \, s \sin s \, \mathrm{d}s \\ &\qquad (3.5) \\ &\quad + 4\sqrt{\frac{2}{\pi}} \int_{\frac{\pi}{2}}^\infty \cos s \left\{ \frac{s^2 \left[\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1\right]}{[(k\pi)^2 - s^2]^2} + \frac{\zeta_\pm \left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - \frac{1}{2}\left[\left(\frac{s}{k\pi}\right)^{2\zeta_\pm} - 1\right]}{(k\pi)^2 - s^2} \right\} \mathrm{d}s . \end{aligned}$$

The general term $C_\pm$ is composed of $C^{\mu_\pm}$ by the rule

$$\begin{aligned} C_\pm &:= b(\pm 1) C^{\mu_\pm} + \qquad (3.6) \\ &\int_{z:\, \Re z = 0} \frac{1}{2} \left\{ \hat{\mathbf{k}}^\pm\Big(\frac{1}{2} + \frac{\sigma_\pm}{2} - \frac{1}{2} + z\Big) - \hat{\mathbf{k}}^\pm\Big(\frac{1}{2} + \frac{\sigma_\pm}{2} + \frac{1}{2} + z\Big) \right\} C^{\sigma_\pm/2+z} \, \mathrm{d}z . \end{aligned}$$

Note that the operators $A_\pm$ are limit operators of the collocation operator A_n since, for the matrix $(a^n_{j,k})^n_{j,k=1}$ of A_n with respect to the Lagrange basis $\{\tilde{\ell}^\varphi_{kn}\}$, we have the limit relations $\lim_{n\longrightarrow\infty} a^n_{(j+1),(k+1)} = a^+_{j,k} := b^+_{j,k} + c^+_{j,k}$ and $\lim_{n\longrightarrow\infty} a^n_{n-j,n-k} = a^-_{j,k} := b^-_{j,k} + c^-_{j,k}$ (cf. the existence of the strong limits $W_\omega\{A_n\}, \omega = 3,4$ in Lemma 4.1), where $A_\pm = ((j+1)^{1/2-\sigma_\pm} a^\pm_{j,k} (k+1)^{-1/2+\sigma_\pm})_{j,k=0}^\infty \in \mathcal{L}(\ell^2)$ and where $c^\pm_{j,k}$ is defined from $c^{\mu_\pm}_{j,k}$ as $C_\pm$ from $C^{\mu_\pm}$ (cf. (3.6)). Finally, we note that the curves of the Gohberg-Krupnik symbol (cf. Theorem 5.1) describing the Fredholm properties of the operators $A_\pm$ are given by

$$\begin{aligned} \tilde{\Gamma}_\pm &:= \Gamma_\pm \cup \Gamma_\pm^\sim , \\ \Gamma_\pm^\sim &:= \left\{ \frac{a(\pm 1) \pm b(\pm 1)(-\mathbf{i}) \cot\Big(\pi[z - \varrho_\pm]\Big)}{a(\pm 1) - b(\pm 1)} : \; z = \frac{1}{2} + \frac{\sigma_\pm}{2} - \mathbf{i}\nu, \; \nu \in \mathbb{R} \right\} . \qquad (3.7) \end{aligned}$$

Theorem 3.1 *Suppose the conditions (2.4), (2.6), and (2.7) are satisfied. Then the collocation method is convergent and stable if and only if*

i) The operator $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ is invertible (cf. Theorem 2.1).

ii) The closed continuous curves $\tilde{\Gamma}_\pm$ do not contain the zero point and their winding numbers with respect to zero vanish.

iii) The null spaces of the operators $A_\pm \in \mathcal{L}(\ell^2)$ are trivial.

The rest of this paper is devoted to the proof of this main theorem. We introduce the Banach algebra setting necessary for the proof in Sect. 4, and, applying the algebra technique, we complete the proof in Sect. 5. Note that the condition ii) is equivalent to the Fredholm property with index zero for the operators $A_\pm$. Hence, conditions ii) and iii) together are equivalent to the invertibility of the operators $A_\pm$. Consequently, the collocation is stable if and only if the operators A and $A_\pm$ are invertible. This appears to be a natural condition since all these operators can be approximated using the operator sequence A_n.

In principle the conditions i)-ii) are easy to verify if the Mellin symbols $\hat{\mathbf{k}}^\pm$ are known. To check condition iii), however, seems to be hopeless for the general case. Therefore, it is good to know that condition iii) is not "essential" in the sense that the case of condition ii) fulfilled but condition iii) violated is very rare and exceptional:

Remark 3.1 *Fix $\mathbf{k}^\pm$, $b(\pm 1)$, μ, and σ. Consider the set Σ of all complex numbers $z = a(\pm 1)$ such that condition ii) holds. The set of points in Σ such that condition iii) is violated is finite. Note that this fact is a simple consequence of the general theory of analytic families of Fredholm operators.*

Remark 3.2 *If the exceptional case should occur, then the numerical method should be modified slightly. One way to do this is the so called i_* modification introduced in [6] and used also e.g. in [21, 31]. For a stability proof of such a modified method, we refer to [31], Sects. 11.30 and 12.46.*

Remark 3.3 *Define the real numbers $\kappa_\pm$ by*

$$\kappa_\pm := \Im\left\{\frac{1}{2\pi}\log\frac{a(\pm 1)\mp b(\pm 1)}{a(\pm 1)\pm b(\pm 1)}\right\} = \frac{1}{2\pi}\arg\frac{a(\pm 1)\mp b(\pm 1)}{a(\pm 1)\pm b(\pm 1)} \in (-0.25, 0.75]\ .$$

In the particular case of singular integral operators $A = aI + b\mu^{-1}S\mu I + K_0$ the condition ii) in Theorem 3.1 is equivalent to the inequalities $|\kappa_\pm - (\sigma_\pm/2 - \mu_\pm)| < 1/2$ (cf. [17, Theorem 1.1]).

Remark 3.4 *In the particular case of singular integral operators $A = aI + b\mu^{-1}S\mu I$ and of $\mu(x) = \sigma(x)^{1/2}\varphi(x)^{1/2}$ the condition iii) is satisfied whenever condition ii) holds (cf. [17] and the original proofs in [20, 18]).*

Remark 3.5 *There exist compact operators $K_\pm \in \mathcal{L}(\ell^2)$ such that the limit operators $A_\pm$ take the form*

$$\begin{aligned} A_\pm \;=\; & \Big(a(\pm 1)\delta_{j,k}\Big)_{j,k=0}^\infty \pm b(\pm 1)\left(\frac{1}{\pi\mathbf{i}}\frac{2(k+1)\,\tilde{\delta}_{j,k}}{(j+1)^2-(k+1)^2}\right)_{j,k=0}^\infty \\ & + \left(\mathbf{k}^\pm\left(\frac{(j+1)^2}{(k+1)^2}\right)\frac{1}{k+1}\right)_{j,k=0}^\infty \\ & \pm b(\pm 1)\left(\frac{1}{\pi\mathbf{i}}\left[\frac{(k+1)^{2\mu_\pm-2\varrho_\pm}}{(j+1)^{2\mu_\pm-2\varrho_\pm}}-1\right]\frac{2(k+1)(1-\delta_{j,k})}{(j+1)^2-(k+1)^2}\right)_{j,k=0}^\infty + K_\pm . \end{aligned} \tag{3.8}$$

This follows from the Gohberg-Krupnik symbol of $A_\pm$ which will be determined in the proof of Lemma 5.1. In fact, following [31, Sections 11.26 and 11.27] it can easily be shown that the right-hand side of (3.8) has the same symbol.

