Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Existence and approximation of solutions to an anisotropic phase field system of Penrose–Fife type

Olaf Klein

Dipartimento di Matematica
Universita' di Milano
Via C. Saldini 50
20133 Milano
Italy
E-mail: klein@mat.unimi.it

submitted: August 14, 2000

Preprint No. 603
Berlin 2000

1991 *Mathematics Subject Classification.* 65M12, 35K50, 80A22, 35K60, 35R35.

Key words and phrases. Phase–field model, Penrose–Fife model, anisotropie, semidiscretization, convergence.

Supported by the TMR Network "Viscosity solutions and their applications" and the Weierstrass Institute for Applied Analysis and Stochastics, Berlin..

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper is concerned with a phase field system of Penrose–Fife type for a non–conserved order parameter χ with a kinetic relaxation coefficient depending on the gradient of χ. This system can be used to model the dendritic solidification of liquids. A time discrete scheme for an initial–boundary value problem to this system is presented. By proving the convergence of this scheme, the existence of a solution to the problem is shown.

1 Introduction

A class of phase–field systems modeling the dynamics of diffusive phase transitions has been derived by Penrose and Fife in [34]. Dealing with a non–conserved order parameter, one of these systems is generalized to the following form:

$$c_0\theta_t + \lambda'(\chi)\chi_t + \kappa\nabla q = g, \quad q = \kappa\nabla\left(\frac{1}{\theta}\right), \tag{1.1}$$

$$\zeta(\nabla\chi)\chi_t - \varepsilon\Delta\chi + s'(\chi) = -\frac{\lambda'(\chi)}{\theta}. \tag{1.2}$$

In this system an energy balance (1.1) is coupled with an evolution equation (1.2) for the order parameter χ. These equations determine the evolution of the absolute temperature θ and the order parameter. Here, c_0 is the specific heat and κ is the thermal conductivity, both supposed to be positive constants. The heat flux $q = \kappa\nabla\frac{1}{\theta} = -\frac{\kappa}{\theta^2}\nabla\theta$ considered in (1.1) does not to correspond to the classical Fourier law, but to a Fourier law with a temperature dependent thermal conductivity $\frac{\kappa}{\theta^2}$. A heat flux of this form is considered in a number of papers dealing with Penrose–Fife systems. The function $\lambda'(\chi)$ represents the latent heat of the phase transition, and the datum g represents heat sources or sinks.

Moreover, ζ stands for a kinetic relaxation coefficient, depending on the gradient of the order parameter, the positive constant ε is a relaxation coefficient, representing the energy of the phase interfaces, and s' is the derivative of some potential on $\mathbb{R}$.

In the context of a solid–liquid phase transition with a critical temperature θ_C, one typically has a quadratic or linear function λ and the potential $s(r)$ is the sum of $\frac{\lambda(r)}{\theta_C}$ and the *double well potential* $\rho\left(r^2-1\right)^2$ with some positive constant ρ. To ensure that the order parameter attains only values in the interval $[-1,1]$, also the *double obstacle potential* $I_{[-1,1]}(r)+\rho(1-r^2)$, with $I_{[-1,1]}$ being the indicator function of the interval $[-1,1]$, is used instead of of the double well potential .

In this work, the existence of a solution to an initial–boundary value problem for the Penrose–Fife system (1.1)–(1.2) is proved by considering a time discrete scheme and proving the convergence of the scheme.

The Penrose–Fife system with a constant kinetic relaxation parameter ζ has been investigated in a number of papers, for example in [14, 20, 21, 22, 23, 24, 26, 29, 30, 36, 39], and, for more general heat flux laws, in [9, 10, 12, 31]. In [25, 27, 28], the numerical approximation of Penrose–Fife systems with a space depending kinetic relaxation parameter ζ has been considered.

To the knowledge of the author, a dependence of this kinetic relaxation parameter on the gradient of the order parameter has not been taken into account before in the context of a Penrose–Fife system. This form of the kinetic relaxation parameter ζ allows to model the evolution of a phase interface with a direction dependent kinetic mobility, i.e., a phase interface whose normal velocity depends on orientation of the phase interface. This can be used to model the dendritic solidification of liquids. For the *standard phase field system* (c.f., [6]), i.e., the system (1.1), (1.2) with $\lambda'(\chi) \equiv L$ for some constant $L > 0$ and $\frac{1}{\theta}$ replaced by $-\theta$, this has already been done, see, e.g., in [7, 16].

If only the equation (1.2) with a given right–hand side and s equal to the double well or the double obstacle potential is considered, one is dealing with the Allen–Cahn equation or the double–obstacle Allen–Cahn equation, respectively. In these models, one does not take into account the latent heat of the phase transitions. The double–obstacle Allen–Cahn equation with a kinetic relaxation parameter ζ depending on the direction of $\nabla\chi$ is considered in [17, 18, 19].

To deal with non–smooth potentials, the potential is split in the form $s(r) = \phi(r) - \sigma(r)$, where ϕ represents the convex, maybe not smooth, part of the potential, and σ is a differentiable function, such that $-\sigma$ can represent the non–convex part of the potential. Now, in (1.2), $s'(\chi)$ is replaced by $\xi - \sigma'(\chi)$, where ξ is a representation of the subdifferential $\partial\phi$ of ϕ.

The layout of this paper is as follows: In Section 2, two formulations of the considered initial–boundary value problem for phase–field system with corresponding existence results are presented. In Section 3, the time–discrete scheme is introduced and the approximation results are shown. The remaining sections are dealing with the proof of the results.

2 The Phase–Field system

In this section, an initial–boundary value problem for the phase field system of Penrose–Fife type is investigated. It will be considered on a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2, 3\}$ and a smooth boundary $\Gamma = \partial\Omega$. Let $\Omega_T := \Omega \times (0, T)$ and $\Gamma_T := \Gamma \times (0, T)$, where $T > 0$ stands for a final time.

First, the boundary condition for the temperature is derived. Afterwards, a precise formulation of the initial–boundary value problem and a corresponding existence result are presented. Since this existence result can not be applied for some important versions of the Penrose–Fife system, a second existence result for a weaker formulation of the initial–boundary value problem is presented in Section 2.4, which requires to introduce before a way to deal with $\frac{1}{\theta}$ if θ is not a function on Ω but only a functional on some function space on Ω.

2.1 Boundary condition for the temperature

On the boundary Γ, a heat exchange with an external environment at temperature θ_{ext} is considered. For an energy balance with a heat flux q satisfying the Fourier law with a constant thermal conductivity $\kappa_{\text{Four}} > 0$, i.e., $q = -\kappa_{\text{Four}} \nabla \theta$, this is modeled by the boundary condition

$$-\kappa_{\text{Four}} \frac{\partial \theta}{\partial n} = \gamma_{\text{Four}} \left(\theta - \theta_{\text{ext}} \right), \tag{2.1}$$

where γ_{Four} is some positive constant and n is the outward unit normal to Γ.

Now, the derivation of this boundary condition as in [37] is adapted to deal with more general heat fluxes. To avoid technicalities, this derivation is presented in a one dimensional, time independent setup. Hence, we consider a temperature field θ on an interval (x_0, x_1).

The external temperature $\theta_{\text{ext}}(x_1)$ considered on the right–hand side x_1 of the interval does not correspond to a physical temperature of the environment in the point x_1, but to the temperature on the right–hand side of some interface region on the right–hand side of x_1, wherein the physical temperature changes continuously its value from $\theta(x_1)$ to $\theta_{\text{ext}}(x_1)$.

We assume that the interfaces region has the thickness $\delta > 0$. Hence, we can extend the temperature field θ continuously to $[x_0, x_1 + \delta]$ such that

$$\theta(x_1 + \delta) = \theta_{\text{ext}}(x_1). \tag{2.2}$$

Since the heat flux is continuous across $\partial(x_0, x_1)$, we get for the heat flux q_{inter} in the interface region, i.e., in $\Omega_{\text{inter}} := (x_1, x_1 + \delta)$, and the heat flux q in (x_0, x_1):

$$q(x_1) = q_{\text{inter}}(x_1).$$

Assuming that the heat flux stays constant inside the interfaces region, we get

$$q(x_1) = q_{\text{inter}}(x_1 + \tau), \quad \forall\, 0 \leq \tau \leq \delta. \tag{2.3}$$

In the derivation of the boundary condition for a heat flux satisfying the Fourier law as in [37], we have $q = -\kappa_{\text{Four}} \frac{\partial \theta}{\partial x}$ and it is assumed that the heat flux in Ω_{inter} is of the same form, i.e., $q_{\text{inter}} = -\kappa_{\text{inter}} \frac{\partial \theta}{\partial x}$ with some constant $\kappa_{\text{inter}} > 0$. Hence, for $0 \leq \tau \leq \delta$, (2.3) yields that

$$-\kappa_{\text{Four}} \frac{\partial \theta}{\partial x}(x_1) = q(x_1) = -\kappa_{\text{inter}} \frac{\partial \theta}{\partial x}(x_1 + \tau).$$

Therefore, (2.2) implies that

$$-\delta\kappa_{\mathrm{Four}}\frac{\partial\theta}{\partial x}(x_1) = -\kappa_{\mathrm{inter}}\int_{x_1}^{x_1+\delta}\frac{\partial\theta}{\partial x}(\tau)\,\mathrm{d}\tau = -\kappa_{\mathrm{inter}}\left(\theta_{\mathrm{ext}}(x_1) - \theta(x_1)\right).$$

Performing an analogous calculation also for the left–hand side of the interval, we find that (2.1) holds on $\partial(x_0, x_1)$ with $\gamma_{\mathrm{Four}} := \frac{\kappa_{\mathrm{inter}}}{\delta}$.

To derive the boundary condition for (1.1), we consider in (x_0, x_1) the corresponding heat flux $q = \kappa\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)$ and in Ω_{inter} a heat flux of the same form, i.e., we have $q_{\mathrm{inter}} = \kappa_{\mathrm{inter}}\frac{\partial}{\partial x}\left(\frac{1}{\theta}\right)$ with some positive constant κ_{inter}. Therefore, (2.3) yields that

$$\kappa\frac{\partial(1/\theta)}{\partial x}(x_1) = q(x_1) = \kappa_{\mathrm{inter}}\frac{\partial(1/\theta)}{\partial x}(x_1+\tau), \quad \forall\, 0 \le \tau \le \delta.$$

Hence, considering the integral over $[x_1, x_1+\delta]$ and using (2.2), we observe that

$$\delta\kappa\frac{\partial(1/\theta)}{\partial x}(x_1) = \kappa_{\mathrm{inter}}\left(\frac{1}{\theta_{\mathrm{ext}}(x_1)} - \frac{1}{\theta(x_1)}\right).$$

Defining $\gamma := \frac{\kappa_{\mathrm{inter}}}{\delta}$ and performing an analogous calculation also for the left–hand side of the interval, we get on $\partial(x_0, x_1)$:

$$\kappa\frac{\partial(1/\theta)}{\partial n} = \gamma\left(\frac{1}{\theta_{\mathrm{ext}}} - \frac{1}{\theta}\right). \tag{2.4}$$

Hence, we have shown that the boundary condition (2.4) has a proper physical meaning for the heat flux considered in (1.1). To the knowledge of the author, this has not been pointed out until now, even if this boundary condition has already been used in a number of papers dealing with Penrose–Fife systems, see, e.g., [12, 25, 26, 27, 28]. For heat fluxes of the more general form $q = \kappa\nabla\alpha(\theta)$, with a function $\alpha : (0,\infty) \to \mathbb{R}$, the corresponding generalized version of this boundary condition, i.e.,

$$\kappa\frac{\partial\alpha(\theta)}{\partial n} = \gamma\left(\alpha(\theta_{\mathrm{ext}}) - \alpha(\theta)\right), \tag{2.5}$$

has been used in [9, 10, 14, 23, 24] without discussing their physical meaning. Perform the same computations as above with $1/(\cdot)$ replaced by $\alpha(\cdot)$, we see that also this boundary condition models the heat exchange through a thin interface surrounding the considered domain, with a heat flux inside this interface that is of the same form as the considered heat flux in the domain. The same holds for the boundary condition used in [12] for a heat flux with thermal memory, i.e., a heat flux depending also on former values of the temperature θ.

