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Analysis and optimization of nonsmooth mechanical structures

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Abstract

It is our aim to give a new treatment for some classical models of arches and plates and for their optimization. In particular, our approach allows to study nonsmooth arches, while the standard assumptions from the literature require $W^{3,\infty}$-regularity for the parametric representation. Moreover, by a duality-type argument, the deformation of the arches may be explicitly expressed by integral formulas.

As examples for the shape optimization problems under study, we mention the design of the middle curve of a clamped arch or of the thickness of a clamped plate such that, under a prescribed load, the obtained deflection satisfies certain desired properties. In all cases, no smoothness is required for the design parameters.

1 Introduction

If $\varphi\colon [0,1] \to I\!R^2$ is a smooth clamped arch and c denotes its curvature, then the classical Kirchhoff-Love model (with normalized mechanical constants) is given by:

$$\begin{aligned}&\int_0^1 \left[\frac{1}{\varepsilon}(v_1' - c\,v_2)(u_1' - c\,u_2) + (v_2' + c\,v_1)'(u_2' + c\,u_1)'\right] ds \\ &= \int_0^1 (f_1\,u_1 + f_2\,u_2)\,ds\,, \qquad \forall\ u_1 \in H_0^1(0,1)\,, \quad \forall\ u_2 \in H_0^2(0,1)\,.\end{aligned} \tag{1.1}$$

Here, $\sqrt{\varepsilon}$ is the constant thickness of the arch, $v_1 \in H_0^1(0,1)$, $v_2 \in H_0^2(0,1)$ are the tangential, respectively, the normal components of the deformation in the local coordinate system associated with the arch, and $[f_1, f_2]$ is a similar representation of the forces, including the internal and external loading of the arch, which are assumed to act in the same plane.

A thorough presentation via Dirichlet's principle and Korn's inequality of the existence and the uniqueness of the solution for (1.1) may be found in Ciarlet [12, p. 432]. In Chenais and Paumier [9] the "locking" problem, in connection with the numerical approximation of (1.1) and of shells, is discussed: if the discretization parameter is of the same order as ε, then the obtained numerical approximation may be meaningless, and special finite element schemes are necessary in order to solve (1.1).

In **Section 1**, we introduce a new variational formulation for (1.1), based on optimal control theory, which is valid also for Lipschitz (or, by reparametrization — see **Remark 2.5** — absolutely continuous) mappings φ. Using duality-type arguments, we derive explicit integration rules for (1.1). If φ is smooth, we show that our solution satisfies (1.1). In the general case, if φ is approximated by a sequence of smooth functions φ_δ with $\delta \to 0$ (obtained by a regularization via Friedrichs mollifiers), the approximation remains valid for the corresponding solutions, as well.

This shows that our variational formulation is a natural extension of (1.1) to the case of nonsmooth arches. It also provides, by its explicit character, a complete solution of the above mentioned "locking" problem in dimension one. We also study the behaviour for $\varepsilon \to 0$ and obtain, under the weak optimal control formulation of (1.1), the analogue of flexural models in the sense of Ciarlet [13]. Some of the results of this section were announced without proofs in Sprekels and Tiba [23]. Our arguments neither use the Dirichlet principle nor the Korn inequality. Moreover, although the arch may have an infinity of corners, we do not impose transmission conditions as in Geymonat and Sanchez-Palencia [15] — they are implicitly contained in our approach. Models for shells and rods, under low geometrical regularity conditions, are also discussed in Blouza and Le Dret [6], Chapelle [8].

In **Section 3**, we use the optimal control formulation from the previous section in its equivalent form obtained by a variant of Pontryagin's maximum principle. For given $[f_1, f_2]$, we study the shape optimization problem of finding φ in a closed bounded subset of the space of Lipschitz arches, such that the obtained deflection $[v_1, v_2]$ has certain desired properties.

It should be noted that in this setting the considered optimization problem appears as a nonconvex control-into-coefficients problem. We prove the existence of the minimizer and we derive the first order optimality conditions, by computing the directional derivative of the cost. Similar problems were studied by Rousselet, Piekarski and Myslinski [18], Chenais and Rousselet [10], Chenais, Rousselet and Benedict [11], under differentiability assumptions.

In **Section 4**, we consider the case of nonhomogeneous clamped plates with variable thickness $u \in L^\infty(\Omega)$ in a smooth domain $\Omega \subset I\!R^N$:

$$
\begin{aligned}
&\Delta(u^3\,\Delta y) = f && \text{in } \Omega\,,\\
&y = z && \text{on } \partial\Omega\,,\\
&\frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} && \text{on } \partial\Omega\,,
\end{aligned}
\tag{1.2}
$$

where the load $f \in L^2(\Omega)$ and $z \in H^2(\Omega)$ are given. A characterization of the solution of (1.2) via an optimal control problem is obtained as in **Section 2**. For shape optimization problems associated to (1.2), existence for u in closed bounded sets of $L^\infty(\Omega)$ was established in Sprekels and Tiba [21]. Here, we derive the first order optimality conditions without imposing differentiability assumptions on u. They are used to prove bang-bang type results in some applications.

The last section collects numerical experiments related to arches and to their optimization. For simple input functions, the deformations can be computed by MAPLE. In the optimization case, local gradient methods are combined with some global search, due to the nonconvexity of the problem. We have succeeded in finding, in some examples, global minimum points which have been theoretically justified a posteriori.

Finally, we point out that the core of our methods are various special decompositions of (1.1) or (1.2) obtained via the first order optimality conditions for appropriately defined control problems. In this respect, the present work continues the investigations from Sprekels and Tiba [19, 20, 21, 22, 23]. The main tools that we are using are control theory and duality.

2 The control approach

Let $\theta(t)$ denote the angle between the tangent vector to the arch (given by φ') and the horizontal axis. If φ is smooth, then $\theta' = c$, see Ciarlet [12, p. 432]. If $\varphi \in (W^{1,\infty}(0,1))^2$, then $\theta \in L^\infty(0,1)$ and this is the assumption we impose in the sequel. Note that in this case the variational formulation (1.1) is not meaningful.

Now, we introduce the fundamental matrix W of the homogeneous linear ODE system $v_1' = c\,v_2$, $v_2' = -c\,v_1$ (which is suggested by (1.1)), namely

$$W(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix}, \tag{2.1}$$

and the functions l, h, g_1, g_2 that are constructed from $f_1, f_2 \in L^2(0,1)$ as follows.

$$g_1 = \varepsilon\, l, \qquad -g_2'' = h, \qquad g_2(0) = g_2(1) = 0, \tag{2.2}$$

$$\begin{bmatrix} l \\ h \end{bmatrix}(t) = -\int_0^t W(t)\,W^{-1}(s) \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds\,. \tag{2.3}$$

We then define the constrained control problem

$$(\mathbf{P}_\varepsilon) \qquad \operatorname{Min} \left\{ \frac{1}{2\varepsilon} \int_0^1 u^2\, ds + \frac{1}{2} \int_0^1 (z')^2\, ds \right\},$$

subject to $u \in L^2(0,1)$, $z \in H_0^1(0,1)$, such that the mappings $[v_1, v_2] \in (L^\infty(0,1))^2$,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(t) := \int_0^t W(t)\,W^{-1}(s) \begin{bmatrix} u + g_1 \\ z + g_2 \end{bmatrix}(s)\, ds\,, \tag{2.4}$$

satisfy $v_1(1) = v_2(1) = 0$ in the sense that

$$\int_0^1 W^{-1}(s) \begin{bmatrix} u(s) + g_1(s) \\ z(s) + g_2(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.5}$$

Clearly, $u = -g_1$, $z = -g_2$ give an admissible control pair for (P_ε). By the coercivity and strict convexity of the cost, there follows the existence of a unique minimizer $[u_\varepsilon, z_\varepsilon] \in L^2(0,1) \times H_0^1(0,1)$.

Denote by $S \subset L^2(0,1) \times H_0^1(0,1)$ the closed subspace of admissible variations for (P_ε). Then, $[\mu, \xi] \in S$ if and only if

$$\int_0^1 W^{-1}(s) \begin{bmatrix} \mu(s) \\ \xi(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.6}$$

The Euler equation associated with $[u_\varepsilon, z_\varepsilon]$ is

$$\frac{1}{\varepsilon} \int_0^1 u_\varepsilon \, \mu \, ds + \int_0^1 z_\varepsilon' \, \xi' \, ds = 0 \quad \forall \, [\mu, \xi] \in S. \tag{2.7}$$

In particular, (2.7) says that $[u_\varepsilon, z_\varepsilon] \in S_\varepsilon^\perp$, where $S_\varepsilon^\perp$ denotes the orthogonal subspace of $S \subset L^2(0,1) \times H_0^1(0,1)$ with respect to the modified scalar product defined by the left-hand side of (2.7).

Remark 2.1 If $\theta \in W^{1,1}(0,1)$, then $c \in L^1(0,1)$ and relation (2.4) can be written in differential form as

$$v_1' - c\, v_2 = u + g_1 \qquad \text{a.e. in } (0,1), \tag{2.8}$$

$$v_2' + c\, v_1 = z + g_2 \qquad \text{a.e. in } (0,1). \tag{2.9}$$

Relation (2.4) gives the "mild" solution of (2.8), (2.9) with null initial conditions in the sense of semigroup theory, Bénilan [5], Barbu [3]. If (2.8), (2.9) give the state equations of the control problem (P_ε), then (2.5) is a state constraint. It is expressed directly in the form of a control constraint, since the system (2.8), (2.9) is integrated by (2.4), and $W(t)$ is a nonsingular matrix.

We denote by $[v_1^\varepsilon, v_2^\varepsilon] \in (L^\infty(0,1))^2$ the optimal state of (P_ε), obtained from $[u_\varepsilon, z_\varepsilon]$ via (2.4).

Theorem 2.1 *If* $\varphi \in (W^{3,\infty}(0,1))^2$ *then* $[v_1^\varepsilon, v_2^\varepsilon]$ *is the solution to* (1.1).

Proof. Under this regularity assumption, (2.4) can be written in the form (2.8), (2.9).

For any $u_1 \in H_0^1(0,1)$, $u_2 \in H_0^2(0,1)$, we introduce

$$\tilde{\mu} = u_1' - c\, u_2 \in L^2(0,1), \tag{2.10}$$

$$\tilde{\xi} = u_2' + c\, u_1 \in H_0^1(0,1), \tag{2.11}$$

and we have, consequently,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(t) = \int_0^t W(t)\, W^{-1}(s) \begin{bmatrix} \tilde{\mu} \\ \tilde{\xi} \end{bmatrix}(s)\, ds. \tag{2.12}$$

Since u_1, u_2 vanish at both ends of $[0,1]$, it follows from (2.12) and (2.6) that $[\tilde{\mu}, \tilde{\xi}] \in S$. Hence they may be used in (2.7). Taking into account relations (2.8), (2.9) satisfied by v_1^ε, v_2^ε, as well as (2.10), (2.11), and (2.2), we obtain that

$$0 = \frac{1}{\varepsilon}\int_0^1 \Big((v_1^\varepsilon)' - c\,v_2^\varepsilon - g_1\Big)(u_1' - cu_2)ds + \int_0^1 \Big((v_2^\varepsilon)' + c\,v_1^\varepsilon - g_2\Big)'(u_2' + c\,u_1)'\,ds$$

$$= \frac{1}{\varepsilon}\int_0^1 \big((v_1^\varepsilon)' - c\,v_2^\varepsilon\big)(u_1' - c\,u_2)\,ds + \int_0^1 \Big((v_2^\varepsilon)' + c\,v_1^\varepsilon\Big)'(u_2' + c\,u_1)'\,ds$$

$$- \int_0^1 l(u_1' - c\,u_2)\,ds - \int_0^1 h(u_2' + c\,u_1)\,ds\,.$$

By the regularity assumption, (2.3) can be rewritten in the differential form (2.8), (2.9), and we can infer that

$$\int_0^1 l(u_1' - c\,u_2)\,ds + \int_0^1 h(u_2' - c\,u_1)\,ds$$

$$= - \int_0^1 u_1(l' - c\,h)\,ds - \int_0^1 u_2(h' + c\,l)\,ds = \int_0^1 (f_1\,u_1 + f_2\,u_2)\,ds\,.$$

The last two relations give (1.1) and the proof is finished. □

Remark 2.2 The approach via problem (P_ε) is constructive and does not use either Dirichlet's principle or Korn's inequality. As the formulation of (P_ε) is valid for $\theta \in L^\infty(0,1)$, this method may give solutions even in nonsmooth situations when Korn's inequality is not valid. For such cases, we refer to Geymonat and Gilardi [14].

In the general case, the following extension of **Theorem 2.1** holds true.