4 A Banach Algebra Setting for the Stability Analysis

In this section we introduce the Banach algebra of approximate operators together with some auxiliary notation. We formulate the theorem of Roch and Silbermann on the stability of operator sequences in this algebra. To apply this theorem we recall some facts on collocation methods for singular integral operators and the local principle of Allan and Douglas.

4.1 The Algebra of Approximate Operators Defined by Limit Operators

For the definition of the algebra, we need some further spaces and operator sequences defined with the help of special basis functions. By U_n, $n = 0, 1, 2, \ldots$, we denote the normalized Chebyshev polynomials

$$U_n(\cos s) := \sqrt{\frac{2}{\pi}} \frac{\sin(n+1)s}{\sin s}, \quad n = 0, 1, 2, \ldots ,$$

of the second kind. These U_n are the orthogonal polynomials with respect to the Chebyshev weight of second kind $\varphi(x)$, and the points x^{φ}_{jn} are the zeros of U_n. We set

$$\widetilde{u}_n(x) := \vartheta(x) U_n(x), \quad n = 0, 1, 2, \ldots ,$$

with $\vartheta := \sqrt{\sigma^{-1}\varphi} = v^{\frac{1}{4}-\frac{\sigma_-}{2}, \frac{1}{4}-\frac{\sigma_+}{2}}$. Then the solution of (3.1) can be represented by

$$u_n(x) = \sum_{k=0}^{n-1} \xi_{kn} \widetilde{u}_k(x),$$

and, with respect to the orthonormal system $\{\widetilde{u}_n\}_{n=0}^{\infty}$ in $\mathbf{L}^2_\sigma$, the orthogonal projection L_n takes the form

$$L_n u = \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\sigma \widetilde{u}_k .$$

The projection M_n is the weighted interpolation operator $M_n := \vartheta L^{\varphi}_n \vartheta^{-1} I$, where L^{φ}_n denotes the classical polynomial Lagrange interpolation operator with respect to the nodes x^{φ}_{jn}, $j = 1, \ldots, n$. By ℓ^2 we denote the Hilbert space of all square summable sequences $\xi := \{\xi_k\}_{k=0}^{\infty}$ of complex numbers equipped with the inner product $\langle \xi, \eta \rangle_{\ell^2} := \sum_{k=0}^{\infty} \xi_k \overline{\eta_k}$. Finally, we retain the definition of the discrete weights

$$\omega_{kn} := \sqrt{\frac{\pi}{n+1} \varphi(x^{\varphi}_{kn}) \sigma(x^{\varphi}_{kn})} = \sqrt{\frac{\pi}{n+1}}\, v^{\frac{1}{4}+\frac{\sigma_-}{2}, \frac{1}{4}+\frac{\sigma_+}{2}}(x^{\varphi}_{kn}) .$$

Now we define four limit operators. We introduce the index set $T := \{1,2,3,4\}$, and, for $\omega \in T$, we define projections $L_n^{(\omega)}$ on the Hilbert spaces $\mathbf{X}_\omega$ and operators $E_n^{(\omega)} : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n^{(\omega)}$. In particular, we define the spaces $\mathbf{X}_\omega$, the projections $L_n^{(\omega)}$, and the operators $E_n^{(\omega)}$ by $\mathbf{X}_1 := \mathbf{X}_2 := \mathbf{L}^2_\sigma$, $\mathbf{X}_3 := \mathbf{X}_4 := \ell^2$, $L_n^{(1)} := L_n^{(2)} := L_n$, $L_n^{(3)} := L_n^{(4)} := P_n$, $E_n^{(1)} := L_n$, $E_n^{(2)} := W_n$, $E_n^{(3)} := V_n$, $E_n^{(4)} := \widetilde{V}_n$, where

$$\begin{aligned} W_n u &:= \sum_{k=0}^{n-1} \langle u, \widetilde{u}_{n-1-k} \rangle_\sigma \, \widetilde{u}_k \,, \\ P_n\{\xi_0, \xi_1, \xi_2, \ldots\} &:= \{\xi_0, \ldots, \xi_{n-1}, 0, 0, \ldots\} \,, \\ V_n u &:= \{\omega_{1n} u(x_{1n}^\varphi), \ldots, \omega_{nn} u(x_{nn}^\varphi), 0, 0, \ldots\} \,, \\ \widetilde{V}_n u &:= \{\omega_{nn} u(x_{nn}^\varphi), \ldots, \omega_{1n} u(x_{1n}^\varphi), 0, 0, \ldots\} \,. \end{aligned} \tag{4.1}$$

For an arbitrary sequence $\{B_n\}$ with $B_n \in \mathcal{L}(\operatorname{im} L_n)$, the limits in $\mathcal{L}(\mathbf{X}_\omega)$ are defined as the strong limits $W_\omega\{B_n\} := \lim_{n\to\infty} E_n^{(\omega)} B_n (E_n^{(\omega)})^{-1} L_n^{(\omega)}$. In particular, for the collocation sequence $\{A_n\}$, these four limit operators $W_\omega\{A_n\}$ for $\omega \in T$ exist (cf. the subsequent Lemma 4.1).

Next we define the algebra of operator sequences - the basis of our further considerations. By $\mathcal{F}$ we denote the set of all sequences $\{B_n\} = \{B_n\}_{n=1}^\infty$ of linear operators $B_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$, for which there exist operators $W_\omega\{B_n\} \in \mathcal{L}(\mathbf{X}_\omega)$ such that, for all $\omega \in T$,

$$E_n^{(\omega)} B_n (E_n^{(\omega)})^{-1} L_n^{(\omega)} \longrightarrow W_\omega\{B_n\}, \quad \left(E_n^{(\omega)} B_n (E_n^{(\omega)})^{-1} L_n^{(\omega)}\right)^* \longrightarrow W_\omega\{B_n\}^*$$

holds in the sense of strong convergence for $n \to \infty$. If we define

$$\lambda_1\{B_n\} + \lambda_2\{C_n\} := \{\lambda_1 B_n + \lambda_2 C_n\}\,,\ \{B_n\}\{C_n\} := \{B_n C_n\}\,,\ \{B_n\}^* := \{B_n^*\}\,,$$

and

$$\|\{B_n\}\|_{\mathcal{F}} := \sup\left\{ \|B_n L_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} : n = 1, 2, \ldots \right\},$$

then it is not hard to see that $\mathcal{F}$ is a C^*-algebra with the unit element $\{L_n\}$. We introduce the set $\mathcal{J}$ of all sequences of the form

$$\sum_{\omega=1}^{4} \left\{ (E_n^{(\omega)})^{-1} L_n^{(\omega)} T_\omega E_n^{(\omega)} \right\} + \{C_n\}\,,$$

where $T_\omega \in \mathcal{K}(\mathbf{X}_\omega)$ and where $\|C_n L_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \longrightarrow 0$. Due to [17, Cor. 2.2], $\mathcal{J}$ is a subset of $\mathcal{F}$. The following theorem of Roch and Silbermann is crucial for our stability and convergence analysis.

