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für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Regularity results for interface problems in 2D

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submitted: 16 Mar 2000

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Preprint No. 565
Berlin 2000

1991 Mathematics Subject Classification. 35B65, 35J25, 35J20, 34L15.

Key words and phrases. elliptic equations, regularity, interface problems, transmission problems, singularities, discontinuous diffusion coeffcients, Sturm-Liouville eigenvalue problem.

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Abstract

We investigate the regularity of solutions of interface problems in 2D. Our objective are regularity results which are independent of global bounds of the data (the diffusion). Therefore we introduce a criterion on the data,the quasi-monotonicity condition, which we show to be sufficient and necessary to provide regularity better then H^1. In the proof we use estimates of eigenvalues of a related Sturm-Liouville eigenvalue problem. This approach allows to derive sharp regularity results for quite a large class of configurations. Additionally we give a regularity result depending on the global bounds of the data.

Contents

1 Introduction and Outline

We are interested in elliptic interface problems in 2D. These are elliptic problems with piecewise constant data k. The data are constant on subdomains and can be interpreted as a diffusion term. The strong form of the problem is

$$\nabla \cdot k(x) \nabla u(x) = f(x) \quad ,$$

with some boundary conditions.

In this article we investigate the regularity of solutions of interface problems independent of global bounds of k. Known results are listed in section 2.2. All of the known results which yield regularity $H^s, 1 < s$ (with s independent on global bounds on k) are restricted to case that the maximal number n of subdomains which meet in a point is $n = 2$.

We use a criterion called quasi-monotonicity which is defined independently of this number (section 2.3). Roughly speaking the function k is quasi-monotone, if all of its traces on small spheres around singular points have only one local maximum. If a singular point belongs to the boundary, there is an additional condition.

The quasi-monotonicity condition was introduced by [12]. This condition was shown to be sufficient [12] and necessary [13] to define robust interpolation operators onto finite element spaces which are stable in norms weighted with $\sqrt{k}$.

We show that quasi-monotonicity is sufficient and necessary to yield regularity H^s for some $1 < s$ independent of the global bounds on k and show $s = 1 + 1/4$. For this we use the approach of Kellogg [5], who showed that the regularity is restricted by eigenvalues of a Sturm-Liouville eigenvalue problem (section 2.4). The bounds of the eigenvalues are derived in section 3.1. In some special situations one can further improve the bounds for the eigenvalues, see section 3.2.

The main result is given in section 3.3, where we show regularity $H^{1+1/4}$. In some special cases we are able to show better regularity. The reader interested in the regularity results may skip the preceding sections 2.4,3.1,3.2. We show that the situations covered by known results are a special case of our approach.

In section 4.1 we discuss the necessity of the quasi-monotonicity condition and show an example with deterioating regularity. In section 4.2 we derive regularity results when no restrictions on the weight function are imposed. Here the regularity is restricted by the global bounds on the weight function.

2 The problem and its properties

2.1 Definition of the problem

Let a domain $\Omega \subset R^2$ with polygonal boundary be given. Here we allow also slits. The domain Ω can be decomposed into disjoint subdomains $\Omega_i, i = 1,..,n_d$ with polygonal boundaries: $\bar{\Omega} = \bigcup_{i=1,..,n_d} \bar{\Omega}_i$. We denote by $\partial\Omega$ be the boundary of Ω. We define the interface $\Gamma = Closure(\bigcup_i \partial\Omega_i/\partial\Omega)$. For each domain Ω_i let a positive weight k_i be given. We can assume that for subdomains Ω_i, Ω_j with $meas_1(\Omega_i \cap \Omega_j) > 0$ it yields $k_i \neq k_j$. Otherwise merge Ω_i, Ω_j. Here $meas_1(\cdot)$ denotes the one dimensional Lebesgue measure.

Denote with $k = k_i\chi_{\Omega_i} \in L^2(\Omega)$ the global weight function, which is constant on subdomains Ω_i. We impose the global bounds

$$0 < M_k^{-1} \leq k(x) \leq M_k \quad , \quad \forall x \in \Omega \quad .$$

We introduce the Dirichlet boundary $\Gamma_D, meas(\Gamma_D) > 0$ and the Neumann boundary $\Gamma_N = \partial\Omega/\Gamma_D$ and demand that they are made up by sums of $\partial\Omega_i \cap \partial\Omega$.

