Branching systems with long living particles at the critical dimension

Kaus Fleischmann*, Vladimir A. Vatutin†,

and Anton Wakolbinger‡

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Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
D–10117 Berlin, Germany
e-mail: fleischmann@wias-berlin.de
fax: 49-30-204 49 75
URL: http://www.wias-berlin.de/~fleischm

Department of Discrete Mathematics
Steklov Mathematical Institute
8 Gubkin Street
117 966 Moscow, GSP-1, Russia
e-mail: vatutin@mi.ras.ru

Fachbereich Mathematik (12)
Johann Wolfgang von Goethe-Universität
D-60054 Frankfurt am Main, Germany
e-mail: wakolbin@math.uni-frankfurt.de

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Abstract

A spatial branching process is considered in which particles have a life time law with a tail index smaller than one. Other than in classical branching particle systems, at the critical dimension the system does not suffer local extinction when started from a spatially homogenous initial population. In fact, persistent convergence to a mixed Poissonian system is shown. The random limiting intensity is characterized in law by the random density in a space point of a related age-dependent superprocess at a fixed time. The proof relies on a refined study of the system starting from asymptotically large but finite initial populations.

1 Introduction and statement of results

1.1 Motivation and purpose

The study of spatially homogeneous (critical) branching particle systems in R^d with long living particles has been initiated in [SW93] and [VW99], and touched in [KS98]. Here *'long living'* means that the lifetime distribution of a particle has a tail of index $\gamma \in (0,1)$, implying that the mean lifetime is infinite. There a new phenomenon has been revealed: Starting from a homogeneous Poissonian system of particles, in *supercritical* dimensions the persistent limit in law is homogeneous *Poissonian* again, opposed to the usual non-Poissonian limits of systems of particles with finite expected lifetimes. (See also Lemma 3 (b) below.) This has the following intuitive reason: Due to the long lifetimes, all the siblings of a particle in the limit population were born so long ago that they moved out of the finite window of observation. Therefore, only "completely mixed" populations can be observed in the limit.

Let us manifest this by some *heuristic calculation* based on backward tree considerations. Assume the particles' motion index is $\alpha \in (0,2]$, and the index of the branching is $\beta \in (0,1]$ (see Hypothesis 1 below). By renewal theory ([Fel71, § 11.5 and § 11.3]), the number of branching points along an ancestral line of a particle, called "ego", picked at time t "at random" is asymptotically in law (as $t \uparrow \infty$) of the form κt^γ for some random variable $\kappa > 0$, and the time between t and *any* of these earlier branching time points is of order t (as opposed to the case of exponentially distributed branching times). Consider such a branching time point. The number $\gimel$, say, of particles (additionally) generated at this branching point has moment generating function[1] $\Phi(s) = 1 - c\,(1-s)^\beta$ (differentiate the offspring generating function in (3) below). Consider *any* of these $\gimel$ offspring. Let q denote the probability that its descendants at time t populate a (fixed) ball B around "ego". The arguments in ([VW99, Lemmas 2 and 3]) yield that q

[1] With c we always denote a positive constant which may vary from place to place.

decays at least with the order $t^{-\theta}$, where $\theta := (d/\alpha+\gamma)/(1+\beta)$. Therefore, the probability that none of the $\gimel$ particles has an offspring that populates B at time t is asymptotically $\Phi(1-q) = 1 - c\,q^{\beta} \geq 1 - c\,t^{-(d/\alpha+\gamma)\beta/(1+\beta)}$. Consequently, the probability that none of the relatives of "ego" populates a ball around "ego" asymptotically equals

$$\begin{aligned}\mathbf{E}\big(1 - c\,t^{(-\frac{d}{\alpha}-\gamma)\frac{\beta}{1+\beta}}\big)^{\kappa t^{\gamma}} &\approx \mathbf{E}\exp\big[-\,c\,\kappa t^{\gamma-(\frac{d}{\alpha}+\gamma)\frac{\beta}{1+\beta}}\big] \\ &= \mathbf{E}\exp\big[-\,c\,\kappa t^{(\gamma-\frac{d\beta}{\alpha})\frac{1}{1+\beta}}\big]. \qquad (1)\end{aligned}$$

This indicates that the number $d_{\mathrm{c}} := \gamma\alpha/\beta$ should be critical for the dimension d of space. Indeed, for $d > d_{\mathrm{c}}$, the expression in (1) converges to 1, showing that in supercritical dimensions "ego" asymptotically has no relatives around. In other words, relationships of particles in our observation window vanish as $t \uparrow \infty$, which corresponds to a convergence of the original populations towards a Poissonian system.

If $d < d_{\mathrm{c}}$ instead, then (1) converges to 0. In other words, provided that $t^{-\theta}$ is the correct order of decay of q, then for $d < d_{\mathrm{c}}$ each small ball around "ego" is populated by a relative of "ego" with asymptotic probability one. This can be taken as an indication for the usual clumping of the original particle system in subcritical dimensions, and is in fact in line with the local extinction of the branching particle system for $d < d_{\mathrm{c}}$ which was proved in [VW99] (see also Lemma 3 (a) below).

At the critical dimension d_{c} however, (1) suggests that at late times t the probability that relatives of "ego" show up in B might be strictly between 0 and 1. Since the latest common ancestor of any of these relatives lived way back in the past of order t, the spatial correlations of these relatives should vanish as $t \uparrow \infty$. In fact the population should tend to a homogeneous mixed Poissonian system, where the randomness of the intensity comes from the randomness of the branching at very early times.

In this paper we prove that such a picture indeed is true, and hereby essentially strengthen a result of ([VW99]) where it was shown that at the critical dimension there exist non-trivial limit points. According to our *main result* the following convergence statement holds for the process $Z = \{Z_t : t \geq 0\}$ under consideration. At the critical dimension, Z_t converges in law as $t \uparrow \infty$, to a limit Z_∞ which is again of full intensity (*persistence*) and is in fact a *mixed homogeneous Poissonian* particle system whose random intensity i_∞ is non-degenerate. In particular, the limit is *not ergodic*. (See Theorem 11 below.)

Note that the picture is reminiscent of the situation in *catalytic* branching models (for a recent survey, see [DF00]). From there one knows already the phenomenon of persistent convergence in critical dimensions: In a superprocess setting, starting from the Lebesgue measure ℓ, the limit in law of a super-Brownian reactant with a super-Brownian catalyst is a random multiple $i_\infty \ell$ of ℓ, where by self-similarity the random factor i_∞ equals in law the random density of reactant's mass at time 1 at the origin ([FK99]).

In the analogous particle model, one has a mixed homogeneous Poissonian limit instead ([GKW99]), just as already described above for the Z process.

In the particle system we are considering, it is convenient to refine the description by taking into account the *residual lifetimes* of the particles, thus arriving at a Markovian description $\bar{Z}$ of the model. Under a suitable mass-time-space rescaling (Lemma 6 below), one can pass to a corresponding superprocess $\bar{X}$, whose spatial component X then is the corresponding rescaling limit of the process Z. (The process X appears also in a recent work of Kaj and Sagitov [KS98] under the notion of "projected superprocess"; one should note, however, that X is not Markov and therefore is no superprocess in the classical sense.) As it turns out, the process X has absolutely continuous states $X_t(\mathrm{d}a) = X_t(a)\,\mathrm{d}a$ in all dimensions $d < \alpha/\beta$, covering the critical dimension d_{c} (Proposition 7 below). This corresponds to the intuition that the long lifetimes of the infinitesimal particles could have some additional smoothing effect on the mass distribution (compared with the case of lifetimes with finite mean).

At the same time, at the critical dimension d_{c}, starting from a Lebesgue initial state, we have a self-similarity property (Proposition 8 (b) below). For $d = d_{\mathrm{c}}$, this turns the absolute continuity of states into the existence of a large time limit X_∞, which is a random multiple $i_\infty \ell$ of the Lebesgue measure, and where i_∞ coincides in law with $X_1(0)$, the random density of X_1 at the origin (see Corollary 10 (b) below).

Finally, it is well-known that in the case of exponential lifetimes the branching particle systems arise from the corresponding superprocesses via a Poissonization (cf. [GRW90, p.277]). In our case of long living particles, we will show that (for homogeneous initial states, or even for a large initial pile of mass concentrated in a remote starting point) this property holds asymptotically in the long-term limit, which leads to the claimed convergence towards a mixed Poissonian process.

Next we will introduce the mentioned (refined) branching particle system $\bar{Z}$ in more detail.

1.2 The $(d, \alpha, \beta, \gamma)$–branching particle system $\bar{Z}$

We are dealing with a model of (critical) branching in R^d which is a spatial generalization of the so-called age-dependent (critical) branching process or (critical) Bellman-Harris branching process. It is based on the following ingredients which, for convenience, we expose as a hypothesis.

Hypothesis 1 (Ingredients of the branching particle system)

(a) (Particle motion process ξ) Fix $\alpha \in (0, 2]$. Consider the symmetric α*–stable* process $(\xi, P_a\,,\ a \in \mathsf{R}^d)$ in R^d, (cf. [Bre68, p.317] and [Ber96, Ch. VIII]), that is, the (time-homogeneous) Markov process with generator $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, the fractional Laplacian ([Yos74, p.260]), and with càdlàg paths. Denote by $\mathrm{p} = \{\mathrm{p}_t(b) : \ t > 0,\ b \in \mathsf{R}^d\}$ the continuous transition densities of this *particle motion process* ξ.

(b) (Particles' lifetime τ) Introduce the non-lattice distribution function G of a random variable $\tau > 0$ with tails

$$P(\tau > u) \;=\; 1 - G\,(u) \;\sim\; c_G\, u^{-\gamma} \quad \text{as } u \uparrow \infty \tag{2}$$

for some index $\gamma \in (0,1)$ and a constant $c_G > 0$. That is, the *lifetime distribution* G of particles is in the normal domain of attraction of a stable law of index γ. In particular, τ has infinite expectation $E\tau = \infty$.

(c) (Critical branching mechanism) Fix attention to the *offspring generating function*

$$f(s) \;:=\; Es^{\zeta} \;=\; s + c_f\,(1-s)^{1+\beta}\,, \qquad 0 \le s \le 1, \tag{3}$$

of the random number ζ of offspring of a particle, where $\beta \in (0,1]$ and $c_f \in \bigl(0, \frac{1}{1+\beta}\bigr]$. Consequently, $E\zeta = 1$ (*criticality*), and we consider a branching mechanism in the normal domain attraction of a stable law of index $1+\beta$. Note that $E\zeta^2 < \infty$ if and only if $\beta = 1$.

(d) (Phase space E) A point $e = (u,a) \in \mathsf{E} := \mathsf{R}_+ \times \mathsf{R}^d$ describes the *residual lifetime* u and the *position* a of a particle. With the metric

$$\mathrm{d}_{\mathsf{E}}(e_1,e_2) \;:=\; 1 \wedge |u_1 - u_2| \,+\, |a_1 - a_2|, \tag{4}$$

$e_i = (u_i\,, a_i) \in \mathsf{E},\ i = 1,2,$ we get a Polish space $(\mathsf{E}, \mathrm{d}_{\mathsf{E}})$.

(e) (Test functions) Fix a number $p \in (d, d+\alpha]$ (recall that α is the motion index), and introduce the reference function

$$\phi_p(a) \;:=\; \bigl(1 + |a|^2\bigr)^{-p/2}, \qquad a \in \mathsf{R}^d. \tag{5}$$

Let $\bar{\mathcal{C}}_p = \bar{\mathcal{C}}_p(\mathsf{E})$ denote the set of all continuous functions $\psi : \mathsf{E} \to \mathsf{R}$ such that

$$\|\psi\| \;:=\; \sup_{(u,a)\in\mathsf{E}} \left|\frac{\psi(u,a)}{\phi_p(a)}\right| \;<\; \infty, \tag{6}$$

and such that the map

$$(u,a) \;\mapsto\; \frac{\psi(u,a)}{\phi_p(a)} \quad \text{on } \mathsf{E} \tag{7}$$

can continuously be extended to a function on $\mathsf{R}_+ \times \dot{\mathsf{R}}^d$, where $\dot{\mathsf{R}}^d$ is the one-point compactification of R^d. Then $\bigl(\bar{\mathcal{C}}_p\,, \|\cdot\|\bigr)$ is a separable Banach space.