Theorem 4.1 ([31],Theorem 10.33 and [17],Theorem 2.3) *The set $\mathcal{J}$ forms a two-sided closed ideal of $\mathcal{F}$. A sequence $\{B_n\} \in \mathcal{F}$ is stable if and only if the operators $W_\omega\{B_n\} \in \mathcal{L}(\mathbf{X}_\omega)$, $\omega \in T$, are invertible and if the coset $\{B_n\} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$.*

Now, for the singular integral operator $A \in \mathcal{L}(\mathbf{L}^2_\sigma)$ (cf. (2.1)), we show that the sequence $\{A_n := M_n A L_n\}$ belongs to the algebra $\mathcal{F}$, and we compute $W_\omega\{A_n\}$. For multiplication operators, singular integral operators $\mu^{-1} S \mu I$ with a real valued weight μ, and for compact integral operators, this fact is well known (cf. [17, Lemmata 3.8 – 3.10]). If the weight μ is defined with general complex $\mu_\pm$ such that

$$|2\Re\mu_\pm - \sigma_\pm| < 1, \qquad \Re\mu_\pm - \frac{\sigma_\pm}{2} \neq -\frac{1}{4}, \tag{4.2}$$

then we get $\{A_n\} \in \mathcal{F}$ as well. In other words, it is not hard to derive

Lemma 4.1 *Suppose $a, b \in$ **PC** and $\mu = v^{\mu_-,\mu_+}$, $\varrho = v^{\varrho_-,\varrho_+}$ with (4.2) and $\varrho_\pm := 1/4 + \sigma_\pm/2$. For the analyticity strips $\{z \in \mathbb{C} : \alpha_\pm < \Re z < \beta_\pm\}$ of the Mellin symbols $\hat{\mathbf{k}}^\pm$ of the Mellin kernel functions $\mathbf{k}_\pm$, suppose (2.6). Furthermore, consider the collocation approximation $A_n = M_n A L_n$ for $A = aI + b\mu^{-1}S\mu I + \chi^- K^- + \chi^+ K^+ + K_0$ and suppose that $A_\pm$ is defined as for the Theorem 3.1. Then $\{A_n\} \in \mathcal{F}$ and*

$$W_1\{A_n\} = A, \qquad W_2\{A_n\} = \tilde{A} := aI - b\varrho^{-1}S\varrho I, \tag{4.3}$$

$$W_3\{A_n\} = A_+, \qquad W_4\{A_n\} = A_-. \tag{4.4}$$

Proof: i) The special case of A with real valued μ and with no fixed singularities, i.e. $K^- = K^+ = 0$, has been treated already in [17, Lemmata 3.8 – 3.10].

ii) Moreover, the corresponding results remain valid if we replace the real exponents $\mu_\pm$ of the weight function μ by complex numbers provided that (4.2) holds. Indeed, we set $B := \mu^{-1}S\mu I - \varrho^{-1}S\varrho I$ and $B_n := M_n B L_n$ and prove the uniform boundedness and the strong convergences of B_n as follows.

First we recall from [17, Equations (3.24), (3.25)] the representation

$$\begin{aligned} E_n^{(3)} B_n (E_n^{(3)})^{-1} &= \left(\frac{\omega_{(j+1)n}}{\omega_{(k+1)n}} \left(B\widetilde{\ell}^{\varphi}_{(k+1)n} \right) (x^{\varphi}_{(j+1)n}) \right)_{j,k=0}^{n-1} \\ &= \mathbf{B}_n + \mathbf{D}_n \mathbf{A}_n \mathbf{D}_n^{-1} - \mathbf{A}_n - \mathbf{D}_n \mathbf{A}_n \mathbf{W}_n \mathbf{V}_n \mathbf{D}_n^{-1} + \mathbf{V}_n \mathbf{A}_n \mathbf{W}_n \end{aligned} \tag{4.5}$$

with

$$\mathbf{B}_n := \left(\left(B\widetilde{\ell}^{\varphi}_{(j+1)n} \right) (x^{\varphi}_{(j+1)n}) \delta_{k,j} \right)_{j,k=0}^{n-1}, \quad \mathbf{A}_n := \left(\frac{\varphi(x^{\varphi}_{(k+1)n})}{\mathbf{i}(n+1)} \frac{1 - \delta_{j,k}}{x^{\varphi}_{(k+1)n} - x^{\varphi}_{(j+1)n}} \right)_{j,k=0}^{n-1},$$

$$\mathbf{W}_n := \left(\frac{(-1)^{j+1}}{\sqrt{2\pi}} \delta_{k,j} \right)_{j,k=0}^{n-1}, \quad \mathbf{V}_n := \left(d^n_{j+1} \delta_{k,j} \right)_{j,k=0}^{n-1}, \quad \mathbf{D}_n := \left(\frac{\rho(x^{\varphi}_{(j+1)n})}{\mu(x^{\varphi}_{(j+1)n})} \delta_{k,j} \right)_{j,k=0}^{n-1},$$

where the diagonal elements in $\mathbf{A}_n$ are equal to zero by definition. For the other entries $a_{j,k}$ of $\mathbf{A}_n$, we conclude

$$a_{j,k} = \begin{cases} \dfrac{1}{\pi\mathbf{i}} \dfrac{2k}{j^2 - k^2} + \mathcal{O}\left(\dfrac{k}{n^2} \right) & \text{if } j, k \leq \frac{3n}{4} \\ \dfrac{1}{\pi\mathbf{i}} \dfrac{2(n+1-k)}{(n+1-j)^2 - (n+1-k)^2} + \mathcal{O}\left(\dfrac{(n+1-k)}{n^2} \right) & \text{if } j, k \geq \frac{n}{4} \\ \mathcal{O}\left(\dfrac{(n+1-k)}{n^2} \right) & \text{if } j \leq \frac{3n}{4},\ k \geq \frac{n}{4} \\ \mathcal{O}\left(\dfrac{k}{n^2} \right) & \text{if } j \geq \frac{n}{4},\ k \leq \frac{3n}{4}. \end{cases} \tag{4.6}$$