Considering (2.4) on Γ_T and defining $\mu : \Gamma_T \to \mathbb{R}$ by $\mu := \gamma\frac{1}{\theta_{\mathrm{ext}}}$, we get the boundary condition that is used in this work.

2.2 The phase–field system

We consider now the following initial–boundary value problem for the phase field system of Penrose–Fife type:

(PF): **Find** (θ, u, χ, ξ) **fulfilling**

$$\theta \in H^1(0,T;H^1(\Omega)^*) \cap L^\infty(0,T;L^2(\Omega)), \quad u \in L^2(0,T;H^1(\Omega)), \tag{2.6a}$$

$$\chi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \quad \xi \in L^2(0,T;L^2(\Omega)), \tag{2.6b}$$

$$\theta > 0, \quad u = \frac{1}{\theta}, \quad \chi \in D(\beta), \quad \xi \in \beta(\chi), \quad \text{a.e. in} \quad \Omega_T, \tag{2.6c}$$

$$\begin{aligned} &\langle c_0\theta_t(t) + \lambda'(\chi(t))\chi_t(t), v\rangle_{H^1(\Omega)^*\times H^1(\Omega)} - \kappa \int_\Omega \nabla u(t) \bullet \nabla v \,\mathrm{d}x - \gamma \int_\Gamma u(t) v \,\mathrm{d}\sigma \\ &= \int_\Omega g(t) v \,\mathrm{d}x - \int_\Gamma \mu(t) v \,\mathrm{d}\sigma\,, \quad \forall\, v \in H^1(\Omega), \quad \text{for a.e. } t \in (0,T), \end{aligned} \tag{2.6d}$$

$$\zeta(\nabla\chi)\chi_t - \varepsilon\Delta\chi + \xi - \sigma'(\chi) = -\lambda'(\chi) u, \quad \text{a.e. in} \quad \Omega_T, \tag{2.6e}$$

$$\frac{\partial\chi}{\partial n} = 0, \quad \text{a.e. in} \quad \Gamma_T, \tag{2.6f}$$

$$\theta(\cdot,0) = \theta^0, \quad \text{in} \quad V^*, \qquad \chi(\cdot,0) = \chi^0, \quad \text{a.e. in} \quad \Omega. \tag{2.6g}$$

For dealing with this system, the following assumptions will be used:

(A1): Let β be a maximal monotone graph on $\mathbb{R}$ and $\phi : \mathbb{R} \to [0,\infty]$ a convex, lower semicontinuous function satisfying

$$\beta = \partial\phi, \quad 0 \in D(\beta), \quad 0 \in \beta(0), \quad \text{int } D(\beta) \neq \emptyset.$$

(A2): There are positive constants $\lambda_{\text{fac}}, \sigma'_{\text{fac}}, \lambda''_{\text{sup}}, \sigma''_{\text{sup}}$ such that

$$\begin{gathered} \lambda \in W^{2,\infty}_{\text{loc}}(\mathbb{R}), \quad \sigma \in W^{2,\infty}_{\text{loc}}(\mathbb{R}), \\ -\lambda(s) \le \lambda_{\text{fac}}(\phi(s)+1), \quad (\sigma'(s))^2 \le \sigma'_{\text{fac}}(\phi(s)+1), \quad \forall\, s \in D(\beta), \\ |\lambda''(s)| \le \lambda''_{\text{sup}}, \quad |\sigma''(s)| \le \sigma''_{\text{sup}}, \quad \text{for a.e. } s \in \mathbb{R}. \end{gathered}$$

(A3): We have

$$\begin{gathered} g \in L^2(0,T;L^2(\Omega)), \\ \mu \in L^2(0,T;H^{\frac{1}{2}}(\Gamma)), \quad \mu \ge 0, \quad \text{a.e. in} \quad \Gamma_T. \end{gathered}$$

(A4) We consider initial data $\theta^0 \in L^2(\Omega)$, $\chi^0 \in H^2(\Omega)$, such that

$$\phi(\chi^0) \in L^1(\Omega), \qquad \theta^0 > 0, \quad \chi^0 \in D(\beta), \quad \text{a.e. in} \quad \Omega.$$

(A5): We have positive constants $\zeta_{\inf}, \zeta_{\sup}$ such that the function $\zeta : \mathbb{R}^N \to [\zeta_{\inf}, \zeta_{\sup}]$ is continuous on $\mathbb{R}^N$.

(A6): We have a positive constant $\lambda'_{\sup}$ such that

$$|\lambda'(s)| \leq \lambda'_{\sup}, \quad \forall\, s \in D(\beta).$$

We have the following existence result

Theorem 2.1. *Assume that **(A1)–(A6)** hold. Then there is a solution (θ, u, χ, ξ) to the Penrose–Fife system **(PF)**.*

Remark 2.1. For a quadratic λ, the assumption **(A6)** is only satisfied if $D(\beta)$ is bounded. This is the case for the double obstacle potential, but not for the double well potential. For dealing with this situation, one has to consider the existence result presented in Theorem 2.2.

Remark 2.2. Similar to the existence result for a standard phase field system with a kinetic relaxation parameter depending on the gradient of the order parameter in [8], no uniqueness result is known for the solution to **(PF)**.

Remark 2.3. If one starts to model the evolution of a phase interface with a kinetic mobility term depending on orientation of the interface, one would like to use a kinetic relaxation parameter ζ which depends only on the direction of $\nabla\chi$, and has therefore a discontinuity in 0, but neither one of the theorem in this section or Theorem 3.2, apply to this situation.

By extending the concept of $L^p(\Omega)$–viscosity solution as in [5] with considerations similar to Chapter 9 in [13], we get a $L^p(\Omega)$–viscosity solution formulation for (2.6e), which is also valid if u and ζ are not continuous. But, also in this formulation, χ has to be continuous on Ω_T, and this does not even hold for the solution to the Penrose–Fife system **(PF)** with continuous ζ derived in Theorem 2.1.

2.3 New formulation to deal with $\frac{1}{\theta}$

To prepare the weak formulation of the Penrose–Fife system, the compatibility condition between θ and u has to be replaced by a weaker one, which can also be applied if $\theta(t)$ is not function on Ω but only a functional in $H^1(\Omega)^*$. For $\tilde{u}$ and $\tilde{\theta}$ in $L^2(\Omega)$, the conditions $\tilde{\theta} > 0$ and $\tilde{u} = \frac{1}{\tilde{\theta}}$ a.e. on Ω are equivalent to $-\tilde{u} \in \partial j_0(\tilde{\theta})$ in $L^2(\Omega)$, where $j_0 : L^2(\Omega) \mapsto \mathbb{R} \cup \{\infty\}$ is the $L^2(\Omega)$–representation of the convex function $-\ln(\cdot) : (0, \infty) \to \mathbb{R}$, i.e., we have for $\psi \in L^2(\Omega)$

$$j_0(\psi) := \begin{cases} \int\limits_\Omega -\ln(\psi(x))\,\mathrm{d}x\,, & \text{if } \psi > 0 \text{ a.e. in } \Omega \text{ and } \ln(\psi(\cdot)) \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.7}$$

In [14, 15, 23], Damlamian, Kenmochi, and Kubo extend this function to a function on $H^1(\Omega)^*$ and replace the $L^2(\Omega)$–compatibility condition between θ and u by a condition in this space. Following their formulation, we denote by V the Hilbert space, arising by considering $H^1(\Omega)$ with the inner product $(\cdot,\cdot)_V$ defined by

$$(w, v)_V = \kappa \int\limits_\Omega \nabla w \bullet \nabla v\,\mathrm{d}x + \gamma \int\limits_\Gamma wv\,\mathrm{d}\sigma\,, \quad \forall\, w, v \in H^1(\Omega), \tag{2.8}$$

and the corresponding norm $\|\cdot\|_V$. Thanks to the trace theorem and Poincaré's inequality, we see that the norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent. Hence, V^* can be identified with $H^1(\Omega)^*$ and the $H^1(\Omega)^*$–norm is equivalent to the induced norm $\|\cdot\|_{V^*}$ on V^* as dual space of V.

Let $F : V \to V^*$ be the duality mapping:

$$\langle Fw, v\rangle_{V^*\times V} = (w, v)_V\,, \quad \forall\, w, v \in V. \tag{2.9}$$

We see that V^* is a Hilbert space with the inner product $(\cdot,\cdot)_*$

$$(w, v)_* := \left\langle w, F^{-1}v\right\rangle_{V^*\times V} = \left(F^{-1}w, F^{-1}v\right)_V\,, \quad v, w \in V^*, \tag{2.10}$$

satisfying

$$\|f\|_{V^*} = \sqrt{(f,f)_*} = \left\|F^{-1}f\right\|_V\,, \quad \forall\, f \in V^*. \tag{2.11}$$

Now, j_0 is extended to work on the whole V^* by considering the corresponding Γ–regularization j of j_0 on V^*, i.e., we have

$$j(w) = \inf\left\{\liminf_{n\to\infty} j_0(z_n) \,:\, (z_n)_{n\in\mathbb{N}} \subset L^2(\Omega), \quad z_n \to z \text{ in } V^*\right\}, \quad \forall\, w \in V^*. \tag{2.12}$$

We have, see [15, Theorem 1.5, Corollary 1.6] with $\partial_* j$ denoting the subdifferential of j in the Hilbert space V^*:

Lemma 2.1. *With the above definitions hold:*

1. $j = j_0$ *on* $L^2(\Omega)$.
2. *For* $\tilde\theta \in L^2(\Omega)$*:*

$$\tilde\theta \in D(\partial_* j) \Leftrightarrow \tilde\theta > 0 \text{ a.e. in } \Omega \quad \text{and} \quad \exists\, \tilde u \in H^1(\Omega) : \tilde u = \frac{1}{\tilde\theta}, \quad \text{a.e. in} \quad \Omega. \tag{2.13}$$

3. *For* $\tilde\theta \in L^2(\Omega) \cap D(\partial_* j)$*:*

$$\partial_* j(\tilde\theta) = \{-F\tilde u\}\,, \quad \textit{with } \tilde u \textit{ as in (2.13)}.$$

2.4 Weak formulation of the Penrose–Fife system

Now, we can define the weak formulation $(\mathbf{PF})^*$ of phase field system of Penrose–Fife type.