Theorem 2.2 *If $\varphi \in (W^{1,\infty}(0,1))^2$, then we have for any $[\mu, \xi] \in S$:*

$$\frac{1}{\varepsilon}\int_0^1 (u_\varepsilon + g_1)\mu\,ds + \int_0^1 (z_\varepsilon + g_2)'\,\xi'\,ds = \int_0^1 (f_1\,u_1 + f_2\,u_2)\,ds\,, \tag{2.13}$$

with $u_1, u_2 \in L^\infty(0,1)$ given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(s) = - \int_s^1 W(s)\,W^{-1}(t)\begin{bmatrix} \mu(t) \\ \xi(t) \end{bmatrix} dt, \qquad \textit{for a.e. } s \in (0,1)\,. \tag{2.14}$$

Proof. Since $[u_\varepsilon, z_\varepsilon] \in S_\varepsilon^\perp$ (see (2.7)), we obtain that

$$\frac{1}{\varepsilon}\int_0^1 (u_\varepsilon + g_1)\mu\, ds + \int_0^1 (z_\varepsilon + g_1)'\xi'\, ds = \frac{1}{\varepsilon}\int_0^1 g_1\,\mu\, ds - \int_0^1 \xi\, g_2''\, ds = \int_0^1 [\mu,\xi] \begin{bmatrix} l \\ h \end{bmatrix} dt$$

$$= -\int_0^1 [\mu,\xi](t) \int_0^t W(t)\, W^{-1}(s) \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds\, dt$$

$$= -\int_0^1\int_0^t \big[f_1(s), f_2(s)\big] W(s)\, W^{-1}(t) \begin{bmatrix} \mu(t) \\ \xi(t) \end{bmatrix} ds\, dt\,,$$

due to the orthogonality of the matrix $W(t)$ and to (2.2), (2.3). Fubini's theorem and (2.14) imply the result. $\square$

Remark 2.3 It is possible to prove **Theorem 2.1** via **Theorem 2.2**. These results show that the problem (P_ε) provides a notion of weak solution for the arch problem which is a natural extension of the classical one. This will be further justified below in **Theorem 3.1** and **Remark 3.4**, via an approximation argument.

We introduce now the mappings $w_1, w_2 \in H^2(0,1) \cap H_0^1(0,1)$ given by

$$w_1''(s) = \sin\theta(s), \qquad \text{a.e. in } (0,1)\,. \tag{2.15}$$

$$w_2''(s) = -\cos\theta(s), \qquad \text{a.e. in } (0,1)\,. \tag{2.16}$$

Then, relation (2.6) may be rewritten as

$$\frac{1}{\varepsilon}\int_0^1 \varepsilon\, \cos\theta(s)\, \mu(s)\, ds + \int_0^1 w_1'(s)\, \xi'(s)\, ds = 0\,, \tag{2.17}$$

$$\frac{1}{\varepsilon}\int_0^1 \varepsilon\, \sin\theta(s)\, \mu(s)\, ds + \int_0^1 w_2'(s)\, \xi'(s)\, ds = 0\,. \tag{2.18}$$

From the definition of S using the modified scalar product from (2.7) it follows that the (linearly independent) vectors $[\varepsilon\cos\theta(\cdot), w_1(\cdot)]$ and $[\varepsilon\sin\theta(\cdot), w_2(\cdot)]$ provide a basis of the two-dimensional space $S_\varepsilon^\perp$.

Besides, from relations (2.5) and (2.6), we can infer that $[u_\varepsilon + g_1\,,\, z_\varepsilon + g_2] \in S$. Consequently, relation (2.7) gives that

$$[u_\varepsilon, z_\varepsilon] = -\operatorname{proj}_{S_\varepsilon^\perp}[g_1, g_2]\,, \tag{2.19}$$

where the projection is computed in the norm generated by the modified scalar product from (2.7).

Then, (2.17) to (2.19) yield that

$$[u_\varepsilon, z_\varepsilon] \;=\; \lambda_1^\varepsilon\,[\varepsilon\,\cos\theta\,,\,w_1] + \lambda_2^\varepsilon\,[\varepsilon\,\sin\theta\,,\,w_2]\,, \tag{2.19$'$}$$

for some $\lambda_1^\varepsilon, \lambda_2^\varepsilon \in I\!R$. By virtue of the definition of the projection operator, and owing to (2.19), (2.19)$'$, we see that $(\lambda_1^\varepsilon\,,\,\lambda_2^\varepsilon)$ is the unique minimizer of the unconstrained optimization problem

$$(\mathbf{D}_\varepsilon) \qquad \operatorname*{Min}_{\lambda_1,\lambda_2\in I\!R}\left\{\frac{1}{2\varepsilon}\int_0^1\Big(\lambda_1\,\varepsilon\,\cos\theta(s)\;+\;\lambda_2\,\varepsilon\,\sin\theta(s)\;+\;\varepsilon\,l(s)\Big)^2\,ds\right.$$
$$\left.+\;\frac{1}{2}\int_0^1\Big[(\lambda_1\,w_1\;+\;\lambda_2\,w_2\;+\;g_2)'\Big]^2\,ds\right\}.$$

Problem (D_ε) can be solved explicitly using the system of necessary optimality conditions, which is a linear algebraic system with a strictly positive determinant (by the Cauchy-Schwarz inequality and the structure of the basis of $S_\varepsilon^\perp$). We indicate the system for subsequent use:

$$\begin{aligned}
&\varepsilon\,\lambda_1\int_0^1\cos^2\,\theta(s)\,ds\;+\;\lambda_1\,\big|w_1\big|^2_{H_0^1(0,1)}\;+\;\varepsilon\,\lambda_2\int_0^1\cos\theta(s)\,\sin\theta(s)\,ds\\
&\qquad+\;\lambda_2\int_0^1 w_1'(s)\,w_2'(s)\,ds\;+\;\varepsilon\int_0^1 l(s)\,\cos\theta(s)\,ds\;+\;\int_0^1 g_2'(s)\,w_1'(s)\,ds\;=\;0\,,\\
&\varepsilon\,\lambda_1\int_0^1\cos\theta(s)\,\sin\theta(s)\,ds\;+\;\lambda_1\int_0^1 w_1'(s)\,w_2'(s)\,ds\;+\;\varepsilon\,\lambda_2\int_0^1\sin^2\theta(s)\,ds\\
&\qquad+\;\lambda_2\big|w_2\big|^2_{H_0^1(0,1)}\;+\;\varepsilon\int_0^1 l(s)\sin\theta(s)\,ds\;+\;\int_0^1 g_2'(s)\,w_2'(s)\,ds\;=\;0\,.
\end{aligned} \tag{2.20}$$

We have proved the following result.

Theorem 2.3 *The solution of* (1.1) *(or of* (P_ε)*, if* $\theta \in L^\infty(0,1)$ *) is given by* (2.19)$'$ *and* (2.4)*, with* $(\lambda_1^\varepsilon\,,\,\lambda_2^\varepsilon)$ *being the unique solution of* (D_ε)*, and with* $w_1, w_2,\ g_1, g_2$ *defined by* (2.2), (2.3), (2.15), (2.16).

Remark 2.4 In optimization theory, (D_ε) is the dual problem of (P_ε). Its complete solution is possible since the constraints from (P_ε) are affine and finite dimensional. In simple examples of mappings θ, f_1, f_2, explicit formulas can be derived for the deformation $[v_1, v_2]$. In the general situation, numerical approximation is needed just to evaluate the occurring integrals. See **Section 5**, for examples. In particular,

Theorem 2.3 provides a complete solution of the "locking" problem discussed by Chenais and Paumier [9], in dimension one.

Remark 2.5 We also notice that, if $\tilde{\varphi}\colon [a,b] \to I\!R^2$ is an absolutely continuous Jordan arc of length one such that $\tilde{\varphi}' \neq 0$ a.e. in (a,b), then, by the usual reparametrization via the arc length function $s\colon [a,b] \to [0,1]$, $s(0) = 0$, $s'(\cdot) = \left|\tilde{\varphi}'(\cdot)\right|_{I\!R^2}$, we get that $\varphi(t) = \tilde{\varphi}(s^{-1}(t))$ satisfies $\left|\varphi'(t)\right|_{I\!R^2} = 1$ for a.e. $t \in (0,1)$, i.e. it is Lipschitzian, and our results still apply.

Remark 2.6 If $\theta \in L^\infty(0,1)$, then v_1^ε, v_2^ε as defined by **Theorem 2.3** (see (2.4)) belong to $L^\infty(0,1)$. However, their global cartesian representation is

$$W(t)^{-1}\begin{bmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{bmatrix}(t)$$

and belongs to $(W^{1,2}(0,1))^2$. This means that the lack of smoothness is due to the local coordinates (θ is defined a.e. and may have jumps), and that the constructed deformation is continuous.

The next result gives a characterization of the solution of the problem (P_ε) (or, equivalently, of the problem (D_ε)) as a system of first order differential equations which will be used frequently in the sequel. Implicitly, it provides a nonstandard decomposition of equation (1.1) in the case of nonsmooth coefficients. Basically, this is given by the first order necessary conditions for (P_ε), but the form is different from the classical Pontryagin principle.

Theorem 2.4 *The optimality system for the problem* (P_ε) *is given by*

$$\begin{bmatrix} v_1^\varepsilon \\ v_2^\varepsilon \end{bmatrix}(t) = \int_0^t W(t)\,W^{-1}(s)\begin{bmatrix} u_\varepsilon(s) + g_1(s) \\ z_\varepsilon(s) + g_2(s) \end{bmatrix} ds\,, \quad \textit{for a.e. } t \in (0,1)\,, \tag{2.21}$$

$$\int_0^1 W^{-1}(s)\begin{bmatrix} u_\varepsilon(s) + g_1(s) \\ z_\varepsilon(s) + g_2(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.22}$$

$$\begin{bmatrix} p_\varepsilon \\ q_\varepsilon \end{bmatrix}(t) = W(t)\begin{bmatrix} \lambda_1^\varepsilon \\ \lambda_2^\varepsilon \end{bmatrix}, \quad \textit{for a.e. } t \in (0,1)\,, \tag{2.23}$$

$$u_\varepsilon = \varepsilon\, p_\varepsilon \qquad \textit{a.e. in } (0,1)\,, \tag{2.24}$$

$$z_\varepsilon'' = -\,q_\varepsilon \quad \textit{a.e. in } (0,1)\,, \quad z_\varepsilon(0) = z_\varepsilon(1) = 0\,. \tag{2.25}$$

Proof. Assume first that u_ε, z_ε satisfy (2.21)–(2.25) with some $\lambda_1^\varepsilon, \lambda_2^\varepsilon \in I\!R$, p_ε, q_ε, v_1^ε, $v_2^\varepsilon \in L^\infty(0,1)$. Then clearly, $[u_\varepsilon + g_1\,,\ z_\varepsilon + g_2] \in S$, i.e. $[u_\varepsilon\,,\ z_\varepsilon]$ is admissible for (P_ε). Using (2.23)–(2.25), the definition of S, and the orthogonality of $W(t)$, we find that for any $[\mu, \xi] \in S$ it holds

$$\frac{1}{\varepsilon} \int_0^1 u_\varepsilon \, \mu \, ds + \int_0^1 z_\varepsilon' \, \xi' \, ds = \int_0^1 p_\varepsilon \, \mu \, ds + \int_0^1 q_\varepsilon \, \xi \, ds = \int_0^1 [\mu, \xi] W(s) \begin{bmatrix} \lambda_1^\varepsilon \\ \lambda_2^\varepsilon \end{bmatrix} ds$$

$$= [\lambda_1^\varepsilon, \lambda_2^\varepsilon] \int_0^1 W(s)^{-1} \begin{bmatrix} \mu \\ \xi \end{bmatrix} (s) \, ds = 0 .$$

Consequently, $[u_\varepsilon \, , \, z_\varepsilon] \in S_\varepsilon^\perp$. Together with the admissibility of $[u_\varepsilon, z_\varepsilon]$, noticed above, this gives immediately that $[u_\varepsilon \, , \, z_\varepsilon]$ is the unique minimizer of (P_ε).

Conversely, we remark that (2.23)–(2.25) give a complete description of the two-dimensional space $S_\varepsilon^\perp$, when $\lambda_1, \lambda_2 \in I\!R$ are arbitrary. By (2.6), we know that the optimal control $[u_\varepsilon \, , \, z_\varepsilon]$ belongs to $S_\varepsilon^\perp$. Hence, there are $\lambda_1^\varepsilon, \lambda_2^\varepsilon \in I\!R$ such that $[u_\varepsilon \, , \, z_\varepsilon]$ can be represented via (2.23)–(2.25) (this is, in fact, the same representation as in (2.19)$'$). Moreover, $[u_\varepsilon \, , \, z_\varepsilon]$ also satisfy (2.21), (2.22) by their admissibility for (P_ε). This ends the proof. □

As a first application of **Theorem 2.4**, we study the behaviour for $\varepsilon \to 0$ of the problem (P_ε). Since arches are special cases of cylindrical shells, after passing to the limit a "flexural" model will be obtained, Ciarlet [13]. The treatment that we indicate below is valid under the weak regularity condition $\theta \in L^\infty(0,1)$. We shall also assume that θ is nonconstant in $[0,1]$, i.e. the arch is not a bar. Also for constant θ the results remain valid, but some adaption of the argument is necessary, since the dimension of $S_\varepsilon^\perp$ reduces to one in this case.

Theorem 2.5 *As* $\varepsilon \searrow 0$, *the mappings* v_1^ε, v_2^ε, p_ε, q_ε *are bounded in* $L^\infty(0,1)$, λ_1^ε, λ_2^ε *are bounded in* $I\!R$, z_ε *is bounded in* $H^2(0,1)$, *and* u_ε *strongly converges to* 0 *in* $L^\infty(0,1)$. *If we denote without* ε *their weak or weak* limits (on a subsequence) in the corresponding spaces, then these satisfy the conditions*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} (t) = \int_0^t W(t) \, W^{-1}(s) \begin{bmatrix} 0 \\ z(s) + g_2(s) \end{bmatrix} ds ,$$

$$\int_0^1 W^{-1}(s) \begin{bmatrix} 0 \\ z(s) + g_2(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

$$\begin{bmatrix} p \\ q \end{bmatrix} (t) = W(t) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} ,$$

$$z'' = -q , \qquad z(0) = z(1) = 0 .$$

Proof. The explicit calculus indicated in **Theorem 2.3**, (2.20), shows directly that λ_1^ε, λ_2^ε are bounded in $I\!R$ for $\varepsilon \to 0$. It is here that the assumption that θ is nonconstant is necessary, since for $\varepsilon = 0$ and θ constant the vectors used in $(2.19)'$ become proportional (in this case only one parameter λ is necessary and a simpler argument works).

