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Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Asymptotics of Solutions to Joukovskii–Kutta–Type Problems at Infinity

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submitted: 17 Mar 2000

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Preprint No. 567
Berlin 2000

1991 Mathematics Subject Classification. 76B05, 35J05, 35B40.

Key words and phrases. Airfoil theory, Joukovskii–Kutta condition, Laplace equation, Helmholtz equation, asymptotic farfield behavior.

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Abstract

We investigate the behavior at infinity of solutions to Joukovskii–Kutta–type problems, arising in the linearized lifting surface theory. In these problems one looks for the perturbation velocity potential induced by the presence of a wing in a basic flow within the scope of a linearized theory and for the wing circulation. We consider at first the pure two–dimensional case, then the three–dimensional case, and finally we show in the case of a time–harmonically oscillating wing in $I\!R^3$ in a weakly damping gas the exponential decay of solutions of the Joukovskii–Kutta problem.

1 Introduction

In this article we consider within the scope of a linearized theory problems for a perturbation velocity potential Φ which is generated by the presence of a wing of an aeroplane in a basic flow:

$$\text{Total potential} \;=\; \text{potential without wing} + \underbrace{\text{perturbation potential } \Phi}_{''small''}.$$

For describing these problems we denote by $x = (y, z) = (y_1, y_2, z) \in I\!R^3$ a point in the three–dimensional space $I\!R^3$. We assume, that an inviscid, barotropic, and compressible gas flows with a constant subsonic velocity into the positive y_1–direction and that the wing as an obstacle is thin and weakly cambered.

Let L denote the projection of the wing onto the y-plane and $\Pi := \{(y, z) : z = 0,\ -l < y_2 < l\}$ the strip which contains L and is of minimal width. We assume L to be of a trapezoidal shape, i.e.,

$$\overline{L} = \{(y, z) : z = 0,\ y_2 \in [-l, l]\,,\ -Y_-(y_2) \le y_1 \le Y_+(y_2) \text{ for } y_2 \in [-l, l]\}$$

where $-Y_-(y_2)$, $Y_+(y_2)$, $y_2 \in (-l, l)$ denote parametrizations of the leading edge $-Y_-$, respectively the trailing edge Y_+ of L, with smooth functions $Y_\pm$. Especially $Y(y_2) = Y_-(y_2) + Y_+(y_2)$ is assumed to be positive for all $y_2 \in [-l, l]$.

The set $\Pi\backslash\overline{L}$ is double–connected. The wake W is defined by $W := \{(y, z) \in \Pi : y_1 > Y_+(y_2), y_2 \in (-l, l)\} \subset \Pi\backslash\overline{L}$.

Reissner [15] formulated 1949 boundary conditions for the perturbation potential Φ which has to be looked for in $\Omega = I\!R^3 \backslash \overline{L \cup W}$. These conditions have been reformulated 1983 by Meister [11] in a mathematically more handable model in the two–dimensional situation. Especially Meister considered a time–harmonically oscillating wing with reduced wavenumber k under the assumption of a weakly damping gas, i.e., Re $k > 0$, Im $k > 0$. The resulting model has been extended 1988 by Hebeker [5] under Meister's assumption of a weakly damping gas to the three–dimensional situation and has been investigated in [6], [12], [13] in more detail.

The full problem is: Find the perturbation velocity potential Φ as the solution of the problem

$$\begin{array}{rcl} (\triangle_x + k^2)\Phi(x) & = & 0 \text{ for } x \in \Omega,\ \operatorname{Re} k > 0,\ \operatorname{Im} k > 0 \ , \\ \partial_z \Phi(y, \pm 0) & = & g_\pm(y) \text{ for } y \in L \ , \\ \left[\Phi\right](y) & = & \Gamma(y_2)\exp\left(ikM^{-1}(y_1 - Y_+(y_2))\right) \text{ for } y \in W \ , \\ \left[\partial_z \Phi\right](y) & = & 0 \text{ for } y \in W \ . \end{array} \tag{1.1}$$

Here $g_\pm$ denote prescribed functions defined on L which results from the requirement of vanishing total normal velocities on the wing and which we assume to be smooth, M is the Machnumber (here: $M < 1$), and Γ denotes the wing circulation as a function defined on the trailing edge Y_+ of L where we agree upon $[\Phi] = \Phi(y, -0) - \Phi(y, +0)$. In the full Joukovskii–Kutta problem, besides Φ also Γ has to be found by the requirement of vanishing intensity factors of Φ at the borderline Y_+ between L and W.

We assume here that $\Gamma \in H^{1/2}(Y_+)$ is prescribed. This assumption is consistent with the results [12], [13], which prove Γ to be a continuous function even in interior angular points of the trailing edge Y_+ of L and to possess singularities of the order $O(\sqrt{l - |y_2|})$ at the endpoints of Y_+.