(f) (State space $\mathcal{N}_p^{\circ}$) Let $\mathcal{M}_p = \mathcal{M}_p(\mathsf{E})$ denote the set of all *p–tempered measures* on $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$, that is measures μ on E such that the integral

$$\langle \mu, \phi_p \rangle \;:=\; \int_{\mathsf{E}} \mu\bigl(\mathrm{d}(u,a)\bigr)\, \phi_p(a) \tag{8}$$

is finite. Introduce the weakest topology in $\mathcal{M}_p$ such that for each $\psi \in \bar{\mathcal{C}}_p$ the mapping

$$\mu \ \mapsto \ \langle \mu, \psi \rangle \ := \ \int_{\mathsf{E}} \mu(\mathrm{d}e)\, \psi(e) \tag{9}$$

is continuous. Note that $\nu \times \ell$ belongs to $\mathcal{M}_p$ for each finite measure ν on R_+. Write $\mathcal{N}_p = \mathcal{N}_p(\mathsf{E})$ for the subset of all those measures μ in $\mathcal{M}_p$ with values in $\{0, 1, ..., \infty\}$ (counting measures), and $\mathcal{N}_p^\circ = \mathcal{N}_p^\circ(\mathsf{E})$ if additionally $\mu\left(\{0\} \times \mathsf{R}^d\right) = 0$. We let both sets $\mathcal{N}_p$ and $\mathcal{N}_p^\circ$ inherit the topologies of $\mathcal{M}_p$. The set $\mathcal{N}_p^\circ$ will serve as a *state space* of our branching particle system. In particular, the Dirac delta measure $\delta_{(u,a)} \in \mathcal{N}_p^\circ$ describes a single particle having residual lifetime $u > 0$ and sitting at position $a \in \mathsf{R}^d$. ◇

Definition 2 (Branching particle system $\bar{Z}$) Abstaining from a more detailed definition, the process $\bar{Z} = \left\{ \bar{Z}_t : t \geq 0 \right\}$ we are interested in can now be described by the following properties:

- Given a particle $\delta_{(u,a)} \in \mathcal{N}_p^\circ$ at time $r \geq 0$, its further path in $(0, \infty) \times \mathsf{R}^d$ is $t \mapsto \left(u - (t - r), \, \xi_{t-r}\right)$, $r \leq t < r + u$, where ξ is distributed according to P_a.
- If a particle reaches the residual lifetime $0+$, it immediate dies, but still before that it reproduces the random number ζ of offspring.
- Newly born particles get independent residual lifetimes, all distributed as $\tau > 0$.
- At time $t = 0$, we start with a system of particles described by a measure $\mu \in \mathcal{N}_p^\circ$.
- Write $\mathbf{P}_\mu$ for the law of $\bar{Z}$. It is considered as a measure on the set $\mathcal{D}(\mathsf{R}_+, \mathcal{N}_p)$ of all $\mathcal{N}_p$–valued *càdlàg* paths ω satisfying additionally $\omega_t \in \mathcal{N}_p^\circ$, $t \geq 0$.

For convenience, we call this process $(\bar{Z}, \mathbf{P}_\mu, \mu \in \mathcal{N}_p^\circ)$ a $(d, \alpha, \beta, \gamma)$–*branching particle system.* Note that $\bar{Z}$ is a time-homogeneous Markov process.

Note also that we put maximal independence assumptions in defining the model. The main dependence assumption is that newly born particles start their evolution from the ancestor's death place.

Integrating out the residual lifetimes, we get back the non-Markovian process

$$Z_t \ = \ \bar{Z}_t\big(\mathsf{R}_+ \times (\cdot)\big), \qquad t \geq 0, \tag{10}$$

already mentioned in Subsection 1.1. ◇

We are interested in the long-term behavior of this (d,α,β,γ)–branching particle system $\bar{Z}$. For simplicity, we take Poissonian particle systems with intensity measure [2)] $i_0\, G(\mathrm{d}u)\ell(\mathrm{d}a)$ as initial states, where $i_0 > 0$ is a fixed constant. Note that in this case all the initial residual lifetimes are independent copies of the random variable τ, which means that all initial particles are just newly born.

Recall that a random counting measure π on E is called a *Poissonian* particle system with intensity measure $\mu \in \mathcal{M}_p$, if it has the log-Laplace transform

$$-\log \mathbf{E} \exp \langle \pi, -\psi \rangle \;=\; \left\langle \mu, 1 - \mathrm{e}^{-\psi} \right\rangle, \qquad \psi \in \bar{\mathcal{C}}_p^{\,+}, \tag{11}$$

[the index $+$ on a set refers to all of its non-negative members, as we already used $\mathsf{R}_+ = [0,\infty)$]. We write π_μ for such a Poissonian particle system with intensity measure μ. If $\mu = i_0\, \nu \times \ell$ with a constant $i_0 \geq 0$, a probability law ν on $(0,\infty)$ (and ℓ the Lebesgue measure on R^d), then $\pi_{i_0\, \nu\times\ell}$ is a *homogeneous* Poissonian particle system with intensity i_0 and residual lifetimes distributed according to ν. If i_0 is additionally random, then $\pi_{i_0\, \nu\times\ell}$ is said to be a *mixed homogeneous Poissonian* particle system with random intensity i_0 ("double stochastic" particle system). Analogous terminology is used for Poissonian particle systems on R^d only.

1.3 Detour: Lifetimes with finite mean and the extinction-persistence dichotomy

Let us assume that $\bar{Z}_0$ is the homogeneous Poissonian particle system $\pi_{i_0\, G\times\ell}$ of intensity $i_0 > 0$ and residual lifetime distributed according to G (recall that in this case only newly born particles are considered in the beginning).

First we contrast our model of a (d,α,β,γ)–branching particle system $\bar{Z}$ with the case where the assumption (2) on long tails is replaced by that of a (non-lattice) lifetime distribution function G with *finite* expectation. Then there is a dichotomy between persistent convergence and local extinction depending on whether $d > \frac{\alpha}{\beta}$ holds or is violated (cf. [VW99]). This kind of picture is of course known from other variants of spatially homogeneous (critical) branching processes, see, for instance, [Lie69, Daw77, Kal77, DF85, Fle88], and [GW91]. An intuitive reason for this dichotomy is the following: By the critical branching, the offspring of any considered finite subpopulation always goes to extinction. Because of the smaller mobility in low dimensions, this effect leads finally to a local extinction of the infinite population. In large dimensions, however, the higher mobility allows some of the particles (coming from far away) to enter the finite window of observation and thus to show up in the limit population. Consequently, in low dimensions the local fluctuations (coming from the

[2)] We do not distinguish in notation between the distribution function G and its related probability measure $G(\mathrm{d}t)$ on $(0,\infty)$.

critical branching) prevail, whereas in higher dimensions the effect of the motion is dominating. Altogether, it depends on the dimension of space and the indices of branching and motion, which of the competing features (local mass fluctuation or spatial dispersion) wins in the long run.

The same statement holds true for $(d, \alpha, \beta, \gamma)$–branching particle system: below the critical dimension $d_c = \gamma\alpha/\beta$ there is local extinction, and above it there is persistence (see Subsection 1.4 below). However, at the critical dimension itself, the behavior is completely different from that of systems with finite expected lifetime of particles (see Subsection 1.9).

1.4 Long living particles in non-critical dimensions

Let us return to our $(d, \alpha, \beta, \gamma)$–branching particle system $\bar{Z}$ according to Definition 2. Before coming to our main case, we will briefly recall the picture in non-critical dimensions. For this purpose, we take results from [SW93] and [VW99], where the present extensions to the case with residual lifetimes included in the description can easily be provided.

Lemma 3 (Long-term behavior in non-critical dimensions) *Consider the $(d, \alpha, \beta, \gamma)$–branching particle system $\bar{Z}$ starting from a homogeneous Poissonian particle system of intensity $i_0 > 0$ and with residual lifetimes distributed according to G, that is, $\bar{Z}_0 = \pi_{i_0\, G \times \ell}$. For $K > 0$, introduce the time-scaled process $\bar{Z}^{(K)}$:*

$$\bar{Z}_t^{(K)}\big((\,\cdot\,) \times (\,\cdot\,)\big) \;:=\; \bar{Z}_{Kt}\big(K^{-1}(\,\cdot\,) \times (\,\cdot\,)\big), \qquad t \geq 0, \tag{12}$$

and fix $t > 0$. Then the following statements hold.

- **(a) (Local extinction in subcritical dimensions)** ([VW99]) *If $d < \frac{\alpha}{\beta}$, then $\bar{Z}_t^{(K)}$ suffers local extinction as $K \uparrow \infty$, that is, $\bar{Z}_t^{(K)} \to 0$ in probability.*
- **(b) (Persistent convergence in supercritical dimensions)** ([SW93, VW99]) *On the other hand, if $d > \frac{\gamma\alpha}{\beta}$, then $\bar{Z}_t^{(K)}$ converges in law as $K \uparrow \infty$ to a homogeneous Poissonian limit population $\bar{Z}_t^{(\infty)} = \pi_{i_0\, G_\infty^t \times \ell}$ of full intensity i_0 and with residual lifetimes distributed according to*

$$G_\infty^t(\mathrm{d}u) \;:=\; \frac{\sin \pi\gamma}{\pi}\, \frac{t^\gamma\, \mathrm{d}u}{u^\gamma\,(t+u)}\,, \qquad u > 0, \tag{13}$$

(*for the occurrence of G_∞^t, see* [Fel71, *Theorem in* §14.3]).

As pointed out already in Subsection 1.1, the striking new feature relative to the case of a lifetime distribution with finite mean (Subsection 1.3) is that in supercritical dimensions the local dependencies between relatives are lost in the limit. This is caused by the heavily tailed lifetime distribution of the particles – siblings are too fare away in the long run.

According to a result of [VW99], at the critical dimension $d_c = \gamma\alpha/\beta$ local extinction does *not* hold (recall our discussion in Subsection 1.1). Our purpose is to enlighten this situation. In order to describe this in detail, we first need to introduce a superprocess variant $\bar{X}$ of the (d,α,β,γ)–branching particle system $\bar{Z}$, since the detailed description of the long-term behavior of $\bar{Z}$ depends on a quantity derived from $\bar{X}$.

1.5 The (d,α,β,γ)–superprocess $\bar{X}$

Since [Fel51], it is common to ask also for diffusion type approximations of branching particle systems. For our present (d,α,β,γ)–model, at least starting with a finite initial system, such approximation had been provided by [KS98], who dropped, however, the residual lifetimes from the description. In this way one looses the Markov property, in fact in both the particle systems as well as in its high density limits (despite the independence assumptions between motion and aging). In other words, one never will end up with a Markov superprocess (not even with a time-inhomogeneous one, contrasting a statement on p.149 in [KS98]). This is the reason why we insist in keeping the residual lifetimes in our description also in the superprocess limits.

In order to introduce the limiting (d,α,β,γ)–superprocess $\bar{X}$, we first need to describe the basic "motion" process for this superprocess.