Thus, by (2.23), p_ε and q_ε are bounded in $L^\infty(0,1)$. Relation (2.24) gives $u_\varepsilon \to 0$ strongly in $L^\infty(0,1)$, and (2.25) shows that z_ε is bounded in $H^2(0,1)$, for instance. By (2.21), we see that v_1^ε, v_2^ε are bounded in $L^\infty(0,1)$, as well. Definition (2.2) gives that g_1 depends on ε (and has the strong limit 0 in $L^\infty(0,1)$), while g_2 is independent of ε.

Finally, we can pass to the limit in (2.21)–(2.25) on a subsequence, and we obtain the desired conclusion. □

Remark 2.7 The system obtained by **Theorem 2.5** characterizes, in the sense of **Theorem 2.4**, the following constrained optimal control problem:

$$\operatorname{Min}\left\{\frac{1}{2}|z|^2_{H_0^1(0,1)}\right\},$$

subject to $z \in H_0^1(0,1)$, such that the mappings

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(t) = \int_0^t W(t)\,W^{-1}(s)\begin{bmatrix} 0 \\ z(s) + g_2(s) \end{bmatrix} ds$$

satisfy $v_1(1) = v_2(1) = 0$ in the sense that

$$\int_0^1 W^{-1}(s)\begin{bmatrix} 0 \\ z(s) + g_2(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The structure of this problem is very similar to (P_ε), and the proof follows closely that of **Theorem 2.4**, by considering the subspace $Z \subset H_0^1(0,1)$, defined by

$$\int_0^1 W^{-1}(s)\begin{bmatrix} 0 \\ \xi(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and its orthogonal subspace $Z^\perp$. If $\theta \in L^\infty(0,1)$ is not constant, $Z^\perp$ has dimension two, and we can argue as above.

Remark 2.8 If $\theta \in W^{2,\infty}(0,1)$, then one can show, as in **Theorem 2.1**, that v_1, v_2 defined in **Remark 2.7** satisfy the "flexural" arch model:

$$\int_0^1 (v_2' + cv_1)'\,(u_2' + c\,u_1)'\,ds = \int_0^1 (f_1\,u_1 + f_2\,u_2)\,ds$$

$$\forall\ (u_1, u_2) \in V_F = \left\{(u_1, u_2) \in H_0^1(0,1) \times H_0^2(0,1)\,;\, u_1' - c\,u_2 = 0\right\},$$

$$(v_1, v_2) \in V_F.$$

Such asymptotic properties have been discussed in detail by Ciarlet [13] for the case of shells. **Theorem 2.5** shows that they remain valid for nonsmooth arches under our variational formulation via optimal control theory.

3 Optimization of nonsmooth arches

One advantage of the method presented in the previous section is that in the study of related optimization problems, a large class of nonsmooth arches may be taken into consideration. Let $\mathcal{K} \subset L^\infty(0,1)$ be a closed subset. We shall study the model problem:

$$\textbf{(Q)} \qquad \underset{\theta \in \mathcal{K}}{\mathrm{Min}} \left\{ \frac{1}{2} \int_0^1 v_2^2 \, ds \right\},$$

subject to

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(t) = \int_0^t W_\theta(t)\, W_\theta^{-1}(s) \begin{bmatrix} u(s) + g_1(s) \\ z(s) + g_2(s) \end{bmatrix} ds\,, \quad \text{for a.e. } t \in (0,1)\,, \tag{3.1}$$

$$\int_0^1 W_\theta^{-1}(s) \begin{bmatrix} u(s) + g_1(s) \\ z(s) + g_2(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.2}$$

$$\begin{bmatrix} p \\ q \end{bmatrix}(t) = W_\theta(t) \begin{bmatrix} \lambda_1^\varepsilon \\ \lambda_2^\varepsilon \end{bmatrix}, \quad \text{for a.e. } t \in (0,1)\,, \tag{3.3}$$

$$u = \varepsilon\, p\,, \quad \text{a.e. in } (0,1)\,, \tag{3.4}$$

$$z'' = -\,q\,, \quad \text{a.e. in } (0,1)\,, \quad z(0) = z(1) = 0\,. \tag{3.5}$$

The matrix W_θ is given by (2.1), and the new notation just puts into evidence the dependence on the arch (characterized by θ). The state system (3.1)–(3.5) is exactly the decomposition of the Kirchhoff-Love model provided by **Theorem 2.4**. It should be noted that all the quantities appearing in it (including the data g_1, g_2 defined by (2.2), (2.3)) depend on θ. This is due to W_θ and to the fact that $[f_1, f_2]$ (the load) depends on θ by the local choice of the coordinates system. In the sequel, we shall write $v_1(\theta)$, $v_2(\theta)$, $\lambda_1(\theta)$, $\lambda_2(\theta)$, etc. (ε is fixed now).

Remark 3.1 The shape optimization problem **(Q)** is a nonconvex control-into-coefficients problem. In the given subset $\mathcal{K}$, the arch that minimizes the normal deflection (in the L^2-norm) is sought. This is a natural safety requirement. Various other cost functionals may be studied as well.

Theorem 3.1 *If $\theta_n \to \theta$ in $L^\infty(0,1)$ and $f_i(\theta_n) \to f_i(\theta)$ in $L^1(0,1), i = 1,2$, then $W_{\theta_n} \to W_\theta$ in $(L^\infty(0,1))^4$, $\lambda_1(\theta_n) \to \lambda_1(\theta)$, $\lambda_2(\theta_n) \to \lambda_2(\theta)$, $g_1(\theta_n) \to g_1(\theta)$,*

$h(\theta_n) \to h(\theta)$ and $l(\theta_n) \to l(\theta)$ in $L^\infty(0,1)$, $g_2(\theta_n) \to g_2(\theta)$ in $W^{2,\infty}(0,1)$, $p(\theta_n) \to p(\theta)$, $u(\theta_n) \to u(\theta)$ and $q(\theta_n) \to q(\theta)$ in $L^\infty(0,1)$, $z(\theta_n) \to z(\theta)$ in $W^{2,\infty}(0,1)$, and $v_1(\theta_n) \to v_1(\theta)$, $v_2(\theta_n) \to v_2(\theta)$ in $L^\infty(0,1)$. If $\theta_n \to \theta$ in $C[0,1]$, then the above convergences are also valid in $C[0,1]$ and $C^2[0,1]$, respectively.

Proof. If $\theta_n \to \theta$ in $L^\infty(0,1)$, then $\cos\theta_n \to \cos\theta$ and $\sin\theta_n \to \sin\theta$ in $L^\infty(0,1)$. Consequently, $W_{\theta_n} \to W_\theta$, $W_{\theta_n}^{-1} \to W_\theta^{-1}$, strongly in $(L^\infty(0,1))^4$. Moreover, (2.15), (2.16) show that $w_1(\theta_n) \to w_1(\theta)$ and $w_2(\theta_n) \to w_2(\theta)$ in $W^{2,\infty}(0,1)$. If $\Delta(\theta_n)$ is the determinant associated with the system (2.20) (written for θ_n), a direct calculus gives that $\Delta(\theta_n) \to \Delta(\theta)$.

From the relation (2.3) we infer that for a.e. $t \in (0,1)$ it holds

$$\begin{aligned}
&\left| \begin{bmatrix} l(\theta_n) \\ h(\theta_n) \end{bmatrix}(t) - \begin{bmatrix} l(\theta) \\ h(\theta) \end{bmatrix}(t) \right|_{I\!R^2} \\
&\leq \left| W_{\theta_n} - W_\theta \right|_{(L^\infty(0,1))^4} \left| W_{\theta_n}^{-1} \right|_{(L^\infty(0,1))^4} \left| \begin{bmatrix} f_1(\theta_n) \\ f_2(\theta_n) \end{bmatrix} \right|_{(L^1(0,1))^2} \\
&+ \left| W_\theta \right|_{(L^\infty(0,1))^4} \left| W_{\theta_n}^{-1} - W_\theta^{-1} \right|_{(L^\infty(0,1))^4} \left| \begin{bmatrix} f_1(\theta_n) \\ f_2(\theta_n) \end{bmatrix} \right|_{(L^1(0,1))^2} \\
&+ \left| W_\theta \right|_{(L^\infty(0,1))^4} \left| W_\theta^{-1} \right|_{(L^\infty(0,1))^4} \left| \begin{bmatrix} f_1(\theta_n) \\ f_2(\theta_n) \end{bmatrix} - \begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix} \right|_{(L^1(0,1))^2} .
\end{aligned} \tag{3.6}$$

It follows that $l(\theta_n) \to l(\theta)$, $h(\theta_n) \to h(\theta)$, strongly in $L^\infty(0,1)$. By (2.2), the same is valid for $g_1(\theta_n) \to g_1(\theta)$, while $g_2(\theta_n) \to g_2(\theta)$ strongly in $W^{2,\infty}(0,1)$. Then, one obtains $\lambda_1(\theta_n) \to \lambda_1(\theta)$ and $\lambda_2(\theta_n) \to \lambda_2(\theta)$ from (2.20).

The equations (3.3)–(3.5) give the assertion for $p(\theta_n)$, $q(\theta_n)$, $u(\theta_n)$, $z(\theta_n)$. The argument for the convergence $v_1(\theta_n) \to v_1(\theta)$, $v_2(\theta_n) \to v_2(\theta)$, strongly in $L^\infty(0,1)$, is similar to that in the inequality (3.6). If $\theta_n \to \theta$ in $C[0,1]$, the proof follows the same lines, with minor modifications. □

Corollary 3.2 *The shape optimization problem* (Q) *has at least one solution if $\mathcal{K}$ is compact in $L^\infty(0,1)$.*

Proof. This is a direct consequence of **Theorem 3.1**, by noticing that it is possible to pass to the limit in (3.2) and in the cost functional, if $\theta_n \to \theta$ strongly in $L^\infty(0,1)$. □

Remark 3.2 In completion to **Remark 2.6**, we notice that the convergence of the global cartesian representation of the displacement

$$W_{\theta_n}^{-1}(t) \begin{bmatrix} v_1(\theta_n) \\ v_2(\theta_n) \end{bmatrix}(t)$$

is valid in $(W^{1,\infty}(0,1))^2$. Here, we also use the fact that by (3.4) the solution $[u, z]$ of the problem (P_ε) belongs to $(L^\infty(0,1))^2$.

Remark 3.3 If the curvature c corresponding to the arches associated with $\theta \in \mathcal{K}$ is bounded in some $L^r(0,1)$, $r > 1$, then $\mathcal{K}$ is compact in $C[0,1]$. This shows that the compactness assumption from the **Theorem 3.1** and **Corollary 3.2** is very weak in comparison with the existing literature.

Remark 3.4 For any $\theta \in L^\infty(0,1)$, we may define a smooth sequence θ_n converging to θ in $L^r(0,1), \forall\, r \geq 1$, by a regularization process with a Friedrichs mollifier. Then, keeping $[f_1, f_2] \in (L^2(0,1))^2$ fixed, it is possible to modify (3.6) and the other arguments in the proof of **Theorem 3.1** to show that for the corresponding solutions we have $v_n^1 \to v^1$, $v_n^2 \to v^2$ in $L^r(0,1), \forall\, r \geq 1$. If θ is continuous, the obtained convergences are uniform. We also note that the global cartesian representation

$$W_{\theta_n}^{-1}(t) \begin{bmatrix} v_n^1 \\ v_n^2 \end{bmatrix} (t)$$

is convergent in $(W^{1,r}(0,1))^2, \forall\, r \geq 1$. As for θ_n the corresponding solution of (P_ε) then coincides with the solution of (1.1) (by **Theorem 2.1**), we see that for any $\theta \in L^\infty(0,1)$ the optimal state of (P_ε) can be approximated by usual solutions of (1.1).

The remainder of this section is devoted to the sensitivity analysis of the Kirchhoff-Love model. We proceed in two steps. First, we assume that $c \in L^1(0,1)$ and that, consequently, $\theta \in W^{1,1}(0,1)$, and we compute the gradient of the cost in this case. Then, we use an approximation argument to reduce the general case $\theta \in L^\infty(0,1)$ to the previous one.

Under the assumption $c \in L^1(0,1)$ the state system (3.1)–(3.5) for the problem (Q) can be written in differential form, namely

$$v_1' - c\,v_2 = u + g_1\,, \tag{3.7}$$

$$v_2' + c\,v_1 = z + g_2\,, \tag{3.8}$$

$$v_1(0) = v_2(0) = 0\,, \tag{3.9}$$

$$v_1(1) = v_2(1) = 0\,, \tag{3.10}$$

$$p' - c\,q = 0\,, \tag{3.11}$$

$$q' + c\,p = 0\,, \tag{3.12}$$

$$p(0) = \lambda_1\,, \quad q(0) = \lambda_2\,, \tag{3.13}$$

$$u = \varepsilon\, p\,, \tag{3.14}$$

$$z'' = -q\,, \tag{3.15}$$

$$z(0) = z(1) = 0\,. \tag{3.16}$$

We shall denote by $v_1(c)$, $v_2(c)$, $\ldots$ the dependence of the solution of (3.7)–(3.16) on $c \in L^1(0,1)$, which is now considered instead of the related dependence on θ.