We focus here in finding the asymptotic behavior of the solution Φ of (1.1) at infinity. We consider at first in section 2 the two–dimensional case with $k = 0$. Here, L is an interval $(-a_-, a_+)$ where we assume $0 \in (-a_-, a_+)$ and $W = (a_+, \infty)$. In this situation the problem (1.1) reduces to the following one:

$$\begin{array}{rcl} \triangle_{(y_1, z)}\Phi(y_1, z) & = & 0 \text{ for } (y_1, z) \in I\!R^2 \backslash [-a_-, \infty) \ , \\ \partial_z \Phi(y_1, \pm 0) & = & g_\pm(y_1) \text{ for } y_1 \in L \ , \\ \left[\Phi\right](y_1) = \Gamma \text{ for } y_1 \in W \ , & & \left[\partial_z \Phi\right](y_1) = 0 \text{ for } y_1 \in W \ , \end{array} \tag{1.2}$$

where Γ is a constant.

In section 3 we consider the three–dimensional case with $k = 0$, i.e., we look for the asymptotic behavior of solutions of the problem

$$
\begin{aligned}
\triangle_x \Phi(x) &= 0 \text{ for } x \in \Omega \,, \\
\partial_z \Phi(y, \pm 0) &= g_\pm(y) \text{ for } y \in L \,, \\
\left[\Phi\right](y) = \Gamma(y_2) \text{ for } y \in W \,, \qquad & \left[\partial_z \Phi\right](y) = 0 \text{ for } y \in W \,.
\end{aligned} \tag{1.3}
$$

Finally we prove in section 4 the exponential decay for solutions Φ of the problem (1.1) in the case of a time–harmonically oscillating wing with $\mathrm{Re}\, k > 0$, $\mathrm{Im}\, k > 0$, i.e., for a weakly damping gas.

2 The two–dimensional case with a vanishing wave-number

In this section we present the asymptotic formula for solutions Φ of (1.2) at infinity.

By virtue of $(1.2)_3$, a solution Φ of (1.2) cannot decay at infinity for $\Gamma \neq 0$. The case $\Gamma = 0$ can be neglected by physical reasons. Hence, (1.2) possesses no solution in the Sobolev space $W_2^1(I\!R^2 \backslash [-a_-, \infty))$.

We now want to reduce (1.2) to a problem with a compactly supported right–hand side. To do this, we use polar coordinates (r, φ), centered in the point $(0,0)$ with $r = \sqrt{y_1^2 + z^2}$, $\varphi = 0$ for $z = 0$, $y_1 > 0$ and a cut–off function $\chi \in C_0^\infty(I\!R)$ with $\chi(r) = 1$ for r sufficiently small and $\chi(r) = 0$ for $r > \min\{a_-, a_+\}$.

THEOREM 2.1 *There exists a solution Φ of (1.2) which possesses for $r \longrightarrow \infty$ the representation*

$$
\Phi(y_1, z) = (1 - \chi(r)) \left(\frac{1}{2\pi} \int_{-a_-}^{a_+} (g_+(y_1) - g_-(y_1)) dy_1 \cdot \ln \frac{1}{r} + c + \frac{\Gamma}{2\pi} \varphi \right) + O(r^{-1}) \tag{2.1}
$$

where c denotes an arbitrary constant. Any solution fulfilling the relation $\Phi(y_1, z) = o(1 + r)$ takes the form (2.1).

<u>Proof.</u> Let us perform in (1.2) the substitution

$$
\Phi(y_1, z) = (1 - \chi(r)) \left(\frac{\mathbf{a}}{2\pi} \ln \frac{1}{r} + \mathbf{c} + \frac{\Gamma}{2\pi} \varphi \right) + \Phi^0(y_1, z) \tag{2.2}
$$

with an arbitrary constant $\mathbf{c}$ and an unknown constant $\mathbf{a}$ which has to be fixed. Φ^0 is also unknown. Note that the first term in the second factor in (2.2) covers all harmonic functions which grow at infinity not faster than $o(1 + r)$, the second term is constant, and the third term takes care of the jump of Φ on the wake (a_+, ∞).