Indeed, if $j,k \le 3n/4$, then we introduce the smooth function ϕ by $\cos(x)=\phi(x^2)$ and conclude $\sin(x)=-\phi'(x^2)2x$ as well as

$$\begin{aligned}
a_{j,k} &= \frac{1}{\mathbf{i}(n+1)}\frac{\sin\left(\pi\frac{k}{n+1}\right)}{\cos\left(\pi\frac{k}{n+1}\right)-\cos\left(\pi\frac{j}{n+1}\right)} = \frac{2\left(\pi\frac{k}{n+1}\right)}{\mathbf{i}(n+1)} \quad\times \\
&\left\{\frac{1}{\left(\pi\frac{j}{n+1}\right)^2-\left(\pi\frac{k}{n+1}\right)^2}+\left[\frac{\phi'\left(\left[\pi\frac{k}{n+1}\right]^2\right)}{\phi\left(\left[\pi\frac{j}{n+1}\right]^2\right)-\phi\left(\left[\pi\frac{k}{n+1}\right]^2\right)}-\frac{1}{\left(\pi\frac{j}{n+1}\right)^2-\left(\pi\frac{k}{n+1}\right)^2}\right]\right\} \\
&= \frac{2\left(\pi\frac{k}{n+1}\right)}{\mathbf{i}(n+1)}\left\{\frac{1}{\left(\pi\frac{j}{n+1}\right)^2-\left(\pi\frac{k}{n+1}\right)^2}\right. \\
&\quad\left.+\frac{\phi'\left(\left[\pi\frac{k}{n+1}\right]^2\right)\left[\left(\pi\frac{j}{n+1}\right)^2-\left(\pi\frac{k}{n+1}\right)^2\right]-\left[\phi\left(\left[\pi\frac{j}{n+1}\right]^2\right)-\phi\left(\left[\pi\frac{k}{n+1}\right]^2\right)\right]}{\left[\phi\left(\left[\pi\frac{j}{n+1}\right]^2\right)-\phi\left(\left[\pi\frac{k}{n+1}\right]^2\right)\right]\left[\left(\pi\frac{j}{n+1}\right)^2-\left(\pi\frac{k}{n+1}\right)^2\right]}\right\} \\
&= \frac{1}{\pi\mathbf{i}}\frac{2k}{j^2-k^2}+\mathcal{O}\left(\frac{k}{n^2}\right).
\end{aligned}$$

The other bounds in (4.6) follow analogously or are even easier. Now, from the representation in [17, Equation (7.3)] we obtain that the matrix $(j^{2\gamma}[2k(1-\delta_{j,k})]/[j^2-k^2]k^{-2\gamma})_{j,k}$ is bounded in the space ℓ^2 for $\Re\gamma=\rho_\pm-\mu_\pm$ and $\Re\gamma=0$. This fact, Estimate (4.6), and simple Frobenius norm estimates prove that the matrices A_n and $D_nA_nD_n^{-1}$ are uniformly bounded. On the other hand, the decay estimates [17, Equations (3.31), (3.32)] for the diagonal entries of B_n and V_n remain valid for the case of complex exponents $\mu_\pm$. Consequently, due to (4.5), we obtain that the matrix B_n is uniformly bounded.

Using the uniform boundedness, the limit relations (4.4) and the corresponding limits for the adjoint sequences follow analogously to the real case treated in [17, Proof of Lemma 3.10, points iv) and v)]. To prove the first relation of (4.3), we remark that it is sufficient to show $M_nBL_nf\longrightarrow Bf$ for an arbitrary but fixed polynomial f such that f vanishes at the end points ±1. For n larger than the degree of f, however, we get $M_nBL_nf=M_nBf$, and Bf takes the form $\mu(x)^{-1}[S(\mu f)](x)-\varrho(x)^{-1}[S(\varrho f)](x)$. Due to well-known mapping properties of the Cauchy singular operator we get the continuity of $S(\mu f)$ and $S(\varrho f)$, and $M_nBf\longrightarrow Bf$ follows (cf. [20, Lemma 3.1]).

To show the limit $(M_nBL_n)^*L_nf\longrightarrow B^*f$, we only have to consider smooth functions f vanishing in neighbourhoods of the end points ±1. Thus we may introduce a smooth cut-off function χ, $0\le\chi\le1$ vanishing in some neighbourhoods of ±1 and such that $\chi f=f$. Due to the uniform boundedness it remains to prove $(M_nBL_n)^*M_nf=(M_nBL_n)^*M_n\chi L_nM_nf\longrightarrow B^*f$. Using $(M_n\chi L_n)^*=M_n\chi L_n$ (cf. [17, Lemma 3.8]), we have to derive the convergence $(M_nBL_n)^*(M_n\chi L_n)^*M_nf=(M_n\chi L_nM_nBL_n)^*M_nf\longrightarrow B^*f$. This is equivalent to showing $(M_n\chi BL_n)^*M_nf\longrightarrow B^*f$. Since χB maps $\mathbf{L}^2_\sigma$ compactly into $C[-1,1]$, we conclude $\|(M_n-L_n)\chi BL_n\|\longrightarrow0$, and it remains to show that $(L_n\chi BL_n)^*M_nf\longrightarrow(\chi B)^*f=B^*f$. This, however, is obvious from the strong convergence of $L_n^*=L_n$ to I and from the smoothness of f.

The limit relation $W_n(M_nBL_n)^*W_nL_nf\longrightarrow0$ follows analogously. Indeed, using the commutativity of W_n and $M_n\chi L_n$ (cf. [17, Equations (2.2) and (3.5)]), it is sufficient to derive $W_n(M_n\chi BL_n)^*W_nL_nf\longrightarrow0$ or $W_n(L_n\chi BL_n)^*W_nL_nf=W_n(\chi B)^*W_nf\longrightarrow0$.

Since $W_n = W_n L_n$ tends weakly to zero (cf. [17, Lemma 2.1]) and since $\chi B : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ is compact, the sequence $W_n(\chi B)^* W_n$ converges strongly to zero (cf. [31, Section 1.1, (f)]), and the relation $W_n(M_n B L_n)^* W_n L_n f \longrightarrow 0$ is proved.