We study its Gâteaux differentiability, and we take variations of the form $c + \delta d$ with $d \in L^1(0,1)$, $\delta \in I\!R$ "small".

The definitions of g_1, g_2, given in (2.2) and (2.3), can also be rewritten in differential form:

$$g_1 = \varepsilon l, \tag{3.17}$$

$$g_2'' = -h, \tag{3.18}$$

$$g_2(0) = g_2(1) = 0, \tag{3.19}$$

$$l' - ch = -f_1, \tag{3.20}$$

$$h' + cl = -f_2, \tag{3.21}$$

$$l(0) = h(0) = 0. \tag{3.22}$$

We have

$$\frac{l(c+\delta d)' - l(c)'}{\delta} - (c+\delta d)\frac{h(c+\delta d) - h(c)}{\delta} = d\,h(c) - \frac{f_1(c+\delta d) - f_1(c)}{\delta}, \tag{3.23}$$

$$\frac{l(c+\delta d)' - h(c)'}{\delta} + (c+\delta d)\frac{l(c+\delta d) - l(c)}{\delta} = -d\,l(c) - \frac{f_2(c+\delta d) - f_2(c)}{\delta}. \tag{3.24}$$

We interpret $f_1, f_2 : L^1(0,1) \to L^1(0,1)$ as nonlinear operators, and we assume that they are Gâteaux differentiable. Multiplying (3.23), (3.24) by $\left[\frac{l(c+\delta d) - l(c)}{\delta}\right.$, $\left.\frac{h(c+\delta d) - h(c)}{\delta}\right]$, and integrating over $[0,t]$, we find that

$$\frac{1}{2}\left|\left[\begin{array}{c}\dfrac{l(c+\delta d) - l(c)}{\delta}\\ \dfrac{h(c+\delta d) - h(c)}{\delta}\end{array}\right](t)\right|^2_{I\!R^2}$$

$$\leq \int_0^t \left\langle \begin{array}{c} d\,h(c) - \dfrac{f_1(c+\delta d) - f_1(c)}{\delta} \\ -d\,l(c) - \dfrac{f_2(c+\delta d) - f_2(c)}{\delta}\end{array}; \begin{array}{c}\dfrac{l(c+\delta d) - l(c)}{\delta}\\ \dfrac{h(c+\delta d) - h(c)}{\delta}\end{array}\right\rangle_{I\!R^2} ds \tag{3.25}$$

with obvious notations for the norm and the scalar product in $I\!R^2$.

The Brezis [7] variant of Gronwall's lemma and (3.25) imply that $\left\{\frac{l(c+\delta d) - l(c)}{\delta}\right\}$, $\left\{\frac{h(c+\delta d) - h(c)}{\delta}\right\}$ are bounded in $L^\infty(0,1)$ for $\delta \to 0$. From (3.23), (3.24), we see that the boundedness is even valid in $W^{1,1}(0,1)$, and we also have equi-uniform continuity due to the equi-absolute integrability of $\left\{ \dfrac{f_i(c + \delta\, d) - f_i(c)}{\delta} \right\}$, $i = 1, 2$. Consequently, by taking a subsequence, we get convergence and the Gâteaux differentiability of $l(c)$, $h(c)$ in $L^2(0,1)$, for instance. Relations (3.17)–(3.19) then show that $g_1(\cdot) : L^1(0,1) \to L^2(0,1)$, $g_2(\cdot) : L^1(0,1) \to W^{2,2}(0,1)$ are also Gâteaux differentiable.

The auxiliary mappings w_1, w_2 defined in (2.15), (2.16), are clearly Gâteaux differentiable, and if $\bar{w}_1$, $\bar{w}_2$ denote the directional derivatives at c in the direction d, we see that

$$\bar{w}_1'' = \left(\int_0^t d(s)\, ds \right) \cos \left(\int_0^t c(s)\, ds \right), \qquad \bar{w}_1(0) = \bar{w}_1(1) = 0, \tag{3.26}$$

$$\bar{w}_2'' = \left(\int_0^t d(s)\, ds \right) \sin \left(\int_0^t c(s)\, ds \right), \qquad \bar{w}_2(0) = \bar{w}_2(1) = 0. \tag{3.27}$$

The choice $\theta(t) = \int_0^t c(s)\, ds$ in (3.26), (3.27) just means that the global coordinates system is such that $\theta(0) = 0$. Other choices are possible as well. Next, we recall, by (2.20), that $\lambda_1(c)$, $\lambda_2(c)$ are solutions of an affine system with $\Delta(c) > 0$ and coefficients which are Gâteaux differentiable, by (3.26), (3.27). Then, $\lambda_1(c)$, $\lambda_2(c)$ are as well Gâteaux differentiable from $L^1(0,1)$ into $I\!R$. Moreover, (3.12), (3.13) imply the Gâteaux differentiability of $p, q : L^1(0,1) \to L^2(0,1)$, for instance. It follows immediately that $u : L^1(0,1) \to L^2(0,1)$ and $z : L^1(0,1) \to W^{2,2}(0,1)$ are Gâteaux differentiable. Finally, applying arguments similar to (3.23)–(3.25) to (3.7)–(3.9), we obtain that also $v_1, v_2 : L^1(0,1) \to L^2(0,1)$ are Gâteaux differentiable.

We denote by $\bar{v}_1$, $\bar{v}_2$, ... the directional derivatives of the mappings defined by (3.7)–(3.16) with respect to $c \in L^1(0,1)$ and in the direction $d \in L^1(0,1)$.

We thus have established the following result.

Theorem 3.3 *The mappings defined in* (3.7)–(3.16) *are Gâteaux differentiable, and the directional derivatives satisfy the system*

$$\bar{v}_1' - c\,\bar{v}_2 = d\, v_2(c) + \bar{u} + \bar{g}_1, \tag{3.28}$$

$$\bar{v}_2' + c\,\bar{v}_1 = -\,d\, v_1(c) + \bar{z} + \bar{g}_2, \tag{3.29}$$

$$\bar{v}_1(0) = \bar{v}_2(0) = 0, \tag{3.30}$$

$$\bar{v}_1(1) = \bar{v}_2(1) = 0, \tag{3.31}$$

$$\bar{p}' - c\,\bar{q} = d\, q(c), \tag{3.32}$$

$$\bar{q}' + c\,\bar{p} = -\,d\,p(c)\,, \tag{3.33}$$

$$\bar{p}(0) = \bar{\lambda}_1\,, \quad \bar{q}(0) = \bar{\lambda}_2\,, \tag{3.34}$$

$$\bar{u} = \varepsilon\,\bar{p}\,, \tag{3.35}$$

$$\bar{z}'' = -\,\bar{q}\,, \tag{3.36}$$

$$\bar{z}(0) = \bar{z}(1) = 0\,. \tag{3.37}$$

Remark 3.5 The system (3.28)–(3.37) admits a unique solution, since its homogeneous variant may be reformulated in the language of the control problem (P_ε). Here, homogeneous means that $\bar{g}_1 = 0$, $\bar{g}_2 = 0$, $d = 0$, and the corresponding solution of (P_ε) is in this situation clearly identically zero in $[0,1]$. Consequently, the limits defining $\bar{v}_1$, $\bar{v}_2$, ... are valid without taking subsequences; we have convergence of the entire sequences.

Next, we introduce the adjoint system associated with (3.28)–(3.37):

$$P_1' - c\,P_2 = 0\,, \tag{3.38}$$

$$P_2' + c\,P_1 = -\,v_2(c)\,, \tag{3.39}$$

$$P_3' - c\,P_4 = R\,, \tag{3.40}$$

$$P_4' + c\,P_3 = Q\,, \tag{3.41}$$

$$Q'' = -\,P_2\,, \tag{3.42}$$

$$R = \varepsilon\,P_1\,, \tag{3.43}$$

$$Q(0) = Q(1) = P_3(0) = P_3(1) = P_4(0) = P_4(1) = 0\,. \tag{3.44}$$

Proposition 3.4 *The system* (3.38) – (3.44) *has a unique solution such that* P_1, P_2, P_3, P_4, $R \in W^{1,1}(0,1)$ *and* $Q \in W^{2,\infty}(0,1)$.

Proof. Let μ_1, $\mu_2 \in I\!R^2$ be some arbitrary initial conditions for the equations (3.38), (3.39). Then

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}(t) = W_c(t)\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix},$$

where

$$\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = \int_0^t W_c(t)\,W_c^{-1}(s)\begin{bmatrix} 0 \\ -\,v_2(c) \end{bmatrix}(s)\,ds\,,$$

and $P_1, P_2 \in W^{1,1}(0,1)$ if $c \in L^1(0,1)$. Here, W_c is a new notation for the matrix W that puts into evidence its dependence on c.

Consequently, $R(t) = \varepsilon P_1$ and $Q(t)$ depend in an affine manner on μ_1, μ_2 and belong to $W^{1,1}(0,1)$ and $W^{2,\infty}(0,1)$, respectively. Then,

$$\begin{bmatrix} P_3 \\ P_4 \end{bmatrix}(t) = -\int_t^1 W_c(t) W_c^{-1}(s) \begin{bmatrix} R(s) \\ Q(s) \end{bmatrix} ds$$

belongs to $(W^{1,1}(0,1))^2$. We have used the final null conditions. Notice that the constraint

$$\int_0^1 W_c^{-1}(s) \begin{bmatrix} R(s) \\ Q(s) \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.45}$$

should be fulfilled to obtain the initial null conditions (3.44) for P_3, P_4. By writing (3.45) explicitly, we obtain a linear system as (2.20) for μ_1, μ_2. Since its determinant is positive, it has a unique solution, and the proof is finished. $\square$

Theorem 3.5 *The directional derivative of the cost functional in the problem* (Q) *at the point* $c \in L^1(0,1)$ *and in the direction* $d \in L^1(0,1)$ *is given by*

$$\int_0^1 d\Big(P_1\, v_2(c) - P_2\, v_1(c) + g_1'(c)^* P_1 + g_2'(c)^* P_2 - P_3\, q(c) + P_4\, p(c)\Big)\, ds\,. \tag{3.46}$$

Here, $g_i'(c)$, $i = 1,2$, denote the Gâteaux derivative of g_i at $c \in L^1(0,1)$, and $g_i'(c)^* : L^2(0,1) \to L^\infty(0,1)$ is the adjoint operator.

Proof. We have (by (3.38), (3.39), partial integration, etc.) that

$$\lim_{\delta \to 0} \frac{1}{2\delta} \left[\int_0^1 (v_2(c + \delta\, d))^2\, ds - \int_0^1 (v_2(c))^2\, ds \right] = \int_0^1 v_2(c)\, \bar v_2\, ds$$

$$= -\int_0^1 (P_2' + c\, P_1) \bar v_2\, ds - \int_0^1 (P_1' - c\, P_2)\, \bar v_1\, ds$$

$$= \int_0^1 P_1(\bar v_1' - c\, \bar v_2)\, ds + \int_0^1 P_2(\bar v_2' + c\, \bar v_1)\, ds$$

$$= \int_0^1 d(P_1\, v_2(c) - P_2\, v_1(c))\, ds + \int_0^1 P_1(\bar u + \bar g_1)\, ds + \int_0^1 P_2(\bar z + \bar g_2)\, ds\,,$$

owing to (3.28), (3.29). Now recall that

$$\bar g_1 = g_1'(c)\, d\,, \quad \bar g_2 = g_2'(c)\, d\,.$$

Hence, using (3.40) and (3.41), we can write

$$\int_0^1 v_2(c)\,\bar v_2\,ds = \int_0^1 d(P_1\,v_2(c) - P_2\,v_1(c) + g_1'(c)^* P_1 + g_2'(c)^* P_2)\,ds$$

$$+ \int_0^1 \varepsilon^{-1} R\,\bar u\,ds - \int_0^1 Q''\,\bar z\,ds = \int_0^1 d(\ldots)\,ds + \int_0^1 R\,\bar p\,ds + \int_0^1 Q\,\bar q\,ds$$

$$= \int_0^1 d(\ldots)\,ds + \int_0^1 \bar p(P_3' - c\,P_4)\,ds + \int_0^1 \bar q(P_4' + c\,P_3)\,ds\,.$$

From this, again using partial integration together with (3.32), (3.33), we obtain (3.46), and the proof is finished. □

Next, we shall study the differentiability properties of (Q) in the general case $\theta \in L^\infty(0,1)$. We consider variations of the form $\theta + \delta\,\eta$, $\eta \in L^\infty(0,1), \delta \in I\!R$ small. We assume that $f_i : L^\infty(0,1) \to L^2(0,1), i = 1,2$, depend directly on θ and are Gâteaux differentiable. A direct calculus starting from (2.3) and taking into account the dependence of $W(t)$ on θ, leads to

$$\begin{bmatrix} \bar l \\ \bar h \end{bmatrix}(t) = -\int_0^t W_\theta(t)\,W_\theta^{-1}(s) \begin{bmatrix} \bar f_1(s) \\ \bar f_2(s) \end{bmatrix} ds - \begin{pmatrix} 0 & \eta(t) \\ -\eta(t) & 0 \end{pmatrix} \begin{bmatrix} l(\theta) \\ h(\theta) \end{bmatrix}(t)$$

$$+ \int_0^t \begin{pmatrix} 0 & \eta(s) \\ -\eta(s) & 0 \end{pmatrix} W_\theta(t)\,W_\theta^{-1}(s) \begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix}(s)\,ds\,. \tag{3.47}$$

By (2.2), it holds

$$\bar g_1 = \varepsilon\,\bar l\,, \quad -\bar g_2'' = \bar h\,, \quad \bar g_2(0) = \bar g_2(1) = 0\,. \tag{3.48}$$

Comparing (3.47) with (3.20)–(3.22), we see that the integral formulation is more difficult to handle.