Defining

$$
[\triangle, \chi](\psi) = \triangle(\chi \cdot \psi) - \chi \cdot \triangle \psi \,, \tag{2.3}
$$

where confusions with the jumps of functions are excluded, and the compactly supported function f by

$$f(y_1, z) = [\triangle_x, \chi(r)] \left(\frac{\mathbf{a}}{2\pi} \ln \frac{1}{r} + \mathbf{c} + \frac{\Gamma}{2\pi}\varphi \right) \tag{2.4}$$

we arrive at the following problem for Φ^0:

$$\begin{array}{c} \triangle_{(y_1, z)} \Phi^0(y_1, z) = f(y_1, z) \text{ for } (y_1, z) \in I\!R^2 \setminus [-a_-, \infty) \ , \\ \partial_z \Phi^0(y_1, \pm 0) = g_\pm(y_1) + (1 - \chi(r)) \frac{\Gamma}{2\pi r} \text{ for } y_1 \in (-a_-, a_+) \ , \\ [\partial_z \Phi^0] (y_1) = [\Phi^0] (y_1) = 0 \text{ for } y_1 > a_+ \ . \end{array} \tag{2.5}$$

It is wellknown (e.g. [4]) that problem (2.5) possesses a solution Φ^0 which vanishes at infinity if and only if the compatibility condition

$$\int\limits_{I\!R^2 \setminus [-a_-, \infty)} f(y_1, z) \, dy_1 dz + \sum_\pm \mp \int\limits_{-a_-}^{a_+} \left(g_\pm(y_1) + (1 - \chi(r)) \frac{\Gamma}{2\pi y_1} \right) dy_1 = 0 \tag{2.6}$$

is fulfilled.

Hence we must fix the constant $\mathbf{a}$ in the ansatz (2.2) by condition (2.6). Some simple calculations show

$$\int\limits_{I\!R^2 \setminus [-a_-, \infty)} f(y_1, z) dy_1 dz = \mathbf{a}, \quad \sum_\pm \mp \int\limits_{-a_-}^{a_+} (1 - \chi(r)) \frac{\Gamma}{2\pi y_1} dy_1 = 0$$

which yields

$$\mathbf{a} = \sum_\pm \pm \int\limits_{-a_-}^{a_+} g_\pm(y_1) dy_1 .$$

Inserting the last equality into (2.2) proves the assertion. ∎

REMARK 2.2 *It is known that only the antisymmetric part of $g_\pm$ with respect to the y_1–axis produces a contribution to the lifting force. The fundamental solution $-(2\pi)^{-1} \ln r^{-1}$ of the two–dimensional Laplace equation is symmetric and also the constant $\mathbf{c}$. Hence the lifting force appears due to the term $(2\pi)^{-1}\Gamma\varphi$ in (2.2).*

3 The three–dimensional case with a vanishing wave-number

In this section we investigate the asymptotic behavior of solutions Φ of the problem (1.3) at infinity where we assume the circulation $\Gamma \in H^{1/2}(-l, l)$ as being taken for granted in consistence with the results in [12], [13], mentioned in the introduction. Furthermore, we denote in the rest of this article by r the expression $\sqrt{y_2^2+z^2}$, the polar–radius in planes, perpendicular to the wake.

For finding the asymptotics of Φ we note at first a tool.

LEMMA 3.1 *Let $\Gamma \in H^{1/2}(-l,l)$ and define*

$$G := I\!R^2 \backslash \{(y_2,z) : |y_2| \leq l, z=0\}.$$

The problem

$$\begin{array}{rcl} \triangle_{(y_2,\,z)} V(y_2,z) & = & 0 \text{ in } G\,, \\ {[V]}\,(y_2) = \Gamma(y_2)\,, & & [\partial_z V]\,(y_2) = 0 \text{ for } y_2 \in (-l,l)\,, \\ |\nabla^{\alpha}_{(y_2,\,z)} V(y_2,z)| & \leq & c\,r^{-1\,-\,|\alpha|} \text{ for } r \gg 1,\ \alpha \in N_0^2 \end{array} \tag{3.1}$$

possesses in $W_2^1(G)$ a solution

$$V(y_2,z) = -\frac{1}{2\pi}\int_{-l}^{l} \frac{z\,\Gamma(t)}{(y_2-t)^2+z^2}\,dt. \tag{3.2}$$

This solution admits at infinity the representation

$$V(y_2,z) = -\frac{z}{2\pi r^2}\Gamma_0 \,+\, \tilde{V}(y_2,z) \tag{3.3}$$

where

$$\Gamma_0 = \int_{-l}^{l} \Gamma(y_2)\,dy_2\,, \tag{3.4}$$

and $\tilde{V}$ satisfies for $r \gg 1$ the estimate

$$|\nabla^{\alpha}_{(y_2,\,z)} \tilde{V}(y_2,z)| \leq c\,r^{-2\,-\,|\alpha|} \qquad \forall\, \alpha \in I\!N_0^2. \tag{3.5}$$

<u>Proof.</u> That the right–hand side in (3.2) delivers a solution of (3.1) in $W_2^1(G)$ follows from wellknown results from potential theory (see e.g. [3]) because for the transversal derivative of the fundamental solution of the two–dimensional Laplace equation the following relation holds true:

$$-\frac{\partial}{\partial z}\frac{1}{2\pi}\ln\frac{1}{r} = \frac{z}{2\pi r^2}.$$