Recall that we are working with the phase space $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$. In the R^d–component, we are keeping the particles' α–stable motion process $(\xi, P_a,\ a \in \mathsf{R}^d)$. Additionally we need an independent (limiting) residual lifetime process $t \mapsto \vartheta_t$. To introduce it, for our $0 < \gamma < 1$, we start from an independent γ*–stable subordinator* $\eta = \{\eta_t : t \geq 0\}$ in R_+ which is a càdlàg time-homogeneous Markov process with stationary independent increments having log-Laplace transition function

$$-\log E\left\{\exp\left[-\theta\eta_t\right] \,\middle|\, \eta_0 = u\right\} \;=\; u\theta + c_\eta\, t\, \theta^\gamma, \qquad \theta, t, u \geq 0, \tag{14}$$

where it is convenient for us to normalize the constant $c_\eta > 0$ to

$$c_\eta \;:=\; 1/\Gamma(1+\gamma). \tag{15}$$

From this we deduce two other processes, the process

$$k_t \;:=\; \inf\{s > 0 :\ \eta_s > t\}, \qquad t \geq 0, \tag{16}$$

inverse to η, and the (limiting) *residual lifetime process*

$$\vartheta_t \;:=\; \inf\{\eta_s - t :\ \eta_s > t,\ s \geq 0\} \;=\; \eta_{k_t} - t, \qquad t \geq 0. \tag{17}$$

Note that ϑ is a càdlàg time-homogeneous Markov process, and that $\mathrm{d}k_t$ is a continuous and time-homogeneous additive functional of ϑ. (See [Ber96, p.123] for background.) Also recall that in the special case $\gamma = \frac{1}{2}$ the process $t \mapsto k_{\eta_0 + t}$ is just the local time at 0 of a Brownian motion started at 0, and ϑ describes the residual lifetimes of the Brownian excursions from 0.

We pass now to the pair $\bar{\xi} := (\vartheta, \xi)$ of independent processes whose laws we denote by $\bar{P}_e$, $e \in \mathsf{E}$, thus arriving at a càdlàg time-homogeneous Markov process $(\bar{\xi}, \bar{P}_e, e \in \mathsf{E})$ in $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$. This $\bar{\xi}$ will serve as the "motion" process of an intrinsic particle in the superprocess $\bar{X}$ we now will introduce. The time-homogeneous continuous additive functional $\mathrm{d}k_t$ of $\bar{\xi}$ will moreover serve as the branching functional for $\bar{X}$.

Recall that $\mathcal{D} = \mathcal{D}(\mathsf{R}_+, \mathcal{M}_p)$ denotes the set of all càdlàg measure-valued paths $\omega : \mathsf{R}_+ \to \mathcal{M}_p = \mathcal{M}_p(\mathsf{E})$ endowed with the Skorohod topology. Write $\mathcal{P}$ for the set of all probability laws on $\mathcal{D}$ furnished with the topology of weak convergence.

Lemma 4 (Unique existence of the $(d, \alpha, \beta, \gamma)$–superprocess $\bar{X}$) *Fix constants*

$$0 < \alpha \leq 2, \quad 0 < \beta \leq 1, \quad 0 < \gamma < 1, \quad \textit{and} \quad \varrho > 0. \tag{18}$$

To each measure $\mu \in \mathcal{M}_p$, there is a unique law $\mathbb{P}_\mu \in \mathcal{P}$ of a time-homogeneous Markov process $\bar{X}$ with log-Laplace transition functional

$$-\log \mathbb{E}_\mu \exp \langle \bar{X}_t, -\psi \rangle = \langle \mu, \bar{V}_t \psi \rangle, \qquad t \geq 0, \quad \psi \in \bar{\mathcal{C}}_p^+, \tag{19}$$

where the function $\bar{v} = \bar{V}\psi = \{\bar{V}_t \psi(e) : t \geq 0, e \in \mathsf{E}\} \geq 0$ uniquely solves the integral equation

$$\bar{v}_t(e) = \bar{E}_e \left[\psi(\bar{\xi}_t) - \varrho \int_0^t \mathrm{d}k_s \, \bar{v}_{t-s}^{1+\beta}(\bar{\xi}_s) \right], \qquad t \geq 0, \quad e \in \mathsf{E}. \tag{20}$$

We call $(\bar{X}, \mathbb{P}_\mu, \mu \in \mathcal{M}_p)$ the $(d, \alpha, \beta, \gamma)$–*superprocess* with branching rate ϱ. The construction of such a process is nowadays standard and we will abstain from this (see [Sch99]).

A good interpretation of this measure-valued process $\bar{X}$ is provided by the particle system approximation worked out in the next subsection.

Note that $\bar{X}$ is *critical:*

$$\mathbb{E}_\mu \langle \bar{X}_t, \psi \rangle = \int_{\mathsf{E}} \mu(\mathrm{d}e) \, \bar{E}_e \psi(\bar{\xi}_t), \qquad \mu \in \mathcal{M}_p, \quad t \geq 0, \quad \psi \in \bar{\mathcal{C}}_p^+. \tag{21}$$

Remark 5 (Decoupling) Note that the integral in (20) can be rewritten as

$$\int_0^t \mathrm{d}k_s \, \bar{v}_{t-s}^{1+\beta}(0, \xi_s) \tag{22}$$

since

$$\int_0^\infty \mathrm{d}k_s \, 1\{\vartheta_s > 0\} = 0, \quad \bar{P}_e\text{–a.s.}, \ e \in \mathsf{E}, \tag{23}$$

which follows just from the definitions of the processes k and ϑ. But k and ξ are independent, hence from

$$\bar{E}_e k_s \;=\; E_u k_s \;=\; 1_{[u,\infty)}(s)\,(s-u)^\gamma, \qquad e=(u,a), \quad s\geq 0, \tag{24}$$

we conclude that (20) can be rewritten as [3)]

$$\bar{v}_t(e) \;=\; \big[\bar{E}_e \psi(\bar{\xi}_t)\big] \;-\; \varrho \int_{u\wedge t}^{t} \mathrm{d}_s(s-u)^\gamma \; E_a\, \bar{v}_{t-s}^{1+\beta}(0,\xi_s), \tag{25}$$

$t\geq 0, \quad e=(u,a)\in \mathsf{E}$. ◇

1.6 Approximation of $\bar{X}$ by particle systems

As already indicated, $\bar{X}$ arises from the (d,α,β,γ)–branching particle system $\bar{Z}$ via a diffusion type approximation, which we now want to make precise (the proof follows the lines in [KS98] with the obvious changes incorporating the residual lifetimes):

Lemma 6 (Approximation of $\bar{X}$ by particle systems) *For $n\geq 1$, let $\bar{Z}^{(n)}$ denote a (d,α,β,γ)–branching particle system but with lifetime distribution function G of Hypothesis* 1 (b) *replaced by $G^{(n)}$:*

$$G^{(n)}(u) \;:=\; G\big(n^{\beta/\gamma} u\big), \qquad u\geq 0. \tag{26}$$

Moreover, assume that the $\bar{Z}^{(n)}$ start with (deterministic) initial populations $\bar{Z}_0^{(n)} \in \mathcal{N}_p^\circ$ satisfying

$$\frac{1}{n}\,\bar{Z}_0^{(n)} \;\underset{n\uparrow\infty}{\longrightarrow}\; \mu \quad \textit{in } \mathcal{M}_p\,. \tag{27}$$

Then

$$\frac{1}{n}\,\bar{Z}^{(n)} \;\underset{n\uparrow\infty}{\Longrightarrow}\; \bar{X} \quad \textit{in law}, \tag{28}$$

where $\bar{X}$ is the (d,α,β,γ)–superprocess from Lemma 4 *starting from μ, and with special branching rate*

$$\varrho \;=\; \frac{\sin \pi\gamma}{\pi}\,\frac{c_f}{c_G} \tag{29}$$

with c_f from (3) *and c_G from* (2).

In simple terms, if in the (d,α,β,γ)–branching particle system $\bar{Z}$ the particles' lifetimes τ are replaced by the rescaled (and smaller) lifetimes $n^{-\beta/\gamma}\tau$, if their unit masses are replaced by the small masses $\frac{1}{n}$, and if the initial populations $\bar{Z}_0^{(n)}$ are chosen such that the rescaled measures $\frac{1}{n}\,\bar{Z}_0^{(n)}$ converge as in (27), then the whole rescaled branching particle systems $\frac{1}{n}\,\bar{Z}^{(n)}$ tend to the (d,α,β,γ)–superprocess $\bar{X}$ of Lemma 4 with special branching rate ϱ as defined in (29).

[3)] Here d_s indicates that the Stieltjes integral is formed with respect to the variable s [of the monotone function $s\mapsto(s-u)^\gamma$].

1.7 The marginal measure process X

The residual lifetime process ϑ, starting from a non-zero state, is *deterministic* until it reaches the state 0. Thus, the "motion" process $\bar{\xi}$ does not have transition densities, hence the superprocess $\bar{X}$ will also not have "smooth" states. But integrating out the residual lifetimes, that is, passing to

$$X_t(\cdot) := \bar{X}_t\big(\mathsf{R}_+ \times (\cdot)\big), \qquad t \geq 0, \tag{30}$$

the situation changes, and this will be enough for our description for the long-term behavior of $\bar{Z}$. In fact, in dimensions $d < \frac{\alpha}{\beta}$, the measure states X_t are absolutely continuous. This we will verify by modifying of a general criterion in [Kle00]. For convenience, we expose this as a proposition,[4] the proof is postponed to Subsection 2.2.

For this purpose, we need to introduce further notation. If F is a function or a generalized function on R^d, we denote by $\bar{F}$ the "constant extension" from R^d to $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$:

$$\bar{F}(e) := F(a), \qquad e = (u,a) \in \mathsf{E}. \tag{31}$$

We apply this in particular to the constantly extended δ–functions $\overline{\delta_a}$. As a non-negative generalized function can be identified with a measure, for $a \in \mathsf{R}^d$ fixed, $\overline{\delta_a}$ can be identified with the measure

$$B \mapsto 1\{B_a \neq \emptyset\}, \qquad \text{Borel set } B \subseteq \mathsf{E}, \tag{32}$$

on E, where B_a denotes the section of B at $a \in \mathsf{R}^d$. Note that the random density $X_t(a) = \langle X_t, \delta_a \rangle$ at $a \in \mathsf{R}^d$, if it makes sense, coincides with $\langle \bar{X}_t, \overline{\delta_a} \rangle$. Finally, let $\chi \geq 0$ denote a bounded continuous function on R^d with $\langle \ell, \chi \rangle = 1$, and consider the regularization

$$\chi_\varepsilon(a) := \varepsilon^{-d} \chi(\varepsilon^{-1} a), \qquad a \in \mathsf{R}^d, \quad \varepsilon > 0, \quad \varepsilon \downarrow 0, \tag{33}$$

of the δ–function δ_0 on R^d.

Proposition 7 (Marginal measure process X) *Let $d < \frac{\alpha}{\beta}$, and fix $t > 0$. Consider the marginal process $X = \bar{X}\big(\mathsf{R}_+ \times (\cdot)\big)$ of the (d,α,β,γ)–superprocess $\bar{X}$ with law $\mathbb{P}_\mu$, $\mu \in \mathcal{M}_p(\mathsf{E})$.*

(a) (Random density at a point) *For all $a \in \mathsf{R}^d$, the limit in law*

$$\lim_{\varepsilon \downarrow 0} \langle X_t, \delta_a * \chi_\varepsilon \rangle =: X_t(a) = \langle X_t, \delta_a \rangle \tag{34}$$

exists. Moreover, the random density $X_t(a)$ of X_t at site a has the log-Laplace transform

$$-\log \mathbb{E}_\mu \exp\big[-\theta X_t(a)\big] = \big\langle \mu, \bar{V}_t(\theta \overline{\delta_a}\,)\big\rangle, \qquad \theta \geq 0, \tag{35}$$

[4] In [Kle00], Achim Klenke invited to think about interesting new examples meeting his criterion. In a sense, our proposition can be counted as such an example.

where (for $\theta \geq 0$ *and* $a \in \mathsf{R}^d$ *fixed)*

$$\bar{v} = \bar{V}(\theta\overline{\delta_a}) = \left\{\bar{V}_t(\theta\overline{\delta_a})(e') : \ t > 0, \ e' = (u', a') \in \mathsf{E}\right\} \geq 0 \quad (36)$$

uniquely solves equation (20) *with* ψ *replaced by* $\theta\overline{\delta_a}$ *, that is,*

$$\bar{v}_t(e') = \theta\, \mathrm{p}_t(a - a') - \varrho\, \bar{E}_{e'} \int_0^t \mathrm{d}k_s\, \bar{v}_{t-s}^{1+\beta}(0, \xi_s), \quad (37)$$

$t > 0, \ e' = (u', a') \in \mathsf{E}.$

(b) (Absolutely continuous measure states) $\mathbb{P}_\mu$*–almost surely,* X_t *is absolutely continuous.*

Note that the index γ does not enter into the dimension restriction $d < \frac{\alpha}{\beta}$ of this proposition. Roughly speaking, the enlargement of the lifetime of an "intrinsic particle" in the superprocess does not have an effect to the local issue of smoothness of the marginal measure states X_t. In particular, concerning absolute continuity of the states, there is no difference between the "classical" case corresponding to $\gamma = 1$ and our cases $\gamma < 1$. Of course, one expects singularity of the marginal measure states if $d \geq \frac{\alpha}{\beta}$.