For the auxiliary mappings w_1, w_2 defined in (2.15), (2.16), we easily obtain that

$$\begin{aligned} \bar w_1'' &= \eta\cos\theta\,, \quad \bar w_2'' = \eta\sin\theta\,, \\ \bar w_i(0) &= \bar w_i(1) = 0\,, \quad i = 1,2\,. \end{aligned} \tag{3.49}$$

Relations (3.47)–(3.49) also show the continuous dependence in $L^2(0,1)$ of $\bar g_i$, $\bar w_i$, $i = 1,2$, and $\bar l$, $\bar h$ with respect to regularizations of η and θ, if the same is assumed for f_i, $\bar f_i$, $i = 1,2$. For $\bar w_i$, $i = 1,2$, and $\bar g_2$, this is valid even in $H^2(0,1)$. An elementary calculus, starting from (2.20), shows that the same continuity property remains valid for $\bar\lambda_1$, $\bar\lambda_2$.

From relation (3.3), we obtain that

$$\begin{bmatrix} \bar p \\ \bar q \end{bmatrix}(t) = \begin{pmatrix} 0 & \eta(t) \\ -\eta(t) & 0 \end{pmatrix} W_\theta(t) \begin{bmatrix} \lambda_1(\theta) \\ \lambda_2(\theta) \end{bmatrix} + W_\theta(t) \begin{bmatrix} \bar\lambda_1 \\ \bar\lambda_2 \end{bmatrix}, \tag{3.50}$$

with the same continuity property in $(L^2(0,1))^2$ with respect to regularizations of η and θ. By (3.4), (3.5), this property is preserved by $\bar{u}$, $\bar{z}$, and we have

$$\bar{u} = \varepsilon\,\bar{p}, \quad \bar{z}'' = -\bar{q}, \quad \bar{z}(0) = \bar{z}(1) = 0. \tag{3.51}$$

Finally, equation (3.1) gives

$$\begin{aligned}\begin{bmatrix}\bar{v}_1\\ \bar{v}_2\end{bmatrix}(t) &= \int_0^t W_\theta(t)\,W_\theta^{-1}(s)\begin{bmatrix}\bar{u}(s) + \bar{g}_1(s)\\ \bar{z}(s) + \bar{g}_2(s)\end{bmatrix} ds + \begin{pmatrix}0 & \eta(t)\\ -\eta(t) & 0\end{pmatrix}\begin{bmatrix}v_1(\theta)\\ v_2(\theta)\end{bmatrix}(t)\\ &- \int_0^t \begin{pmatrix}0 & \eta(s)\\ -\eta(s) & 0\end{pmatrix} W_\theta(t)\,W_\theta^{-1}(s)\begin{bmatrix}u(\theta) + g_1(\theta)\\ z(\theta) + g_2(\theta)\end{bmatrix}(s)\,ds\end{aligned} \tag{3.52}$$

with the same conclusion on the continuous dependence on η, θ. Let us now introduce explicitly the regularizations of θ and η,

$$\theta_\delta(t) = \int_R \theta(t - \delta\,y)\,\rho(y)\,dy, \quad \eta_\delta\,(t) = \int_R \eta(t - \delta\,y)\,\rho(y)\,dy, \tag{3.53}$$

where θ and y are extended by 0 outside the interval $[0,1]$, $\delta > 0$, and where $\rho \in C_0^\infty(I\!R)$ is a Friedrichs mollifier. We also denote $d_\delta = \eta'_\delta$, $c_\delta = \theta'_\delta$ which exist in $L^1(0,1)$, but have no good convergence properties for $\delta \to 0$. Then, the systems (3.7)–(3.16), (3.28)–(3.37) and (3.38)–(3.44) can be solved for the data c_δ, d_δ. Let us denote the corresponding solutions with an index or an exponent δ. Then we can prove the following result.

Theorem 3.6 *The gradient of the cost functional of the problem* (Q) *at the point* $\theta \in L^\infty(0,1)$ *and in the direction* $\eta \in L^\infty(0,1)$ *is given by*

$$\begin{aligned}\int_0^1 v_2(\theta)\,\bar{v}_2\,ds &= \int_0^1 \eta\Big[g_1'(\theta)^* P_1 + g_2'(\theta)^* P_2 - v_1(\theta)\,v_2(\theta)\\ &- P_1(\theta)\,(z(\theta) + g_2(\theta)) + P_2(\theta)\,(u(\theta) + g_1(\theta)) + q(\theta)\,R(\theta) - p(\theta)\,Q(\theta)\Big]\,ds.\end{aligned} \tag{3.54}$$

Here, $v_1(\theta)$, $v_2(\theta)$, $u(\theta)$, $z(\theta)$, $p(\theta)$, $q(\theta)$ are obtained by (3.1)–(3.5) with $g_1(\theta)$, $g_2(\theta)$ given by (2.2), (2.3), and P_1, P_2, P_3, P_4, R, Q are computed via (3.38)–(3.44) rewritten in integral form (which is obvious).

Proof. By (3.52), (3.53), we can write:

$$\int_0^1 v_2(\theta)\,\bar{v}_2\,ds = \lim_{\delta\to 0}\int_0^1 v_2^\delta\,\bar{v}_2^\delta\,ds. \tag{3.55}$$

From **Theorem 3.5**, we obtain that

$$\int_0^1 v_2^\delta\,\bar{v}_2^\delta\,ds = \int_0^1 d_\delta\left(P_1^\delta\,v_2^\delta - P_2^\delta\,v_1^\delta - P_3^\delta\,q^\delta + P_4^\delta\,p^\delta + P_1^\delta\,\bar{g}_1^\delta + P_2^\delta\,\bar{g}_2^\delta\right)\,ds.$$

Using the boundary conditions and the differentiability properties, we first compute

$$
\begin{aligned}
& \int_0^1 d_\delta \left(P_1^\delta \, v_2^\delta \; - \; P_2^\delta \, v_1^\delta \; - \; P_3^\delta \, q^\delta \; + \; P_4^\delta \, p^\delta \right) \, ds \\
= & \; - \int_0^1 \int_0^t d_\delta \left((P_1^\delta)' \, v_2^\delta \; + \; P_1^\delta \, (v_2^\delta)' \; + \; \ldots \; + \; (P_4^\delta)' \, p^\delta \; + \; P_4^\delta \, (p^\delta)' \right) \, ds \\
= & \; - \int_0^1 \eta_\delta \left(v_1^\delta \, v_2^\delta \; + \; P_1^\delta \, (z^\delta \; + \; g_2^\delta) \; - P_2^\delta (u^\delta \; + \; g_1^\delta) \; - \; q^\delta \, R^\delta \; + \; p^\delta \, Q^\delta \right) \, ds \, . \quad (3.56)
\end{aligned}
$$

We indicate only a partial calculation on how the last equality in (3.56) is established:

$$
\begin{aligned}
& (P_4^\delta)' \, p^\delta \; + \; P_4^\delta \, (p^\delta)' \; - \; (P_3^\delta)' \, q^\delta \; - \; P_3^\delta \, (q^\delta)' \\
= & \; (P_4^\delta)' \, p^\delta \; + \; P_4^\delta \, c_\delta \, q^\delta \; - \; (P_3^\delta)' \, q^\delta \; + \; P_3^\delta \, c_\delta \, p^\delta \\
= & \; q^\delta (- \, (P_3^\delta)' \; + \; c_\delta \, P_4^\delta) \; + \; p^\delta ((P_4^\delta)' \; + \; c_\delta \, P_3^\delta) \\
= & \; - \, q^\delta \, R^\delta \; + \; p^\delta \, Q^\delta \, ,
\end{aligned}
$$

by (3.11), (3.12), and (3.40), (3.41).

We also consider the term:

$$
\begin{aligned}
& \int_0^1 \left(P_1^\delta \, \bar{g}_1^\delta \; + \; P_2^\delta \, \bar{g}_2^\delta \right) \, ds \; = \; \int_0^1 \left(P_1^\delta \, g_1'(\theta_\delta) \, \eta_\delta \; + \; P_2^\delta \, g_2'(\theta_\delta) \eta_\delta \right) \, ds \\
= & \; \int_0^1 \eta_\delta \left[(g_1^\delta)'(\theta_\delta)^* P_1^\delta \; + \; (g_2^\delta)'(\theta_\delta)^* P_2^\delta \right] \, ds \, . \qquad (3.57)
\end{aligned}
$$

The derivatives of g_1, g_2 may be taken directly with respect to θ. This can be clearly seen from (3.23)–(3.25), where f_i may depend on θ, without modifying the argument.

We combine (3.55)–(3.57), and we pass to the limit as $\delta \to 0$. The continuity properties with respect to both η_δ and θ_δ have been explained in (3.47)–(3.52). We remark that the continuous dependence on $\delta \to 0$ is valid for $P_1^\delta, P_2^\delta, P_3^\delta, P_4^\delta, R^\delta, Q^\delta$ since the system (3.38)–(3.44) can be put into integral (mild) form as well. $\square$

Remark 3.6 The gradient provided by **Theorem 3.6** will be used in **Section 5** in the computation of numerical examples of shape optimization. It is also possible to write the first order optimality conditions for the problem (Q) by imposing (3.54) to be positive in the admissible directions of variation.

4 The nonhomogeneous clamped plate

In this section, we study the nonhomogeneous fourth-order boundary value problem

$$\Delta(u^3\,\Delta y) = f \quad \text{in } \Omega\,, \tag{4.1}$$

$$y = z \quad \text{on } \partial\Omega\,, \tag{4.2}$$

$$\frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} \quad \text{on } \partial\Omega\,, \tag{4.3}$$

where Ω is a smooth domain in $\mathbb{R}^N, u \in L^\infty(\Omega)$, $0 < \alpha \leq u(x) \leq \beta$ a.e. in Ω, $f \in L^2(\Omega)$, and z (giving the boundary conditions) is in $H^2(\Omega)$. In dimension two, (4.1)–(4.3) is a simplified model of a clamped plate with variable thickness u and with load f, Bendsoe [4].

We show that the method used in **Section 2** can also be applied in this case. We also study optimization problems associated to (4.1)–(4.3) as in **Section 3**. In this way, one can see that our methods have a large range of applications.

We formulate the optimal control problem:

$$\operatorname*{Min}_{h\in L^2(\Omega)} \left\{ \frac{1}{2}\int_\Omega l\,h^2\,dx \right\}, \tag{4.4}$$

subject to

$$\Delta y = l\,h + l\,g \quad \text{in } \Omega\,, \tag{4.5}$$

$$y = z \qquad \text{on } \partial\Omega\,, \tag{4.6}$$

and to the state constraints

$$\frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} \quad \text{on } \partial\Omega\,. \tag{4.7}$$

Here $g \in H^2(\Omega) \cap H_0^1(\Omega)$ is the solution of

$$\Delta g = f \quad \text{in } \Omega\,, \tag{4.8}$$

$$g = 0 \quad \text{on } \partial\Omega\,, \tag{4.9}$$

and $l \in L^\infty(\Omega)$ is given by

$$l = u^{-3} \in [\beta^{-3}, \alpha^{-3}] \quad \text{a.e. in } \Omega\,. \tag{4.10}$$

It is clear that $y = z$, $h = u^3(\Delta z - lg)$ is an admissible pair for the problem (4.4)–(4.7). By the coercivity of the cost, due to (4.10), and by its strict convexity, we obtain the existence of a unique optimal pair $[y^*, h^*] \in H^2(\Omega) \times L^2(\Omega)$.

It is obvious that $y - z \in H_0^2(\Omega)$ for any admissible state y, and thus

$$\left\{h \in l^{-1}\Delta\left[H_0^2(\Omega)\right] + l^{-1}\Delta z - g\right\} \subset L^2(\Omega) \tag{4.11}$$

provides a complete description of the admissible control set.

By (4.11), the control problem (4.4)–(4.7) can be reformulated as a mathematical programming problem, namely

$$\underset{s \in l^{-1}\Delta[H_0^2(\Omega)]}{\text{Min}} \left\{\frac{1}{2}\int_\Omega l(s + l^{-1}\Delta z - g)^2\, dx\right\}. \tag{4.12}$$

We introduce the new unknown $\xi = l^{\frac{1}{2}} s \in l^{-\frac{1}{2}}\Delta[H_0^2\Omega)]$. Then a simple transformation yields the solution of (4.12),

$$s^* = l^{-\frac{1}{2}}\xi^*, \quad \xi^* = \text{proj}_{l^{-\frac{1}{2}}\Delta[H_0^2(\Omega)]}\left(l^{\frac{1}{2}} g - l^{-\frac{1}{2}}\Delta z\right), \tag{4.13}$$

where the projection onto the closed linear subspace $Z = l^{-\frac{1}{2}}\Delta[H_0^2(\Omega)] \subset L^2(\Omega)$ is to be computed in the $L^2(\Omega)$-norm.

From (4.13) it follows that

$$h^* = s^* + l^{-1}\Delta z - g = l^{-\frac{1}{2}}\text{proj}_{Z^\perp}\left\{l^{-\frac{1}{2}}\Delta z - l^{\frac{1}{2}} g\right\}, \tag{4.14}$$

with the orthogonal complement of Z defined with respect to the inner product in $L^2(\Omega)$.