Finally, (3.3)–(3.5) can be verified by some elementary calculations and estimates. ∎

REMARK 3.2 *Due to the jump $\Gamma(y_2)$ in (3.1)$_2$, the function V does not belong to $W^1_{2,\,loc}(\mathbb{R}^2)$. Nevertheless, $V \in L_{2,\,loc}(\mathbb{R}^2)$, $\dfrac{\partial V}{\partial y_2} \in L_2(\mathbb{R}^2)$ and the function*

$$(y_2, z) \mapsto \partial_z V(y_2, z) \; - \; \Gamma(y_2)\theta_l(y_2) \; \delta(z) \in L_2(G) \tag{3.6}$$

where θ_l denotes the characteristic function of the interval $[-l, l]$, i.e., $\theta_l(y_2) = 1$ for $|y_2| \leq l$ and $\theta_l(y_2) = 0$ for $|y_2| > l$. δ denotes the Dirac functional.

However, in the following it is convenient to ignore the Dirac functional in (3.6) and to regard $\partial_z V$ as an element of $L_2(G)$.

In order to find out informations about the farfield, we consider two cases. We investigate the behavior of Φ in a narrow conical neighborhood of the wake W, respectively outside of such a neighborhood. For describing a conical neighborhood of W we use as a cut–off function the expression $(1 - \chi(y_1))\,\chi(r/y_1)$ where χ is a cut–off function similar to that introduced at the beginning of section 2 and we shall specify Φ in the form

$$\Phi(x) = (1 - \chi(y_1))\,\chi(r/y_1)\,V(y_2, z) + \tilde{\Phi}(x). \tag{3.7}$$

THEOREM 3.3 *Let $\rho = \sqrt{y_1^2 + y_2^2 + z^2}$. Then the representation (3.7) of the solutions Φ of (1.3) takes inside the conical neighborhood of W, i.e., in the region where $(1 - \chi(y_1))\,\chi(r/y_1) \; = 1$ the form*

$$\Phi(x) = -\frac{1}{2\pi} \int\limits_{-l}^{l} \frac{z\Gamma(t)}{(y_2 - t)^2 + z^2}\, dt + O(\rho^{-1}) \textit{ for } \rho \longrightarrow \infty. \tag{3.8}$$

Outside of this cone the expansion

$$\Phi(x) = \frac{1}{4\pi\rho} \int\limits_{L} (g_+(y) - g_-(y))\, dy + \frac{\Gamma_0 z}{4\pi r^2}(1 + \frac{y_1}{\rho}) + O(\rho^{-2} \ln \rho) \textit{ for } \rho \longrightarrow \infty \tag{3.9}$$

holds true.

<u>Proof.</u> Performing the substitution (3.7) in (1.3)$_1$ we obtain according to (1.3)$_3$ the equation

$$\triangle_x \tilde{\Phi}(x) = -[\triangle_x, (1 - \chi(y_1))\,\chi(r/y_1)]V(y_2, z) \text{ in } \Omega \tag{3.10}$$

as the equation for $\tilde{\Phi}$.

(3.3) and (3.5) ensure the right–hand side F in (3.10) to take the form

$$F(x) = (1 - \chi(y_1))\,[\triangle_x, \chi(\frac{r}{y_1})]\frac{\Gamma_0 z}{2\pi r^2} + \tilde{F}(x) \tag{3.11}$$

with

$$|\nabla_x^\alpha \tilde{F}(x)| \le c\,\rho^{-4-|\alpha|}\ \forall\ \alpha \in I\!N_0^3. \tag{3.12}$$

Denoting by $(\rho,\theta) = (\rho,\theta_1,\theta_2)$ spherical coordinates where θ belongs to the unit sphere $S^2 \subset I\!R^3$ we can prove by elementary calculations the identity

$$-[\triangle x, \chi(\frac{r}{y_1})]\frac{z}{2\pi r^2} = \rho^{-3} h(\theta) \tag{3.13}$$

where $h \in C^\infty(S^2)$. Hence, (3.10), (3.11), (3.12), [7], [10], and [14, Theorems 3.5.6 and 3.5.12] yields $\tilde{\Phi}$ to possess a representation of the form

$$\tilde{\Phi}(x) = \frac{1}{\rho}(\frac{a}{4\pi} + \Gamma_0(b\ln\rho + \Psi(\theta))) + \Phi^*(x) \tag{3.14}$$

where Φ^* satisfies the estimate

$$|\nabla_x^\alpha \Phi^*(x)| \le c(\alpha)\rho^{-2-|\alpha|}\ln\rho\ \forall\ \alpha \in I\!N_0^3. \tag{3.15}$$

Here, a and b denote unknown constants and Ψ and b are coupled by the equation

$$\tilde{\triangle}_\theta \Psi(\theta) = h(\theta) - b,\ \theta \in S^2 \tag{3.16}$$

with the Beltrami–Laplace operator $\tilde{\triangle}_\theta$. Especially, a depends on the whole data of the problem (1.3).

