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Duality principle for discrete linear inclusions

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Abstract

Two properties of finite sets $\{A_j\}$ of $n \times n$-matrices are introduced: P-stability and BV-stability. These properties can be interpreted as two kinds of robustness of orbits of the form $x_{i+1} = A_{j_i} x_i + u_i$ with respect to disturbances $\{u_i\}$. Duality between these properties is established, which proves that they are equivalent, respectively, to the right convergent product (RCP) property and the left convergent product (LCP) property of finite sets of matrices. The results can be applied, in particular, in the theory of polyhedral Skorokhod problems and sweeping processes with oblique reflection.

Introduction

We consider here finite sets $\Sigma = \{A_1, \ldots, A_k\}$ of $n \times n$-matrices and their finite and infinite products, see [1, 2, 6, 8]. Following [8], by the *discrete linear inclusion* DLI(Σ) we understand the set of all sequences (trajectories of Σ) $x_{j+1} = A_{i_j} x_j$, $j = 0, 1, \ldots$, of vectors in R^n, where matrices $A_{i_j} \in \Sigma$ are taken in an arbitrary order.

In applications such as control theory, wavelet analysis or Markov chains, it is often important whether trajectories of the DLI stay bounded or may diverge to infinity. A DLI whose trajectories are bounded is called *product bounded*. Another important property of the DLI is convergence of any trajectory to some point in R^n (not necessarily to 0). This property is stronger than product boundedness and is called left convergent products property or just LCP. As is known [1], LCP is equivalent to convergence of any left-infinite matrix product of the form $A_{j_0} A_{j_1} \ldots$, $A_{j_i} \in \Sigma$, $i = 0, 1, \ldots$, to some $n \times n$-matrix A^*. The dual property RCP (right convergent products) is not equivalent to LCP, but it is, obviously, equivalent to the LCP property of the dual set $\Sigma^* = \{A_j^*\}$, where A^* is the transversed matrix A.

In the theory of polyhedral Skorokhod problems with oblique reflection [3, 4, 5, 13], it is often convenient to write conditions of existence and uniqueness of solutions, and also of Lipschitz continuity of corresponding Skorokhod operators, in terms of different kinds of stability of a special set Σ of matrices. For a Skorokhod problem characterized by a finite number of pairs (n_j, d_j), where n_j is the outward normal to j-th face of the characteristic polyhedral set $Z \subseteq R^n$ and d_j is the corresponding vector of oblique projection, the *associated projection set* Σ is considered. The set Σ consists of all matrices P_j of oblique projection onto $L_j = n_j^{\perp}$ along the line $\mathrm{span}(d_j)$:

$$P_j x = x - \frac{d_j \langle n_j, x \rangle}{\langle n_j, d_j \rangle}, \quad i = j, \ldots, k.$$

Apart from product boundedness, two other stability-type properties of the ASP Σ play an important role in verifying regularity of the Skorokhod operator. First of them, the *BV-stability*, consists in finiteness of length of any trajectory of Σ, that is, the inequality

$$\sum_{i=0,1,\dots} \|x_{i+1} - x_i\| < \infty$$

should hold for any trajectory $\{x_j\}$ of Σ. This property, in particular, ensures convergence of discrete-time approximation methods to continuous-time solutions of the Skorokhod problem.

The second property, the *P-stability*, see [10], needs an additional construction for its formulation. Together with any $P_i \in \Sigma$, let us consider two affine operators P_i^+ and P_i^- of projection along d_i onto parallel hyperplanes $L_i^+ = L_i + n_i$ and $L_i^- = L_i - n_i$, respectively. The set Σ is called P-stable if all sequences $\{x_0, x_1, \dots\}$ of vectors in R^n satisfying

$$x_{j+1} = \alpha_j P_i^+ x_j + (1 - \alpha_j) P_i^- x_j,$$

for some $i = i(j) \in J$ and some $\alpha_j \in [0,1]$ at each step j, are bounded. This property is a sufficient condition of Lipschitz continuity of the Skorokhod operator acting in the space of continuous inputs and outputs [5].