We introduced the random densities $X_t(a)$, since in some special cases they will enter into the description of some mixed homogeneous Poissonian particle systems occurring in our main result (Theorem 11 below). But before returning to particle systems, we expose some other interesting properties of the superprocess $\bar{X}$. In particular, the long-term behavior of $\bar{X}$ will be "parallel" to our branching particle system case.

1.8 Scaling properties of $\bar{X}$ at the critical dimension

The next property follows from the log-Laplace representation in Lemma 4 by standard arguments (we skip the details).

Proposition 8 (Scaling properties) *Let* $d = \frac{\gamma\alpha}{\beta}$*. Then the* $(d, \alpha, \beta, \gamma)$*–superprocess has the following properties.*

(a) (Scaling for finite initial masses) *For each* $a \in \mathsf{R}^d$ *and* $K > 0$*, the process*

$$\left\{ \left(K^{-d/\alpha} \bar{X}_{Kt}\big(K^{-1}(\,\cdot\,) \times K^{1/\alpha}(\,\cdot\,)\big) \right)_{t \geq 0} \;\Big|\; \bar{X}_0 = \delta_{(0,a)} \right\} \quad (38)$$

coincides in law with the process

$$\left\{ \bar{X} \;\Big|\; \bar{X}_0 = K^{-d/\alpha} \delta_{(0, K^{-1/\alpha} a)} \right\}. \quad (39)$$

(b) (Self-similarity) *If instead* $\bar{X}_0 = i_0\, \delta_0 \times \ell$*, then* $\bar{X}$ *is self-similar:*

$$\left\{ K^{-d/\alpha} \bar{X}_{Kt}\big(K^{-1}(\,\cdot\,) \times K^{1/\alpha}(\,\cdot\,)\big) : \ t \geq 0 \right\} \overset{\mathcal{L}}{=} \bar{X}, \qquad K > 0. \quad (40)$$

Scaling properties are often a very useful tool. In the present case, we may combine them with Proposition 7, in order to get immediately the following result. Define $\bar{X}^{(K)}$ just as we introduced $\bar{Z}^{(K)}$ in (12). Recall the limiting residual lifetime distribution G^t_∞ introduced in (13).

Corollary 9 (Limiting scaling behavior of $\bar{X}$) *Let $d = \frac{\gamma\alpha}{\beta}$, and fix $t > 0$. Then the $(d, \alpha, \beta, \gamma)$–superprocess $\bar{X}$ has the following limiting scaling behavior.*

(a) (Asymptotics of $\bar{X}^{(K)}$ for finite initial masses) *Fix $a \in \mathsf{R}^d$. Assume that $\bar{X}_0 = K^{d/\alpha}\delta_{(0,K^{1/\alpha}a)}$. Then*

$$\bar{X}^{(K)}_t \underset{K\uparrow\infty}{\Longrightarrow} X'_t(0)\, G^t_\infty \times \ell \quad \textit{in law,} \tag{41}$$

where $\bar{X}'$ is the $(d, \alpha, \beta, \gamma)$–superprocess starting from $\bar{X}'_0 = \delta_{(0,a)}$, and $X'_t(0)$ is the random density at time t at the origin from the marginal process $X' := \bar{X}'\big(\mathsf{R}_+ \times (\,\cdot\,)\big)$ according to Proposition 7 (a).

(b) (Persistent convergence of $\bar{X}^{(K)}$) *If instead $\bar{X}_0 = i_0\,\delta_0 \times \ell$ for $i_0 > 0$, then*

$$\bar{X}^{(K)}_t \underset{K\uparrow\infty}{\Longrightarrow} X_t(0)\, G^t_\infty \times \ell \quad \textit{in law,} \tag{42}$$

where $X_t(0)$ is again the random density at time t at 0 of the marginal process $X = \bar{X}\big(\mathsf{R}_+ \times (\cdot)\big)$.

Specifying first to $t = 1$ and then to $K = t$ in the previous corollary and paying attention only to the spatial marginal measures leads to the following result.

Corollary 10 (Long-term behavior of X) *Let $d = \frac{\gamma\alpha}{\beta}$. The marginal X of the $(d, \alpha, \beta, \gamma)$–superprocess $\bar{X}$ has the following long-term behavior.*

(a) (Asymptotics of X_t for t–dependent finite initial masses) *Consider a whole family $\{{}^t\bar{X} : t \ge 0\}$ of $(d, \alpha, \beta, \gamma)$–superprocesses starting from ${}^t\bar{X}_0 = t^{d/\alpha}\delta_{(0,t^{1/\alpha}a)}$, $a \in \mathsf{R}^d$. Then for the spatial marginal processes ${}^tX := {}^t\bar{X}\,\big(\mathsf{R}_+ \times (\,\cdot\,)\big)$ we have*

$${}^tX_t \underset{t\uparrow\infty}{\Longrightarrow} X'_1(0)\,\ell \quad \textit{in law,} \tag{43}$$

where $X'_1(0)$ is the random density at time 1 at the origin from the marginal process $X' := \bar{X}'\big(\mathsf{R}_+ \times (\,\cdot\,)\big)$ according to Proposition 7 (a) *of the $(d, \alpha, \beta, \gamma)$–superprocess $\bar{X}'$ starting from $\bar{X}'_0 = \delta_{(0,a)}$.*

(b) (Persistent convergence of X_t) *If instead $\bar{X}_0 = i_0\,\delta_0 \times \ell$, $i_0 > 0$, then*

$$X_t \underset{t\uparrow\infty}{\Longrightarrow} X'_1(0)\,\ell \quad \textit{in law,} \tag{44}$$

where $X'_1(0)$ is as in (a) *but with $\bar{X}'_0 = i_0\,\delta_0 \times \ell$.*

Note that in the superprocess setting with $\bar{X}_0 = i_0\,\delta_0 \times \ell$ as in (b) it is easy to understand why in the critical dimension mixed Lebesgue measures occur in the spatial component in the limit as $K \uparrow \infty$, instead of the zero measure in the "classical" case of a (d,α,β)–superprocess. This behavior is caused by the coincidence that one has nice scaling properties *and* the existence of absolutely continuous states. While (d,α,β)–superprocesses have singular states at their critical dimension $d_{\rm c} = \frac{\alpha}{\beta}$, in contrast the spatial projections of (d,α,β,γ)–superprocesses have absolutely continuous states at their critical dimension $d_{\rm c} = \frac{\alpha\gamma}{\beta}$ $(< \frac{\alpha}{\beta})$.

Recall also that the statement in Corollary 10 (b) is quite analogous to a result on the super-Brownian reactant with a super-Brownian catalyst at the critical dimension $d = 2$, see [FK99]. (Note that also there an analog of part (a) of Corollary 10 can be established.)

1.9 Persistent convergence of Z at the critical dimension

Now we return to our marginal branching particle system $Z = \bar{Z}\big(\mathsf{R}_+ \times (\cdot)\big)$. First we state the analog of Corollary 10 (b). Note that the following theorem deepens the statement in [VW99] that non-trivial limit points exist.

Theorem 11 (Persistent convergence to a mixed Poisson system) *Consider the (d,α,β,γ)–branching particle system $\bar{Z}$ with $\bar{Z}_0 = \pi_{i_0\,G\times\ell}$. Assume that $d = \frac{\gamma\alpha}{\beta}$. Then*

$$Z_t \underset{t\uparrow\infty}{\Longrightarrow} \pi_{X_1(0)\,\ell} \quad \textit{in law,} \tag{45}$$

where $X_1(0)$ is the random density at time 1 at 0 of the marginal process $X = \bar{X}\big(\mathsf{R}_+ \times (\cdot)\big)$ of the (d,α,β,γ)–superprocess $\bar{X}$ with $\bar{X}_0 = i_0\,\delta_0 \times \ell$.

In other words, the limit population is mixed homogeneous Poisson, and its random intensity is just $X_1(0)$, the random density at time 1 at the origin of the marginal process X of the (d,α,β,γ)–superprocess $\bar{X}$ starting from $i_0\,\delta_0 \times \ell$.

Theorem 11 will actually be derived from our next theorem, and this will be provided in the end of the next subsection.

1.10 Refined asymptotics

The progress relative to [VW99], exhibited in Theorem 11, was possible by revealing a refined asymptotics as $K \uparrow \infty$ for the marginal $Z^{(K)} := \bar{Z}^{(K)}\big(\mathsf{R}_+ \times (\cdot)\big)$ of the rescaled (d,α,β,γ)–branching particle system $\bar{Z}^{(K)}$ starting from *finite* and asymptotically large initial populations [the particle analog of Corollary 9 (a)], which we now want to deal with.

For this purpose, we need to introduce further notation. Let $\mathcal{C}_p = \mathcal{C}_p(\mathsf{R}^d)$ denote the set of all continuous functions $\varphi : \mathsf{R}^d \to \mathsf{R}$ such that $|\varphi| \le c_\varphi\,\phi_p$

for some constant c_φ and such that φ/ϕ_p can continuously extended to a function on the one-point compactification $\dot{\mathsf{R}}^d$. Note that the "constant extension" $\bar{\varphi}$ of $\varphi \in \mathcal{C}_p$ belongs to $\bar{\mathcal{C}}_p$.