Let us comment (3.14). Since $\tilde{\triangle}_\theta$ is a formally self–adjoint differential operator and $\tilde{\triangle}_\theta w(\theta) = 0$ yields $w = const$, due to results coming up from the Fredholm theory, the compatibility condition

$$\int_{S^2} (h(\theta) - b)\, ds_\theta = 0\,,\ \text{i.e.,}\ b = \frac{1}{4\pi}\int_{S^2} h(\theta)\, ds_\theta \tag{3.17}$$

has to hold. This formula would enable us to calculate b. However, we shall show by using an other method that b vanishes. Especially we want to avoid in the following to fix Ψ by solving (3.16). Finally we can calculate a using the weight function technique (e.g. [2], [10]).

The fact $b = 0$ proves of course (3.8) by reason of (3.7), (3.2), (3.14), (3.15) and the smoothness of Ψ.

At the same time, (3.14) indicates the behavior of Φ outside a narrow conical neighborhood of W. We choose a function Υ which is harmonic in $I\!R^3\backslash\{x : r = 0, y_1 \ge 0\}$ such that

$$u(x) = (1 - \chi(y_1))(\Upsilon(x) - \frac{1}{2\pi}\chi(r)\frac{z}{r^2}) \tag{3.18}$$

leaves an additional discrepency which has to be compensated by a solution of a problem similar to (3.1). The solution v of this problem, which takes care both on the jump Γ and the above–mentioned discrepancy decays suitable for $r \longrightarrow \infty$ which ensures, that the main term (3.13) in (3.11) disappears. Collecting all these facts will yield the representation

$$\Phi(x) = -\Gamma_0 u(x) + (1-\chi(y_1))\chi(\frac{r}{y_1})v(y_2, z) + \Phi^{**}(x) \tag{3.19}$$

where

$$\Phi^{**}(x) = \frac{a}{4\pi\rho} + O(\rho^{-2}\ln\rho). \tag{3.20}$$

We define the desired function Υ by

$$\Upsilon(x) = \frac{1}{4\pi}\int_0^\infty \frac{z}{(r^2+(t-y_1)^2)^{3/2}}\,dt = \frac{z}{4\pi r^2}(1+\frac{y_1}{\rho}). \tag{3.21}$$

Note that the integrand in (3.21) is just the derivative with respect to z of the three–dimensional fundamental solution of the Laplace equation. Therefore, Υ is a harmonic function in $I\!R^3\backslash\{x : r = 0, y_1 \geq 0\}$. Moreover, $\Upsilon(x) - \frac{z}{2\pi r^2}$ is a smooth function in the cylinder $\{x : r \leq r_0, y_1 \geq c_0\}$ where r_0, c_0 denote positive numbers. Because

$$\triangle u(x) = -\frac{1}{2\pi}(1-\chi(y_1))[\triangle_{(y_2,\,z)}, \chi(r)]\frac{z}{r^2} + g(x) \tag{3.22}$$

with a function $g \in C_0^\infty(I\!R^3)$, we consider

$$\begin{gathered} \triangle_{(y_2,\,z)}v(y_2, z) = -\tfrac{A}{2\pi}[\triangle_{(y_2,\,z)}, \chi(r)]\tfrac{z}{r^2} \text{ in } G\,, \\ [v]\,(y_2) = \Gamma(y_2)\,,\,[\partial_z v]\,(y_2) = 0 \text{ for } y_2 \in (-l, l) \end{gathered} \tag{3.23}$$

as the problem for v. Here A denotes an unknown constant which we have to choose lateron.

There exists a solution v of (3.23) with the following behavior at infinity:

$$\begin{gathered} v(y_2, z) = \tfrac{\beta}{2\pi}\ln\tfrac{1}{r} + \text{ const } - \frac{\gamma z}{2\pi r^2} - \frac{\delta y_2}{2\pi r^2} + \tilde{v}(y_2, z)\,, \\ |\nabla^\alpha_{(y_2,\,z)}\tilde{v}(y_2, z)| \leq c(\alpha)\, r^{-2\,-\,|\alpha|}\,. \end{gathered} \tag{3.24}$$