$(\mathbf{PF})^*$: **Find** (θ, u, χ, ξ) **fulfilling (2.6b), (2.6d)–(2.6g), and**

$$\theta \in H^1(0,T;V^*), \quad u \in L^2(0,T;V), \tag{2.14a}$$

$$-Fu(t) \in \partial_* j(\theta(t)) \quad \text{in} \quad V^*, \quad \text{for a.e.} \quad t \in (0,T), \tag{2.14b}$$

$$\chi \in D(\beta), \quad \xi \in \beta(\chi), \quad \text{a.e. in} \quad \Omega_T. \tag{2.14c}$$

Remark 2.4. We see that every solution to the Penrose–Fife system **(PF)** is also a solution to the Penrose–Fife system in the weak formulation $(\mathbf{PF})^*$. On the other hand, Lemma 2.1 yields that a solution to the Penrose–Fife system in the weak formulation $(\mathbf{PF})^*$ with $\theta \in L^\infty(0,T;L^2(\Omega))$ is also a solution to the Penrose–Fife system **(PF)**.

We have the following existence result:

Theorem 2.2. *Assume that **(A1)**–**(A5)** hold. Then there is a solution (θ, u, χ, ξ) to the Penrose–Fife system in the weak formulation $(\mathbf{PF})^*$.*

3 The time discrete scheme

In this section, a time discrete scheme is introduced to prepare numerical computations. Moreover, this scheme is used to prove the existence results in the last section. We consider time–steps sizes that do not need to be uniform, but satisfy the following assumption, where $c_{\text{up}} \geq 1$ is a fixed constant.

(A7): The vector $H = (h_1, \dots, h_K) \in \mathbb{R}^K$ of time–steps sizes, with $K \in \mathbb{N}$, fulfills

$$\sum_{m=1}^{K} h_m = T, \qquad h_m \leq c_{up} h_{m-1}, \quad \forall\, 1 < m \leq K,$$

$$0 < h_m < \frac{\zeta_{\inf}}{3\sigma''_{\sup}}, \quad \forall\, 1 \leq m \leq K.$$

We define $h_{\max}(H) := \max\limits_{1\leq m\leq K} h_m$, $t_0 := 0$, and, for $1 \leq m \leq K$:

$$t_m := t_{m-1} + h_m = \sum_{i=1}^{m} h_i, \tag{3.1}$$

$$g_m(x) := \frac{1}{h_m} \int_{t_{m-1}}^{t_m} g(x,t)\,\mathrm{d}t\,, \quad \mu_m(\sigma) := \frac{1}{h_m} \int_{t_{m-1}}^{t_m} \mu(\sigma,t)\,\mathrm{d}t\,, \quad \forall\, x \in \Omega,\ \sigma \in \Gamma. \tag{3.2}$$

Now, an Euler scheme in time for the Penrose–Fife systems is presented, which is implicit, except for the treatment of the nonlinearities λ', σ', and ζ:

(D): **Let**

$$\theta_0 := \theta^0, \quad \chi_0 := \chi^0, \tag{3.3a}$$

and, for $1 \leq m \leq K$**, find**

$$\theta_m \in L^2(\Omega), \qquad u_m, \chi_m \in H^2(\Omega), \qquad \xi_m \in L^2(\Omega) \tag{3.3b}$$

such that

$$0 < \theta_m, \quad u_m = \frac{1}{\theta_m}, \quad \chi_m \in D(\beta), \quad \xi_m \in \beta(\chi_m), \quad \text{a.e. in} \quad \Omega, \tag{3.3c}$$

$$c_0 \frac{\theta_m - \theta_{m-1}}{h_m} + \lambda'(\chi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} + \kappa \Delta u_m = g_m, \quad \text{a.e. in} \quad \Omega, \tag{3.3d}$$

$$\zeta\left(\nabla\chi_{m-1}\right)\frac{\chi_m-\chi_{m-1}}{h_m}-\varepsilon\Delta\chi_m+\xi_m-\sigma''(\chi_{m-1})\chi_m$$
$$=-\lambda'\left(\chi_{m-1}\right)u_m-\sigma''(\chi_{m-1})\chi_{m-1}+\sigma'(\chi_{m-1}),\quad\text{a.e. in}\quad\Omega, \tag{3.3e}$$
$$-\kappa\frac{\partial u_m}{\partial n}=\gamma u_m-\mu_m,\quad\frac{\partial\chi_m}{\partial n}=0,\quad\text{a.e. in}\quad\Gamma. \tag{3.3f}$$

Remark 3.1. The time–discrete scheme **(D)**, especially the approximation used for the coupling terms, is chosen in such a way that one can use discrete versions of the *a priori* estimates derived by Sprekels and Zheng (cf. [36]).

The approximation for $\sigma'(\chi_m)$ used in (3.3e) is linear with respect to χ_m, i.e., with respect to the implicit part, and involves an approximation error which is less or equal $\sigma''_{\sup}\left(\chi_m-\chi_{m-1}\right)^2$. This approximation is equal to $\sigma'(\chi_m)$, if $\sigma'(\cdot)$ is a linear function.

Theorem 3.1. *Assume that **(A1)**–**(A5)**, and **(A7)** hold. Then there exists a unique solution to **(D)**.*

We use the solution to **(D)** to construct an approximate solution $\left(\widehat{\theta}^Z,\overline{u}^Z,\widehat{\chi}^Z,\overline{\xi}^Z\right)$ in $\left(L^\infty(0,T;L^2(\Omega))\right)^4$ to the Penrose–Fife system. The function $\widehat{\theta}^Z$ is defined to be linear in time on $[t_{m-1},t_m]$ for $m=1,\dots,K$ such that $\widehat{\theta}^Z(t_k)=\theta_k$ holds for $k=0,\dots,K$. The function $\widehat{\chi}^Z$ is defined analogously. We define $\overline{u}^Z$ piecewise constant in time by $\overline{u}^Z(t)=u_m$ for $t\in(t_{m-1},t_m]$ and $m=1,\dots,K$, and $\overline{\xi}^Z$ is defined analogously.

We have the following convergence result:

Theorem 3.2. *Assume that **(A1)**–**(A5)** hold. Let a sequence $\left\{H^{(n)}\right\}_{n\in\mathbb{N}}$ of vectors of time–step sizes with **(A7)** and $h_{\max}\left(H^{(n)}\right)\xrightarrow[n\to\infty]{}0$ be given.*

Denote by $\left(\left(\widehat{\theta}^{(n)},\overline{u}^{(n)},\widehat{\chi}^{(n)},\overline{\xi}^{(n)}\right)\right)_{n\in\mathbb{N}}$ the corresponding sequence of approximations. Hence, there is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and a solution (θ,u,χ,ξ) to the weak formulation $(\mathbf{PF})^*$ of the Penrose–Fife system such that

$$\widehat{\theta}^{(n_k)}\xrightarrow[k\to\infty]{}\theta,\quad\textit{strongly in}\quad C([0,T];H^1(\Omega)^*), \tag{3.4}$$
$$\textit{weakly in}\quad H^1(0,T;H^1(\Omega)^*), \tag{3.5}$$
$$\overline{u}^{(n_k)}\xrightarrow[k\to\infty]{}u,\quad\textit{weakly in}\quad L^2(0,T;H^1(\Omega)), \tag{3.6}$$
$$\widehat{\chi}^{(n_k)}\xrightarrow[k\to\infty]{}\chi,\quad\textit{weakly in}\quad H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)), \tag{3.7}$$
$$\textit{weakly–star in}\quad L^\infty(0,T;H^1(\Omega)), \tag{3.8}$$
$$\overline{\xi}^{(n_k)}\xrightarrow[k\to\infty]{}\xi,\quad\textit{weakly in}\quad L^2(0,T;L^2(\Omega)). \tag{3.9}$$

*If also **(A6)** is satisfied, then (θ,u,χ,ξ) is a solution to the Penrose–Fife system **(PF)**, and we have*

$$\widehat{\theta}^{(n_k)}\xrightarrow[k\to\infty]{}\theta,\quad\textit{weakly–star in}\quad L^\infty(0,T;L^2(\Omega)). \tag{3.10}$$

Remark 3.2. The upper bound for λ'' and σ'' used in **(A2)** can be weakened to some growth condition by using ideas similar to [27].

If one replaces $L^2(0,T;H^2(\Omega))$ in (3.7) by $L^2(t,T;H^2(\Omega))$ for all $0 < t < T$, and uses a more technical argumentation (c.f. [8]) to prove the strong convergence (6.16) for the approximation of χ, one can weaken the assumptions for χ^0 in **(A4)** to $\chi^0 \in H^1(\Omega)$ and $\phi(\chi^0) \in L^1(\Omega)$.

Remark 3.3. If **(A2)** holds, there is some $n_0 \in \mathbb{N}$ satisfying $3T\sigma''_{\text{sup}} \leq n_0\zeta_{\text{inf}}$. For $n \in \mathbb{N}$, we can consider the vector $H^{(n)} = \left(h^{(n)}, h^{(n)}, \ldots, h^{(n)}\right) \in \mathbb{R}^{n+n_0}$ of time–step sizes with $h^{(n)} := \frac{T}{n_0+n}$, such that **(A7)** is satisfied. Hence, it follows from Theorem 3.2 that Theorem 2.2 and Theorem 2.1 hold.

In the sequel, Theorem 3.2 will be proved. The existence of a unique solution to the scheme is proved in Section 4, and uniform estimates for the solutions to the scheme are derived in Section 5. In Section 6, the convergence of the solutions to the time discrete scheme and the existence of a solution to the considered Penrose–Fife system is proved. Therein, the notation $\|\cdot\|_p$ will be used for the $L^p(\Omega)$–norm and the notation $\|\cdot\|_{p,N}$ will be used for the $(L^2(\Omega))^N$–norm for all $p \in [1, \infty]$.

Remark 3.4. As mentioned above in Remark 2.3, one is interested in weaken the assumption **(A5)** on the kinetic relaxation parameter ζ by allowing ζ to be discontinuous in 0. If this weaker version of **(A5)** is used, Theorem 3.1 still holds and the estimates in Section 5 can be performed for the corresponding solutions to the scheme. Hence, one can get all convergences results in Section 6 except of (6.22) and (6.25), and is therefore not able to prove in this way that (2.6e) is satisfied.

4 The proof of Theorem 3.1

Proof. Assume that **(A1)**–**(A5)**, and **(A7)** hold. Now, the existence of a unique solution to the scheme will be shown by induction.

Thanks to (3.3a), **(A4)**, and Sobolev's embedding theorem, we have $\theta_0 \in L^2(\Omega), \chi_0 \in L^\infty(\Omega)$.

Let $\theta_{m-1} \in L^2(\Omega)$ and $\chi_{m-1} \in L^\infty(\Omega)$ be given for some $m \in \{1, \ldots, K\}$. Because of **(A2)** and **(A5)**, we obtain $\lambda'(\chi_{m-1}) \in L^\infty(\Omega)$ and $\zeta \circ \nabla\chi_{m-1} \in L^\infty(\Omega)$.