For $\varphi \in \mathcal{C}_p^+$, set [5)]

$$Q_t\varphi\,(a) \;:=\; 1 - \mathbf{E}_{\delta_{(\tau,a)}} \exp\big\langle \bar{Z}_t\,, -\bar{\varphi}\big\rangle \;=\; 1 - \mathbf{E}_{\delta_{(\tau,a)}} \exp\langle Z_t\,, -\varphi\rangle, \qquad (46)$$

$t \geq 0,\ \ a \in \mathsf{R}^d$. Note that $Q_t\varphi\,(a)$ occurs in the log-Laplace functional of the state $\bar{Z}_t$ of the (d,α,β,γ)–branching particle system starting from the homogeneous Poissonian particle system $\bar{Z}_0 = \pi_{i_0\,G\times\ell}$:

$$-\log \mathbf{E}_{\pi_{i_0\,G\times\ell}} \exp\big\langle \bar{Z}_t\,, -\bar{\varphi}\big\rangle \;=\; \langle i_0\,\ell, Q_t\varphi\rangle. \qquad (47)$$

Applied to the rescaled process $\bar{Z}^{(K)}$ introduced in (12) we get

$$\begin{aligned} -\log \mathbf{E}_{\pi_{i_0\,G\times\ell}} \exp\big\langle \bar{Z}_t^{(K)}, -\bar{\varphi}\big\rangle \;&=\; -\log \mathbf{E}_{\pi_{i_0\,G\times\ell}} \exp\big\langle Z_t^{(K)}, -\varphi\big\rangle \\ &=\; i_0 \int_{\mathsf{R}^d} \mathrm{d}a\; K^{d/\alpha}\, Q_{Kt}\varphi\,(K^{1/\alpha}a). \qquad (48) \end{aligned}$$

The asymptotics we are interested in concerns the quantity

$$V_t^{(K)}\varphi\,(a) \;:=\; K^{d/\alpha}\, Q_{Kt}\varphi\,(K^{1/\alpha}a) \qquad (49)$$

occurring in the integrand in (48). Here is our *key result*:

Theorem 12 (Refined asymptotics) *Assume that* $d = \frac{\gamma\alpha}{\beta}$ *. Then, for each* $\varphi \in \mathcal{C}_p^+,\ t > 0,$ *and* $a \in \mathsf{R}^d$,

$$V_t^{(K)}\varphi\,(a) \underset{K\uparrow\infty}{\longrightarrow} \bar{V}_t\big(\theta(\varphi)\overline{\delta_0}\,\big)\,(0,a) \;=:\; v_t(a), \qquad (50)$$

where

$$\theta(\varphi) \;:=\; \int_{\mathsf{R}^d} \mathrm{d}b\, \big[1 - \mathrm{e}^{-\varphi(b)}\big], \qquad (51)$$

and $\bar{V}\big(\theta(\varphi)\overline{\delta_0}\,\big)$ *is the unique solution to equation* (37) *in the case* $\theta = \theta(\varphi)$ *and* $a = 0$, *and with special* ϱ *as in* (29). *Consequently, writing*

$$v_t(a) \;:=\; \bar{V}_t\big(\theta(\varphi)\overline{\delta_0}\,\big)\,(0,a), \qquad t > 0, \quad a \in \mathsf{R}^d, \qquad (52)$$

then $v = \{v_t(a) : t > 0,\ a \in \mathsf{R}^d\} \geq 0$ *solves*

$$v_t(a) \;=\; \theta(\varphi)\,\mathrm{p}_t(a) \;-\; \varrho \int_0^t \mathrm{d}(s^\gamma)\; E_a v_{t-s}^{1+\beta}(\xi_s) \qquad (53)$$

$t > 0,\ \ a \in \mathsf{R}^d$.

[5)] By an abuse of notation, $\mathbf{P}_{\delta_{(\tau,a)}}$ denotes the (deterministic) law $\int_0^\infty \mathrm{d}G(u)\,\mathbf{P}_{\delta_{(u,a)}}$ of $\bar{Z}$ started from a single particle with position a and residual life time τ distributed by G. Also in other cases of "mixed" initial states we will use such type of notation.

The proof of this theorem will be given in Subsection 2.6 below, after preparations in the Subsections 2.3 – 2.5.

We mention that a refined asymptotics for some Markov branching models (concerning exponential lifetimes) with infinite variance was recently also given by [Kle98].

Now we want to explain how Theorem 12 is related to the mentioned asymptotics for the case of large but finite initial populations.

Corollary 13 (Asymptotics starting from large finite populations) *Let* $d = \frac{\gamma\alpha}{\beta}$. *Assume*[6] $\bar{Z}_0 = [K^{d/\alpha}]\delta_{(\tau, K^{1/\alpha}a)}$ *with* $a \in \mathsf{R}^d$. *Then for the marginal rescaled processes* $Z^{(K)} := \bar{Z}^{(K)}\bigl(\mathsf{R}_+ \times (\cdot)\bigr)$ *and* $t > 0$ *we have*

$$Z_t^{(K)} \underset{K\uparrow\infty}{\Longrightarrow} \pi_{X_t(0)\,\ell} \quad \textit{in law}, \tag{54}$$

where $X_t(0)$ *is the random density at time* t *at* 0 *of the marginal process* X *of the* (d,α,β,γ)*–superprocess* $\bar{X}$ *with* $\bar{X}_0 = \delta_{(0,a)}$.

Clearly, the previous corollary can be restated in terms of a long-term behavior analogously to Corollary 10 (a).

Proof of Corollary 13 Fix $t > 0$, $a \in \mathsf{R}^d$, and $\varphi \in \mathcal{C}_p^+$. By the branching property and the definition (12) of rescaling, for $K > 0$,

$$\log \mathbf{E}_{[K^{d/\alpha}]\delta_{(\tau,K^{1/\alpha}a)}} \mathrm{e}^{-\langle Z_t^{(K)},\varphi\rangle} \;=\; [K^{d/\alpha}] \log \mathbf{E}_{\delta_{(\tau,K^{1/\alpha}a)}} \mathrm{e}^{-\langle Z_{Kt},\varphi\rangle}. \tag{55}$$

Using the definition (46) of $Q\varphi$, the limit as $K \uparrow \infty$ of the right hand side of the latter equation equals

$$\lim_{K\uparrow\infty} [K^{d/\alpha}] \log\bigl(1 - Q_{Kt}\varphi\,(K^{1/\alpha}a)\bigr) \;=\; -\lim_{K\uparrow\infty} V_t^{(K)}\varphi\,(a) \;=\; -v_t(a), \tag{56}$$

where we used notation (49) and Theorem 12. But by the definition (50) of $v_t(a)$, the log-Laplace formula (35) in Proposition 7,

$$-v_t(a) \;=\; \log \mathbb{E}_{\delta_{(0,a)}} \exp\bigl[-\theta(\varphi)\, X_t(0)\bigr]. \tag{57}$$

However by formula (51), this is the log-Laplace transform of the mixed homogeneous Poissonian particle system $\pi_{X_t(0)\ell}$ as desired. This finishes the proof (based on Theorem 12). ■

Proof of Theorem 11 We need to combine (48) and (50) to conclude for

$$-\log \mathbf{E}_{\pi_{i_0\,G\times\ell}} \exp\langle Z_t\,, -\varphi\rangle \underset{t\uparrow\infty}{\longrightarrow} i_0 \int_{\mathsf{R}^d} \mathrm{d}a\; \bar{V}_1\bigl(\theta(\varphi)\overline{\delta_0}\,\bigr)\,(0,a) \tag{58}$$

[6] $[z]$ denotes the integer part of z.

with $\varphi \in \mathcal{C}_p^+$ and $\theta(\varphi)$ as in (51), since the right hand side in (58) equals the log-Laplace transform of the mixed homogeneous Poissonian particle system $\pi_{X_1(0)\ell}$. For this we only have to justify that in

$$\int_{\mathsf{R}^d} \mathrm{d}a\ t^{d/\alpha}\, Q_t\varphi\,(t^{1/\alpha}a) \tag{59}$$

from (48) the limit can be provided under the integral. In fact, by the definition (46) of Q,

$$Q_t\varphi\,(a)\ \le\ \mathbf{E}_{\delta_{(\tau,a)}}\langle Z_t\,,\varphi\rangle\ =\ E_a\varphi(\xi_t), \tag{60}$$

where we used the criticality of the process Z [see (86) below]. Hence

$$t^{d/\alpha}\, Q_t\varphi\,(t^{1/\alpha}a)\ \le\ t^{d/\alpha}\int_{\mathsf{R}^d} \mathrm{d}b\ \mathrm{p}_t(b-t^{1/\alpha}a)\,\varphi(b). \tag{61}$$

But by the *self-similarity*

$$K^{d/\alpha}\,\mathrm{p}_{Ks}(K^{1/\alpha}b)\ =\ \mathrm{p}_s(b),\qquad K,s>0,\quad b\in\mathsf{R}^d, \tag{62}$$

of the stable transition densities of Hypothesis 1 (a), the right hand side of (61) can be rewritten as

$$\int_{\mathsf{R}^d} \mathrm{d}b\ \mathrm{p}_1(t^{-1/\alpha}b-a)\,\varphi(b) \tag{63}$$

which converges to

$$\mathrm{p}_1(a)\int_{\mathsf{R}^d} \mathrm{d}b\ \varphi(b)\ =:\ \mathrm{p}_1(a)\,\|\varphi\|_1 \tag{64}$$

as $t\uparrow\infty$ (where we also used the symmetry of p_s). Finally, integrating the expressions in (63) and (64) with respect to $\mathrm{d}a$ we get identically $\|\varphi\|_1$. Thus, we can apply the extended dominated convergence theorem to justify that the limit in (59) can be provided under the integral. This finishes the proof (based on Theorem 12). ∎

2 Remaining proofs

After some preparation, we will demonstrate in Subsection 2.2 that X has random densities in each point. The main purpose however is to prove the refined asymptotics Theorem 12, which is done in the Subsections 2.3 – 2.6.

2.1 Preliminaries: Some notation

With c we will always denote a positive constant which might vary from place to place. A c with some additional mark (as c_f or c_G) will however denote a specific constant. A constant of the form $c_{(\#)}$ means, this constant first occurred related to formula line (#).

Recall that p denotes the stable transition density function from Hypothesis 1 (a). From the self-similarity (62) we immediately get the following simple estimate

$$\mathrm{p}_r(a) \ \leq \ c_{(65)}\, r^{-d/\alpha}, \qquad r>0, \quad a \in \mathsf{R}^d. \tag{65}$$

On the other hand, we also have

$$\mathrm{p}_r(a) \ \leq \ c_{(66)}\, r\, |a|^{-d-\alpha}, \qquad r>0, \quad a \in \mathsf{R}^d, \tag{66}$$

see, for instance, [FG86, formula (A.8)].

We denote by $\bar{S} = \{\bar{S}_t : t \geq 0\}$ the semigroup of the Markov process $\bar{\xi}$.

Recall also that the boldface letter $\mathbf{P}$ is related to a law of a particle system, the blackboard letter $\mathbb{P}$ to superprocesses, and the italics P to a law of a "basic" random object as ξ, τ, η. Corresponding expectations are expressed as $\mathbf{E}, \mathbb{E}, E$, respectively.

2.2 Absolutely continuous measure states of X (proof of Proposition 7)

Here we will *prove Proposition* 7. It will be based on some modification of a general criterion for absolutely continuous measure states of catalytic superprocesses given in [Kle00]. Recall that $d < \frac{\alpha}{\beta}$. Fix $\mu \in \mathcal{M}_p(\mathsf{E})$ and $t > 0$.

1° (*Absolutely continuous expectation*) By the expectation formula (21), the measure $\mathbb{E}_\mu X_t$ on R^d is absolutely continuous with continuous density function

$$a \ \mapsto \ \int_{\mathsf{E}} \mu\big(\mathrm{d}(u,b)\big)\ \mathrm{p}_t(b-a) \ =: \ \mu * \mathrm{p}_t\,(a). \tag{67}$$

2° (*Absolutely continuous states*) Assume for the moment, that

$$\lim_{\varepsilon \downarrow 0} \Big(-\log \mathbb{E}_\mu \exp\big\langle X_t\,,\, -\theta\, \delta_a * \chi_\varepsilon\big\rangle\Big) \ =: \ w(a,\theta) \tag{68}$$

exists for each $a \in \mathsf{R}^d$ and $\theta > 0$, and that

$$\frac{\partial}{\partial\theta} w(a,\theta)\big|_{\theta=0+} \ = \ \mu * \mathrm{p}_t\,(a), \qquad a \in \mathsf{R}^d. \tag{69}$$

Then, for each $a \in \mathsf{R}^d$, the random density $X_t(a) = \langle X_t\,, \delta_a\rangle$ exists as claimed in (34), has log-Laplace transform

$$-\log \mathbb{E}_\mu \exp\big[-\theta X_t(a)\big] \ = \ w(a,\theta), \qquad \theta > 0, \tag{70}$$

and "full" expectation

$$\mathbb{E}_\mu X_t(a) \ = \ \mu * \mathrm{p}_t\,(a). \tag{71}$$

Hence, combined with step 1°, we recognize that the singular measure component of the random measure X_t must disappear a.s., that is, we get the absolute continuity claim (b) (see, for instance, [Kle00, Lemma 2.2] for a more careful formulation).