Next we turn to the proof of $W_n M_n B L_n W_n L_n f \longrightarrow 0$. Surely, there is an $\varepsilon^b > 0$ such that the support of χ is contained in $[-1+\varepsilon^b, 1-\varepsilon^b]$. We choose $\varepsilon^a < \varepsilon^b$ and consider a second cut-off function $\tilde{\chi}$ such that the support of $1-\tilde{\chi}$ is contained in the union $[-1,-1+\varepsilon^a] \cup [1-\varepsilon^a, 1]$. Then, for any prescribed $\varepsilon > 0$, there is an ε^a small in comparison to ε^b such that $\|M_n(1-\tilde{\chi})L_n M_n B L_n M_n \chi L_n\| \leq \varepsilon$. Indeed, this is a consequence of the estimate (4.6), of simple Frobenius norm estimates, and of the corresponding estimate with $M_n B L_n$ replaced by the matrix $(1/(\pi\mathbf{i})\, 2k/(j^2-k^2))_{j,k=1}^n$. This last estimate follows from [17, Lemma 4.1, i)] and from the fact that $(1/(\pi\mathbf{i})\, 2k/(j^2-k^2))_{j,k=1}^\infty$ is in the algebra *alg* $\mathcal{T}(\mathbf{PC})$ of operators in $\mathcal{L}(\ell^2)$ generated by the Toeplitz matrices with piecewise continuous symbols (cf. [17, Lemma 7.1, ii)] and use the representation analogous to [17, Equation (7.3)]). Now, in view of the estimate $\|M_n(1-\tilde{\chi})L_n M_n B L_n M_n \chi L_n\| \leq \varepsilon$, we only have to show $W_n M_n \tilde{\chi} L_n M_n B L_n W_n L_n f = W_n M_n \tilde{\chi} B W_n L_n f \longrightarrow 0$ or $W_n L_n \tilde{\chi} B L_n W_n L_n f = W_n \tilde{\chi} B W_n f \longrightarrow 0$, and we are in the same situation as above.

iii) Now we turn to the Mellin operators. For similarity reasons, we restrict our consideration to K^-. Due to the assumption (2.2) and (2.6), we get

$$\begin{aligned}
\mathbf{k}^-(s) &= \frac{1}{2\pi\mathbf{i}} \int_{\Re z = 1/2+\sigma_-/2} s^{-z}\hat{\mathbf{k}}^-(z)\,\mathrm{d}z, \\
\mathbf{k}^-(s)s^{\pm 1/2} &= \frac{1}{2\pi\mathbf{i}} \int_{\Re z = 1/2+\sigma_-/2} s^{-(z\mp 1/2)}\hat{\mathbf{k}}^-(z)\,\mathrm{d}z \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\Re z = 1/2+\sigma_-/2} s^{-z}\hat{\mathbf{k}}^-(z\pm 1/2)\,\mathrm{d}z, \\
\mathbf{k}^-(s)\left[s^{-1/2}-s^{1/2}\right] &= \frac{1}{2\pi\mathbf{i}} \int_{\Re z = 1/2+\sigma_-/2} s^{-z}\left\{\hat{\mathbf{k}}^-(z-1/2)-\hat{\mathbf{k}}^-(z+1/2)\right\}\,\mathrm{d}z, \qquad (4.7) \\
\mathbf{k}^-(s) &= \int_{\Re z = 1/2+\sigma_-/2} \left[\frac{1}{\pi\mathbf{i}}\frac{s^{-z+1/2}}{1-s}\right]\frac{1}{2}\left\{\hat{\mathbf{k}}^-(z-1/2)-\hat{\mathbf{k}}^-(z+1/2)\right\}\,\mathrm{d}z, \qquad (4.8) \\
\mathbf{k}^-\left(\frac{x}{t}\right)\frac{1}{t} &= \int_{\Re z = 1/2+\sigma_-/2} \left[\frac{1}{\pi\mathbf{i}}\frac{\frac{x^{-z+1/2}}{t^{-z+1/2}}}{t-x}\right]\frac{1}{2}\left\{\hat{\mathbf{k}}^-(z-1/2)-\hat{\mathbf{k}}^-(z+1/2)\right\}\,\mathrm{d}z.
\end{aligned}$$

Whereas the first Mellin transform integral in the last sequence of formulas holds in the L^2 sense (compare the definition of the Fourier transform for L^2 functions), the last integrand behaves like $\mathcal{O}(|z|^{-2})$ for $|z| \longrightarrow \infty$, i.e. the last integrand is absolutely summable. Note that (4.7) for $s=1$ implies

$$0 = \frac{1}{2\pi\mathbf{i}} \int_{\Re z = 1/2+\sigma_-/2} \left\{\hat{\mathbf{k}}^-(z-1/2)-\hat{\mathbf{k}}^-(z+1/2)\right\}\,\mathrm{d}z. \qquad (4.9)$$

From (4.8) and (4.9) we obtain

$$\mathbf{k}^-\left(\frac{1+x}{1+t}\right)\frac{1}{1+t} = \int_{\Re z = 1/2+\sigma_-/2} \frac{1}{2}\left\{\hat{\mathbf{k}}^-\left(z-\frac{1}{2}\right)-\hat{\mathbf{k}}^-\left(z+\frac{1}{2}\right)\right\}\times$$

$$\left[\frac{1}{\pi\mathbf{i}}\frac{(1+x)^{1/2-z}(1+t)^{z-1/2}}{t-x} - \frac{1}{\pi\mathbf{i}}\frac{\varrho(x)^{-1}\varrho(t)}{t-x}\right] \,\mathrm{d}z,$$

$$K^- = \int_{\Re z=0} \frac{1}{2}\left\{\hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z-\frac{1}{2}\right) - \hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z+\frac{1}{2}\right)\right\} \times \left[[\delta_-^z]^{-1}S\delta_-^z - \varrho^{-1}S\varrho\right] \mathrm{d}z, \tag{4.10}$$

$$\delta_-^z(x) := (1+x)^{\sigma_-/2+z}(1-x)^0.$$

This representation of K^- is an integral in the sense of Bochner of an absolutely integrable and continuous function, and K^- is the limit of finite Riemann sums with respect to the operator norm. If we discretize the last integral representation of the operator by polynomial collocation, then we get

$$M_nK^-L_n = \int_{\Re z=0} \frac{1}{2}\left\{\hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z-\frac{1}{2}\right) - \hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z+\frac{1}{2}\right)\right\} \times \left[M_n[\delta_-^z]^{-1}S\delta_-^zL_n - M_n\varrho^{-1}S\varrho L_n\right] \mathrm{d}z. \tag{4.11}$$

Applying the results and formulae from part ii) of the present proof to the integrand, we arrive at the limits introduced before Theorem 3.1. Note that here part ii) is applied with the choice $''\mu_-'' = \sigma_-/2+z$ and $''\mu_+'' = 0$. However, if $\sigma_+ = 1/2$, then the corresponding assumption (2.7) is violated. In this case we choose a small $\varepsilon > 0$, set $''\mu_+'' = \varepsilon$, and start from a representation for $(1-x)^{-\varepsilon}K^-(1-t)^\varepsilon$. The difference of $(1-x)^{-\varepsilon}K^-(1-t)^\varepsilon$ and K^- is a compact integral operator with a sufficiently smooth kernel. ∎