Let $f \in L^2(\Omega)$ be given. With the space $V = \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}$ we look for $u \in V$ and satisfying:

$$\int_\Omega k(x)\nabla u \nabla v = \int_\Omega fv \quad , \quad \forall v \in V \quad . \tag{1}$$

We define the energy (semi-)norm $|v|_{kH^1(\Omega)}$ and the (semi-)norm $|v|_{H^1(\Omega)}$

$$|v|^2_{H^1(\Omega)} := \int_\Omega k(x)\nabla v \nabla v \quad , \quad |v|^2_{kH^1(\Omega)} := \int_\Omega \nabla v \nabla v \quad .$$

Existence of the solution follows as in the case of a Laplace equation. The equivalence $|v|_{kH^1(\Omega)} \approx \|v\|_{H^1(\Omega)}$ follows from the bounds on k and a Poincare inequality which proves since $meas\,(\Gamma_D) > 0$ that $|v|_{H^1(\Omega)} \approx \|v\|_{H^1(\Omega)}$. We use Riesz' theorem to prove existence and uniqueness of a solution $u \in H^1(\Omega)$.

2.2 Known regularity results

Let us discuss regularity of the solution of problem (1). Due to $f \in L^2(\Omega)$ regularity is not greater then $H^2(\Omega)$. The jumps of the normal derivatives of u on the interface restrict global regularity to $u \notin H^{3/2}(\Omega)$. But it is also interesting to know the regularity in Ω_i. This may be important for instance for Finite Element applications. For the definition of Sobolev Spaces H^s see [3].

Usually from regularity on subdomains $H^{1+\lambda}(\Omega_i), i = 1, 2$ does not follow regularity on the sum of these subdomains $H^{1+\lambda}(\Omega_1 \cup \Omega_2)$. This may be true for $0 \leq \lambda < 1/2$.

Lemma 2.1 *Let the polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ be decomposed into disjoint polygonal subdomains Ω_1, Ω_2. Let $0 \leq \lambda < 1/2$,$v \in H^{1+\lambda}(\Omega_i), i = 1,2$ and $v \in H^1(\Omega)$. Then $v \in H^{1+\lambda}(\Omega)$.*

PROOF The proof follows from definition 1.2.4 and theorem 1.2.16 [3]. It suffices to prove $\nabla v \in \left(H^\lambda(\Omega)\right)^2$. Denote with $v_{j,i} = \frac{\partial v}{\partial x_j}$ the partial derivatives of v in Ω_i. Since $v_{j,i} \in H^\lambda(\Omega_i), i = 1,2$ and due to the implication given after theorem 1.2.16 [3] one can extend $v_{j,i}$ by zero to $v^+_{j,i} \in H^\lambda(\Omega)$. By Gauss' theorem one checks $\frac{\partial v}{\partial x_j} = v^+_{j,1} + v^+_{j,2}$ and hence $\frac{\partial v}{\partial x_j} \in H^\lambda(\Omega)$. ■

To discuss regularity we classify certain geometrical situations and introduce the following definition

Definition 1 *A point $x \in \partial\Omega$ is a homogeneous singular point if in a neighborhood of x the weight function k is constant and*

- *the intersection of the domain Ω with a convex neighborhood of x is not convex*
- *the boundary condition change in x and Ω coincides in a neighborhood of x with a cone with angle $> \pi/2$.*

Definition 2 *A point on the interface $x \in \Gamma$ is an heterogeneous singular point if*

- *x is an interior point $x \in \Gamma \; \partial\Omega$ and in any neighborhood of x the interface is not a straight line.*
- *x lies on the boundary $x \in \partial\Omega$*

Definition 3 *If x is a homogeneous or a heterogeneous singular point we call x a singular point.*

Interior heterogeneous singular points are also called crosspoints.

We illustrate the singular points in figure 1.

In the case $k = 1$ usual regularity theorems ([3], [1]) state that the regularity is $u \in H^2(U \cap \Omega)$ if the open subdomain U contains no singular points (Theorem 2.1.4 [3]). But there is less regularity if U contains a homogeneous singular point:

Lemma 2.2 *Let $k = 1$ and u be a solution of problem (1). Then for any neighborhood of a singular point from 1,2 $x \in U_x$, such that U_x contains no other singular points and such that the interior angle of Ω at x is smaller then 2π the solution has regularity $u \in H^{1+1/2}(U_x \cap \Omega)$ if the boundary conditions do not change in x and $u \in H^{1+1/4}(U_x \cap \Omega)$ if they do.*

In case of a slit domain regularity regularity is $u \in H^{1+1/4-\varepsilon}(U_x \cap \Omega), \varepsilon > 0$.