As follows from Proposition 1.3 below, both properties of BV-stability and P-stability of sets of oblique projections are stronger than product boundedness. In this paper we extend these notions to sets of general matrices and study their basic properties. In particular, the following duality principle is proved: The set $\{A_j\}$ is BV-stable if and only if the dual set $\{A_j^*\}$ is P-stable.

1 Discrete linear inclusions

We will consider finite sets $\Sigma = \{A_1, \dots, A_k\}$ of $n \times n$-matrices and their finite and infinite products. Following [8], by the *discrete linear inclusion* DLI(Σ) we will understand the set of all sequences $\{x_i\}$, $i = 0, 1, \dots$, of vectors in R^n such that

$$x_{i+1} = A_{j_i} x_i \tag{1.1}$$

for some $A_{j_i} \in \Sigma$. These sequences will be called *trajectories* of Σ; finite initial segments of trajectories will be called *finite trajectories.*

Let us also define a more general class of sequences in R^n:

Definition 1.1 A sequence $\{x_i \in R^n : i = 0, 1, \dots,\}$ is a *partial trajectory* of Σ if

$$x_{i+1} = \alpha_i x_i + (1 - \alpha_i) A_{j_i} x_i,$$

for some $\alpha_i \in [0,1]$ and some $j_i \in J = \{1, \dots, k\}$, $i = 0, 1, \dots$.

The following notion is central in the theory of discrete linear inclusions.

Definition 1.2 A set of matrices $\Sigma = \{A_1, \dots, A_k\}$ is *product bounded* if there exists a $C > 0$ such that $\|A_{j_1} \dots A_{j_m}\| < C$ for all finite sequences $j_i \in J$, $i = 1, \dots, m$.

The following assertion is an easy consequence of well-known results in the theory of DLIs (see, for instance, [1]).

Proposition 1.3 *For a finite matrix set* Σ, *the following four properties are equivalent:*

(1) Σ *is product bounded;*

(2) all trajectories of Σ *are bounded;*

(3) all partial trajectories of Σ *are bounded;*

(4) there exists a norm $\|\cdot\|_\Sigma$ *in* R^n *such that*

$$\|Ax\|_\Sigma \leq \|x\|_\Sigma, \quad A \in \Sigma,\ x \in R^n. \tag{1.2}$$

Any norm $\|\cdot\|_\Sigma$ satisfying (1.2) will be called a *contraction norm* for the set Σ.

2 BV-stability of matrix sets

Let $\Sigma = \{A_1, \dots, A_k\}$ be a finite set of $n \times n$-matrices.

Definition 2.1 The set Σ is called *BV-stable* if all its trajectories $\{x_0, x_1, \dots\}$ have bounded variation (finite length), that is, if

$$\sum_{i=0,1,\dots} \|x_{i+1} - x_i\| < +\infty. \tag{2.1}$$

Now, let us derive some basic properties of BV-stable sets of matrices.

Proposition 2.2 *The set* Σ *is BV-stable if and only if any left-infinite sequence* $\{M_m\}$, $m = 0, 1, \dots,$ *of products*

$$M_m = A_{j_m} A_{j_{m-1}} \dots A_{j_0}, \quad A_{j_i} \in \Sigma,\ i = 0, 1, \dots, m,$$

has bounded variation, that is, if and only if

$$\sum_{i=0}^{\infty} \|M_{i+1} - M_i\| < +\infty, \tag{2.2}$$

where $\|\cdot\|$ *is some matrix norm (they are all equivalent, thus, the result does not depend on the choice of* $\|\cdot\|$*).*

Proof. Obviously, (2.2) implies (2.1). Now, let us consider the norm

$$\|x\| = \sum_{s=1,\dots,n} |x_s|, \quad \text{where } x = (x_1, \dots, x_n).$$

If the variation of the sequence M_i of matrices is infinite, then the sum of variations of vector sequences $\{M_0 e_j, M_1 e_j, \dots\}$ over $j = 1, \dots, n$ (here e_j are the coordinate vectors) is also infinite because

$$\|(M_{i+1} - M_i)\| \leq \sum_{j=1,\dots,n} \|(M_{i+1} - M_i) e_j\|.$$

Hence, at least one of these sequences has infinite variation which is a contradiction to the BV-stability of the set Σ. ■

For completeness, let us also prove that BV-stability of Σ implies uniform boundedness of lengths of all trajectories of Σ starting in any bounded set.