Finally, we introduce a new class of sequences belonging to $\mathcal{F}$. We consider the C^* algebra $\mathcal{L}(\ell^2)$ of continuous operators in ℓ^2. By $alg\,\mathcal{T}(\mathbf{PC})$ we denote the closed subalgebra generated by the Toeplitz matrices $(\hat{g}_{k-j})_{k,j=0}^\infty$ with piecewise continuous symbols $g(t) := \sum_{l\in\mathbb{Z}}\hat{g}_l t^l$ defined on

$$\mathbb{T} := \{t \in \mathbb{C} : \ |t| = 1\}$$

and continuous on $\mathbb{T}\backslash\{\pm 1\}$. For $R \in alg\,\mathcal{T}(\mathbf{PC})$, we use the projections P_n from (4.1) and define the finite sections $R_n := P_nR|_{\mathrm{im}\,P_n} \in \mathcal{L}(\mathrm{im}\,P_n)$. Furthermore, using the notation from the beginning of this section, we form the operators

$$R_n^\omega := (E_n^{(\omega)})^{-1}R_nE_n^{(\omega)}, \quad \omega \in \{3,4\}, \tag{4.12}$$

mapping $\mathrm{im}\,L_n$ into $\mathrm{im}\,L_n$. We infer from [17, Lemma 4.1, ii)]

Lemma 4.2 *For any $R \in alg\,\mathcal{T}(\mathbf{PC})$, the sequences $\{R_n^\omega\}_{n=0}^\infty$, $\omega \in \{3,4\}$, belong to $\mathcal{F}$.*

4.2 The Local Principle of Allan and Douglas

For a local analysis, the algebra $\mathcal{F}$ is still too large. By $\mathcal{A}$ we denote the smallest C^*-subalgebra of $\mathcal{F}$ generated by all sequences of the ideal $\mathcal{J}$, by all sequences $\{R_n^\omega\}$ with $\omega \in \{3,4\}$ and $R \in alg\,\mathcal{T}(\mathbf{PC})$, and by all sequences of the form

$$\{M_n(aI + b\mu^{-1}S\mu I)L_n\}, \quad a, b \in \mathbf{PC},$$

where $\mu := v^{\mu_-,\mu_+}$ satisfies (2.4) and (2.7). Note that the operator sequences $\{M_nK^{\pm}L_n\}$ are automatically in $\mathcal{A}$ due to the representation (4.11) for K^- and the corresponding representation for K^+. While proving Theorem 3.1, we have to show the invertibility of the coset, corresponding to the collocation sequence, in the quotient algebra $\mathcal{F}/\mathcal{J}$ (cf. Theorem 4.1). This will be done by showing the invertibility in the quotient algebra $\mathcal{A}/\mathcal{J}$. If $\{A_n\} \in \mathcal{F}$, then we write $\{A_n\}^o$ for the coset $\{A_n\} + \mathcal{J}$ of $\mathcal{F}/\mathcal{J}$.

The starting point for the local principle is a large subalgebra of the centre of $\mathcal{A}$. We borrow from [17, Lemma 5.1]

Lemma 4.3 *The cosets $\{M_nfL_n\}^o$, where $f \in \mathbf{C}[-1,1]$, belong to the center of $\mathcal{A}/\mathcal{J}$.*

Using this, we introduce the central subalgebra $\mathcal{C} := \{\{M_nfL_n\}^o : f \in \mathbf{C}[-1,1]\}$. This algebra is $*$-isomorphic to $\mathbf{C}[-1,1]$ via the isomorphism $\{M_nfL_n\}^o \mapsto f$, and, consequently, the maximal ideal space of $\mathcal{C}$ is equal to $\{\mathcal{I}_\tau : \tau \in [-1,1]\}$ with the ideals $\mathcal{I}_\tau := \{\{M_nfL_n\}^o : f \in \mathbf{C}[-1,1],\ f(\tau) = 0\}$. By $\mathcal{J}_\tau$ we denote the smallest closed ideal of $\mathcal{A}/\mathcal{J}$ which contains $\mathcal{I}_\tau$, i.e.

$$\mathcal{J}_\tau := \operatorname{clos}_{\mathcal{A}/\mathcal{J}} \left\{ \sum_{j=1}^{m} \{A_n^jM_nf_jL_n\}^o : \{A_n^j\} \in \mathcal{A},\ f_j \in \mathbf{C}[-1,1],\ f_j(\tau) = 0,\ m = 1,2,\dots \right\}. \tag{4.13}$$

The local principle of Allan and Douglas claims

Theorem 4.2 ([7] and [31], Theorem 1.21) *The ideal $\mathcal{J}_\tau$ is a proper ideal in $\mathcal{A}/\mathcal{J}$ for all $\tau \in [-1,1]$. Suppose $\{A_n\}^o$ is an arbitrary element of $\mathcal{A}/\mathcal{J}$. Then $\{A_n\}^o$ is invertible if and only if $\{A_n\}^o + \mathcal{J}_\tau$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ for all $\tau \in [-1,1]$.*

5 The Proof of the Main Theorem

To prove Theorem 3.1 we apply Theorem 4.1 with $\{B_n\} = \{A_n\}$. Due to this theorem and Lemma 4.1 the collocation is stable if and only if the operators A, $\tilde{A}$, and $A_\pm$ are invertible and if the coset $\{A_n\} + \mathcal{J}$ is invertible in $\mathcal{F}/\mathcal{J}$. These conditions will be verified in the next two subsections.

5.1 The Invertibility of the Limit Operators

The invertibility of A corresponds to condition i) in Theorem 3.1. Moreover, the conditions ii) and iii) are equivalent to the invertibility of the operators $A_\pm$. Indeed, iii) means that the null spaces of the operators are trivial, and we shall show next that ii) is equivalent to the Fredholm property and to a vanishing index. To this end we need the following symbol calculus of Gohberg and Krupnik.

Theorem 5.1 ([22] and [32] or Lemma 11.4 of [31]) *There is a continuous mapping* **Symb** *from alg $\mathcal{T}(\mathbf{PC})$ to a set of functions defined over $\mathbb{T} \times [0,1]$. For each $R \in$ alg $\mathcal{T}(\mathbf{PC})$, the corresponding function $\mathbf{Symb}_R(t,\nu)$ will be called the symbol of R. This symbol satisfies:*

1) For any $t \neq \pm 1$, the value $\mathbf{Symb}_R(t,\nu)$ is independent of ν, and the function $t \mapsto \mathbf{Symb}_R(t,0)$ is continuous on $\{t \in \mathbb{T} : \Im t \geq 0\}$ and on $\{t \in \mathbb{T} : \Im t \leq 0\}$ with the limits

$$\begin{aligned}
\mathbf{Symb}_R(1+0,0) &:= \lim_{t \longrightarrow +1,\, \Im t>0} \mathbf{Symb}_R(t,0) = \mathbf{Symb}_R(1,1)\,, \\
\mathbf{Symb}_R(1-0,0) &:= \lim_{t \longrightarrow +1,\, \Im t<0} \mathbf{Symb}_R(t,0) = \mathbf{Symb}_R(1,0), \\
\mathbf{Symb}_R(-1+0,0) &:= \lim_{t \longrightarrow -1,\, \Im t<0} \mathbf{Symb}_R(t,0) = \mathbf{Symb}_R(-1,1), \\
\mathbf{Symb}_R(-1-0,0) &:= \lim_{t \longrightarrow -1,\, \Im t>0} \mathbf{Symb}_R(t,0) = \mathbf{Symb}_R(-1,0).
\end{aligned}$$

Moreover, the function $\nu \mapsto \mathbf{Symb}_R(\pm 1,\nu)$ is continuous on $[0,1]$.