Theorem 4.1 *h^* is the optimal state of the unconstrained boundary control problem*

$$\underset{\tau \in H^{-\frac{1}{2}}(\partial\Omega)}{\text{Min}} \left\{\frac{1}{2}\int_\Omega l(h + g - l^{-1}\Delta z)^2\, dx\right\}, \tag{4.15}$$

$$\Delta h = 0 \quad \textit{in } \Omega, \tag{4.16}$$

$$h = \tau \quad \textit{on } \partial\Omega. \tag{4.17}$$

Proof. It is a simple exercise to verify that

$$Z^\perp = l^{\frac{1}{2}}\left\{\Delta[H_0^2(\Omega)]\right\}^\perp,$$

$$\left\{\Delta H_0^2(\Omega)\right\}^\perp = \left\{w \in L^2(\Omega);\, \Delta w = 0 \text{ in } \mathcal{D}'(\Omega)\right\}. \tag{4.18}$$

The form (4.14) of h^* shows that h^* solves

$$\underset{h \in [\Delta H_0^2]^\perp}{\text{Min}} \left\{\frac{1}{2}\int_\Omega l(h + g - l^{-1}\Delta z)^2\, dx\right\}. \tag{4.19}$$

Relations (4.18), (4.19) may be reformulated as (4.15)–(4.17), and the proof is finished. □

Remark 4.1 The problem (4.15)–(4.17) may be interpreted as the dual of the problem (4.4)–(4.7). Its advantage is that it has no constraints. The solution of (4.16), (4.17) should be understood in the transposition sense. Relations (4.14) and (4.5) provide an explicit reduction of (4.1)–(4.3) to second order elliptic equations.

Theorem 4.2 *The problem* (4.15)–(4.17) *or, equivalently, the problem* (4.4)-(4.7), *solve the equation* (4.1)–(4.3).

Proof. We know already the equivalence of the problems (4.15)–(4.17) and (4.4)–(4.7) in the sense of **Theorem 4.1**. Take any $k \in L^2(\Omega)$ satisfying (4.18) and consider variations of the form $h^* + \lambda k$, $\lambda \in I\!R$. Then

$$\int_\Omega l(h^* + g - l^{-1}\Delta z)k\,dx = 0\,, \quad \forall\ k \in [\Delta H_0^2(\Omega)]^\perp . \tag{4.20}$$

We define the adjoint system for the problem (4.15)–(4.17) by

$$\Delta p^* = l(h^* + g - l^{-1}\Delta z) \quad \text{in } \Omega\,, \tag{4.21}$$

$$p^* = 0 \quad \text{on } \partial\Omega\,. \tag{4.22}$$

Clearly $p^* \in H^2(\Omega) \cap H_0^1(\Omega)$, and

$$0 = \int_\Omega k\,\Delta p^*\,dx = \int_{\partial\Omega} v\,\frac{\partial p^*}{\partial n}\,, \tag{4.23}$$

by the definition of the transposition solution, and with $v \in H^{-\frac{1}{2}}(\partial\Omega)$ being the "trace" of k on $\partial\Omega$, in the sense of Lions [17, § 4.2]. Then, (4.23) gives (as v is arbitrary in $H^{-\frac{1}{2}}(\partial\Omega)$):

$$\frac{\partial p^*}{\partial n} = 0 \quad \text{on } \partial\Omega\,. \tag{4.24}$$

A simple calculus based on the definition of l, g, z shows that $p^* + z$ is exactly the solution of (4.1)–(4.3) and the proof is finished. □

Remark 4.2 Related arguments, using also the penalization of (4.7) (or, equivalently, of (4.3)) were employed by Sprekels and Tiba [19], [21] and by Arnăutu, Langmach, Sprekels and Tiba [2]. Comparing with **Section 2**, we see that the constraint (4.7) in the problem (4.4)–(4.6) is affine, but no longer finite dimensional. Therefore, the dual problem (4.15)–(4.17) cannot be solved explicitly, since it remains infinite dimensional.

We shall now discuss shape optimization problems associated to (4.1)–(4.3). As a first step, we analyze the continuity and the differentiability of the mapping $l \to y$, defined by (4.4)–(4.7). Notice that, although (4.10) gives a very simple relation between u and l, no continuity properties are valid in the weak* topology of $L^\infty(\Omega)$, for instance. The reformulations (4.4)–(4.7) or (4.15)–(4.17) have the advantage to introduce l as the main unknown and to remove the inconvenience caused by (4.10). For shape optimization problems it is enough to analyze the behaviour with respect to $l \in L^\infty(\Omega)$ and to transpose just the result into the language of $u \in L^\infty(\Omega)$, the thickness of the plate.

Theorem 4.3 *If $l_n \to l$ weakly* in $L^\infty(\Omega)$, then the solutions of* (4.4)–(4.7) *associated to l_n, l, satisfy $y_n = y(l_n) \to y = y(l)$ weakly in $H^2(\Omega)$.*

Proof. Denote by $h_n = h(l_n)$, $p_n = p(l_n)$ the other unknown mappings appearing in (4.4)–(4.7) and in (4.21), (4.22). By (4.10) and (4.4), we see that $\{h_n\}$ is bounded in $L^2(\Omega)$, and (4.5), (4.6) give that $\{y_n\}$ is bounded in $H^2(\Omega)$. By (4.16), we have $\Delta h_n = 0$ in $\mathcal{D}'(\Omega)$. Then, if $h_n \to h$ weakly in $L^2(\Omega)$ on a subsequence, it follows that $h_n(x) \to h(x)$ for any $x \in \Omega$ due to the mean value property of harmonic functions. The Vitali theorem shows that $h_n \to h$ strongly in $L^s(\Omega)$, for any $s \in [1, 2[$. Then, clearly $l_n h_n \to l\, h$ weakly in $L^2(\Omega)$, the identification of the limit being possible due to the strong convergence of h_n established above.

Finally, it is possible to pass to the limit in (4.5)–(4.7) and in (4.21)–(4.24). As we noticed before, this is the optimality system for the control problem (4.4), and its unique solution $[y, h, p]$, associated to l after passing to the limit, provides the optimal pair $[y, h]$ and the adjoint state p for the control problem (4.4) defined by l. □

Remark 4.3 A variant of **Theorem 4.3** was proved by Sprekels and Tiba [21].

Remark 4.4 By (4.5)–(4.7) and (4.16), one immediately recovers (4.1)–(4.3) and conversely. We shall study the differentiability properties of the mapping $l \to y(l)$ in this system formulation.

Theorem 4.4 *The mappings $l \to y$ and $l \to h$ are Gâteaux differentiable from $L^\infty(\Omega)$ to $H^2(\Omega)$ and $L^2(\Omega)$, respectively, and the directional derivatives at l in the direction v satisfy*

$$\Delta \bar{y} = l\,\bar{h} + v(h + g) \quad \text{in } \Omega, \tag{4.25}$$

$$\bar{y} = \frac{\partial \bar{y}}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{4.26}$$

$$\Delta \bar{h} = 0 \qquad \text{in } \Omega. \tag{4.27}$$

The solution $[\bar{y}, \bar{h}]$ is unique in $H_0^2(\Omega) \times L^2(\Omega)$.

Proof. Let $v \in L^\infty(\Omega)$, $\lambda \in I\!R$ "small", and $l \in L^\infty(\Omega)$ satisfy (4.10). Then, we may assume that

$$l + \lambda\, v \geq \frac{1}{2}\beta^{-3} > 0 \quad \text{a.e. in } \Omega.$$

Denote by y_λ, h_λ the mappings associated to $l + \lambda v$ by the system (4.5)–(4.7), (4.16). We can write:

$$\Delta \frac{y_\lambda - y}{\lambda} = (l + \lambda v)\frac{h_\lambda - h}{\lambda} + v(h + g) \quad \text{a.e. in } \Omega,$$

$$\frac{y_\lambda - y}{\lambda} = 0, \quad \frac{\partial}{\partial n}\left(\frac{y_\lambda - y}{\lambda}\right) = 0 \quad \text{on } \partial\Omega,$$

$$\Delta \frac{h_\lambda - h}{\lambda} = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Relation (4.18) yields that $\lambda^{-1}(h_\lambda - h) \in [\Delta H_0^2(\Omega)]^\perp$, while, from above, we get $\lambda^{-1}(y_\lambda - y) \in H_0^2(\Omega)$. This shows that

$$\int_\Omega \Delta \frac{y_\lambda - y}{\lambda} \frac{h_\lambda - h}{\lambda} \, dx = 0.$$

Multiplying by $\frac{h_\lambda - h}{\lambda}$ in the equation of $\frac{y_\lambda - y}{\lambda}$, we infer that $\left\{\frac{h_\lambda - h}{\lambda}\right\}$ is bounded in $L^2(\Omega)$:

$$\left|\frac{h_\lambda - h}{\lambda}\right|_{L^2(\Omega)} \le \frac{2}{\beta^3} |v|_{L^\infty(\Omega)} |h + g|_{L^2(\Omega)}.$$

The equation for $\frac{y_\lambda - y}{\lambda}$ shows, consequently, that $\left\{\frac{y_\lambda - y}{\lambda}\right\}$ is bounded in $H_0^2(\Omega)$. The passage to the limit is obvious, and the proof of (4.25)–(4.27) is finished. If the homogeneous variant of (4.25)–(4.27) is considered, i.e. with $v(h+g)$ replaced by zero, it may be rewritten as

$$\Delta(u^3 \Delta \bar{y}) = 0 \quad \text{in } \Omega$$

and has the unique null solution by (4.26). This proves the uniqueness of the solution $\bar{y}$, $\bar{h}$ as well. □

Remark 4.5 Equations (4.25), (4.27) may be, formally, rewritten as a fourth order equation

$$\Delta(u^3 \Delta \bar{y}) = \Delta(u^3 v(h + g)) \quad \text{in } \Omega$$

with boundary conditions (4.26). It should be noted that the right-hand side is nonsmooth due to $u \in L^\infty(\Omega)$, $v \in L^\infty(\Omega)$.

Next we study some optimal shape design problems in which the minimization parameter is the thickness $u \in L^\infty(\Omega)$, or equivalently, $l \in L^\infty(\Omega)$:

$$\textbf{(R)} \qquad \operatorname*{Min}_{l \in \mathcal{K}} \left\{ \frac{1}{2} \int_\Omega y^2(x)\, dx \right\},$$

$$\mathcal{K} = \left\{ l \in L^\infty(\Omega)\,;\, 0 < \beta^{-3} \le l(x) \le \alpha^{-3} \quad \text{a.e. in } \Omega \right\}, \tag{4.28}$$

subject to (4.5)–(4.7), (4.16).

The existence of at least one optimal pair $[y^*, l^*]$ for problem **(R)** is a direct consequence of **Theorem 4.3**.

Theorem 4.5 *If $f \neq 0$ a.e. in Ω, then $u^*(x) \in \{\alpha, \beta\}$ a.e. in Ω, where $u^*(x) = l^*(x)^{-\frac{1}{3}}$ is the optimal thickness for the problem* **(R)**.

Proof. We introduce the adjoint system for $p \in L^2(\Omega)$, $q \in H_0^2(\Omega)$, namely

$$\Delta p = y^* \quad \text{in } \Omega,$$

$$\Delta q = l^* p \quad \text{in } \Omega,$$

$$q = \frac{\partial q}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Taking admissible variations v for l^* and using **Theorem 4.4**, we find that

$$0 \le \int_\Omega y^* \bar{y}\, dx = \int_\Omega \Delta p\, \bar{y}\, dx = \int_\Omega p\, \Delta \bar{y}\, dx = \int_\Omega p\, l^*\, \bar{h}\, dx$$

$$+ \int_\Omega p\, v (h^* + g)\, dx = \int_\Omega \Delta q\, \bar{h}\, dx + \int_\Omega p\, v(h^* + g)\, dx = \int_\Omega p\, v(h^* + g)\, dx,$$

as $\bar{h}$ is orthogonal to $\Delta[H_0^2(\Omega)]$ in $L^2(\Omega)$.

The Pontryagin maximum principle for the problem **(R)** is therefore

$$0 \le \int_\Omega p(w - l^*)\,(h^* + g)\, dx, \quad \forall\, w \in \mathcal{K},$$

or, equivalently,

$$-p(h^* + g) \in \partial I_{\mathcal{K}}(l^*) \quad \text{a.e. in } \Omega,$$

with $\partial I_{\mathcal{K}}$ denoting the subdifferential of the indicator function $I_{\mathcal{K}}$ of $\mathcal{K}$ in $L^\infty(\Omega)$.

As $f \neq 0$ a.e. in Ω and $\Delta(h^* + g) = f$, we have $h^* + g \neq 0$ a.e. in Ω by the interior regularity properties of h^* and the maximal regularity of g (see Brezis [7, p. 195]). Then, (4.5) implies that $\Delta y^* \neq 0$ a.e. in Ω, i.e. $y^* \neq 0$ a.e. Ω. Similarly, $\Delta p \neq 0$ a.e. in Ω and $p \neq 0$ a.e. in Ω, i.e. $p(h^* + g) \neq 0$ a.e. in Ω. The Pontryagin maximum principle and (4.28) give that $l^*(x) \in \{\beta^{-3}, \alpha^{-3}\}$ a.e. in Ω ($\partial I_{\mathcal{K}}$ is different from zero only in the endpoints of the constraints interval), and the proof is finished. $\square$

Remark 4.6 This is a bang-bang result for the problem **(R)**. In the sequel, we shall discuss a more realistic example involving pointwise state constraints, and we shall establish a partial result of the same type.