3° (*Uniform regularity*) To attack the proof of (68) and (69), we go back from the marginal measure $X_t = \bar{X}_t\big(\mathsf{R}_+ \times (\cdot)\big)$ to $\bar{X}_t$, and use the log-Laplace representation (19). This means we have to study the log-Laplace equation (20) with the function ψ replaced by

$$(u,b) \;\mapsto\; \theta\,\delta_a * \chi_\varepsilon(b) \;=:\; \theta\,\overline{\delta_a}^{\,\varepsilon}(u,b), \tag{72}$$

which is constant in the first coordinate of $(u,b) \in \mathsf{E}$, and we have to let $\varepsilon \downarrow 0$. Note that $\overline{\delta_a}^{\,\varepsilon}$ approaches the "constantly extended δ–function" $\overline{\delta_a}$ on E, which of course does not belong to the set $\bar{\mathcal{C}}_p^+$ of test functions occurring in the log-Laplace equation (20), so we have to justify its usage in order to come to equation (37).

First we show that the family $\big\{\overline{\delta_a} : a \in \mathsf{R}^d\big\}$ of measures (generalized functions) on E has the following *regularity* property, *uniformly* in space: For all $u \geq 0$ and $0 \leq r \leq t$,

$$\sup_{a,b\,\in\,\mathsf{R}^d} \bar{E}_{(u,b)} \int_r^t \mathrm{d}k_s\ \big[\bar{S}_{t-s}\overline{\delta_a}\,(\bar{\xi}_s)\big]^{1+\beta} \tag{73}$$

$$\leq\; c_{(74)}\, t^{-d/\alpha} \int_{\{u\vee r\,\leq\, s\,\leq\, t\}} \mathrm{d}_s(s-u)^\gamma\ (t-s)^{-\beta d/\alpha} \;<\; \infty. \tag{74}$$

In fact,

$$\bar{S}_{t-s}\overline{\delta_a}\,(\bar{\xi}_s) \;=\; \delta_a * \mathrm{p}_{t-s}\,(\xi_s) \;=\; \mathrm{p}_{t-s}(\xi_s - a), \tag{75}$$

and, since k and ξ are independent, by using (24), the expectation expression in (73) is bounded from above by

$$\varrho \int_{\{u\vee r\,\leq\, s\,\leq\, t\}} \mathrm{d}_s(s-u)^\gamma\ E_b\,\mathrm{p}_{t-s}^{1+\beta}(\xi_s - a)$$

$$\leq\; c\,t^{-d/\alpha} \int_{\{u\vee r\,\leq\, s\,\leq\, t\}} \mathrm{d}_s(s-u)^\gamma\ (t-s)^{-\beta d/\alpha}, \tag{76}$$

where we first applied the simple estimate (65) to p^β, then Chapman-Kolmogorov to the expectation on p, and finally again (65). But the integral in (76) is finite since γ and $\beta d/\alpha$ belong to $(0,1)$. Consequently, (74) is true.

4° (*Uniform integrability in the regularization*) Next we verify that for $u \geq 0$ and $a,b \in \mathsf{R}^d$,

$$\lim_{r\uparrow t}\ \limsup_{\varepsilon\downarrow 0}\ \bar{E}_{(u,b)} \int_r^t \mathrm{d}k_s \big[\bar{S}_{t-s}\overline{\delta_a}^{\,\varepsilon}\,(\bar{\xi}_s)\big]^{1+\beta} \;=\; 0 \tag{77}$$

[recall notation (72)]. Indeed, by Jensen's inequality,

$$\left[\bar{S}_{t-s}\overline{\delta_a}^{\varepsilon}\,(\bar{\xi}_s)\right]^{1+\beta} \;\le\; \int_{\mathsf{R}^d} \mathrm{d}a'\;\chi_\varepsilon(a')\left[\bar{S}_{t-s}\overline{\delta_{a+a'}}\,(\bar{\xi}_s)\right]^{1+\beta}. \tag{78}$$

Therefore, the expectation expression in (77) can be bounded from above by

$$\sup_{a',b\,\in\,\mathsf{R}^d} \bar{E}_{(u,b)} \int_r^t \mathrm{d}k_s\;\left[\bar{S}_{t-s}\overline{\delta_{a'}}\,(\bar{\xi}_s)\right]^{1+\beta} \tag{79}$$

which does not dependent on ε and, moreover, tends to 0 as $s \uparrow t$ by the estimate in (74). This gives (77).

5° (*Conclusions*) Since according to step 3°, for a in R^d fixed, $\overline{\delta_a}$ is a regular measure on E, then, by [Kle00, Proposition 1.2], equation (20) with ψ replaced by $\theta\overline{\delta_a}$ where $0<\theta\le 1$, that is equation (37), has exactly one solution $\bar{v} = \bar{V}(\theta\overline{\delta_a}\,)$, and

$$\frac{\partial}{\partial\theta}\bar{v}_t(e')\big|_{\theta=0+} \;=\; \mathrm{p}_t\,(a-a'), \qquad e'=(u',a')\in\mathsf{E}. \tag{80}$$

Moreover, as in the proof of formula line (2.12) in [Kle00],

$$\bar{V}_t\big(\theta\,\overline{\delta_a}^{\varepsilon}\,\big)(e') \;\underset{\varepsilon\downarrow 0}{\longrightarrow}\; \bar{V}_t(\theta\overline{\delta_a}\,)(e'), \qquad e'\in\mathsf{E}. \tag{81}$$

Hence, from the general *domination* formula

$$0 \;\le\; \bar{V}_t\psi\,(e) \;\le\; \bar{E}_e\psi(\bar{\xi}_t), \qquad t\ge 0, \quad e\in\mathsf{E}, \quad \psi\in\bar{\mathcal{C}}_p^{\,+}, \tag{82}$$

and dominated convergence, we obtain

$$\big\langle\mu, \bar{V}_t(\theta\,\overline{\delta_a}^{\varepsilon}\,)\big\rangle \;\underset{\varepsilon\downarrow 0}{\longrightarrow}\; \big\langle\mu, \bar{V}_t(\theta\overline{\delta_a}\,)\big\rangle.$$

But this gives the needed statements (68) and (69) in step 2° with $w(a,\theta) = \big\langle\mu, \bar{V}_t(\theta\overline{\delta_a}\,)\big\rangle$, and at the same time claims (35) – (37), finishing the proof of Proposition 7. ■

2.3 Renewal type equation and its scaling

Our starting point for the asymptotic properties of $V^{(K)}\varphi$ as needed for Theorem 12 is the fact, that it satisfies an integral equation (Lemma 15 below). For this we need first some further notation.

Recall that τ with law G denotes the lifetime of a newly born particle in our branching particle system $\bar{Z}$. Let $\tau_1, \tau_2, \ldots$ denote independent copies of τ, and set $\sigma_n := \sum_{1\le i\le n}\tau_i$, $n\ge 0$. With

$$g_t \;:=\; \sum_{n\ge 1}\mathbf{1}_{\{\sigma_n\,\le\, t\}}, \qquad t\ge 0, \tag{83}$$

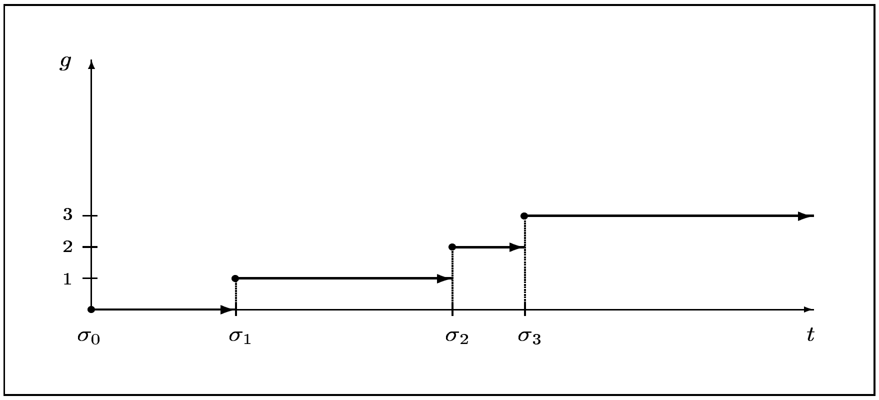

Figure 1: Generation number process g

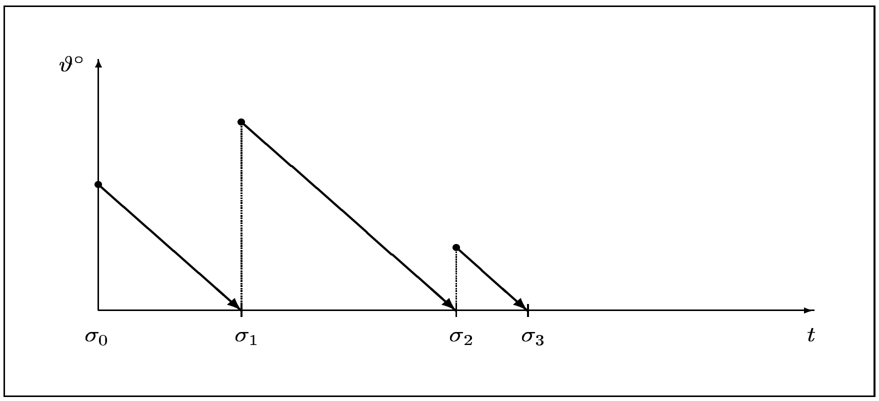

Figure 2: Residual lifetime process ϑ°

we get the "*generation number process*" (see Figure 1), and with

$$\vartheta_t^\circ \;:=\; \sigma_{g_t+1} - t, \qquad t \geq 0, \tag{84}$$

the *residual lifetime process* related to a single particle in our branching particle system (see Figure 2). Recalling that ξ denotes the particles' motion process, we pass now to the pair $\bar{\xi}^\circ := (\vartheta^\circ, \xi)$ of independent processes which laws we denote by $\bar{P}_e^\circ$, $e \in \mathsf{E}$. Note that $(\bar{\xi}^\circ, \bar{P}_e^\circ, \, e \in \mathsf{E})$ is a càdlàg time-homogeneous Markov process in $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$, and that the laws $\bar{P}_e^\circ$, $e \in \mathsf{E}$, describe the renewal process σ and the generation number process g as well. Recall the notation $Q\varphi$ introduced in (46).

Lemma 14 (Renewal type equation) *Fix* $\varphi \in \mathcal{C}_p^+$. *Then*

$$Q_t\varphi(a) \;=\; E_a\big[1-\mathrm{e}^{-\varphi(\xi_t)}\big] - c_f\,\bar{E}^{\circ}_{(\tau,a)}\int_{(0,t]}\mathrm{d}g_s\,\big[Q_{t-s}\varphi(\xi_s)\big]^{1+\beta}, \qquad (85)$$

$t \geq 0,\ a \in \mathsf{R}^d$.

Proof Follow the proof of [KS98, Lemma 3] with the obvious changes to adapt to our setting with the residual lifetimes in the description. In particular, first distinguish between $\tau \geq t$ and $\tau < t$; then, in the latter case further iterate. ■

Consequently, despite the lifetime distribution is not exponential, one can exploit the renewal in the time points of death of a particle, and some invariance properties related to these points.

Note that (85) yields the *expectation formula*

$$\mathbf{E}_{\delta_{(\tau,a)}}\big\langle \bar{Z}_t\,,\bar{\varphi}\big\rangle \;=\; E_a\varphi(\xi_t), \qquad a\in\mathsf{R}^d,\quad t\geq 0,\quad \varphi\in\mathcal{C}_p^+, \qquad (86)$$

(replace, for instance, φ by $\theta\varphi$ and differentiate with respect to $\theta > 0$ at $\theta = 0+$).