To rewrite the conditions in the scheme, let the nonlinear operators A_m and D_m on $L^2(\Omega)$ and the linear operator $B_m : L^2(\Omega) \to L^2(\Omega)$ be defined by

$$
\begin{aligned}
A_m u = & -\frac{c_0}{u} - h_m \kappa \Delta u + c_0 \theta_{m-1} + h_m g_m + \lambda'(\chi_{m-1})\chi_{m-1}, \quad \text{a.e. in} \quad \Omega, \\
& \forall\, u \in D(A_m),
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
D(A_m) = \Big\{ & u \in H^2(\Omega) \mid -\frac{\partial u}{\partial n} = \gamma u - \mu_m, \quad \text{a.e. in} \quad \Gamma, \\
& u > 0, \quad \text{a.e. in} \quad \Omega, \quad \frac{1}{u} \in L^2(\Omega) \Big\},
\end{aligned} \tag{4.2}
$$

$$
B_m \chi = \left(\frac{\zeta(\nabla \chi_{m-1})}{h_m} - \frac{\zeta_{\inf}}{2h_m} - \sigma''(\chi_{m-1}) \right) \chi, \quad \text{a.e. in} \quad \Omega, \quad \forall\, \chi \in L^2(\Omega), \tag{4.3}
$$

$$
D_m \chi = -\varepsilon \Delta \chi + \{ \xi \in L^2(\Omega) \mid \xi \in \beta(\chi), \quad \text{a.e. in} \quad \Omega \}, \tag{4.4}
$$

$$
\begin{aligned}
D(D_m) = \Big\{ & \chi \in H^2(\Omega) \mid \frac{\partial \chi}{\partial n} = 0, \quad \text{a.e. in} \quad \Gamma, \quad \chi \in D(\beta), \quad \text{a.e. in} \quad \Omega, \\
& \exists \xi \in L^2(\Omega) \,:\, \xi \in \beta(\chi), \quad \text{a.e. in} \quad \Omega \Big\}.
\end{aligned} \tag{4.5}
$$

Thanks to **(A5)**, **(A2)**, **(A7)**, and $\chi_{m-1} \in L^2(\Omega)$, we conclude that B_m is a maximal monotone linear operator, and [4, Corollary 13] yields that D_m is a maximal monotone operator on $L^2(\Omega)$. By translating the proof of [4, Corollary 13], we see that the operator A_m is maximal monotone. By showing that this operator is also coercive, we obtain that the operator is also surjective. By finally estimating the difference between two given solutions, we have shown that A_m is one–to–one as operator from $D(A_m) \to L^2(\Omega)$. Details can be found in [25, Lemma 5.1].

Defining $f_m \in L^2(\Omega)$ by

$$
f_m := \frac{1}{h_m} \zeta(\nabla \chi_{m-1})\chi_{m-1} - \sigma''(\chi_{m-1})\chi_{m-1} + \sigma'(\chi_{m-1}), \quad \text{a.e. in} \quad \Omega \tag{4.6}
$$

we can see that the conditions (3.3b)–(3.3f) are satisfied if and only if

$$
A_m u_m = \lambda'(\chi_{m-1})\chi_m, \tag{4.7}
$$

$$
\frac{\zeta_{\inf}}{2h_m}\chi_m + B_m \chi_m + D_m \chi_m + \lambda'(\chi_{m-1}) u_m \ni f_m, \tag{4.8}
$$

and the functions ξ_m and θ_m are defined by (3.3e) and $\theta_m = \frac{1}{u_m}$ respectively.

Using $\lambda'(\chi_{m-1}) \in L^\infty(\Omega)$ and that A_m is maximal monotone and one–to–one as operator from $D(A_m) \to L^2(\Omega)$, we deduce that $E_m : L^2(\Omega) \to L^2(\Omega)$ with

$$
E_m \chi := \lambda'(\chi_{m-1}) A_m^{-1}(\lambda'(\chi_{m-1})\chi), \quad \text{a.e. in} \quad \Omega, \quad \forall\, \chi \in L^2(\Omega),
$$

is maximal monotone and (4.7) yields $\lambda'(\chi_{m-1})u_m = E_m \chi_m$.

Hence, we can replace in the system (4.7)–(4.8) the second condition by

$$
\frac{\zeta_{\inf}}{2h_m}\chi_m + (B_m + D_m + E_m)\chi_m \ni f_m. \tag{4.9}
$$

Applying a theorem on summing maximal monotone operators (see, e.g., [3, Chap. II, The. 1.7]), we observe that $B_m + D_m + E_m$ is a maximal monotone operator on $L^2(\Omega)$. Therefore, we conclude that (4.9) has a unique solution $\chi_m \in H^2(\Omega) \subset L^\infty(\Omega)$. Now, $u_m \in H^2(\Omega)$ and $\xi_m \in L^2(\Omega)$ are uniquely defined by (4.7) and (3.3e), and $\theta_m \in L^2(\Omega)$ is uniquely defined by $\theta_m := \frac{1}{u_m}$.

□

5 Uniform estimates

In this section, uniform estimates for the solution to the time–discrete scheme are derived.

Assume that **(A1)**–**(A5)**, and **(A7)** hold. Thanks to Theorem 3.1, there exists a unique solution to the scheme **(D)**.

Remark 5.1. Applying (3.3d), Green's formula, (3.3f), and (2.8), we deduce that

$$\begin{aligned}\int_\Omega \left(c_0 \frac{\theta_m - \theta_{m-1}}{h_m} + \lambda'(\chi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right) v \,\mathrm{d}x \;-\; (u_m, v)_V \\ = \int_\Omega g_m v \,\mathrm{d}x \;-\; \int_\Gamma \mu_m v \,\mathrm{d}\sigma\,, \quad \forall\, v \in H^1(\Omega), \quad 1 \le m \le K.\end{aligned} \tag{5.1}$$

In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of the vector H of time–step sizes. To prepare the *a priori* estimates, we estimate the data and their approximations:

Lemma 5.1. *There exist positive constants C_1, C_2, C_3, C_4 such that,*

$$|\lambda'(s)| + |\sigma'(s)| \le C_1(|s| + 1), \quad \forall\, s \in \mathbb{R}, \tag{5.2}$$

$$\begin{aligned}\left| \int_\Omega g_m v \,\mathrm{d}x \right| + \left| \int_\Gamma \mu_m v \,\mathrm{d}\sigma \right| \le C_2 \left\| v \right\|_{H^1(\Omega)} \left(\|g_m\|_2 + \|\mu_m\|_{L^2(\Gamma)} \right) \\ \le C_3 \left\| v \right\|_V \left(\|g_m\|_2 + \|\mu_m\|_{L^2(\Gamma)} \right), \quad \forall\, v \in H^1(\Omega),\, 1 \le m \le K,\end{aligned} \tag{5.3}$$

$$\sum_{m=1}^{K} h_m \left(\|g_m\|_2^2 + \|\mu_m\|_{L^2(\Gamma)}^2 \right) \le C_4. \tag{5.4}$$

Proof. These estimates follow from **(A2)**, **(A3)**, (3.2), the trace–mapping from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma)$, and the equivalence of the $H^1(\Omega)$–norm and $\|\cdot\|_V$. □

The following Lemmas use ideas from [22, 36, 11, 27].

Lemma 5.2. *There are two positive constants* C_5, C_6 *such that*

$$\max_{0\le m\le K}\Big(\|\theta_m\|_1+\|\ln(\theta_m)\|_1+\|\chi_m\|^2_{H^1(\Omega)}+\|\phi(\chi_m)\|_1\Big)$$
$$+\sum_{m=1}^{K}h_m\,\|u_m\|_V^2+\sum_{m=1}^{K}h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2+\sum_{m=1}^{K}\|\chi_m-\chi_{m-1}\|^2_{H^1(\Omega)}\le C_5, \qquad (5.5)$$
$$\max_{0\le m\le K}\left(\|\lambda'(\chi_m)\|_6+\|\sigma'(\chi_m)\|_6\right)\le C_6. \qquad (5.6)$$

Proof. Testing (3.3e) by $(\chi_m-\chi_{m-1})$, and using **(A5)**, Green's formula, (3.3f), (3.3c), **(A1)**, (AP.6), and Hölder's inequality, we deduce

$$\zeta_{\inf}h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2+\frac{\varepsilon}{2}\|\nabla\chi_m\|^2_{2,N}+\frac{\varepsilon}{2}\|\nabla\chi_m-\nabla\chi_{m-1}\|^2_{2,N}+\|\phi(\chi_m)\|_1$$
$$\le\frac{\varepsilon}{2}\|\nabla\chi_{m-1}\|^2_{2,N}+\|\phi(\chi_{m-1})\|_1-\int_\Omega\lambda'(\chi_{m-1})\,(\chi_m-\chi_{m-1})\,u_m\,\mathrm{d}x$$
$$+\|\sigma''(\chi_{m-1})\|_\infty\,\|\chi_m-\chi_{m-1}\|_2^2+\|\sigma'(\chi_{m-1})\|_2\,\|\chi_m-\chi_{m-1}\|_2\,.$$

Taking the sum from $m=1$ to $m=k$, and applying (3.3a), **(A4)**, **(A2)**, **(A7)**, Schwarz's inequality, and Young's inequality, we deduce

$$\frac{\zeta_{\inf}}{6}\sum_{m=1}^{k}h_m\left\|\frac{\chi_m-\chi_{m-1}}{h_m}\right\|_2^2+\frac{\varepsilon}{2}\|\nabla\chi_k\|^2_{2,N}+\frac{\varepsilon}{2}\sum_{m=1}^{k}\|\nabla\chi_m-\nabla\chi_{m-1}\|^2_{2,N}+\|\phi(\chi_k)\|_1$$
$$\le C_7-\sum_{m=1}^{k}\int_\Omega\lambda'(\chi_{m-1})\,(\chi_m-\chi_{m-1})\,u_m\,\mathrm{d}x+\frac{1}{2\zeta_{\inf}}\sigma'_{\mathrm{fac}}\sum_{m=2}^{k}h_m\,\|\phi(\chi_{m-1})\|_1\,. \qquad (5.7)$$

For $1\le m\le K$ and $\alpha>0$ to be specified later, we insert $v=h_m\alpha-h_m u_m$ in (5.1), use (3.3c), take into account that $\frac{-1}{s}$ is the derivative of the convex function $-\ln(s)$, and apply (5.3), and Young's inequality, to conclude that

$$\alpha c_0\,\|\theta_m\|_1-\alpha c_0\,\|\theta_{m-1}\|_1+c_0\int_\Omega\left(-\ln(\theta_m)\right)\,\mathrm{d}x\;-c_0\int_\Omega\left(-\ln(\theta_{m-1})\right)\,\mathrm{d}x\;+\frac{1}{2}h_m\,\|u_m\|_V^2$$
$$\le\int_\Omega\lambda'(\chi_{m-1})\,(\chi_m-\chi_{m-1})\,(u_m-\alpha)\,\mathrm{d}x+C_8h_m\left(\|g_m\|_2^2+\|\mu_m\|^2_{L^2(\Gamma)}\right). \qquad (5.8)$$