Next, we fix the constants A, β, γ, and δ. To this end, we insert v and the linear function

$$\Lambda(y_2, z) = \Lambda_0 + \Lambda_1 y_2 + \Lambda_2 z$$

into the 2^{nd} Greens formula, where we integrate over the ball $B_R = \{(y_2, z) : r < R\}$ with some radius $R > 0$ such that $\chi \equiv 0$ on ∂B_R. We obtain the equation

$$I_3 = I_1 + I_2 \tag{3.25}$$

where

$$I_1 = \int\limits_{\partial B_R} \left(\Lambda \frac{\partial v}{\partial r} - v \frac{\partial \Lambda}{\partial r} \right) ds \,,$$

$$I_2 = \sum_{\pm} \mp \int\limits_{-l}^{l} \left(\Lambda(y_2, 0) \frac{\partial v}{\partial z}(y_2, \pm 0) - v(y_2, \pm 0) \frac{\partial \Lambda}{\partial z}(y_2, 0) \right) dy_2 \,, \tag{3.26}$$

$$I_3 = \int\limits_{B_R} (\Lambda \cdot \triangle v - v \cdot \triangle \Lambda) \, dy_2 dz \,.$$

Some elementary calculations show that $I_1 = -\beta\Lambda_0 + \delta\Lambda_1 + \gamma\Lambda_2 + o(1)$ for $R \longrightarrow \infty$ and on account of $(3.23)_2$, $I_2 = \Gamma_0 \cdot \Lambda_2$ holds true. Finally, if we choose a ball of a radius ε such that $\chi \equiv 1$ on ∂B_ε we observe

$$\begin{aligned} I_3 &= -\frac{A}{2\pi} \int\limits_{B_R} \Lambda [\triangle_{(y_2, z)}, \chi(r)] \frac{z}{r^2} \, dy_2 \, dz = -\frac{A}{2\pi} \int\limits_{B_R \setminus B_\varepsilon} \Lambda \, \triangle_{(y_2, z)} \frac{z}{r^2} \, dy_2 \, dz \\ &= \frac{A}{2\pi} \varepsilon \int\limits_0^{2\pi} \left(\Lambda(y_2, z) \frac{\partial}{\partial r} \frac{z}{r^2} - \frac{z}{r^2} \frac{\partial}{\partial r} \Lambda(y_2, z) \right) \Big|_{r = \varepsilon} d\varphi = -A \cdot \Lambda_2 \end{aligned}$$

where φ denotes the argument of polar coordinates. Inserting these results into (3.25) and passing to the limit $R \longrightarrow \infty$ yields $\beta = \delta = 0$ and if we choose $A = -\Gamma_0$, also $\gamma = 0$ holds true.

Searching Φ now in the form of (3.19) yields just

$$\triangle_x \Phi^{**}(x) = -[\triangle_x, \chi(r/y_1)] v(y_2, z) + \tilde{F}(x) \tag{3.27}$$

as the problem for Φ^{**} where $\tilde{F}$ is compactly supported and due to (3.24) with $\beta = \gamma = \delta = 0$

$$|\nabla_x^\alpha [\triangle_x, \chi(r/y_1)] v(y_2, z)| \leq c(\alpha) \rho^{-4 - |\alpha|} \,.$$

A comparison of the last results with (3.14), (3.15), respectively, verifies the identity (3.20) and proves the constant b in (3.14) to vanish.

Finally, we have to calculate the constant a in the representation (3.19) for Φ which appears from the presence of Φ^{**}. We use the weight–function technique with the weight–function $\mathbf{1}$, where $\mathbf{1}(x) = 1 \; \forall \; x \in I\!\!R^3$. The idea is, to insert Φ and $\mathbf{1}$ into the 2^{nd} Greens formula where we integrate over the ball $\mathbf{B}_R = \{x \in I\!\!R^3 : \rho < R\}$ and then we pass to the limit $R \longrightarrow \infty$ which yields an algebraic equation in a. Using $(1.3)_2$, $(1.3)_3$ and $y_1 = \rho \sin\theta_1 \cos\theta_2, y_2 = \rho \sin\theta_1 \cos\theta_2$ and $z = \rho\cos\theta_1$, we obtain the identity

$$
\begin{aligned}
0 &= \int_{\mathbf{B}_R} (\triangle\Phi - \Phi \triangle \mathbf{1})\, dx = \int_{\partial\mathbf{B}_R} \frac{\partial\Phi}{\partial n}\, do + \int_L (g_+(y) - g_-(y))\, dy \\
&= \int_{\partial\mathbf{B}_R} \frac{\partial}{\partial n}\Big(\frac{a}{4\pi\rho} + \frac{\Gamma_0}{4\pi}\frac{z}{r^2}\big(1 + \frac{y_1}{\rho}\big)\Big)\, do + \int_L (g_+(y) - g_-(y))\, dy + o(1) \\
&= R^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta_1 \frac{\partial}{\partial\rho}\Big(\frac{a}{4\pi\rho} + \frac{\Gamma_0}{4\pi}\frac{z}{r^2}\big(1 + \frac{y_1}{\rho}\big)\Big)\Big|_{\rho = R}\, d\theta_1\, d\theta_2 \\
&\qquad + \int_L (g_+(y) - g_-(y))\, dy + o(1) \\
&= \int_0^{2\pi}\int_0^{\pi} -\frac{a}{4\pi}\sin\theta_1 - \frac{\Gamma_0}{4\pi}\frac{\sin\theta_1\cos\theta_1 + \sin^2\theta_1\cos\theta_1\cos\theta_2}{1 - \sin^2\theta_1\cos^2\theta_2}\, d\theta_1\, d\theta_2 \\
&\qquad + \int_L (g_+(y) - g_-(y))\, dy + o(1)\,.
\end{aligned}
$$