Theorem 2.3 *A set Σ is BV-stable if and only if, for any bounded set $B \subseteq R^n$, there exists a $D > 0$ such that*

$$\sum_{i=0,1,\ldots} \|x_{i+1} - x_i\| \leq D$$

for any trajectory $\{x_0, x_1, \ldots\}$ of Σ satisfying $x_0 \in B$.

Proof. Let us introduce auxiliary functions $F_m(x)$ on R^n by the following recursive procedure. Set $F_0(x) = \|x\|$ and then define

$$F_{m+1} = \max_{A\in\Sigma}(\|x - Ax\| + F_k(Ax)) \tag{2.3}$$

if the function $F_m(x)$ is already defined. Thus, for each m, $F_m(x)$ is the length of the longest m-step finite trajectory of Σ from x.

Let us show that all $F_m(x)$ are homogeneous symmetric convex functions on R^n. Indeed, the relation

$$F_m(\alpha x) = |\alpha| F_m(x), \quad \alpha \in R,$$

follows immediately from definitions. The convexity of $F_{m+1}(x)$ follows from the convexity of $F_k(x)$ and convexity of $\|x - A_j x\|$ for each $j \in J$. Indeed, according to (2.3), $F_{m+1}(x)$ is the maximum of convex functions $G_{jm}(x) = \|x - A_j x\| + F_m(A_j x)$ over $j \in J$.

By construction, $F_{m+1}(x) \geq F_m(x)$ for each $x \in R^n$. Let us denote

$$F(x) = \lim_{m\to\infty} F_m(x)$$

(we do not exclude the case $F(x) = +\infty$). The function $F(x)$ is a convex homogeneous symmetric function from R^n to $R \cup \{+\infty\}$. To prove the theorem, it suffices to prove that, for any BV-stable set Σ, the function $F(x)$ is finite for any $x \in R^n$ because this would imply uniform boundedness of F on any bounded subset of R^n.

Suppose the contrary and denote by H the set of $x \in R^n$ for which $F(x) < +\infty$. Obviously, H is a convex set in R^n and, also, $0 \in H$, and $\alpha H = H$ for each real α, as follows from the properties of $F(x)$. Hence, H is a linear subspace of R^n. We will need the following auxiliary result.

Lemma 2.4 *For each $x \notin H$ there exists a finite trajectory $\{x_0, \ldots, x_m\}$ of Σ such that*

$$x_0 = x, \qquad \sum_{i=0,\ldots,m-1} \|x_{i+1} - x_i\| \geq 1, \quad x_m \notin H. \tag{2.4}$$

Proof. Suppose the contrary. Consider a trajectory $\{x_0, x_1, \ldots\}$ from $x = x_0$ such that $\sum \|x_{i+1} - x_i\| > 1$ and let p be the minimal index such that $x_p \in H$. We have $x_{p-1} \notin H$ and, thus, $\|x_0 - x_{p-1}\| \leq 1$. Hence,

$$\|x_p\| \leq (\max_{j\in J} \|A_j\|)\|x_{p-1}\| \leq (\max_j \|A_{j\in J}\|)(\|x_0\| + 1).$$

Denote $S := (\max_{j \in J} \|A_j\|)(\|x_0\| + 1)$. Now, since the function $F(x)$ is convex and finite on a linear subspace H, it is continuous on H and, hence, uniformly bounded from above on the ball

$$B_S = \{x \in H : \|x\| \leq S\}$$

by some constant $Q > 0$. Hence, by definition of $F(\cdot)$, we have $\|x_{p+q}\| \leq S + Q$ for each $q = 0, 1, \ldots$, and this is a contradiction to the assumption $x_0 \notin H$. Lemma 2.4 is proved. ■

Now, choose an arbitrary initial point $x_0 \notin H$ and construct a trajectory $\{x_0, \ldots, x_m\}$ satisfying (2.4). Then take $x_m \notin H$ for a new initial point and repeat the procedure. Composition of all resulting finite trajectories is a trajectory of Σ of infinite length. Theorem 2.3 is completely proved. ■

Let us also formulate a recent result on equivalent definitions of BV-stability [12]. First, we will define more stability-like properties of finite sets of matrices.