Because of **(A2)** and Taylors formula, we have

$$-\lambda'(\chi_{m-1})\,(\chi_m-\chi_{m-1})\le-\lambda(\chi_m)+\lambda(\chi_{m-1})+\frac{\lambda''_{\sup}}{2}\left(\chi_m-\chi_{m-1}\right)^2,\quad \text{a.e. in}\quad\Omega.$$

Hence, summing (5.8) from $m=1$ to $m=k$, and applying (5.4), (3.3c), (3.3a), **(A2)**, and **(A4)**, we conclude that

$$
\begin{aligned}
&\alpha c_0 \left\|\theta_k\right\|_1 + c_0 \int\limits_\Omega \left(-\ln(\theta_k)\right)\,\mathrm{d}x + \frac{1}{2}\sum_{m=1}^{k} h_m \left\|u_m\right\|_V^2 \\
\leq{}& C_9 + \sum_{m=1}^{k} \int\limits_\Omega \lambda'(\chi_{m-1})(\chi_m - \chi_{m-1}) u_m \,\mathrm{d}x + \alpha\lambda_{\mathrm{fac}} \left\|\phi(\chi_k)\right\|_1 \\
&+ \alpha\frac{\lambda''_{\mathrm{sup}}}{2}\sum_{m=1}^{k} h_m^2 \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 .
\end{aligned}
\tag{5.9}
$$

Now, for $\alpha := \min\left(\frac{1}{2\lambda_{\mathrm{fac}}}, \frac{1}{6\lambda''_{\mathrm{sup}}T}\zeta_{\inf}\right)$, we use Lemma AP.7, and add (5.9) to (5.7) to derive

$$
\begin{aligned}
&C_{10}\left\|\theta_k\right\|_1 + c_0\left\|\ln(\theta_k)\right\|_1 + \frac{1}{2}\sum_{m=1}^{k} h_m \left\|u_m\right\|_V^2 + \frac{1}{12}\zeta_{\inf}\sum_{m=1}^{k} h_m \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 + \frac{\varepsilon}{2}\left\|\nabla\chi_k\right\|_{2,N}^2 \\
&+ \frac{\varepsilon}{2}\sum_{m=1}^{k}\left\|\nabla\left(\chi_m - \chi_{m-1}\right)\right\|_{2,N}^2 + \frac{1}{2}\left\|\phi(\chi_k)\right\|_1 \leq C_{11} + \frac{1}{2\zeta_{\inf}}\sigma'_{\mathrm{fac}}\sum_{m=1}^{k-1} h_{m+1}\left\|\phi(\chi_m)\right\|_1 .
\end{aligned}
$$

Thanks to the discrete version of Gronwall's lemma, **(A7)**, (3.3a), and **(A4)**, we have proved that (5.5) is satisfied.

In the light of (5.2), (AP.1), and (5.5), we observe that (5.6) holds. □

Lemma 5.3. *There is a positive constant C_{12} such that*

$$
\sum_{m=1}^{K} h_m \left\|\lambda'(\chi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{\frac{3}{2}}^2 + \sum_{m=1}^{K} h_m \left\|\frac{\theta_m - \theta_{m-1}}{h_m}\right\|_{V^*}^2 \leq C_{12}. \tag{5.10}
$$

Proof. In view of the terms in (5.1) and the estimates (5.3)–(5.5), we deduce that

$$
\sum_{m=1}^{K} h_m \left\|c_0\frac{\theta_m - \theta_{m-1}}{h_m} + \lambda'(\chi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{V^*}^2 \leq C_{13}. \tag{5.11}
$$

Thanks to Hölder's inequality as in Lemma AP.2, we have

$$
\left\|\lambda'(\chi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_{\frac{3}{2}} \leq \left\|\lambda'(\chi_{m-1})\right\|_6 \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2, \quad \forall\, 1 \leq m \leq K. \tag{5.12}
$$

Hence, combining this with (5.6), (5.5), the continuity of the embedding of $L^{\frac{6}{5}}(\Omega)$ in $H^1(\Omega)^*$, the equivalence of the spaces $H^1(\Omega)$ and V, and (5.11), we conclude that (5.10) is satisfied. □

Lemma 5.4. *There exists a positive constant C_{14} such that*

$$\sum_{m=1}^{K} h_m \|\xi_m\|_2^2 + \sum_{m=1}^{K} h_m \|\chi_m\|_{H^2(\Omega)}^2 \le C_{14}. \tag{5.13}$$

Proof. We use (AP.3), (5.5), and (3.3f), and compare the terms in (3.3e), to derive that

$$\|\chi_m\|_{H^2(\Omega)} \le C_{15} + C_{16} \|\Delta\chi_m\|_2 \le C_{15} + C_{16}\frac{1}{\varepsilon} \|f_m^* - \xi_m\|_2 , \tag{5.14}$$

with $f_m^* \in L^2(\Omega)$ defined by

$$f_m^* := -\lambda'(\chi_{m-1})\, u_m - \zeta(\nabla\chi_{m-1})\frac{\chi_m - \chi_{m-1}}{h_m} + \sigma''(\chi_{m-1})(\chi_m - \chi_{m-1}) + \sigma'(\chi_{m-1}). \tag{5.15}$$

Testing formally (3.3e) by ξ_m and using Green's formula, (3.3f), (3.3c), Young's inequality, and (5.15), we observe that

$$\|\xi_m\|_2 \le \|f_m^*\|_2 . \tag{5.16}$$

For a precise derivation of this inequality, one has to consider for $n \in \mathbb{N}$ the nonlinear elliptic problem

$$\chi_{m,n} - \varepsilon\Delta\chi_{m,n} + \beta_{\frac{1}{n}}(\chi_{m,n}) = f_m^* + \chi_m, \quad \text{a.e. in} \quad \Omega,$$

$$\frac{\partial\chi_{m,n}}{\partial n} = 0, \quad \text{a.e. in} \quad \Gamma,$$

with the Yosida approximation $\beta_{\frac{1}{n}}$ of β. This equation is tested by $\chi_{m,n}$ and $\beta_{\frac{1}{n}}(\chi_{m,n})$, using that $\chi_{m,n}$ is an element of $H^{1,6}(\Omega)$ such that the generalized chain rules hold, see [33, Theorem 1] and [32, Lemma 2.1 and Remark 2.1]. Now, a passage to the limit and using [3, Cha. II Prob. 1.1(iv)] lead to (5.16).

Because of (5.15), the discrete Schwarz's inequality, Hölder's inequality, **(A5)**, and **(A2)**, we have

$$\|f_m^*\|_2^2 \le 3\left(\|\lambda'(\chi_{m-1})\|_4^2 \|u_m\|_4^2 + \left(\zeta_{\sup} + \sigma''_{\sup} h_m\right)^2 \left\|\frac{\chi_m - \chi_{m-1}}{h_m}\right\|_2^2 + \|\sigma'(\chi_{m-1})\|_2^2\right).$$

Hence, in the light of (5.6), (5.5), and (AP.1), we observe that

$$\sum_{m=1}^{K} h_m \|f_m^*\|_2^2 \le C_{17}. \tag{5.17}$$

Combining (5.14), (5.16), and (5.17), we see that (5.13) is satisfied. ☐

Lemma 5.5. *We have*

$$\sum_{m=1}^{K} h_m \|\chi_m - \chi_{m-1}\|_{H^1(\Omega)}^6 \le C_{18} h_{\max}(H) \tag{5.18}$$

Proof. We have

$$\sum_{m=1}^{K} h_m \left\| \chi_m - \chi_{m-1} \right\|_{H^1(\Omega)}^6$$

$$\le h_{\max}(H) \max_{1\le m\le K} \left(\|\chi_m\|_{H^1(\Omega)} + \|\chi_{m-1}\|_{H^1(\Omega)} \right)^4 \sum_{m=1}^{K} \|\chi_m - \chi_{m-1}\|_{H^1(\Omega)}^2 .$$

Recalling (5.5), we conclude that (5.18) is satisfied. □

Lemma 5.6. *If **(A6)** holds, there exists a positive constant C_{19} such that*

$$\sum_{m=1}^{K} h_m \|\theta_m \nabla u_m\|_{2,N}^2 + \max_{0\le m\le K} \|\theta_m\|_2 \le C_{19}. \tag{5.19}$$

Proof. We multiply (3.3d) by h_m, sum up the resulting equation for $m = 1$ to $m = i$ and apply (3.3a), to derive that

$$c_0\theta_i + \kappa \sum_{m=1}^{i} h_m \Delta u_m = c_0\theta^0 + \sum_{m=1}^{i} h_m \left(g_m - \lambda'(\chi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right), \quad \text{a.e. in} \quad \Omega. \tag{5.20}$$

Recalling **(A6)** and (5.5), we observe that

$$\sum_{m=1}^{K} h_m \left\| \lambda'(\chi_{m-1}) \frac{\chi_m - \chi_{m-1}}{h_m} \right\|_2^2 \le C_{20}. \tag{5.21}$$

Because of (3.3c), the continuity of u_i on $\overline{\Omega}$, (3.2), and **(A3)**, we see that $u_i \ge 0$ and $\mu_i \ge 0$ a.e. in Γ. Hence, by applying Green's formula, (3.3c), and (3.3f), we get, at least formally,

$$\int_\Omega \theta_i \Delta u_i \, \mathrm{d}x \ \ge \|\theta_i \nabla u_i\|_{2,N}^2 - C_{21}. \tag{5.22}$$

For a precise derivation of this inequality, one has to perform this computation with θ_i replaced by the approximation $\theta_{i,l} \in H^1(\Omega)$ defined by

$$\theta_{i,l} := \left(u_i + \frac{1}{l} \right)^{-1}, \quad \text{a.e. in} \quad \Omega, \quad \forall\, l \in \mathbb{N},$$

and consider afterwards the limit for $l \to \infty$, using that the Lebesgue dominated convergence theorem yields that we have strong convergences for $\theta_{i,l}$ in $L^2(\Omega)$, such that $\theta_{i,l}\nabla u_i \xrightarrow[l\to\infty]{} \theta_i \nabla u_i$ strongly in $(L^1(\Omega))^3$ and weakly in $(L^2(\Omega))^3$.