Because $\int_0^{2\pi}\int_0^{\pi} \frac{a}{4\pi}\sin\theta_1\, d\theta_1\, d\theta_2 = a$ and because

$$\int_0^{\pi} \frac{\sin\theta_1\cos\theta_1 + \sin^2\theta_1\cos\theta_1\cos\theta_2}{1 - \sin^2\theta_1\cos^2\theta_2}\, d\theta_1 = 0$$

holds true due to symmetry properties of the integrand with respect to the point $\pi/2$, we obtain immediately that

$$a = \int_L (g_+(y) - g_-(y))\, dy\ ,$$

which proves (3.9). ■

4 The three–dimensional case with a weakly damping gas

In this section we investigate the asymptotic behavior of solutions Φ of (1.1) at infinity, assuming a weakly damping gas, i.e., Re k, Im $k > 0$ and show an exponential decay of Φ. Here we concentrate to the three–dimensional situation while in the two–dimensional case only some simplifications occur.

Again, based on [12], [13] we assume $\Gamma \in H^{1/2}(-l, l)$ as being taken for granted.

Now we present the weak formulation of the problem (1.1). Let $\chi \in C^\infty(\mathbb{R}^3)$ be a cut–off function such that $\chi \equiv 1$ in a neighborhood of $\overline{W}$ and $\chi(x) = 0$ for $y_2^2 + z^2 > R^2$, $y_1 < -R$, respectively with some large number $R > 0$. Moreover, we assume χ to be independent of y_1 for $y_1 > R$.

We make the ansatz

$$\Phi(x) = u(x) - \chi(x)V(y_2, z)E(y) \tag{4.1}$$

where V denotes the solution (3.2) of (3.1) and $E(y)$ is the exponential factor in $(1.1)_3$.

Performing the substitution (4.1) in (1.1) and multiplying the transformed equation $(1.1)_1$ by a test function $v \in W_2^1(\Omega)$ we obtain after an integration by parts using the transformed boundary conditions $(1.1)_2$ – $(1.1)_4$ the integral identity

$$\begin{aligned}(\nabla u, \nabla v)_\Omega - k^2(u, v)_\Omega &= (g_+, v)_{L_+} - (g_-, v)_{L_-} \\ &-(\nabla \chi V E, \nabla v)_\Omega + k^2(\chi V E, v)_\Omega \ \forall v \in W_2^1(\Omega)\end{aligned} \tag{4.2}$$

as the weak formulation of the problem for u. Here $(\cdot,\cdot)_\Omega$ denotes the scalar product in $L_2(\Omega)$.

The following proposition holds true:

PROPOSITION 4.1 *The problem (4.2) possesses an unique solution $u \in W_2^1(\Omega)$. Moreover, the estimate*

$$\|u; W_2^1(\Omega)\| \leq c \cdot N \tag{4.3}$$

is valid where

$$N := \left(\sum_{\pm} \|g_\pm; L_2(L)\| + \|\Gamma; H^{1/2}(-l, l)\|\right). \tag{4.4}$$

Proof. For proving the assertion we only have to verify the assumptions of the Lax–Milgram theorem. The integrals in the right–hand side of (4.2) exists by reason of the exponential decay of E and this right–hand side represents a continuous linear functional in v which acts on $W_2^1(\Omega)$. Furthermore, owing to the above mentioned conditions Re k, Im k > 0, one can show by some elementary calculations the existence of constants $c > 0$, $\delta_k > 0$ such that

$$|(\nabla u, \nabla u)_\Omega - k^2(u, u)_\Omega| \geq c\,\|\nabla u; L_2(\Omega)\|^2 + \delta_k\|u; L_2(\Omega)\|^2 \tag{4.5}$$

holds true. Clearly, the corresponding bilinearform is continuous. Hence, the application of the usual Lax–Milgram technique proves the assertion. ■

Finally we show u to decay exponentially. We formulate this result in the following proposition.