Definition 2.5 A set Σ is LCP (left convergent products) if any left-infinite matrix product $\ldots A_{j_1} A_{j_0}$ of matrices $A_{j_i} \in \Sigma$ has a limit. This is equivalent to the convergence of any trajectory of Σ (not necessarily to the origin), see [2].

Definition 2.6 The set Σ possesses the *vanishing steps property* (VS) if

$$\lim_{i \to \infty} \|x_{i+1} - x_i\| = 0$$

for any trajectory $\{x_0, x_1, \ldots\}$ of Σ.

Definition 2.7 A matrix A is said to be *paracontracting* with respect to the norm $\|\cdot\|$ in R^n if, for all $x \in R^n$,

$$Ax \neq x \Leftrightarrow \|Ax\| < \|x\|.$$

It is *ℓ-paracontracting* with respect to $\|\cdot\|$ if there exists $\gamma > 0$ such that

$$\|Ax\| \leq \|x\| - \gamma \|Ax - x\|$$

holds for all $x \in R^n$.

A set of matrices is called paracontracting or ℓ-paracontracting with respect to $\|\cdot\|$ if all its matrices possess the respective property; and it is called just paracontracting or ℓ-paracontracting if there exists a norm in R^n such that the set possesses the respective property for this norm. We use the abbreviations PC and LPC, respectively.

The main theorem in [12] reads

Theorem 2.8 *For any finite set Σ of $n \times n$-matrices, the properties BV, LCP, PC, LPC, and VS are equivalent to each other.*

3 P-stability of matrix sets

In order to define P-stability, let us first give the following

Definition 3.1 A sequence $\{x_0, x_1, \ldots\}$ is called an ε*-trajectory* of the set Σ if there exists a sequence of vectors $v_i \in R^n$, $i = 0, 1, \ldots$, and a sequence of indices $j_i \in J$, $i = 0, 1, \ldots$, such that $\|v_i\| \leq \varepsilon$ for any i and

$$x_{i+1} = v_i + A_{j_i}(x_i - v_i), \qquad i = 0, 1, \ldots \tag{3.1}$$

Definition 3.2 A set Σ of $n \times n$-matrices is called *Perron stable* (P-stable) if all its 1-trajectories $\{x_0, x_1, \ldots\}$ are bounded, see [10].

For sets of oblique projection matrices this definition is equivalent to that given in the Introduction.

Obviously, the definition remains the same if instead of 1-trajectories, any other class of ε-trajectories for some $\varepsilon > 0$ is mentioned. Next, we prove some basic results concerning P-stable sets.

Lemma 3.3 *The set of all* 1*-trajectories of* Σ *from* 0 *is uniformly bounded if and only if the class of its* 1*-trajectories from the unit ball is uniformly bounded.*

Proof. If $\{x_0, \ldots, x_m\}$ is a 1-trajectory of Σ then $\{0, x_1 - x_0, \ldots, x_m - x_0\}$ is a $(1 + \|x_0\|)$-trajectory of Σ (this is a direct consequence of definitions). It remains to notice that uniform boundedness of all 1-trajectories of Σ from 0 is equivalent to uniform boundedness of all its 2-trajectories from 0. ■

The following assertion is analogous to Theorem 2.3.

Theorem 3.4 *The set* Σ *is P-stable if and only if, for any bounded subset* $B \subseteq R^n$, *there exists a constant* $D > 0$ *such that* $\|x_m\| \leq D$, $m = 0, 1, \ldots$, *for any* 1*-trajectory* $\{x_0, x_1, \ldots\}$ *of* Σ *satisfying* $x_0 \in B$.