Recall that we are interested in the rescaled quantity $V^{(K)}\varphi$ (where $K > 0$ and $\varphi \in \mathcal{C}_p^+$) introduced in (49). For $t \geq 0$ and $a \in \mathsf{R}^d$, put

$$W_t^{(K)}\varphi(a) \;:=\; K^{d/\alpha}\,E_{K^{1/\alpha}a}\Big(1-\exp\big[-\varphi(\xi_{Kt})\big]\Big), \qquad (87)$$

and introduce the (right-continuous) *renewal function* N corresponding to the lifetime distribution function G :

$$N_s \;:=\; Eg_s \;=\; \sum_{k=1}^{\infty} G^{*k}(s), \qquad s\geq 0, \qquad (88)$$

[recall notation (83)]. Note that by assumption (2) in Hypothesis 1 (b),

$$N_K \;\sim\; D\,K^{\gamma} \quad \text{as } K\uparrow\infty \qquad (89)$$

with the constant

$$D \;:=\; \frac{\sin\pi\gamma}{\pi}\,\frac{1}{c_G}\,. \qquad (90)$$

Lemma 15 (Scaled equation) *Fix* $K > 0$ *and* $\varphi \in \mathcal{C}_p^+$. *The functional* $V^{(K)}\varphi$ *defined in* (49) *satisfies the integral equation*

$$V_t^{(K)}\varphi(a) \;=\; W_t^{(K)}\varphi(a) - c_f\int_{(0,t]}\frac{\mathrm{d}_sN_{Ks}}{K^{\gamma}}\;E_a\big[V_{t-s}^{(K)}\varphi(\xi_s)\big]^{1+\beta}, \qquad (91)$$

$t \geq 0,\ a \in \mathsf{R}^d$.

Proof Scaling (85) as in (49), the left hand side of (85) changes to $V_t^{(K)}\varphi(a)$, and the first part of the right hand side of (85) changes to the $W^{(K)}$–term as in (91). For the remaining part, exploit the independence of the generation number process g and the motion process ξ, and apply the definition (88) of N to get

$$- c_f K^{d/\alpha} E_{K^{1/\alpha}a} \int_{(0,Kt]} \mathrm{d}N_s \left[Q_{Kt-s}\varphi(\xi_s)\right]^{1+\beta} \tag{92}$$

$$= -c_f K^{d/\alpha} \int_{(0,t]} \mathrm{d}_s N_{Ks}\, E_{K^{1/\alpha}a} \left[Q_{K(t-s)}\varphi(\xi_{Ks})\right]^{1+\beta}. \tag{93}$$

Then by the *self-similarity* of the stable process ξ, that is,

$$E_{K^{1/\alpha}a}\big(K^{-1/\alpha}\xi_{Ks} \in \cdot\big) = E_a(\xi_s \in \cdot), \tag{94}$$

$a \in \mathsf{R}^d$, $s \geq 0$, $K > 0$, and the assumed critical parameter constellation $d\beta/\alpha = \gamma$, using once more (49), also the remaining term of (91) occurs, finishing the proof. ■

2.4 Convergence to the limiting equation

Our next task is to let $K \uparrow \infty$ in the integral equation (91), provided that $t > 0$. For this purpose, we first note that by the self-similarity as formulated in (94), $W^{(K)}$ defined in (87) can be written as

$$W_t^{(K)}\varphi(a) = \int_{\mathsf{R}^d} \mathrm{d}b\, \mathrm{p}_t\big(K^{-1/\alpha}b - a\big)\big[1 - \mathrm{e}^{-\varphi(b)}\big]. \tag{95}$$

By dominated convergence, this implies

$$W_t^{(K)}\varphi(a) \underset{K\uparrow\infty}{\longrightarrow} \theta(\varphi)\,\mathrm{p}_t(a), \qquad t > 0, \quad a \in \mathsf{R}^d, \tag{96}$$

with $\theta(\varphi)$ from (51). On the other hand, by (89), for $T > 0$ fixed,

$$\frac{N_{Ks}}{N_K} \underset{K\uparrow\infty}{\longrightarrow} s^\gamma, \qquad \text{uniformly in } s \in [0,T]. \tag{97}$$

In fact, take $\varepsilon \in (0,T)$ and deal with $s \leq \varepsilon$ and $\varepsilon < s \leq T$ separately. Then from (89) and (97),

$$\frac{N_{Ks}}{K^\gamma} = \frac{N_K}{K^\gamma}\frac{N_{Ks}}{N_K} \underset{K\uparrow\infty}{\longrightarrow} D\, s^\gamma \qquad \text{uniformly in } s \in [0,T]. \tag{98}$$

Inserting (96) and (98) into (91) by using the fact that

$$c_f D = \varrho \tag{99}$$

[recall (90) and (29)], we expect that

$$V_t^{(K)}\varphi(a) \underset{K\uparrow\infty}{\longrightarrow} v_t(a) \tag{100}$$

where v solves equation (53).

We start with showing (100) in some uniform $L^1 = \big(L^1(\mathrm{d}a), \|\cdot\|_1\big)$ sense. For some later analytic extension argument, we introduce an additional factor $\lambda \geq 0$ in front of φ, and write

$$v_t^{(\lambda)}(a) \;:=\; \bar{V}_t\big(\theta(\lambda\varphi)\overline{\delta_0}\,\big)(0,a), \qquad t>0, \quad a\in\mathsf{R}^d, \tag{101}$$

instead of $v_t(a)$. Then (53) changes to

$$v_t^{(\lambda)}(a) \;=\; \theta(\lambda\varphi)\,\mathrm{p}_t(a) \;-\; \varrho\int_0^t \mathrm{d}(s^\gamma)\; E_a\big[v_{t-s}^{(\lambda)}(\xi_s)\big]^{1+\beta}, \tag{102}$$

$t>0, \;\; a\in\mathsf{R}^d$.

Lemma 16 (Uniform L^1–convergence for small λ) *Fix* $0<L<T$ *and* $\varphi\in\mathcal{C}_p^+$. *There exists a constant* $\Lambda = \Lambda(T,\varphi) > 0$ *such that for* $V^{(K)}(\lambda\varphi)$ *defined in* (49) *we have*

$$\lim_{K\uparrow\infty}\;\sup_{L\leq t\leq T}\;\sup_{\lambda\,\in\,[0,\Lambda]}\;\Big\|V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\Big\|_1 \;=\; 0, \tag{103}$$

where $v^{(\lambda)}\geq 0$ *solves* (102) *with the constant* $\varrho = c_f D$.

To prepare for the proof of this lemma, by using equations (91) (with φ replaced by $\lambda\varphi$) and (102), we can estimate the L^1–norm expression in the claim (103):

$$\Big\|V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\Big\|_1 \;\leq\; A_1 + c_f\,A_2 + c_f\,A_3 + c_f\,A_4 + c_f D\,A_5\,. \tag{104}$$

Here, by using the identity

$$\int_{\mathsf{R}^d}\mathrm{d}a\;E_aF(\xi_s) \;=\; \|F\|_1\,, \qquad \text{measurable } F\geq 0, \quad s\geq 0, \tag{105}$$

for fixed $t>0$ and $\lambda\geq 0$,

$$A_1 \;:=\; \Big\|W_t^{(K)}(\lambda\varphi) - \theta(\lambda\varphi)\,\mathrm{p}_t\Big\|_1, \tag{106a}$$

$$A_2 \;:=\; \int_{(0,t)}\frac{\mathrm{d}_sN_{Ks}}{K^\gamma}\;\Big\|\big[V_{t-s}^{(K)}(\lambda\varphi)\big]^{1+\beta} - \big[v_{t-s}^{(\lambda)}\big]^{1+\beta}\Big\|_1, \tag{106b}$$

$$A_3 \;:=\; \frac{(N_{Kt}-N_{Kt-})}{K^\gamma}\;\Big\|\big[V_0^{(K)}(\lambda\varphi)\big]^{1+\beta}\Big\|_1, \tag{106c}$$

$$A_4 \;:=\; \Big|\frac{N_K}{K^\gamma} - D\Big|\;\Big\|\int_{(0,t)}\frac{\mathrm{d}_sN_{Ks}}{N_K}\;\big[v_{t-s}^{(\lambda)}\big]^{1+\beta}\Big\|_1, \tag{106d}$$

$$A_5 \;:=\; \Big|\int_{(0,t)}\Big(\frac{\mathrm{d}_sN_{Ks}}{N_K} - \mathrm{d}(s^\gamma)\Big)\Big\|\big[v_{t-s}^{(\lambda)}\big]^{1+\beta}\Big\|_1\Big|. \tag{106e}$$

2.5 Uniform L^1–convergence (proof of Lemma 16)

Here we prove Lemma 16. We proceed in several steps.

1° (*Error term* A_1) First of all, we consider any constant $\Lambda > 0$ and all $\lambda \in [0,\Lambda]$. By (95),

$$A_1 \;\le\; \int_{\mathsf{R}^d} \mathrm{d}a \int_{\mathsf{R}^d} \mathrm{d}b \;\Big|\mathrm{p}_t\big(K^{-1/\alpha}b-a\big)-\mathrm{p}_t(a)\Big|\;\big[1-\mathrm{e}^{-\lambda\varphi(b)}\big] \tag{107}$$

$$\le\; c_\varphi\,\Lambda \int_{\mathsf{R}^d} \mathrm{d}b\;\phi_p(b)\int_{\mathsf{R}^d}\mathrm{d}a\;\Big|\mathrm{p}_t\big(K^{-1/\alpha}b-a\big)-\mathrm{p}_t(a)\Big|. \tag{108}$$

For $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that

$$\int_{|b|\ge C} \mathrm{d}b\;\phi_p(b) \;\le\; \varepsilon, \tag{109}$$

whereas the internal integral in (108) is bounded by 2 since the p_s are probability density functions. On the other hand, by the uniform continuity of the stable transition density functions p on $[L,T]\times\mathsf{R}^d$, for $|b|\le C$ the internal integral in (108) is smaller then ε for K sufficiently large (depending on C). This shows that

$$\lim_{K\uparrow\infty}\;\sup_{\substack{L\,\le\,t\,\le\,T,\\ \lambda\,\in\,[0,\Lambda]}} A_1 \;=\; 0. \tag{110}$$

2° (*Error term* A_3) Use the simply bound

$$\Big\|\big[V_0^{(K)}(\lambda\varphi)\big]^{1+\beta}\Big\|_1 \;\le\; K^{\beta d/\alpha}\,\lambda\|\varphi\|_1\,, \tag{111}$$

which follows from

$$V_0^{(K)}(\lambda\varphi)\,(a) \;\le\; \lambda K^{d/\alpha}\varphi(K^{1/\alpha}a) \tag{112}$$

[by (91) and (87)], criticality $\beta d/\alpha = \gamma$, and

$$\lim_{K\uparrow\infty}\;\sup_{L\le t\le T}\;(N_{Kt}-N_{Kt-}) \;=\; 0, \tag{113}$$

(see, e.g., [Fel71, (9.1.9)]), in order to see that

$$\lim_{K\uparrow\infty}\;\sup_{\substack{L\,\le\,t\,\le\,T,\\ \lambda\,\in\,[0,\Lambda]}} A_3 \;=\; 0, \tag{114}$$

too.