Because of (5.22), we can test (5.20) by $h_i \cdot \Delta u_i$. Taking the sum from $i=1$ to $i=k$ over the resulting equation, and utilizing (AP.5) and (AP.4), we get

$$\begin{aligned} & c_0 \sum_{i=1}^{k} h_i \left\|\theta_i \nabla u_i\right\|_{2,N}^2 + \frac{\kappa}{2} \left\| \sum_{i=1}^{k} h_i \Delta u_i \right\|_2^2 + \frac{\kappa}{2} \sum_{i=1}^{k} h_i^2 \left\|\Delta u_i\right\|_2^2 \\ & \leq C_{22} + \int_\Omega \left(c_0 \theta^0 + \sum_{i=1}^{k} h_i \left(g_i - \lambda'(\chi_{i-1}) \frac{\chi_i - \chi_{i-1}}{h_i} \right) \right) \sum_{i=1}^{k} h_i \Delta u_i \, \mathrm{d}x \\ & \quad - \sum_{i=1}^{k-1} h_{i+1} \int_\Omega \left(g_{i+1} - \lambda'(\chi_i) \frac{\chi_{i+1} - \chi_i}{h_{i+1}} \right) \sum_{m=1}^{i} h_m \Delta u_m \, \mathrm{d}x \, . \end{aligned}$$

Applying Schwarz's inequality, Young's inequality, **(A4)**, (5.4), (5.21), and **(A7)**, we observe that

$$\begin{aligned} & c_0 \sum_{i=1}^{k} k_i \left\|\theta_i \nabla u_i\right\|_{2,N}^2 + \frac{\kappa}{4} \left\| \sum_{i=1}^{k} h_i \Delta u_i \right\|_2^2 + \frac{\kappa}{2} \sum_{i=1}^{k} h_i^2 \left\|\Delta u_i\right\|_2^2 \\ & \leq C_{23} + C_{24} \sum_{i=1}^{k-1} h_i \left\| \sum_{m=1}^{i} h_m \Delta u_m \right\|_2^2 . \end{aligned} \tag{5.23}$$

Thanks to the discrete version of Gronwall's lemma, there is a uniform upper bound for the left–hand side of (5.23). Comparing now the terms in (5.20) and using **(A4)**, (5.21), Schwarz's inequality, (5.4), and (3.3a), we see that (5.19) holds. □

6 Convergence of the time–discrete scheme

In this section, Theorem 3.2 is proved. We assume that **(A1)**–**(A5)** hold, and that we have a sequence $\left\{H^{(n)}\right\}_{n \in \mathbb{N}}$ of vectors

$$H^{(n)} = \left(h_1^{(n)}, h_2^{(n)}, \ldots, h_{K^{(n)}}^{(n)} \right)$$

of time–step sizes with **(A7)** and $h_{\max}\left(H^{(n)}\right) \xrightarrow[n\to\infty]{} 0$.

Hence, Theorem 3.1 yields that for every $H^{(n)}$ there exists a unique solution to the time–discrete scheme **(D)**. Let $\left(\widehat{\theta}^{(n)}, \overline{u}^{(n)}, \widehat{\chi}^{(n)}, \overline{\xi}^{(n)}\right)$ be the corresponding approximations derived from the solution to **(D)** as in Section 3. Moreover, we define the piecewise constant functions $\overline{\theta}^{(n)}$, $\overline{\chi}^{(n)}$, $\overline{\mu}^{(n)}$, $\overline{g}^{(n)}$, analogously to $\overline{u}^{(n)}$, and $\underline{\chi}^{(n)} \in L^\infty(0,T;H^2(\Omega))$ is defined by

$$\underline{\chi}^{(n)}(t) = \widehat{\chi}^{(n)}(t_{m-1}^{(n)}) = \chi_{m-1}^{(n)}, \quad \forall\, t \in (t_{m-1}^{(n)}, t_m^{(n)}),\ 1 \leq m \leq K^{(n)}, \tag{6.1}$$

with $t_0^{(n)} := 0$ and $t_m^{(n)} := \sum_{i=1}^{m} h_i$, for $1 \leq m \leq K^{(n)}$.

Then, by the definition of the approximations, (3.3b)–(3.3a), **(A4)**, and (5.1), we have

$$\widehat{\theta}^{(n)} \in H^1(0,T;L^2(\Omega)), \quad \overline{u}^{(n)} \in L^2(0,T;H^2(\Omega)), \tag{6.2a}$$

$$\widehat{\chi}^{(n)} \in H^1(0,T;H^2(\Omega)), \quad \overline{\chi}^{(n)}, \underline{\chi}^{(n)} \in L^\infty(0,T;H^2(\Omega)), \tag{6.2b}$$

$$\overline{\xi}^{(n)} \in L^\infty(0,T;L^2(\Omega)), \tag{6.2c}$$

$$0 < \overline{u}^{(n)},\ \overline{\theta}^{(n)} = \frac{1}{\overline{u}^{(n)}}, \quad \overline{\chi}^{(n)}, \widehat{\chi}^{(n)}, \underline{\chi}^{(n)} \in D(\beta), \quad \overline{\xi}^{(n)} \in \beta\left(\overline{\chi}^{(n)}\right), \quad \text{a.e. in} \quad \Omega_T, \tag{6.2d}$$

$$\begin{aligned} & c_0 \left\langle \widehat{\theta}_t^{(n)}(t), v \right\rangle_{V^* \times V} + \int_\Omega \lambda'(\underline{\chi}^{(n)}(t)) \widehat{\chi}_t^{(n)}(t) v \,\mathrm{d}x \,-\, \left(\overline{u}^{(n)}(t), v\right)_V \\ &= \int_\Omega \overline{g}^{(n)}(t) v \,\mathrm{d}x \,-\, \int_\Gamma \overline{\mu}^{(n)}(t) v \,\mathrm{d}\sigma\,, \quad \forall\, v \in H^1(\Omega), \quad \text{for a.e.} \quad t \in (0,T), \end{aligned} \tag{6.2e}$$

$$\begin{aligned} & \zeta(\nabla \underline{\chi}^{(n)}) \widehat{\chi}_t^{(n)} - \varepsilon \Delta \overline{\chi}^{(n)} + \overline{\xi}^{(n)} - \sigma''(\underline{\chi}^{(n)})(\overline{\chi}^{(n)} - \underline{\chi}^{(n)}) - \sigma'(\underline{\chi}^{(n)}) \\ &= -\, \lambda'(\underline{\chi}^{(n)}) \overline{u}^{(n)}, \quad \text{a.e. in} \quad \Omega_T, \end{aligned} \tag{6.2f}$$

$$\frac{\partial \overline{\chi}^{(n)}}{\partial n} = 0, \quad \text{a.e. in} \quad \Gamma_T, \tag{6.2g}$$

$$\widehat{\theta}^{(n)}(\cdot,0) = \theta^0, \quad \widehat{\chi}^{(n)}(\cdot,0) = \chi^0, \quad \text{a.e. in} \quad \Omega. \tag{6.2h}$$

From **(A3)** and (3.2), we obtain by a density argument:

$$\left\| g - \overline{g}^{(n)} \right\|_{L^2(0,T;L^2(\Omega))} + \left\| \mu - \overline{\mu}^{(n)} \right\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \xrightarrow[n\to\infty]{} 0. \tag{6.3}$$

In the sequel, C_i, for $i \in \mathbb{N}$, will always denote positive generic constants, independent of n.

We find, from (5.5), (5.10), (5.13), (3.3a), and **(A4)**:

$$\left\| \widehat{\theta}^{(n)} \right\|_{H^1(0,T;V^*) \cap L^\infty(0,T;L^1(\Omega))} + \left\| \overline{\theta}^{(n)} \right\|_{L^\infty(0,T;V^* \cap L^1(\Omega))} + \left\| \overline{u}^{(n)} \right\|_{L^2(0,T;V)} \le C_1, \tag{6.4}$$

$$\begin{aligned} & \left\| \widehat{\chi}^{(n)} \right\|_{H^1(0,T;L^2(\Omega)) \cap C([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} + \left\| \overline{\chi}^{(n)} \right\|_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} \\ & + \left\| \underline{\chi}^{(n)} \right\|_{L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))} + \left\| \overline{\xi}^{(n)} \right\|_{L^2(0,T;L^2(\Omega))} \le C_2. \end{aligned} \tag{6.5}$$

The difference between the different approximations can be estimated, by using (5.5), (5.10), and (5.18):

$$\left\| \widehat{\theta}^{(n)} - \overline{\theta}^{(n)} \right\|_{L^2(0,T;V^*)} \le C_3 h_{\max}\left(H^{(n)}\right) \xrightarrow[n\to\infty]{} 0, \tag{6.6}$$

$$\begin{aligned} & \left\| \widehat{\chi}^{(n)} - \overline{\chi}^{(n)} \right\|_{L^2(0,T;L^2(\Omega))} + \left\| \underline{\chi}^{(n)} - \overline{\chi}^{(n)} \right\|_{L^2(0,T;L^2(\Omega))} + \left\| \widehat{\chi}^{(n)} - \underline{\chi}^{(n)} \right\|^6_{L^6(0,T;H^1(\Omega))} \\ & \quad \le C_4 h_{\max}\left(H^{(n)}\right) \xrightarrow[n\to\infty]{} 0. \end{aligned} \tag{6.7}$$

Thanks to the estimates (6.4)–(6.5), compactness arguments (see, e.g. [38, Prop. 23.7, 23.19, Prob. 23.12]), (6.6), and (6.7), we get a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and functions u, χ, ξ :

$\Omega_T \to \mathbb{R}$ and $\theta : (0,T) \to V^*$, such that we have the convergences

$$\widehat{\theta}^{(n_k)} \xrightarrow[k\to\infty]{} \theta, \qquad \text{weakly in} \quad H^1(0,T;V^*), \tag{6.8}$$

$$\overline{\theta}^{(n_k)} \xrightarrow[k\to\infty]{} \theta, \qquad \text{weakly–star in} \quad L^\infty(0,T;V^*), \tag{6.9}$$

$$\overline{u}^{(n_k)} \xrightarrow[k\to\infty]{} u, \qquad \text{weakly in} \quad L^2(0,T;V), \tag{6.10}$$

$$\widehat{\chi}^{(n_k)} \xrightarrow[k\to\infty]{} \chi, \qquad \text{weakly in} \quad H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{6.11}$$

$$\text{weakly–star in} \quad L^\infty(0,T;H^1(\Omega)), \tag{6.12}$$

$$\overline{\chi}^{(n_k)} \xrightarrow[k\to\infty]{} \chi, \qquad \text{weakly in} \quad L^2(0,T;H^2(\Omega)), \tag{6.13}$$

$$\overline{\xi}^{(n_k)} \xrightarrow[k\to\infty]{} \xi, \qquad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \tag{6.14}$$

Now, we will show that (θ, u, χ, ξ) is a solution to the Penrose–Fife system in the weak formulation $(\mathbf{PF})^*$. Thanks to the convergences above, we see that (2.14a) and (2.6b) are satisfied.