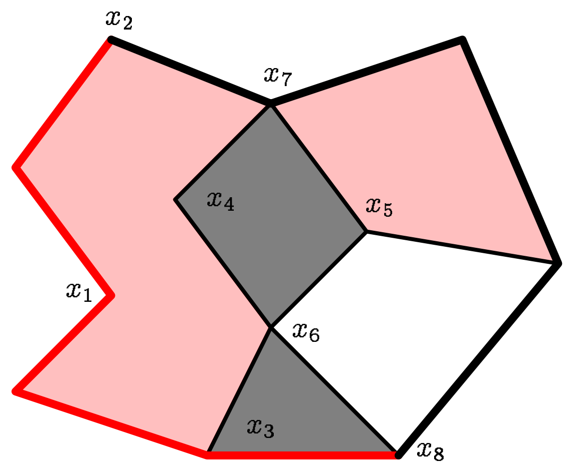

Figure 1: subdomains are shaded with different levels of grey, Dirichlet and Neumann boundaries are shaded differently, $x_l, l = 1, 2$ are homogeneous singular points, $x_l, l > 2$ are heterogeneous singular points (not all singular points are depicted)

PROOF This is corollary 2.4.4. of [3]. For the slit domain one can use $v = r^{1/4}\cos(\frac{1}{4}\varphi)$ to construct a solution u with regularity $u \notin H^{1+1/4}$. Here (r, φ) are polar coordinates with respect to the end of the slit. ■

The lowest regularity is reached in case of a slit domain with Dirichlet and Neumann boundary conditions on either side of the slit.

Our concern is the regularity for heterogeneous singular points. We want to list known facts. To get more detailed results we choose a heterogeneous singular point and classify the geometrical situations according to the number of subdomains neighboring to this singular point.

In the case of a heterogeneous singular point on the boundary the following result is known

Lemma 2.3 *Let u be a solution of problem (1). Let x be a heterogeneous singular point on the boundary and denote with $\in U_x$ a neighborhood containing no other singular points.*

If x belongs to the boundary $\partial\Omega_i$ of only two subdomains then the solution has regularity $u \in H^{5/4}(U_x \cap \Omega)$ if the boundary conditions do not change and $u \in H^1(U_x \cap \Omega)$ if they do.

If x belongs to the boundary $\partial\Omega_i$ of three subdomains then $u \in H^1(U_x \cap \Omega)$.

The regularity bounds are sharp in the sense that without restrictions on k there are no more regular Sobolev Spaces with $u \in H^s(U_x \cap \Omega)$ and $s > 1$.

PROOF This is corollary 1 of [9]. For the case of three subdomains see section 4.1. ■

See also [11],[7] and for the case of two Lipschitz subdomains see [10]. For interior heterogeneous singular points we cite the following results

Lemma 2.4 *Let u be a solution of problem (1). Let x be an interior heterogeneous singular point and denote with $x \in U_x$ a neighborhood containing no other singular points.*

Then the solution has regularity $u \in H^{3/2-\varepsilon}(U_x \cap \Omega), \varepsilon > 0$ if x belongs to two subdomains Ω_i. Further $u \in H^{3/2}(U_x \cap \Omega_i)$.

If x belongs to four subdomains $u \in H^1(U_x \cap \Omega)$.

The regularity bounds are sharp in the sense that without restrictions of k there are examples that u does not belong to more regular Sobolev Spaces $H^s, 1 < s$.

PROOF For case of two subdomains see [6],[5] or [2]. For the second case see the example of [6] and also section 4.1. ■

See also [11],[7],[8]. For the case of two Lipschitz subdomains see [10]. In general it is known

Lemma 2.5 *Let u be a solution of problem (1). Then the solution has regularity $u \in H^{1+\varepsilon(k)}(\Omega)$ where $\varepsilon(k)$ depends on k.*

PROOF See [5]. ■

A similar result covering the case of more general subdomains can be found in [4].