2) For any $R \in alg\,\mathcal{T}(\mathbf{PC})$, the operator R is Fredholm if and only if the symbol $\mathbf{Symb}_R$ does not vanish over $\mathbb{T} \times [0,1]$.

3) For any Fredholm operator $R \in alg\,\mathcal{T}(\mathbf{PC})$, the index of R is the negative winding number of the closed curve

$$\begin{aligned}
\Gamma \;:=\; & \Big\{\mathbf{Symb}_R(e^{\mathbf{i}s},0) :\ 0 < s < \pi\Big\} \cup \Big\{\mathbf{Symb}_R(-1,s) :\ 0 \leq s \leq 1\Big\} \\
& \cup \Big\{\mathbf{Symb}_R(-e^{\mathbf{i}s},0) :\ 0 < s < \pi\Big\} \cup \Big\{\mathbf{Symb}_R(1,s) :\ 0 \leq s \leq 1\Big\}
\end{aligned} \tag{5.1}$$

with respect to the point 0, where the direction of the curve Γ is determined by the parametrizations in (5.1).

Lemma 5.1 *The operators $A_\pm$ belong to the algebra $alg\,\mathcal{T}(\mathbf{PC})$ and the values of their Gohberg-Krupnik symbol are the points of the curves $\tilde{\Gamma}_\pm$ multiplied by $[a(\pm 1) - b(\pm 1)]$. In particular, the operators $A_\pm$ are Fredholm operators with index 0 if and only if condition ii) of Theorem 3.1 is fulfilled.*

Proof: The special case of the present lemma, where the $K^\pm$ vanish, is treated in [17, Lemma 7.2, ii) and Equation (8.1)]. In particular, for the special operator $A^{\mu_-,\mu_+} := \mu^{-1} S \mu$ and the corresponding limit $A_-^{\mu_-,\mu_+} := W_4\{M_n A^{\mu_-,\mu_+} L_n\}$, it was shown that

$$\mathbf{Symb}_{A_-^{\mu_-,\mu_+}}(t,\nu) \;=\; \begin{cases} +1 & \text{if } t \in \mathbb{T},\ \Im t > 0 \\ -1 & \text{if } t \in \mathbb{T},\ \Im t < 0 \\ \mathbf{i}\cot\left(\pi\left(\frac{1}{2} + \frac{\sigma_-}{2} - \mu_- + \frac{\mathbf{i}}{4\pi}\log\left(\frac{\nu}{1-\nu}\right)\right)\right) & \text{if } t = 1 \\ -\mathbf{i}\cot\left(\pi\left(\frac{1}{4} + \frac{\mathbf{i}}{4\pi}\log\left(\frac{\nu}{1-\nu}\right)\right)\right) & \text{if } t = -1. \end{cases}$$

On the other hand, setting $A^M := K^-$ and denoting the corresponding limit $W_4\{M_n K^- L_n\}$ by A_-^M, we pass to the limits in Equation (4.11) to get

$$\begin{aligned}
A_-^M \;&=\; \int_{\Re z=0} \frac{1}{2}\left\{\hat{\mathbf{k}}^-\left(\frac{1}{2} + \frac{\sigma_-}{2} + z - \frac{1}{2}\right) - \hat{\mathbf{k}}^-\left(\frac{1}{2} + \frac{\sigma_-}{2} + z + \frac{1}{2}\right)\right\} \times \\
&\qquad \left[A_-^{\sigma_-/2+z,0} - A_-^{\varrho_-,\varrho_+}\right] \mathrm{d}z \;\in\; alg\,\mathcal{T}(\mathbf{PC}),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\mathbf{Symb}_{A_-^M} \;&=\; \int_{\Re z=0} \frac{1}{2}\left\{\hat{\mathbf{k}}^-\left(\frac{1}{2} + \frac{\sigma_-}{2} + z - \frac{1}{2}\right) - \hat{\mathbf{k}}^-\left(\frac{1}{2} + \frac{\sigma_-}{2} + z + \frac{1}{2}\right)\right\} \times \\
&\qquad \left[\mathbf{Symb}_{A_-^{\sigma_-/2+z,0}} - \mathbf{Symb}_{A_-^{\varrho_-,\varrho_+}}\right] \mathrm{d}z.
\end{aligned}$$

Now we take into account the analyticity assumption (2.2), (2.6) on $\hat{\mathbf{k}}^-$ and apply the residue theorem and (4.9).

$$\mathbf{Symb}_{\underline{A}^M}(t,\nu)$$
$$= \begin{cases} 0 & \text{if } t \in \mathbb{T}\setminus\{1\} \\ \int_{\Re z=0} \left\{\hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z-\frac{1}{2}\right) - \hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z+\frac{1}{2}\right)\right\} & \text{if } t = 1, \\ \qquad \times \frac{\mathbf{i}}{2}\cot\left(\pi\left(\frac{1}{2}-z+\frac{\mathbf{i}}{4\pi}\log\left(\frac{\nu}{1-\nu}\right)\right)\right)\mathrm{d}z & \end{cases}$$

$$\mathbf{Symb}_{\underline{A}^M}(1,\nu)$$
$$= \int_{\Re z=0} \hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z-\frac{1}{2}\right)\frac{\mathbf{i}}{2}\cot\left(\pi\left(\frac{1}{2}-z+\frac{\mathbf{i}}{4\pi}\log\left(\frac{\nu}{1-\nu}\right)\right)\right)\mathrm{d}z$$
$$- \int_{\Re z=1} \hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+z-\frac{1}{2}\right)\frac{\mathbf{i}}{2}\cot\left(\pi\left(\frac{1}{2}-z+\frac{\mathbf{i}}{4\pi}\log\left(\frac{\nu}{1-\nu}\right)\right)\right)\mathrm{d}z$$
$$= \hat{\mathbf{k}}^-\left(\frac{1}{2}+\frac{\sigma_-}{2}+\frac{\mathbf{i}}{4\pi}\log\left(\frac{\nu}{1-\nu}\right)\right).$$

Dividing this symbol by $[a(-1)-b(-1)]$ and taking into account the representation $\mathbb{R} = \{\mathbf{i}/(4\pi)\log[\nu/(1-\nu)] : 0<\nu<1\}$, we get exactly the term $\hat{\mathbf{k}}^-(z)/[a(-1)-b(-1)]$ in the definition of Γ_- and $\tilde{\Gamma}_-$ in the Theorems 2.1 and 3.1. This means that the assertion of the lemma holds true even if a non-zero term K^- is included. The analogous conclusion for K^- replaced by K^+ completes the proof. ∎