Because of (6.5), the Aubin Lemma as in [35, Corollary 8] implies that the sequence $\{\widehat{\chi}^{(n_k)}\}_{k\in\mathbb{N}}$ is relatively compact in $L^6(0,T;H^1(\Omega))$. Therefore, by (6.11),

$$\widehat{\chi}^{(n_k)} \xrightarrow[k\to\infty]{} \chi, \qquad \text{strongly in} \quad L^6(0,T;H^1(\Omega)). \tag{6.15}$$

Recalling (6.7) and the continuous embedding of $H^1(\Omega)$ in $L^6(\Omega)$, we deduce that

$$\underline{\chi}^{(n_k)} \xrightarrow[k\to\infty]{} \chi, \qquad \text{strongly in} \quad L^6(0,T;H^1(\Omega)), \tag{6.16}$$

$$\text{strongly in} \quad L^6(\Omega_T). \tag{6.17}$$

Thus, we can extract a subsequence $\{n_{k_l}\}_{l\in\mathbb{N}}$ from $\{n_k\}_{k\in\mathbb{N}}$ such that we have a.e. cconvergence for $\underline{\chi}^{(n_k)}$ and $\nabla\underline{\chi}^{(n_k)}$. Now, we can assume without losing generality that already $\{n_k\}_{k\in\mathbb{N}}$ has been chosen in such a way, that these convergences are satisfied, i.e., we have

$$\underline{\chi}^{(n_k)} \xrightarrow[k\to\infty]{} \chi, \qquad \text{a.e. in} \quad \Omega, \tag{6.18}$$

$$\nabla\underline{\chi}^{(n_k)} \xrightarrow[k\to\infty]{} \nabla\chi, \qquad \text{a.e. in} \quad \Omega. \tag{6.19}$$

Hence, by applying the generalized Lebesgue dominant convergence theorem (see, e.g., [1, A-1.26]), (5.2), (6.17), and **(A5)**, we conclude that

$$\lambda'(\underline{\chi}^{(n_k)}) \xrightarrow[k\to\infty]{} \lambda'(\chi), \qquad \text{strongly in} \quad L^6(\Omega_T), \tag{6.20}$$

$$\sigma'(\underline{\chi}^{(n_k)}) \xrightarrow[k\to\infty]{} \sigma'(\chi), \qquad \text{strongly in} \quad L^6(\Omega_T), \tag{6.21}$$

$$\zeta(\nabla\underline{\chi}^{(n_k)}) \xrightarrow[k\to\infty]{} \zeta(\nabla\chi), \qquad \text{strongly in} \quad L^p(\Omega_T), \quad \forall\, 1 \le p < \infty. \tag{6.22}$$

Combining this with (6.11), Hölder's inequality, (6.10), and (AP.1), we observe

$$\lambda'(\underline{\chi}^{(n_k)})\widehat{\chi}_t^{(n_k)} \xrightarrow[k\to\infty]{} \lambda'(\chi)\chi_t, \qquad \text{weakly in} \quad L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\Omega)), \tag{6.23}$$

$$\lambda'(\underline{\chi}^{(n_k)})\overline{u}^{(n_k)} \xrightarrow[k\to\infty]{} \lambda'(\chi)u, \qquad \text{weakly in} \quad L^{\frac{3}{2}}(0,T;L^{3}(\Omega)), \tag{6.24}$$

$$\zeta(\nabla\underline{\chi}^{(n_k)})\widehat{\chi}_t^{(n_k)} \xrightarrow[k\to\infty]{} \zeta(\nabla\chi)\chi_t, \qquad \text{weakly in} \quad L^{2-\delta}(0,T;L^{2-\delta}(\Omega)), \quad \forall\, 0<\delta<1. \tag{6.25}$$

Thanks to **(A2)**, (6.7), and (6.21), we have

$$-\sigma''(\underline{\chi}^{(n_k)})(\overline{\chi}^{(n_k)}-\underline{\chi}^{(n_k)})-\sigma'(\underline{\chi}^{(n_k)}) \xrightarrow[k\to\infty]{} -\sigma'(\chi), \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{6.26}$$

Applying (6.2e), (6.8), (6.23), (6.10), and (6.3), we deduce that

$$\begin{aligned} &c_0\,\langle\theta_t(t),v\rangle_{V^*\times V} + \int_\Omega \lambda'(\chi(t))\chi_t(t)v\,\mathrm{d}x \,-\, (u(t),v)_V \\ &= \int_\Omega g(t)v\,\mathrm{d}x \,-\, \int_\Gamma \mu(t)v\,\mathrm{d}\sigma\,, \quad \forall\, v\in H^1(\Omega), \quad \text{for a.e.} \quad t\in(0,T), \end{aligned} \tag{6.27}$$

Hence, by (2.8) and the equivalence of V and $H^1(\Omega)$, we conclude that (2.6d) is satisfied.

Recalling (6.2f), (6.25), (6.13), (6.14), (6.26), and (6.24), we conclude that (2.6e) is satisfied.

Moreover, (6.2g) and (6.13) produces (2.6f), and (2.6g) is satisfied because of (6.2h), (6.8), and (6.11).

Using (6.15), (6.7), and (6.14), we observe that

$$\int_0^T\!\!\int_\Omega \overline{\chi}^{(n_k)}\overline{\xi}^{(n_k)}\,\mathrm{d}x\;\mathrm{d}t \xrightarrow[k\to\infty]{} \int_0^T\!\!\int_\Omega \chi\xi\,\mathrm{d}x\;\mathrm{d}t\,.$$

Now, we combine this with (6.2d), (6.13), (6.14), and [3, Chap. II, Lemma 1.3], to show that (2.14c) is satisfied.

Hence, it remains only to show that (2.14b) is satisfied to prove that (θ,u,χ,ξ) is a solution to the Penrose–Fife system in the weak form $(\mathbf{PF})^*$. This is done by following the calculations in [23, (4.10)–(4.16)].

Inserting $v=F^{-1}\widehat{\theta}^{(n)}(t)$ in (6.2e), we get, by (2.10) and (2.11):

$$\begin{aligned} &\frac{1}{2}c_0\frac{\mathrm{d}}{\mathrm{d}t}\left\|\widehat{\theta}^{(n)}(t)\right\|_{V^*}^2 + \int_\Omega \lambda'(\underline{\chi}^{(n)}(t))\widehat{\chi}_t^{(n)}(t)F^{-1}\hat{\theta}(t)\,\mathrm{d}x \,-\, \Big(F\overline{u}^{(n)}(t),\widehat{\theta}^{(n)}(t)\Big)_* \\ &= \int_\Omega \overline{g}^{(n)}(t)F^{-1}\widehat{\theta}^{(n)}(t)\,\mathrm{d}x \,-\, \int_\Gamma \overline{\mu}^{(n)}(t)F^{-1}\widehat{\theta}^{(n)}(t)\,\mathrm{d}\sigma\,, \quad \text{a.e. in} \quad (0,T). \end{aligned} \tag{6.28}$$

Since V is compactly embedded in $L^3(\Omega)$, the Aubin Lemma yields that $H^1(0,T;V)$ is compactly embedded in $C([0,T];L^3(\Omega))$. Hence, the continuity of the map $F^{-1}: V^* \to V$ and (6.8) yield that

$$F^{-1}\widehat{\theta}^{(n_k)} \xrightarrow[k\to\infty]{} F^{-1}\theta, \qquad \text{strongly in} \quad C([0,T];L^3(\Omega)). \tag{6.29}$$

Hence, integrating (6.28) from 0 to $s \in [0,T]$ and using (6.23), (6.29), Hölder's inequality, and (6.3) produce

$$\frac{1}{2}c_0 \left\|\widehat{\theta}^{(n_k)}(s)\right\|^2_{V*} + \int_0^s \left(-F\overline{u}^{(n_k)}(t), \widehat{\theta}^{(n_k)}(t)\right)_* \,\mathrm{d}t \xrightarrow[k\to\infty]{} \frac{1}{2}c_0 \left\|\theta(0)\right\|^2_{V^*}$$
$$- \int_0^s \int_\Omega \lambda'(\chi(t))\chi_t(t)F^{-1}\theta(t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^s \left(\int_\Omega g(t)F^{-1}\theta(t)\,\mathrm{d}x - \int_\Gamma \mu(t)F^{-1}\theta(t)\,\mathrm{d}\sigma \right) \mathrm{d}t\,. \tag{6.30}$$

Inserting $v = F^{-1}(\theta(t))$ in (6.27), integrating the resulting equation from 0 to s, and applying (2.10), (2.11), (6.30), (6.6), and (6.4), we have proved that

$$\frac{1}{2}c_0 \left\|\widehat{\theta}^{(n_k)}(s)\right\|^2_{V^*} + \int_0^s \left(-F\overline{u}^{(n_k)}(t), \overline{\theta}^{(n_k)}(t)\right)_* \,\mathrm{d}t \xrightarrow[k\to\infty]{} \frac{1}{2}c_0 \left\|\theta(s)\right\|^2_{V^*} + \int_0^s \left(-Fu(t), \theta(t)\right)_* \,\mathrm{d}t\,. \tag{6.31}$$

Hence, (6.8) yields that

$$\limsup_{k\to\infty} \int_0^s \left(-F\overline{u}^{(n_k)}(t), \overline{\theta}^{(n_k)}(t)\right)_* \,\mathrm{d}t \le \int_0^s \left(-Fu(t), \theta(t)\right)_* \,\mathrm{d}t\,, \quad \forall\, 0 \le s \le T. \tag{6.32}$$

Moreover, from (6.2d), (6.2a), and Lemma 2.1, it follows that

$$-F\overline{u}^{(n)}(t) \in \partial_* j(\overline{\theta}^{(n)}(t)) \quad \text{in} \quad V^*, \quad \text{for a.e.} \quad t \in (0,T). \tag{6.33}$$

Combining this with (6.9), (6.10), (6.32) for $s = T$, and [3, Chap. II, Lemma 1.3], we deduce that (2.14b) is satisfied. Hence, we have shown that (θ, u, χ, ξ) is a solution to weak formulation $(\mathbf{PF})^*$ of the Penrose–Fife system.

Thanks to (2.14b), (6.33), (6.9), and (6.10), we observe that

$$0 \le \liminf_{k\to\infty} \int_0^s \left(-F\overline{u}^{(n_k)}(t) + Fu(t), \overline{\theta}^{(n_k)}(t) - \theta(t)\right)_* \,\mathrm{d}t$$
$$= \liminf_{k\to\infty} \int_0^s \left(-F\overline{u}^{(n_k)}(t), \overline{\theta}^{(n_k)}(t)\right)_* \,\mathrm{d}t - \int_0^s \left(-Fu(t), \theta(t)\right)_* \,\mathrm{d}t\,, \quad \forall\, 0 \le s \le T.$$

Hence, (6.31) and (6.32) lead to

$$\left\| \widehat{\theta}^{(n_k)}(s) \right\|_{V^*}^2 \xrightarrow[k\to\infty]{} \|\theta(s)\|_{V^*}^2, \quad \forall\, s \in [0,T].$$

Therefore, we get form (6.8)

$$\widehat{\theta}^{(n_k)}(s) \xrightarrow[k\to\infty]{} \theta(s), \quad \text{strongly in} \quad V^*, \quad \forall\, s \in [0,T]. \tag{6.34}$$

Since the sequence $\left\{\widehat{\theta}^{(n_k)}\right\}_{k\in\mathbb{N}}$ of continuous functions from $[0,T]$ to V^* is uniformly equicontinuous by (6.4), we now recall Ascoli's Theorem and the equivalence of the space V^* and $H^1(\Omega)^*$, to show that (3.4) is satisfied. Moreover, (3.5)–(3.9) hold because of (6.8), (6.10), (6.11), (6.12), (6.14), the equivalence of $H^1(\Omega)$ and V and the equivalence of V^* and $H^1(\Omega)^*$. Hence, we have show the first assertion in Theorem 3.2.

In the sequel, we assume that **(A6)** is satisfied. Hence, (5.19) yields that

$$\left\| \widehat{\theta}^{(n)} \right\|_{L^\infty(0,T;L^2(\Omega))} \le C_5. \tag{6.35}$$

Combining this with (6.8), we observe by compactness, that (3.10) and $\theta \in L^\infty(0,T;L^2(\Omega))$ are satisfied. By Remark 2.4, we deduce that (θ, u, χ, ξ) is also a solution of the Penrose–Fife system **(PF)**.