The problem is the minimization of the volume of the plate such that the deflection in a prescribed point $\tilde{x} \in \Omega$, under the given load f, remains below a given limit. We assume that $\Omega \subset I\!R^3$, so that, consequently, $H^2(\Omega) \subset C(\bar{\Omega})$. We consider the problem

$$\textbf{(S)} \qquad \text{Min} \left\{ \int_\Omega l^{-\frac{1}{3}}(x)\, dx \;=\; \int_\Omega u(x)\, dx \right\}$$

subject to (4.5)–(4.7), (4.16), (4.22) and

$$y(\tilde{x}) \;\geq\; -\,\delta \tag{4.29}$$

with $\delta > 0$ fixed. **(S)** has at least one solution $[y^*, l^*] \in H^2(\Omega) \times \mathcal{K}$ under admissibility hypotheses.

Remark 4.7 If the optimal state satisfies

$$y^*(\tilde{x}) \,+\, \delta \,>\, 0$$

(inactive constraint), then for any $w \in \mathcal{K}$ and for $\lambda \in I\!R$ small enough, $l^*+\lambda(w-l^*)$ is admissible for **(S)** since, by **Theorem 4.3** and embedding properties, we get the corresponding state $y_\lambda(\tilde{x}) > -\delta$.

Introducing these variations into the cost, one obtains easily that

$$0 \,\geq\, \int_\Omega l^{*^{-\frac{4}{3}}}(w \,-\, l^*)\, dx\,, \quad \forall\; w \,\in\, \mathcal{K}\,,$$

that is, $l^* = \alpha^{-3}$ and $u^* = \alpha$ a.e. in Ω, and the solution is explicit.

In the sequel, we shall assume that the state constraint is active: $y^*(\tilde{x}) + \delta = 0$.

We define the adjoint system corresponding to the problem **(S)**:

$$\Delta p \,=\, \delta_{\tilde{x}} \quad \text{in } \Omega\,, \tag{4.30}$$

$$\Delta q \,=\, p\, l^* \quad \text{in } \Omega\,, \tag{4.31}$$

$$q \,=\, 0\,, \quad \frac{\partial q}{\partial n} \,=\, 0 \quad \text{on } \partial\Omega\,, \tag{4.32}$$

where $\delta_{\tilde{x}}$ denotes the Dirac distribution concentrated at $\tilde{x} \in \Omega$. Since $H_0^2(\Omega) \subset C(\bar{\Omega})$ by $\Omega \subset I\!R^3$, we have $\delta_{\tilde{x}} \in H^{-2}(\Omega)$, and the existence of a unique weak solution $p \in L^2(\Omega)$, $q \in H_0^2(\Omega)$ to (4.30)–(4.32) follows by classical variational arguments.

Theorem 4.6 *If $w \in L^\infty(\Omega)$ is an admissible variation of* **(S)** *around $l^* \in \mathcal{K}$, then*

$$\int_\Omega p(w \,-\, l^*)\,(h^* \,+\, g)\, dx \,\geq\, 0\,. \tag{4.33}$$

Conversely, if w satisfies (4.33) with strict inequality sign, it defines an admissible variation around l^*. If $\Omega_1 = \{x \in \Omega\,;\, p(x)(h^*+g)(x) > 0 \text{ a.e. }\}$, then $l^*(x) = \alpha^{-3}$ and $u^*(x) = \alpha$ a.e. in Ω_1.

Proof. For admissible variations $l^* + \lambda(w - l^*)$, $w \in \mathcal{K}$, we have that the corresponding states y_λ satisfy $y_\lambda(\tilde{x}) + \delta \geq 0$. By **Theorem 4.4** and the embedding $H^2(\Omega) \subset C(\bar{\Omega})$, for $\Omega \in I\!R^3$, we get that

$$\bar{y}(\tilde{x}) \;=\; \lim_{\lambda \to 0} \frac{y_\lambda(\tilde{x}) \,-\, y^*(\tilde{x})}{\lambda} \;\geq\; 0\,.$$

We write this as follows:

$$0 \;\leq\; \bar{y}(\tilde{x}) \;=\; \int_\Omega \bar{y}(x)\,\delta_{\tilde{x}} \;=\; \int_\Omega \bar{y}\,\Delta p \;=\; \int_\Omega p\,\Delta\bar{y}\,dx \;=\; \int_\Omega p(w \,-\, l^*)\,(h^* \,+\, g)\,dx\,,$$

as in the proof of **Theorem 4.5**. This shows (4.33).

Conversely, the above calculus shows that this assumption implies that $\bar{y}(\tilde{x}) > 0$, that is, $y_\lambda(\tilde{x}) > -\delta$ for λ small, and the admissibility of the variation $l^* + \lambda(w - l^*)$ follows.

For any $w \in \mathcal{K}$ such that $l^* + \lambda(w - l^*)$ is admissible, a computation similar to **Remark 4.7** gives

$$0 \;\geq\; \int_\Omega {l^*}^{-\frac{4}{3}}(w \,-\, l^*)\,dx\,. \tag{4.34}$$

If $l^*(x) \neq \alpha^{-3}$ on a subset of positive measure of Ω_1, we choose $w(x) = \alpha^{-3}$ in this set and $w(x) = l^*(x)$ otherwise. The above observation and the converse of (4.33) show that such a w will generate an admissible variation around l^*. But this clearly contradicts (4.34). $\Box$

Remark 4.8 Property (4.33) and its converse are valid for any $l \in \mathcal{K}$ which is admissible for **(S)** with active state constraint. That is, **Theorem 4.6** reexpresses the state constraint (4.29) in the language of the admissible control variations, and this is valid in the difficult case of active constraints.

5 Numerical experiments

We have computed several examples using the methods developed in this paper. We have studied in detail the case of arches, including their shape optimization. Numerical examples concerning plates and beams have been reported in the works of Arnăutu, Langmach, Sprekels and Tiba [2], Sprekels and Tiba [23], where different (but related) approaches have been used.

In **Figures 1–4**, deformations of various arches (roman, gothic, closed) with different thicknesses $\varepsilon > 0$ and under certain square integrable loads $[f_1\,,\, f_2]$ are shown.

The algorithm is based on **Theorem 2.3** with explicit solutions of (2.20) obtained via MAPLE. The integrals appearing in the coefficients of (2.20) or else can be computed explicitly in the case of simple arches and simple forces (purely tangential or purely normal, etc.). Otherwise, standard numerical integration procedures on the real line should be applied.

The parametric representation of an arch associated to some function θ on a prescribed interval, is given by $[\varphi_1\,,\,\varphi_2]$ with $\varphi_1' = \cos\theta\,,\,\varphi_2' = \sin\theta\,,$ and with null initial conditions. Notice that in **Figure 1**, θ is discontinuous, and $\varphi = [\varphi_1\,,\,\varphi_2]$ is just Lipschitz, which shows the importance of relaxing the regularity assumptions in (1.1) as is done in the problem (P_ε) in **Section 2**. **Figures 2** and **4** show the same type of arch with similar loading. The difference in the shape of the obtained deformations is due to the fact that the first arch is clamped at both ends, while the closed arch is clamped only in the point $(0,0)$. **Figure 3** refers to the "flexural" model briefly explained in **Theorem 2.5** and **Remark 2.8**. The constant E is the Young modulus of the material, while $S = \varepsilon^{3/2}$ gives the influence of the thickness $\varepsilon > 0$. We indicate, as a short example, the explicit form of the deformation $[v_1\,,\,v_2]$ corresponding to the situation described in **Figure 2**:

$$
\begin{aligned}
v_1(t) \;=\; &(6\,\varepsilon\,\sin t\,+\,4\,\sin t\,+\,2\,\pi\,\varepsilon\,\sin t\,+\,\pi\,\varepsilon^2\,t\,\sin t\,-\,4\,\varepsilon\,t\,\cos t\,-\,2\,\varepsilon^2\,\sin t\\
&-2\,\varepsilon\,t^2\,\sin t\,-\,\varepsilon^2\,t^2\,\sin t\,+\,\pi\,t\,\sin t\,-\,4\,t\,\cos t\,-\,t^2\,\sin t\,-\,2\,\pi\,-\,2\,\varepsilon\,\pi\\
&+2\,\pi\,\cos t\,+\,2\,\pi\,\varepsilon\,\cos t)/4\,\varepsilon^{3/2}(\varepsilon\,+\,1)\,,\\
v_2(t) \;=\; &(\varepsilon\,+\,1)\,(2\,t\,\sin t\,+\,\pi\,t\,\cos t\,-\,\pi\,\sin t\,-\,t^2\,\cos t)/4\,\varepsilon^{3/2}\,.
\end{aligned}
$$

The **Figures 5–9** and the **Tables 1, 2** concern optimization procedures for arches, according to the theory developed in **Section 3**. For the computation of the gradient of the cost functional, as given in (3.54), it is necessary to obtain the numerical solution of the state system (3.1)–(3.5), of the adjoint system (3.38)–(3.44), and the approximation of the mappings $[g_1'(\theta)]^* P_1$ and $[g_2'(\theta)]^* P_2$. It obvious that by the nature of the data an explicit calculation is not possible in the optimization routine.

We have considered an equidistant division of the interval of definition, denoted here by $[0, L]$, into N_0 (a natural number) subintervals $[t_i, t_{i+1}]$, with $t_i = i\,h\,,\ h = \dfrac{L}{N_0}$. The mapping $\theta \in L^\infty(0, L)$ is approximated, in different examples, by piecewise linear splines or by piecewise constant functions. The integrals are computed accordingly by standard quadrature formulas, and the solution of the ordinary differential system is obtained via linear finite elements. The scalars $\lambda_1^\varepsilon\,,\,\lambda_2^\varepsilon$ from (3.3) are found from the algebraic system (2.20). Similarly, the unknown initial conditions $\mu_1\,,\,\mu_2$ for the equations (3.38), (3.39) satisfy a system of the same type as (2.20) with the mappings $l\,,\,g_2$ replaced by $\gamma_1\,,\,\gamma$ with $\gamma'' = -\gamma_2\,,\,\gamma(0) = \gamma(L) = 0$ (see **Proposition 3.4** and its proof). The functions $[g_1'(\theta)]^* P_1$ and $[g_2'(\theta)]^* P_2$ have been approximated in the following way:

$$[g_k'(\theta)]^* P_k(t_i) \simeq \frac{1}{h} \int_{t_i}^{t_{i+1}} [g_k'(\theta)]^* \, P_k(s)\, ds$$

$$= \frac{1}{h} \int_0^L P_k(s) \left(\bar{g}_k \, \chi_{[t_i,t_{i+1}]}\right)(s)\, ds\,, \quad k = 1,2\,, \quad i = \overline{0, N-1}\,,$$

$$[g_k'(\theta)]^* P_k(L) \simeq 0\,, \quad k = 1,2\,.$$

For the determination of $\bar{g}_k$ the relation (3.48) is used, and $\chi_{[t_i,t_{i+1}]}$ is the characteristic function of $[t_i, t_{i+1}]$.

Although the studied optimization problems are nonconvex, adaptations of Rosen's and Uzawa's gradient algorithms with projection, Gruver and Sachs [16], Arnăutu [1], have been used. A maximal number of iterations (between 200 and 300) have been prescribed, and the solution has been chosen as the one which gives the best value of the cost. The algorithm stops as well if the value of the gradient or of the cost is zero.

For a given example, several tests have been performed with various values of the parameters N_0 , α (the parameter from the Rosen algorithm) and with both algorithms. In general, the Rosen algorithm gives better results than the Uzawa algorithm. In the optimization problems, we have fixed $\varepsilon = 0.1$. A typical line search procedure is to subdivide the open-closed interval $]0,1]$ into N_1 equal parts and to give the line search parameter the values $\dfrac{i}{N_1}$, $i = \overline{1, N_1}$. The one which gives the best cost will generate the next iteration. We have avoided, with good numerical results, the usual computation of the line search parameter by a one-dimensional optimization problem, which may be very time-consuming. The used procedure combines in an ad-hoc manner the gradient algorithm principle with a global search. A projection on the admissible set has been performed in each iteration. The optimization problem (Q) looks for the shape of the arch which ensures the minimal normal deformation (in some integral sense) under the action of a prescribed force. We have examined purely tangential ($f_2 = 0$) or normal ($f_1 = 0$) forces (since they give the basis in the local system of axes), as well as forces not depending on the unknown arch. This last case is described in the local system of coordinates by $f_1(t) = \sin(\theta(t))/S$ and $f_2(t) = \cos(\theta(t))/S$ (for the force of modulus one and parallel to the vertical axis), and in converse order for forces parallel to the horizontal axis. It should be noticed that the force is independent of the arch, but its local representation is dependent via θ .

The constraints for θ were given by subintervals of $[0, \pi]$ as indicated in the figures. This suffices for many applications and avoids the self-intersection of arches. However, some degenerate case is still possible, according to **Figure 9**.

In **Figure 5**, under the action of a tangential force, and starting with the initial iteration given by the roman arch, it is seen that the global solution is the beam, which clearly has no normal deflection under such a load. In our representation, two global solutions (beams) are put into evidence, associated to $\theta = 0$ and to $\theta = \pi$. The figure shows some iterations produced by the algorithm and the corresponding values of the cost. In this experiment, we have used $N_0 = 200$, $n_1 = 10$, $\alpha = 0,75$, and the arch close to the beam was obtained in iteration $I = 24$. We underline that in this example, an infinity of global solutions (beams of any slope) exists, and this shows the difficulty of the numerical computations.