PROPOSITION 4.2 *Let u satisfy the identity (4.2) for all compactly supported functions $v \in W_2^1(\Omega)$ and for some small number $\tau_- > 0$ the inclusion*

$$e^{-\tau_-\sqrt{1+\rho^2}}u \in W_2^1(\Omega) \tag{4.6}$$

where $\rho = |x|$. Then there exists a number $\tau_0 > 0$ such that for $\tau_- \in (0, \tau_0)$ the inclusion

$$e^{\tau_+\sqrt{1+\rho^2}} u \in W_2^1(\Omega) \ \forall \tau_+ \in (0, \tau_0) \tag{4.7}$$

is valid. Moreover, there exists a constant $c > 0$ such that the estimate

$$\left(\| e^{\tau_+\sqrt{1+\rho^2}} \nabla u; L_2(\Omega)\|^2 + \delta_k \| e^{\tau_+\sqrt{1+\rho^2}} u; L_2(\Omega)\|^2 \right)^{1/2} \leq c\, N \tag{4.8}$$

holds true. Here δ_k denotes the constant introduced in (4.5).

<u>Proof.</u> It is trivial that the solution $u \in W_2^1(\Omega)$ fulfills the weaker asymptotic assumption (4.6). By proving (4.7), we show u to be also unique assuming the asymptotic behavior described by (4.6). In fact, for a small number $\tau_- > 0$, u is allowed to possess an exponentially growth at infinity.

Taking (4.6) into consideration, the integral identity (4.2) holds still true for exponentially decaying test functions v such that

$$\exp(\tau_-\sqrt{1+\rho^2}) v \ \in \ W_2^1(\Omega)$$

by reason of completion arguments. Especially, we choose

$$v = R_T^2 u \in W_2^1(\Omega) \tag{4.9}$$

where the weight factor R_T is defined by

$$R_T(\rho) = \begin{cases} \exp(\tau_+\sqrt{1+\rho^2}) \text{ for } \rho < T \\ \exp((\tau_+ + \tau_-)\sqrt{1+T^2}) \exp(-\tau_-\sqrt{1+\rho^2}) \text{ for } \rho > T. \end{cases} \tag{4.10}$$

Here, τ_+ denote a small positive number which has to be characterized in more detail and T is a positive parameter which is scheduled to tend to infinity. Note that R_T is a piecewise smooth function on $I\!R^2$ and that $\nabla R_T \in L_{\infty,loc}(I\!R^2)$. Furthermore, the estimate

$$|\nabla R_T(\rho)| \leq c \max\{\tau_+, \tau_-\} R_T(\rho) \tag{4.11}$$

is valid with some constant $c > 0$, independent of T and ρ.

Performing the substitution (4.9), (4.2) takes the form

$$(R_T \nabla u, R_T \nabla u)_\Omega - k^2 (R_T u, R_T u)_\Omega + 2(R_T \nabla u, u \nabla R_T) = I_T(u) \tag{4.12}$$

with

$$\begin{aligned} I_T(u) \ &= \ \sum_{\pm} \pm (R_T g_\pm, R_T u)_L + k^2 (R_T \chi V E, R_T u)_\Omega \\ &\quad -(R_T \nabla \chi V E, R_T \nabla u)_\Omega - 2\, (R_T \nabla \chi V E, u \nabla R_T)_\Omega. \end{aligned} \tag{4.13}$$

If $\tau_+ < M^{-1}$ Im k with the Machnumber M, the integrals in the right–hand side $I_T(u)$ of (4.12) converge by reason of the rate of decay of E and $I_T(u)$ admits the estimate

$$|I_T(u)| \leq c\, N \cdot |||u|||_T$$

where

$$|||u|||^2 = \|R_T \nabla u; L_2(\Omega)\|^2 + \delta_k \|R_T u; L_2(\Omega)\|^2 \tag{4.14}$$

and where the constant $c > 0$ depends neither on T nor on u. Furthermore, by virtue of (4.11), the third term in the left–hand side of (4.12) satisfies the inequality

$$2|(R_T \nabla u, u \nabla R_T)_\Omega| \leq 2\, c \max\{\tau_+, \tau_-\} |||u|||_T^2.$$

Hence, (4.12) and some simple calculations similar to those which led us to (4.5) yield

$$|||u|||_T^2 - 2\, c\, \max\{\tau_+, \tau_-\} |||u|||_T^2 \leq c\, N\, |||u|||_T.$$

Finally, if we choose $\tau_0 > \max\{\tau_+, \tau_-\}$ sufficiently small, we obtain the estimate

$$|||u|||_T \leq 2\, cN. \tag{4.15}$$

The function $T \mapsto |||u|||_T$ is by reason of (4.10) and (4.14) monotone increasing and the limit of this function is for $T \longrightarrow +\infty$ just the left–hand side of (4.8). Hence, (4.8) results from (4.15) and the convergence of the integrals in the left–hand side of (4.8) verifies the exponential decay of u, i.e., (4.7). ■

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