This completes the proof of Theorem 3.2. □

A Appendix

For convenience, we list some inequalities and equalities used throughout this paper.

Lemma AP.1 (Young's inequality). *For $a \ge 0$, $b \ge 0$, $\sigma > 0$, $p > 1$, $q := \frac{p}{p-1}$, it holds*

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q, \quad ab \le \frac{1}{p}\sigma^{-(p-1)}a^p + \frac{1}{q}\sigma b^q.$$

Lemma AP.2 (Hölder's inequality). *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$, $p, p_1, p_2 \in [1,\infty]$, $f_1 \in L^{p_1}(\Omega)$, $f_2 \in L^{p_2}(\Omega)$, with*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p},$$

we have $f_1 \cdot f_2 \in L^p(\Omega)$ and

$$\|f_1 \cdot f_2\|_{L^p(\Omega)} \le \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)}.$$

Thanks to Sobolev's embedding theorem, we have

Lemma AP.3. *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \{2,3\}$ and Lipschitz boundary, and the norm $\|\cdot\|_V$ on $H^1(\Omega)$ defined in Section 2, there are positive constants C_a, C_b such that*

$$\left\| |v|^p \right\|_{L^{\frac{6}{p}}(\Omega)} = \|v\|^p_{L^6(\Omega)} \leq C_a^p \|v\|^p_{H^1(\Omega)} \leq C_b^p \|v\|^p_V, \quad \forall\, v \in H^1(\Omega),\ p \in (0,6]. \tag{AP.1}$$

The following classical elliptic estimate can be found in [2, Remark 9.3 d].

Lemma AP.4. *For a bounded, open domain $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$ and $\partial\Omega$ smooth there is a positive constant C such that*

$$\|v\|^2_{H^2(\Omega)} \leq C \left(\|\Delta v\|^2_{L^2(\Omega)} + \left\| \frac{\partial v}{\partial n} \right\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|v\|^2_{L^2(\Omega)} \right), \quad \forall\, v \in H^2(\Omega). \tag{AP.2}$$

In particular, for all $v \in H^2(\Omega)$ with $\frac{\partial v}{\partial n} = 0$ a.e. on Γ,

$$\|v\|^2_{H^2(\Omega)} \leq C \left(\|\Delta v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right). \tag{AP.3}$$

Elementary calculations lead to

Lemma AP.5. *For $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n$, $b_0, b_1, \ldots, b_n \in \mathbb{R}$, we have*

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} b_j = \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right) - \sum_{j=1}^{n-1} b_{j+1} \sum_{i=1}^{j} a_i, \tag{AP.4}$$

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{i} a_j = \frac{1}{2} \left(\sum_{i=1}^{n} a_i \right)^2 + \frac{1}{2} \sum_{i=1}^{n} a_i^2. \tag{AP.5}$$

Lemma AP.6. *Let H be a Hilbert space with scalar-product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$. Then we have*

$$\langle a, a-b \rangle_H = \frac{1}{2} \|a\|^2_H - \frac{1}{2} \|b\|^2_H + \frac{1}{2} \|a-b\|^2_H, \quad \forall\, a, b \in H. \tag{AP.6}$$

The next lemma follows from elementary analysis.

Lemma AP.7. *Let $a, b > 0$ be given. Then there exists a constant $C > 0$, such that*

$$\frac{a}{2} s + b\,|\ln s| \leq as - b \ln s + C, \quad \forall\, s > 0.$$

Acknowledgement

My thanks are due to Prof. Pierluigi Colli, Prof. Nobuyuki Kenmochi, Prof. Reiner Schätzle, and Prof. Claudio Verdi for fruitful discussions.

I would like acknowledge the support of the Weierstrass Institute for Applied Analysis and Stochastics, Berlin, and the Department of Mathematics in the University of Pavia, where parts of this paper has been written. The financial support of the VIGONI program "Phase transitions, Stefan type problems, and minimizing movements" and the TMR project "Viscosity solutions and their applications" is gratefully acknowledged.

References

[1] ALT, H. *Lineare Funktionalanalysis*, Springer, 1985.

[2] AMANN, H. *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in Function Spaces, Differential Operators and Nonlinear Analysis, H.-J. Schmeisser and H. Triebel, eds., Teubner, Stuttgart, 1993, pp. 9–126.

[3] BARBU, V. *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, Leyden, 1976.

[4] BRÉZIS, H. *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations*, in Contributions to Nonlinear Functional Analysis, E. Zarantonello, ed., Academic Press, New York, 1971, pp. 101–156.

[5] CAFFARELLI, L., CRANDALL, M., LOCAN, M. & ŚWIECH, A. *On viscosity solutions of fully nonlinear equations with measurable ingredients*, Comm. Pure Apll. Math., 49 (1996), pp. 365–397.

[6] CAGINALP, G. *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal., 92 (1986), pp. 205–245.

[7] CAGINALP, G. & LIN, J. *A numerical analysis of an anisotropic phase field model*, IMA J. Appl. Math., 39 (1987), pp. 51–66.

[8] COLLI, P., FRÉMOND, M. & KLEIN, O. *Global existence of a solution to a phase field model for supercooling*, Nonlinear Analysis, TMA, Series B: Real World Applications. to appear.

[9] COLLI, P. & LAURENÇOT, P. *Weak solutions to the Penrose–Fife phase field model for a class of admissible heat flux laws*, Phys. D, 111 (1998), pp. 311–334.

[10] COLLI, P., LAURENÇOT, P., & SPREKELS, J. *Global solution to the Penrose-Fife phase field model with special heat flux laws*, in Variations of domain and free-boundary problems in solid mechanics (Paris, 1997), Kluwer Acad. Publ., Dordrecht, 1999, pp. 181–188.

[11] COLLI, P. & J. SPREKELS, J. *Stefan problems and the Penrose–Fife phase field model*, Adv. Math. Sci. Appl., 7 (1997), pp. 911–934.

[12] COLLI, P. & J. SPREKELS, J. *Weak solution to some Penrose–Fife phase–field systems with temperature–dependent memory*, J. Differential Equations, 142 (1998), pp. 54–77.

[13] CRANDALL, M., ISHII, H. & LIONS, P. *Users's guide to viscosity solution of second order partial differential equations*, Bull. Amer. Math. Soc., 27 (1992), pp. 1–67.

[14] DAMLAMIAN, A. & KENMOCHI, N. *Evolution equations associated with non-isothermal phase transitions*, in Functional analysis and global analysis (Quezon City, 1996), Springer, Singapore, 1997, pp. 62–77.

[15] DAMLAMIAN, A. & KENMOCHI, N. *Evolution equations generated by subdifferentials in the dual space of* $(H^1(\Omega))$, Discrete Contin. Dynam. Systems, 5 (1999), pp. 269–278.

[16] ELLIOTT, C. M. & GARDINER, A. *Double obstacle phase field computations of dendritic growth*, Research report 96–19, University of Sussex, CMAIA, 1996.

[17] ELLIOTT, C. M., GARDINER, A., & KUHN, T. *Generalized double obstacle phase field approximation of the anisotropic mean curvature flow*, Research report 96–17, University of Sussex, CMAIA, 1996.

[18] ELLIOTT, C. M., PAOLINI, M. & SCHÄTZLE, R. *Interface estimates for the fully anisotropic Allen-Cahn equation and anisotropic mean-curvature flow.*, Math. Models Methods Appl. Sci., 6 (1996), pp. 1103–1118.

[19] ELLIOTT, C. M. & SCHÄTZLE, R. *The limit of the fully anisotropic double-obstacle Allen-Cahn equation in the nonsmooth case*, SIAM J. Math. Anal., 28 (1997), pp. 274–303.

[20] HORN, W. *A numerical scheme for the one–dimensional Penrose–Fife model for phase transitions*, Adv. Math. Sci. Appl., 2 (1993), pp. 457–483.

[21] HORN, W., LAURENÇOT, P. & SPREKELS, J. *Global solutions to a Penrose–Fife phase–field model under flux boundary conditions for the inverse temperature*, Math. Methods Appl. Sci., 19 (1996), pp. 1053–1072.

[22] HORN, W., SPREKELS, J. & ZHENG, S. *Global existence of smooth solutions to the Penrose–Fife model for Ising ferromagnets*, Adv. Math. Sci. Appl., 6 (1996), pp. 227–241.

[23] KENMOCHI, N. & KUBO, M. *Weak solutions of nonlinear systems for non–isothermal phase transitions*, Adv. Math. Sci. Appl., 9 (1999), pp. 499–521.

[24] KENMOCHI, N. & NIEZGÓDKA, M. *Systems of nonlinear parabolic equations for phase change problems*, Adv. Math. Sci. Appl., 3 (1993/94), pp. 89–185.

[25] KLEIN, O. *Existence and approximation results for phase–field systems of Penrose–Fife type and Stefan problems*, PhD thesis, Humboldt University, Berlin, 1997.

[26] KLEIN, O. *A semidiscrete scheme for a Penrose–Fife system and some Stefan problems in* $\mathbb{R}^3$, Adv. Math. Sci. Appl., 7 (1997), pp. 491–523.

[27] KLEIN, O. *A class of time discrete schemes for a phase–field system of Penrose–Fife type*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 1361–1292.

[28] KLEIN, O. *A phase-field system with space-dependent relaxation coefficient*, in Free boundary problems: theory and applications (Crete, 1997), Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 327–334.

[29] LAURENÇOT, P. *Etude de quelques Problèmes aux Dérivées Partielles non Linéaires*, PhD thesis, Université de Franche–Comté, 1993.

[30] LAURENÇOT, P. *Weak solutions to a Penrose–Fife model for phase transitions*, Adv. Math. Sci. Appl., 5 (1995), pp. 117–138.

[31] LAURENÇOT, P. *Weak solutions to a Penrose–Fife model with Fourier law for the temperature*, J. Math. Anal. Appl., 219 (1998), pp. 331–343.

[32] MARCUS, M., & . MIZEL, V. J. *Absolute continuity on tracks and mappings of Sobolev spaces*, Arch. Rational Mech. Anal., 45 (1972), pp. 294–320.

[33] MARCUS, M., & . MIZEL, V. J. *Complete characterization of functions which act, via superposition, on Sobolev spaces*, Trans. Amer. Math. Soc., 251 (1979), pp. 187–218.

[34] PENROSE, O., & FIFE, P. *Thermodynamically consistent models of phase–field type for the kinetics of phase transitions*, Phys. D, 43 (1990), pp. 44–62.

[35] SIMON, J. *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.

[36] SPREKELS, J., & ZHENG, S. *Global smooth solutions to a thermodynamically consistent model of phase–field type in higher space dimensions*, J. Math. Anal. Appl., 176 (1993), pp. 200–223.

[37] WILMANSKI, K. Private Communication, 1999.

[38] ZEIDLER, E. *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators*, Springer, New York, 1990.

[39] ZHENG, S. *Nonlinear parabolic equations and hyperbolic–parabolic coupled systems*, vol. 76 of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman, 1995.