Proof. According to Lemma 3.3, it suffices to prove the theorem for $B = \{0\}$. Suppose the contrary. Then, again, by Lemma 3.3, for any $x \in R^n$ and any $L > 0$, there exists a 1-trajectory $\{x, x_1, \ldots, x_m\}$ of Σ such that $\|x_m\| > L$.

Now, let us construct a finite 1-trajectory of Σ from $x_0 = 0$ to some x_1, $\|x_1\| > 1$. Next, construct a 1 trajectory from x_1 to some x_2, $\|x_2\| > 2$, and so on. The resulting composition is an unbounded 1-trajectory of Σ which is a contradiction. ■

Finally, let us give another definition of P-stability which can be interpreted as robustness of a linear control system to perturbations of a special type.

Denote $u_i := v_i - v_{i+1}$ and $y_i = x_i - v_i$ in (3.1). We get the recurrent relation

$$y_{i+1} = A_{j_i} y_i + u_i. \tag{3.2}$$

If the sequence $\|v_i\|$ is uniformly bounded then the uniform boundedness of the sequence $\|x_i\|$ is equivalent to that of the sequence $\|y_i\|$. The uniform boundedness of the sequence $\|v_i\|$ itself is equivalent to that of the sequence $\|\sum_{j=0,\dots,i} u_j\|$, $i = 0, 1, \dots$. Now, we can rewrite the definition of P-stability as follows:

Proposition 3.5 *The set Σ is P-stable if and only if there exists an $M > 0$ such that for any sequence u_i satisfying*

$$\| \sum_{j=0,\dots,i} u_j \| \leq 1, \quad i = 0, 1, \dots,$$

and any $y_0 \in R^n$ satisfying $\|y_0\| \leq 1$, the sequence $\{y_i\}$ constructed by the rule (3.2) satisfies

$$\|y_i\| \leq M, \quad i = 0, 1, \dots.$$

4 Duality of BV-stability and P-stability

As follows from definitions, a set $\Sigma = \{A_j : j \in J\}$ of $n \times n$-matrices is product bounded if and only if the dual set $\Sigma^* = \{A_j^* : j \in J\}$ is product bounded (A^* is the transposed matrix A). It is also clear that the LCP property of Σ is equivalent to the RCP property of Σ^* and vice-versa. The following theorem shows that the notions of BV-stability and P-stability are dual to each other in the same sense.

Theorem 4.1 (Duality principle) *The P-stability of the set Σ is equivalent to the BV-stability of the set Σ^* and vice-versa, The BV-stability of the set Σ is equivalent to the P-stability of the set Σ^*.*

Let us first prove an auxiliary result (it is elementary and, of course, well known, but we provide a proof for convenience of the reader).

Lemma 4.2 *Let $\{h_j\}$, $j \in J$, be a finite set of vectors in R^n. Then there exists a subset $J' \subseteq J$ such that*

$$\| \sum_{j \in J'} h^j \| \geq \alpha(n) \sum_{j \in J} \|h^j\|,$$

where $\alpha(n) > 0$ is a constant for each $n = 1, 2, \dots$.

Proof. Since all the norms in R^n are equivalent, we may assume

$$\|x\| = \max_{s=1,\dots,n} |x_s|, \quad \text{where } x = (x_1, \dots, x_n). \tag{4.1}$$

Denote $L = \sum_{j \in J} \|h^j\|$. Since

$$\sum_{j \in J,\, p=1,\dots,n} |h_p^j| \geq L,$$

there exists a coordinate index p such that

$$\sum_{j \in J} |h_p^j| \geq \frac{L}{n},$$

Hence, for some sequence $s(j) = \pm 1$, we get

$$\|\sum_{j\in J} s(j)h_p^j\| \geq \frac{L}{n}.$$

Denote $J' = \{j \in J : s(j) = 1\}$ and $J'' = \{j \in J : s(j) = -1\}$. We have either

$$\|\sum_{j\in J''} h_p^j\| \geq \frac{L}{2n} \quad \text{or} \quad \|\sum_{j\in J'} h_p^j\| \geq \frac{L}{2n}$$

and, hence, one can choose $\alpha(n) = L/2n$ (for this particular norm (4.1)). ■