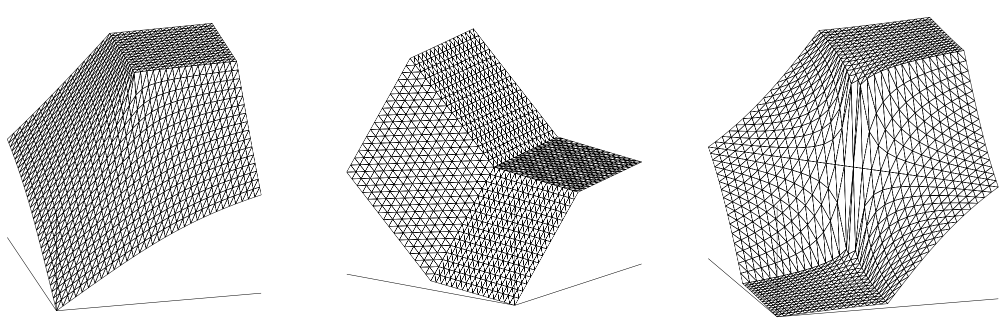

Figure 2: a) $\lambda \approx 0.7$ b) $\lambda \approx 0.99$ c) $\lambda \approx 0.1$

In figure 2 we show plots of different typical solutions of interface problems around an interior heterogeneous singular point. In all cases k varies between 0.1 and 10 but one observes different values for λ where $u \notin H^{1+\lambda}(\Omega_i)$ and $u \in H^{1+\lambda-\varepsilon}(\Omega_i), 0 < \varepsilon$. These functions are special cases of the function u_3 defined in section 4.1.

At first sight the situation for heterogeneous singular points is more complicated. An open question is regularity if three subdomains touch each other in an interior singular point. Further one may be interested in conditions on k such that regularity H^s for some $s > 1$ is guaranteed. In this article we will answer to these questions. We will also give the explicite dependence of ε from lemma 2.5 on the global bounds on k. But first let us look what regularity depends on.

2.3 Notation and quasi-monotonicity condition

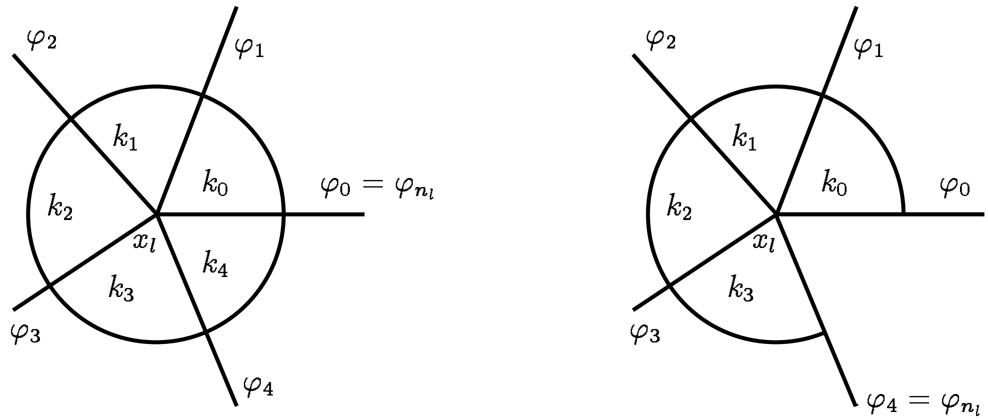

Figure 3: subdomains $\Omega_{l,i}$ coincide with cones $C_{l,i}$ in a neighborhood of an interior (left figure) and a boundary (right figure) heterogeneous singular point x_l

Let x_l be a heterogeneous singular point. We introduce polar coordinates (r, φ) with respect to x_l. Number the subdomains which share the singular point x_l with $\Omega_{l,i}, i = 0, .., n_l - 1$ and choose a radius $r_l > 0$ such that the intersection of the subdomains with the ball centered at x with radius r_l: $\Omega_{l,i} \cap B_{x_l}(r_l), i = 0, .., n_l - 1$ coincides with cones $C_{l,i}$. The cones $C_{l,i}$ are given by the lines $\varphi = \varphi_i$ and $\varphi = \varphi_{i+1}, i = 0, .., n_l - 1$. Let $\varphi_0 < \varphi_1 < ... < \varphi_{n_l}$. Here and in the following the notation $\varphi_0 < \varphi_1 < ... < \varphi_{n_l}$ means that starting from the angle φ_0 and increasing it, one reaches the angles $\varphi_1, \varphi_2, ..., \varphi_{n_l}$ in this order.