3° (*Error term* A_4) From (102), by domination, for $s > 0$,

$$0 \;\le\; v_s^{(\lambda)}(a) \;\le\; \theta(\lambda\varphi)\,\mathrm{p}_s(a) \;\le\; \Lambda\,\|\varphi\|_1\,\mathrm{p}_s(a). \tag{115}$$

Hence, using the simple estimate (65) and the identity $d\beta/\alpha = \gamma$, we get

$$0 \;\leq\; \bigl[v_s^{(\lambda)}(a)\bigr]^{1+\beta} \;\leq\; c\left(\Lambda\,\|\varphi\|_1\right)^{1+\beta}\; \frac{\mathrm{p}_s(a)}{s^\gamma}\,. \tag{116}$$

However, the p_s are probability density functions. Therefore

$$A_4 \;\leq\; c\,\left(\Lambda\,\|\varphi\|_1\right)^{1+\beta}\,\left|\frac{N_K}{K^\gamma} - D\right|\,\int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma}\,. \tag{117}$$

But by substitution, for $0 \leq r < t$,

$$\int_r^t \mathrm{d}(s^\gamma)\,(t-s)^{-\gamma} \;\leq\; \gamma\, t^{1-\gamma}\int_{r/t}^1 \frac{\mathrm{d}s}{s^{1-\gamma}\,(1-s)^\gamma} \;<\; \infty. \tag{118}$$

Thus, from (97), uniformly in $0 \leq r < t \leq T$,

$$\int_{[r,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma} \;\underset{K\uparrow\infty}{\longrightarrow}\; \gamma\int_{r/t}^1 \frac{\mathrm{d}s}{s^{1-\gamma}\,(1-s)^\gamma} \;\leq\; \gamma\,B(\gamma,\gamma+1) \tag{119}$$

with B denoting the Beta function. Indeed, approximate the integral by sums, use monotonicity in time and continuity of the limit in (97). Thus, by (98),

$$\lim_{K\uparrow\infty}\; \sup_{\substack{0<t\leq T,\\ \lambda\in[0,\Lambda]}} A_4 \;=\; 0. \tag{120}$$

4° (*Error term* A_5) Let $\varepsilon \in (0,1)$. We split the integral in A_5 concerning the cases $(1-\varepsilon)t \leq s < t$ and $0 \leq s \leq (1-\varepsilon)t$. In the first case, we pass from the difference to the sum and use the estimate (116). Then apply (119) with $r = (1-\varepsilon)t$ to see that we end up with an ε–term, uniformly in t, λ, and K. In the second case, we use that the map

$$s \;\mapsto\; \left\|\bigl[v_{t-s}^{(\lambda)}\bigr]^{1+\beta}\right\|_1 \quad \text{on} \quad \bigl[0,(1-\varepsilon)t\bigr] \tag{121}$$

is bounded and continuous, and the weak convergence

$$\frac{\mathrm{d}_s N_{Ks}}{N_K} \;\underset{K\uparrow\infty}{\longrightarrow}\; \frac{\gamma\,\mathrm{d}s}{s^{1-\gamma}} \tag{122}$$

on the same interval, uniformly in t. Putting both together, we get

$$\lim_{K\uparrow\infty}\; \sup_{\substack{L\leq t\leq T,\\ \lambda\in[0,\Lambda]}} A_5 \;=\; 0. \tag{123}$$

5° (*A bound for* A_2) Using the elementary inequality

$$\left|x^{1+\beta} - y^{1+\beta}\right| \;\leq\; (1+\beta)\;|x-y|\;\left(x^\beta + y^\beta\right), \qquad x,y \geq 0, \tag{124}$$

the domination

$$0 \;\le\; V_t^{(K)}(\lambda\varphi)\,(a) \;\le\; W_t^{(K)}(\lambda\varphi)\,(a) \tag{125}$$

[by (91)] implying

$$0 \;\le\; V_t^{(K)}(\lambda\varphi)\,(a) \;\le\; c\,\Lambda\,\|\varphi\|_1\,t^{-d/\alpha} \tag{126}$$

[recall (95) and (65)], as well as dominations (115) and (65), we see that, for $0 \le s < t$, the L^1–norm in A_2 is bounded by

$$c\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\,(t-s)^{-\gamma}\,\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 . \tag{127}$$

Thus,

$$A_2 \;\le\; c\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{K^{\gamma}}\,\frac{\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1}{(t-s)^{\gamma}} . \tag{128}$$

6° (*A Gronwall type inequality*) Inserting (110), (114), (120), (123), and the bound (128) into the estimate (104) gives the following statement (recall that $\Lambda > 0$ is fixed and not yet specified): For each $\varepsilon > 0$, there is a constant $K_0 = K_0(\varepsilon,\Lambda) > 0$ such that

$$\begin{aligned}\left\|V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\right\|_1 \;&\le\; \varepsilon \\ &+ c_{(129)}\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{K^{\gamma}}\,\frac{\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1}{(t-s)^{\gamma}}\end{aligned} \tag{129}$$

for all $K \ge K_0$ and $t \in [L,T]$, as well as $\lambda \in [0,\Lambda]$. From the domination formulas (125) and (115) as well as the representation (95) of $W^{(K)}$ we obtain

$$\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 \;\le\; 2\Lambda\,\|\varphi\|_1 , \tag{130}$$

uniformly in $K > 0$, $s < t$ and $\lambda \in [0,\Lambda]$. Hence,

$$\check{D}_t^{(K)} \;:=\; \sup_{\substack{s\,\in\,[0,t),\\ \lambda\,\in\,[0,\Lambda]}}\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 \tag{131}$$

is bounded in $K > 0$ and $t > 0$. Then from the Gronwall type inequality in (129) we get

$$\check{D}_t^{(K)} \;\le\; \varepsilon + c_{(129)}\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\,\check{D}_t^{(K)}\,\frac{N_K}{K^{\gamma}}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^{\gamma}} , \tag{132}$$

for all $K \ge K_0$ and $t \in [L,T]$. But by the convergence and boundedness in (119),

$$\limsup_{K\uparrow\infty}\;\sup_{0\,<\,t\,\le\,T}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^{\gamma}} \;:=\; c_{(133)} \;<\; \infty. \tag{133}$$

Thus, introducing the finite number

$$\check{D} \;:=\; \limsup_{K\uparrow\infty} \sup_{L\le t\le T} \check{D}_t^{(K)}, \tag{134}$$

from (132) and asymptotics (89) we obtain

$$\check{D} \;\le\; \varepsilon + c_{(129)} \left(\Lambda \|\varphi\|_1\right)^{\beta} \check{D}\, D\, c_{(133)}. \tag{135}$$

Since ε is arbitrary, we arrive at

$$\check{D} \;\le\; c_{(136)} \left(\Lambda \|\varphi\|_1\right)^{\beta} \check{D}. \tag{136}$$

Choosing now $\Lambda > 0$ so small that

$$c_{(136)} \left(\Lambda \|\varphi\|_1\right)^{\beta} \;<\; 1, \tag{137}$$

we obtain $\check{D} = 0$. Hence the claim (103) is true. This finishes the proof of Lemma 16. ■

2.6 Pointwise convergence (proof of Theorem 12)

The L^1–convergence implies actually pointwise convergence:

Corollary 17 (Pointwise convergence for small λ) *Fix $T > 0$, φ in $\mathcal{C}_p^+$, and take $\Lambda = \Lambda(T,\varphi) > 0$ as in Lemma 16. Then for each $\lambda \in [0,\Lambda]$,*

$$V_t^{(K)}(\lambda\varphi)\,(a) \;\underset{K\uparrow\infty}{\longrightarrow}\; v_t^{(\lambda)}(a), \qquad t\in(0,T], \quad a\in \mathsf{R}^d. \tag{138}$$

Proof The proof is very similar to the one of Lemma 16 in the previous subsection, so we skip some details. In analogy to (104), for fixed $t \in (0,T]$, $a \in \mathsf{R}^d$, and $\lambda \in \big[0,\Lambda(T,\varphi)\big]$ we have

$$\left|V_t^{(K)}(\lambda\varphi)\,(a) - v_t^{(\lambda)}(a)\right| \;\le\; A_1' + c_f\,[A_2' + A_3' + A_4' + D\,A_5'], \tag{139}$$

where

$$A_1' \;:=\; \left|W_t^{(K)}(\lambda\varphi)\,(a) - \theta(\lambda\varphi)\,\mathrm{p}_t(a)\right|,$$

$$A_2' \;:=\; \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_s(b-a)\, \left|\big[V_{t-s}^{(K)}(\lambda\varphi)\,(b)\big]^{1+\beta} - \big[v_{t-s}^{(\lambda)}(b)\big]^{1+\beta}\right|,$$

$$A_3' \;:=\; \frac{(N_{Kt} - N_{Kt-})}{K^\gamma} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_t(b-a)\big[V_0^{(K)}(\lambda\varphi)\,(b)\big]^{1+\beta},$$

$$A_4' \;:=\; \left|\frac{N_K}{K^\gamma} - D\right| \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_s(b-a)\big[v_{t-s}^{(\lambda)}(b)\big]^{1+\beta},$$

$$A_5' \;:=\; \left|\int_{(0,t)} \Big(\frac{\mathrm{d}_s N_{Ks}}{N_K} - \mathrm{d}(s^\gamma)\Big) \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_s(b-a)\big[v_{t-s}^{(\lambda)}(b)\big]^{1+\beta}\right|.$$

Clearly, $A_1' \to 0$ and $A_5' \to 0$ as $K \uparrow \infty$, by dominated convergence. The same is true for A_4'. In fact, use (116) and Chapman-Kolmogorov to get

$$A_4' \;\le\; c\,\bigl(\Lambda\,\|\varphi\|_1\bigr)^{1+\beta}\,\Bigl|\frac{N_K}{K^\gamma} - D\Bigr|\,p_t(a)\int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma}\,, \tag{141}$$

so we may continue as after (117). Also A_3' will disappear as $K \uparrow \infty$. Indeed, use (112), dominated convergence, criticality, and (113).

It remains to deal with the main term A_2'. Similar to the derivation of the estimate (128),

$$A_2' \;\le\; c\,\bigl(\Lambda\,\|\varphi\|_1\bigr)^{\beta}\times \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma}\int_{\mathsf{R}^d}\mathrm{d}b\;\mathrm{p}_s(b-a)\,\bigl|V_{t-s}^{(K)}(\lambda\varphi)\,(b) - v_{t-s}^{(\lambda)}(b)\bigr|. \tag{142}$$

For $0<\varepsilon<1$, we split the integral on $(0,t)$ in three parts:

$$0<s<\varepsilon t,\quad \varepsilon t \le s \le (1-\varepsilon)t,\quad \text{and}\quad (1-\varepsilon)t<s<t. \tag{143}$$

In the first and last case, we pass from differences to sums in the integrands, use dominations (125) and (115), as well as (95), Chapman-Kolmogorov, and the estimate (65) to see that the internal integral of (142) is bounded in s and K. But by (98),

$$\lim_{\varepsilon\downarrow 0}\,\limsup_{K\uparrow\infty}\int_{(0,\varepsilon t)\,\cup\,\left((1-\varepsilon)t,t\right)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma} \;=\; 0. \tag{144}$$

So it remains to deal with the integral in (142) restricted to $\bigl[\varepsilon t,\,(1-\varepsilon)t\bigr]$, for fixed $0<\varepsilon<1$. Here $\mathrm{p}_s(b-a)\le c$, and we are back to the L^1–norm

$$\bigl\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\bigr\|_1 \;\le\; \sup_{u\,\in\,[\varepsilon t,\,(1-\varepsilon)t]}\bigl\|V_u^{(K)}(\lambda\varphi) - v_u^{(\lambda)}\bigr\|_1 \tag{145}$$

which by Lemma 16 converges to 0 as $K\uparrow\infty$. But by (98),

$$\limsup_{K\uparrow\infty}\int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma} \;<\;\infty, \tag{146}$$

and we verified (138), finishing the proof of Corollary 17. ■

Completion of the proof of Theorem 12 Fix φ,t,a as in the theorem, and let $\lambda\ge 0$. By (101) and (35),

$$v_t^{(\lambda)}(a) \;=\; -\log\,\mathbb{E}_{\delta_{(0,a)}}\exp[-\theta(\lambda\varphi)\,X_t(a)] \tag{147}$$

with $\theta(\lambda\varphi)$ from (51). On the other hand, by (49) and (46),

$$V_t^{(K)}(\lambda\varphi) \;=\; K^{d/\alpha}\Bigl[1 \,-\, \mathbf{E}_{\delta_{(\tau,K^{1/\alpha}a)}}\exp\langle Z_{Kt}\,,-\lambda\varphi\rangle\Bigr]. \tag{148}$$

Note that the expressions in (147) and (148) are *analytic* functions in $\lambda>0$ (or $\Re\lambda>0$). then from the convergence in Corollary 17 for small λ we get the desired convergence for all $\lambda\ge 0$. This finishes the proof. ■

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