Proof of Theorem 4.1. Suppose that the set Σ is P-stable and consider a finite 1-trajectory

$$X = \{x_0, x_1, \dots, x_{q+1}\}.$$

The following relations hold:

$$x_{m+1} = v_m + A_{j_m}(x_m - v_m), \quad \|v_m\| \leq 1, \; m = 0, \dots, q.$$

After transformations, we get

$$x_{m+1} = v_m + M_m^m(v_{m-1} - v_m) + M_{m-1}^m(v_{m-2} - v_{m-1}) + \dots + M_0^m(x_0 - v_0), \quad m = 0, \dots, n, \tag{4.2}$$

where $M_i^m = A_{j_m}A_{j_{m-1}}\dots A_{j_i}$. Let us rewrite (4.2) in the form

$$x_{m+1} = (I - M_m^m)v_m + (M_m^m - M_{m-1}^m)v_{m-1} + \dots + (M_1^m - M_0^m)v_0 + M_0^m x_0.$$

By assumption, the norms of all endpoints x_{m+1} of 1-trajectories from x_0 have to be uniformly bounded for any choice of matrices A_{j_i} from Σ and any sequence of disturbance vectors v_i, $\|v_i\| \leq 1$, $i = 0, 1, \dots$. Let us show that this property (Property P1) is equivalent to the following one (Property P2): All the sums of the form $\|M_m^m - M_{m-1}^m\| + \dots + \|M_1^m - M_0^m\|$ are uniformly bounded from above.

Obviously, P2 implies P1; let us prove the reverse implication. Choose vectors $h_i \in R^n$ such that $\|h_i\| = 1$ and

$$\|(M_m^{i+1} - M_m^i)h_i\| = \|M_m^{i+1} - M_m^i\|\|h_i\|, \quad i = 0, \dots, m-1.$$

Now, using Lemma 4.2, we find a subset $J' \subseteq J = \{0, \dots, m-1\}$ such that

$$\|\sum_{i\in J'}(M_m^{i+1} - M_m^i)h_i\| \geq \alpha(n)\sum_{i\in J}\|M_m^{i+1} - M_m^i\|.$$

It remains to choose

$$v_i = \begin{cases} h_i & \text{if } i \in J', \\ 0 & \text{otherwise.} \end{cases}$$

and get a contradiction with the uniform boundedness of $\|x_i\|$.

Property P2 is equivalent to the fact that the total variation of any right-infinite matrix product $A_{j_0}A_{j_1}\dots$ is uniformly bounded as well. This is, in turn, equivalent to the uniform boundedness of variations of left-infinite products of transposed matrices A_j^*. Finally, the last property is equivalent to the BV-stability of the set Σ^* (see Proposition 2.2).

All the logical implications we made are reversible, thus the required equivalence follows. Since $(\Sigma^*)^* = \Sigma$, the equivalence of the BV-stability of Σ and the P-stability of Σ^* follows immediately. ∎

Corollary 4.3 *The class of P-stable sets coincides with the class of RCP (right convergent products) sets.*

Proof. The assertion follows from Theorem 4.1 and Theorem 2.8 because the set $\Sigma = \{A_j\}$ is RCP if and only if the set Σ^* is LCP. ∎

Corollary 4.4 *The set of projections*

$$\{P_j = P_{\{n_j,d_j\}}\} \text{ where } P_{\{n_j,d_j\}}x = x - \frac{d_j\langle n_j, x\rangle}{\langle n_j, d_j\rangle}, \quad j = 1, \dots, k,$$

is P-stable if and only if the set of dual oblique projections $\{P_j^* = P_{\{d_j,n_j\}}\}$ *is BV-stable.*

Proof. Any projection matrix $P = P_{\{p,d\}}$ can be represented as $P = I - A$, where $A = \{a_{s,q}\}$ and

$$a_{s,q} = \frac{p_s d_q}{\langle p, d\rangle}, \quad 1 \le s, q \le n.$$

Thus, $P^* = I - A^* = P_{\{d,p\}}$ and, hence, $Q_i = P_i^*$. Now, the required statement follows from Theorem 4.1. ∎

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