If x_l is an interior point we see $\varphi_{n_l} = \varphi_0$. If not, the lines φ_0, φ_{n_l} coincide with a part of $\partial\Omega$. We can demand that on neighboring domains $\Omega_{l,i} \cap B_{x_l}(r_l)$ and $\Omega_{i+1} \cap B_{x_l}(r_l)$ the function k takes different values. Otherwise merge these cones and provide renumbering.

Given two angles $a, b \in [0, 2\pi]$ we denote with the interval $[a, b]$ all angles c such that $a \leq c \leq b$.

Denote with $k_{l,i}$ the value of k on $\Omega_{l,i} \cap B_{x_l}(r_l)$. We define a local weight function $k_{x_l}(\varphi)$ on the sector $[\varphi_0, \varphi_{n_l}]$. This function $k_{x_l}(\varphi)$ is piecewise constant on $[\varphi_i, \varphi_{i+1}], i = 0, .., n - 1$ and takes there the value $k_{l,i}$. Chosing a singular point x_l we may drop the subindices l to simplify the notation.

The notation carries over also for homogeneous singular points. There $n_l = 1$.

Now we define the quasi-monotonicity condition for the weight function k. This condition was introduced in [12]. Remember that we assumed that $k_{l,i} \neq k_{l,j}, i \neq j$.

Roughly speaking the quasi-monotonicity condition means that the local weight function $k_{x_l}(\varphi)$, when restricted to the sphere $\partial B_{x_l}(r_l)$, has only one local maximum. Since the restriction of $k_{x_l}(\varphi)$ is a function which is piecewise constant on intervals $[\varphi_i, \varphi_{i+1}]$, it has infinitly many maxima. But we agree to identify all maxima which lie in the same interval $[\varphi_i, \varphi_{i+1}]$. If the sphere $\partial B_{x_l}(r_l)$ intersects with Γ_D, we demand additionally, that the maximum is reached on the intersection with Γ_D.

Definition 4 *Let a heterogeneous singular point x_l be given. The distribution of the weights $k_{l,i}, i = 0, .., n_l - 1$ will be called quasi-monotone with respect to the singular point x_l if the following conditions are fulfilled:*

Denote with N_i the neighboring cones to $C_{l,i}$ that is $N_i := \{j : \bar{C}_{l,j} \cap \bar{C}_{l,i} \neq \emptyset, j \neq i\}$.

There is only one index i_0 such that $k_{l,i_0} > \max_{j \in N_i} \{k_{l,j}\}$.

If $x_l \in \partial\Omega$ and $B_{x_l}(r_l) \cap \Gamma_D \neq \emptyset$ we demand additionally that the closed cone given by $[\varphi_{l,i_0}, \varphi_{l,i_0+1}]$ intersects with $\Gamma_D \cap B_{x_l}(r_l)$.

Definition 5 *The weight function k is quasi-monotone if for all singular points x_l the distribution of weights $k_{l,i}, i = 0, .., n_l - 1$ is quasi-monotone.*

We give conditions for the quasi-monotonicity to hold without restrictions on k but with restrictions on the geometry.

Choose an interior singular point x_l. The distribution of the weights $k_{l,i}, i = 0, .., n_l - 1$ is quasi-monotone with respect to x_l if $n_l \leq 3$. If $x_l \in \partial\Omega$ and $n_l \leq 2$ and the boundary conditions do not change in x_l then the distribution of the weights $k_{l,i}, i = 0, .., n_l - 1$ is quasi-monotone with respect to x_l. Thus the distribution of the weights $k_{l,i}, i = 0, .., n_l - 1$ is always quasi-monotone for points x_1, x_2, x_3, x_4, x_5 from figure 1.

One checks that these bounds on n_l are the maximal ones, if one admits an arbitrary weight function k. That means for points x_6, x_7, x_8 from figure 1 quasi-montonicity depends on k. For instance weights $k_{6,0} = k_{6,2} = 1$ and $k_{6,1} = k_{6,3} = 100$ are not quasi-montone distributed with respect to the singular point x_6.

In figure 2 a), b) we illustrate solutions which may occur in quasi-montone cases. In figure 2 c) we depict a solution which may occur in the non-quasimonotone case.