The operator $\tilde{A} = W_2\{A_n\}$ is a Cauchy singular integral operator for which either the kernel space or the cokernel is trivial. Hence, $\tilde{A}$ is invertible if and only if it is Fredholm with index zero. According to Theorem 2.1 this is equivalent to the vanishing winding number of the curve $\Gamma^\sim := \Gamma_-^\sim \cup \Gamma_1 \cup \Gamma_1' \cup \Gamma_2 \cup \Gamma_2' \cup \ldots \cup \Gamma_N \cup \Gamma_N' \cup \Gamma_{N+1} \cup \Gamma_+^\sim$ (cf. the curve Γ in Theorem 2.1 and (3.7)). Due to condition ii) in Theorem 3.1 the last winding number is equal to that of $\Gamma := \Gamma_- \cup \Gamma_1 \cup \Gamma_1' \cup \Gamma_2 \cup \Gamma_2' \cup \ldots \cup \Gamma_N \cup \Gamma_N' \cup \Gamma_{N+1} \cup \Gamma_+$. This, however, vanishes due to Theorem 2.1 applied to operator A and due to condition i) in Theorem 3.1. In other words, the conditions i) and ii) in Theorem 3.1 imply the invertibility of $\tilde{A} = W_2\{A_n\}$.

5.2 The Invertibility of $\{A_n\} + \mathcal{J}$ in $\mathcal{F}/\mathcal{J}$

Since $\mathcal{F}/\mathcal{J}$ is a C^*-algebra, it is equivalent to prove the invertibility of $\{A_n\}^o := \{A_n\} + \mathcal{J}$ in $\mathcal{A}/\mathcal{J}$. For this, we utilize the local principle of Theorem 4.2. If $-1<\tau<1$ and if the Mellin operators $K^\pm$ are zero, then we infer from [17, Lemma 6.2] that the invertibility of $W_1\{A_n\}$ implies the invertibility of $\{A_n\}^o + \mathcal{J}_\tau$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$. As a consequence of the next Lemma $\{A_n\}^o + \mathcal{J}_\tau$ is invertible in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_\tau$ even for the case of non-zero $K^\pm$.

Lemma 5.2 *The cosets $\{M_n K^\pm L_n\}^o$ are contained in $\mathcal{J}_\tau$ for $-1<\tau<1$.*

Proof: Take a smooth function χ such that χ vanishes in some neighbourhoods around ± 1 and that $\chi(\tau) = 1$. From the definition of the ideal $\mathcal{J}_\tau$ we conclude $\{M_n\chi L_n - L_n\}^o \in \mathcal{J}_\tau$. Hence, it remains to prove $\{M_n\chi K^\pm M_n\chi L_n\} \in \mathcal{J}$. However, the kernel of $K^\pm$ restricted to the support of χ is totally smooth and $[L_n - M_n]\chi K$ tends to zero in the operator norm. It remains to prove $\{L_n\chi K^{\pm}M_n\chi L_n\} \in \mathcal{J}$ which is obvious from the definition of $\mathcal{J}$, from the ideal property of $\mathcal{J}$ (cf. Theorem 4.1), and from the compactness of $\chi K^{\pm}$. ■

In order to prove the local invertibility at $\tau = \pm 1$, we follow the technique of [17, Section 7]. First we restrict our consideration to the case $\tau = 1$ and obtain the missing local invertibility from

Lemma 5.3 i) *Suppose that $R \in alg\,\mathcal{T}(\mathbf{PC})$ is invertible and consider the sequence R_n^3 defined in (4.12). Then the coset $\{[R^{-1}]_n^3\}^o + \mathcal{J}_1$ is the inverse of $\{R_n^3\}^o + \mathcal{J}_1$ in $(\mathcal{A}/\mathcal{J})/\mathcal{J}_1$.*
ii) *Suppose (2.4) - (2.7). Consider $A_n = M_n[aI + b\mu^{-1}S\mu I + \chi^+K^+ + \chi^-K^- + K_0]L_n$ and $R := W_3\{A_n\}$ which is in $alg\,\mathcal{T}(\mathbf{PC})$ due to Lemma 5.1. Then the cosets $\{R_n^3\}^o + \mathcal{J}_1$ and $\{A_n\}^o + \mathcal{J}_1$ coincide. In particular, $\{A_n\}^o + \mathcal{J}_1$ is invertible if R is invertible.*

Proof: The first assertion is taken from Lemma 7.2 in [17]. For the case $K^{\pm} = 0$, even the second assertion has been stated in Lemma 7.2 in [17]. It remains to show the coincidence of $\{R_n^3\}^o + \mathcal{J}_1$ and $\{A_n\}^o + \mathcal{J}_1$ for the special case $A = K^{\pm}$. Since the kernel of K^- restricted to a small neighbourhood of 1 is smooth, the case $A = K^-$ can be treated as in Lemma 5.2. We may suppose $A = K^+$. However, we infer from the proof of [17, Lemma 7.2, ii)] that $\{[A_+]_n^3 - A_n\}^o \in \mathcal{J}_1$ for $A = \mu^{-1}S\mu I$ with μ satisfying (4.2). This together with (cf. the analogous formulae (4.11) and (5.2))

$$\begin{aligned} M_nK^+L_n &= \int_{\Re z=0} \frac{1}{2}\left\{\hat{\mathbf{k}}^+\left(\frac{1}{2}+\frac{\sigma_-}{2}+z-\frac{1}{2}\right) - \hat{\mathbf{k}}^+\left(\frac{1}{2}+\frac{\sigma_-}{2}+z+\frac{1}{2}\right)\right\} \times \\ &\qquad \left[M_n[\delta_+^z]^{-1}S\delta_+^zL_n - M_n\varrho^{-1}S\varrho L_n\right]\mathrm{d}z, \\ &\qquad \delta_+^z(x) := (1+x)^0(1-x)^{\sigma_+/2+z}, \\ W_3\{M_nK^+L_n\} &= \int_{\Re z=0} \frac{1}{2}\left\{\hat{\mathbf{k}}^+\left(\frac{1}{2}+\frac{\sigma_+}{2}+z-\frac{1}{2}\right) - \hat{\mathbf{k}}^+\left(\frac{1}{2}+\frac{\sigma_+}{2}+z+\frac{1}{2}\right)\right\} \times \\ &\qquad \left[W_3\left\{M_n[\delta_+^z]^{-1}S\delta_+^zL_n\right\} - W_3\left\{M_n\varrho^{-1}S\varrho L_n\right\}\right]\mathrm{d}z \end{aligned}$$

proves $\{[A_+]_n^3 - A_n\}^o \in \mathcal{J}_1$ for $A = K^+$. ■

Next we turn to $\tau = -1$. Proceeding analogously to the previous lemma, we observe that the local invertibility at $\tau = -1$ is a consequence of the invertibility of $W_4\{A_n\} = A_-$. The proof of Theorem 3.1 is completed.

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