In **Figure 6**, the initial iteration is again the roman arch, but the force is of constant modulus one and parallel to the vertical axis. The iterations that are represented show how the routine finds again the (unique if θ is constrained in $[0,\pi]$) global solution which is given by a vertical beam characterized by $\theta = \frac{\pi}{2}$. In this configuration, the prescribed force becomes purely tangential to the arch, and the global solution is a special case of the previous example (but not the problem as a whole). We have used $N_0 = 200$, $N_1 = 10$, $\alpha = 1$, and the global optimum was obtained at iteration $I = 139$.

The numerical results from **Figures 5, 6** match perfectly with the physical interpretation. This gives a strong validation of the notion of weak solution that we are using and shows the stability of our methods.

In **Figures 8** and **9**, the case of a purely normal load is discussed, the difference being given by the constraints imposed on θ : $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$, respectively $[0,\pi]$. In **Figure 9**, the "optimal" found θ is represented, not the arch as usual. As the solution is bang-bang, $\theta \in \{0,\pi\}$ a.e. $t \in [0,\pi]$; then the arch degenerates and cannot be graphically represented. Suggested by the bang-bang structure of the obtained solution (the computations were made with $N_0 = 200$, $N_1 = 20$, $\alpha = 1,5$, $I = 27$), we have simply generated a sequence θ^N, by giving to the new parameter N the values listed in **Table 2** and to θ^N the values 0 and π, alternatively on subsequent subintervals. We have directly computed the costs $J(\theta^N)$ associated to such oscillating arches and listed them in **Table 2**. The conclusion is that the sequence θ^N is a very efficient minimizing sequence for this problem, ensuring for $N \geq 50$ lower values of the cost than the one computed by the complete numerical procedure (although this provides a performant result as well). We stress that the oscillatory nature of the minimizing sequence $\{\theta^N\}$ is related to the noncompactness of the constraint set $\{\theta \in L^\infty(\Omega)\,;\, \theta(t) \in [0,\pi]$ for a.e. $t \in (0,\pi)\}$ in $L^\infty(0,\pi)$. This set is only bounded and closed which is not enough to ensure the existence of the optimal θ as discussed in **Theorem 3.1** and **Corollary 3.2**. This numerical example can be interpreted as showing that the assumptions of **Corollary 3.2** are sharp. We also underline that such compactness comments apply to **Figures 5** and **6** as well, although global minimum points exist in these examples.

Figure 8 represents the initial (roman) arch and the obtained solution, in the same problem as in **Figure 9**, with the constraints given by the set $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ in order to avoid degeneracy. The numerical test used $N_0 = 300$, $N_1 = 10$, $\alpha = 1,5$, and the obtained optimum corresponded to the iteration $I = 160$. The bang-bang structure of the solution is again clear (recall that θ is the angle between the tangent to the arch and the horizontal axis). However, **Table 1** shows that the simple sequence $\{\theta^N\}$ constructed as in the previous example but with the values $\pi/6$, $5\pi/6$, is no more a minimizing sequence for this problem. The commuting points for the bang-bang solution are no more equidistant in this example. Finally, in **Figure 7**, a "realistic" example is studied: the construction of a most resistent roof subject to a vertical constant load of modulus one. The reader should pay attention that in this figure we have interchanged the axes to make the representation look more "physical". To perform a more precise calculation, we have fixed $N_0 = 500$, $N_1 = 100$, $\alpha = 10$. Two experiments are reported in **Figure 7**, one with the initial iteration given by a fragment of roman arch, and another with the initial iteration given by two coupled fragments of roman arch. In both cases, the numerical solutions were obtained in the first iteration, $I = 1$, and are very similar. In this example (as in **Figure 8**), the theoretical optimal value is "far" from zero, and the computed values are very good.

We close this presentation by underlining that working with low regularity assumptions was essential for the optimization applications in view of the bang-bang structure of the optimal θ , as found in many examples. However, in **Figure 6**, the global solution is not bang-bang and this property seems just to be related to the applied force. That is why we did not study bang-bang properties in **Section 3**, although such properties are known for plates, according to **Section 4**, or to Sprekels and Tiba [21]. We also underline the nonlocal optimization character of our numerical experiments as this is obvious from the reported results.

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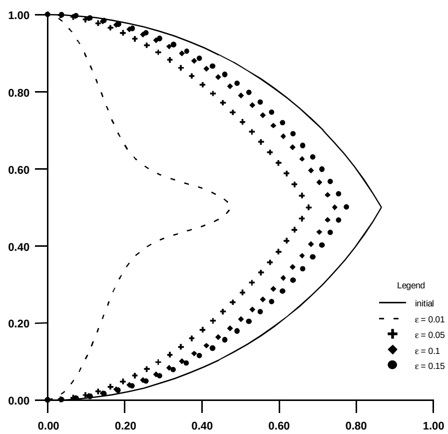

Figure 1 - $\theta(t) = t$, $t \in [0,\pi/3]$, $\theta(t) = t+\pi/3$, $t \in [\pi/3, 2\pi/3]$, $f_1(t) = 0$, $f_2(t) = 1/(S\,E)$, $E = 10$

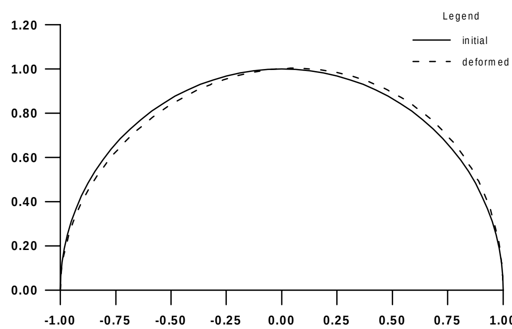

Figure 3 - $\theta(t) = t$, $f_1(t) = \sin(t)$, $f_2(t) = 2\cos(t)$, $t \in [0,\pi]$

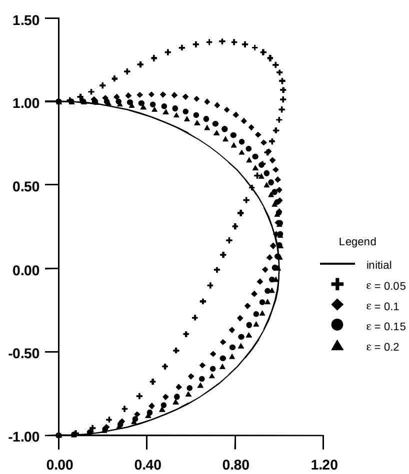

Figure 2 - $\theta(t) = t$, $t \in [0,\pi]$
$f_1(t) = \sin(t)/S$, $f_2(t) = \cos(t)/S$

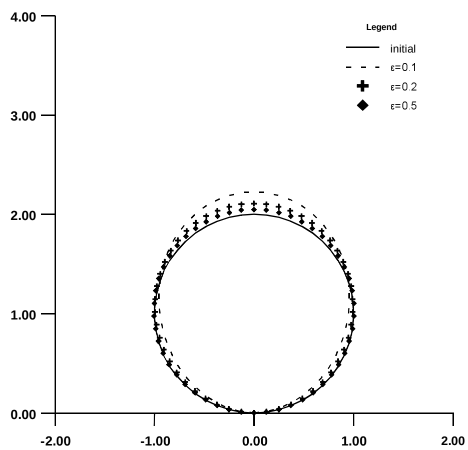

Figure 4 - $\theta(t) = t$, $f_1(t) = \sin(t)/(S\,E)$,
$f_2(t) = \cos(t)/(S\,E)$, $t \in [0,2\pi]$, $E=100$

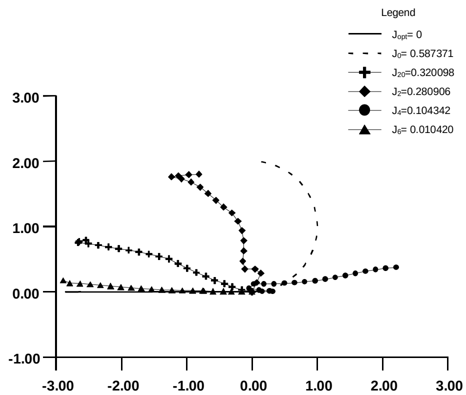

Figure 5 - $\theta(t) \in [0, \pi]$, $f_1(t) = 1/S$, $f_2(t) = 0$,
$\theta_0(t)= t$, $t \in [0, \pi]$

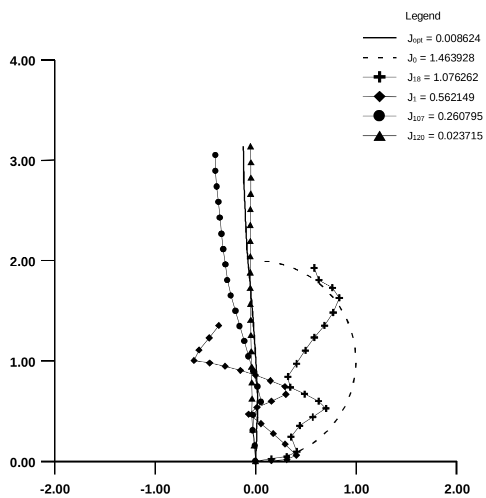

Figure 6 – $\theta(t)\in[0, \pi]$, $f_1(t)=\sin(\theta(t))/S$,
$f_2(t) = \cos(\theta(t))/S$, $\theta_0(t)= t$, $t \in [0, \pi]$

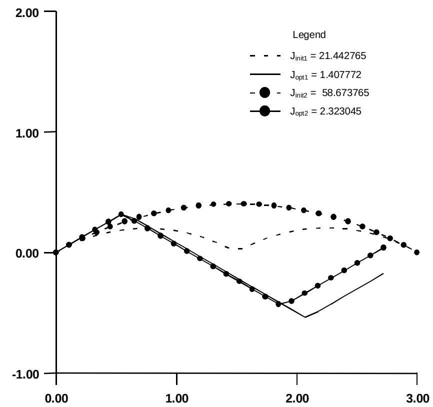

Figure 7 – $\theta(t)\in[\pi/3, 2\pi/3]$
$f_1(t) = \cos(\theta(t))/S$, $f_2(t) = \sin(\theta(t))/S$, $t\in [0, \pi]$,
$\theta_{01}(t)= (2t + \pi)/3$, $t\in [0, \pi/2)$, $\theta_{01}(t)= 2t/3$, $t\in [\pi/2, \pi]$,
$\theta_{02}(t)= (t + \pi)/3$, $t\in [0, \pi]$

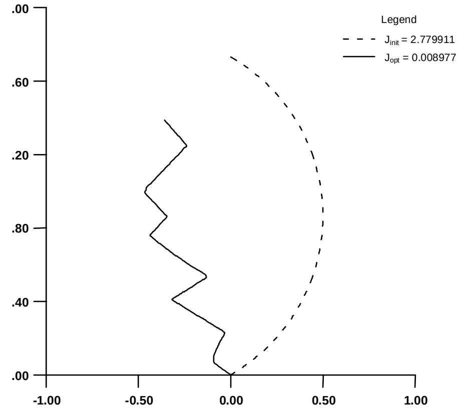

Figure 8 – $\theta(t) \in [\pi/6, 5\pi/6]$, $f_1(t) = 0$, $f_2(t) = 1/S$, $\theta_0(t) = t + \pi/3$, $t \in [0, 2\pi/3]$, $J_{init} = 2.779911$, $J_{opt} = 0.008977$

N	$J(\theta^N)$
30	0.0141367792
50	0.0247750769
100	0.0303698330
200	0.0318697376
300	0.0321519172
500	0.0322969269
800	0.0323467156
1000	0.0323582113

Table 1 - $\theta(t) \in \{\pi/6, 5\pi/6\}$, $f_1(t) = 0$, $f_2(t) = 1/S$, $t \in [0, 2\pi/3]$

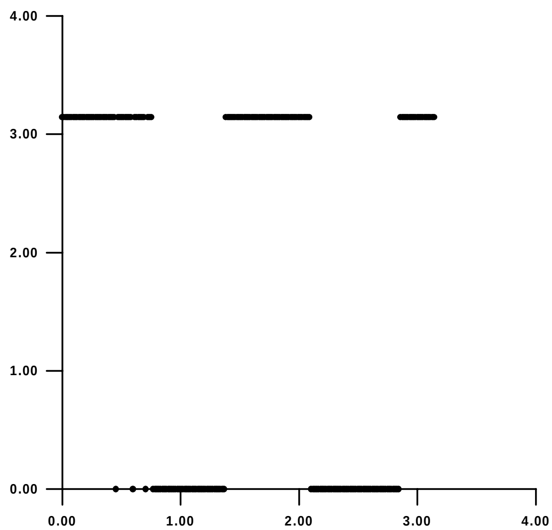

Figure 9 – $\theta(t) \in [0, \pi]$, $f_1(t) = 0$, $f_2(t) = 1/S$, $\theta_0(t) = t$, $t \in [0, \pi]$ $J_{init} = 82.922993$, $J_{opt} = 0.0024772$

N	$J(\theta^N)$
30	0.0095834975
50	0.0012420426
100	0.0000776279
200	0.0000048517
300	0.0000009584
500	0.0000001242
800	0.0000000190
1000	0.0000000078

Table 2 - $\theta(t) \in \{0, \pi\}$, $f_1(t) = 0$, $f_2(t) = 1/S$, $t \in [0, \pi]$