2.4 The Sturm-Liouville eigenvalue problem and regularity

Choose a singular point x. We regard a Sturm-Liouville eigenvalue problem given by

$$-s(\varphi)'' = \lambda^2 s(\varphi) \quad , \quad \varphi \in (\varphi_i, \varphi_{i+1}) \; i = 0, .., n-1 \tag{2}$$

with the interface conditions for i such that the line given by φ_i coincides with a part the interface

$$\begin{aligned} s(\varphi_i - 0) &= s(\varphi_i + 0) \\ k_{i-1} s(\varphi_i - 0)' &= k_i s(\varphi_i + 0)' \end{aligned} \tag{3}$$

and in case $x \in \partial\Omega$ with the boundary conditions

$$\begin{aligned} &\text{either } s(\varphi_0 + 0) = 0 \text{ or } s(\varphi_0 + 0)' = 0 \\ &\text{either } s(\varphi_n - 0) = 0 \text{ or } s(\varphi_n - 0)' = 0 \quad . \end{aligned} \tag{4}$$

For instance we choose $s(\varphi_0 + 0) = 0$ if $meas_1(\partial\Omega_0 \cap \Gamma_D) > 0$. Here we denote with $s(\varphi_i - 0), s(\varphi_i + 0)$ the left resp. right hand side limes of the function s in the point φ.

If x is an interior singular point, the problem is posed in $W = H^1_{per}([0, 2\pi])$. In the case $x \in \partial\Omega$ define W as a subspace of $H^1([\varphi_0, \varphi_n])$ with appropriate homogeneous Dirichlet boundary conditions, depending on whether φ_0 or φ_n coincide with a part of Γ_D.

The next lemma establishes a connection between the above Sturm-Liouville eigenvalue problem and regularity

Lemma 2.6 *The solution u of (1) admits a decomposition into*

$$u = w + \sum_{x_l} \sum_{i=1}^{d_{x_l}} c_{l,i} v_{x_l,i} \quad , \tag{5}$$

where $w \in H^2(\Omega_i), i = 1, .., n_d$ and the sum is over all singular points x_l.

Let $s_{\lambda_{l,i}}(\varphi), i = 1, .., d_{x_l}$ be all eigenfunctions of the respective Sturm-Liouville eigenvalue problem (2),(3),(4) with eigenvalue $\lambda^2_{x_l,i} \leq 1$. Then we call $v_{x_l,i}$ a "singular function" aligned with the point x_l. This function has the form

$$v_{x_l,i} = \eta(r) r^{\lambda_{x_l,i}} s_{\lambda_{l,i}}(\varphi) P_{l,i}(\ln(r)) \quad ,$$

where the function $\eta(r)$ is a smooth cut off function vanishing outside a neighborhood of x_l. Here $P_{l,i}(\ln(r))$ is a polynomial of $\ln(r)$. The singular function $v_{x_l,i}$ does not depend on f.

Let $0 < \gamma < \lambda$ for all nonzero eigenvalues λ^2 of the Sturm-Liouville eigenvalue problem (2), (3), (4) for any singular point x. Then $u \in H^{1+\gamma}(\Omega_i), i = 1, .., n_d$.

PROOF The proof of the representation (5) follows from Theorem 1 [5] and section 3 of [5] with $s = 0$. The representation is also given in theorem 3 of [9].

The regularity result used follows by calculation of $|r^\lambda s_\lambda|_{H^{1+s}(\Omega_i)}$ or by Theorem 1.2.18 [3]. One notices that the logaritmical terms do not influence the regularity. Furthermore we use that the only limit point of $\lambda_{x_l,i}$ is $+\infty$ [5]. ■

Thus if all nonzero eigenvalues are greater then γ, this implies piecewise regularity $H^{1+\gamma}$.

The general solution of equation (2) on an interval $[\varphi_i, \varphi_{i+1}]$ has the form $e_i \cos(\lambda\varphi) + f_i \sin(\lambda\varphi), e_i, f_i \in R$ what can be written as $b_i cos(\lambda(\varphi - c_i))$. We conclude from (2) that the Sturm-Liouville eigenvalue problem (2), (3), (4) is equivalent to the following problem. There are real numbers $b_i, c_i, i = 0, .., n-1$ such that

$$s(\varphi) = b_i \cos(\lambda(\varphi - c_i)) \text{ for } \varphi \in [\varphi_i, \varphi_{i+1}], i = 0, .., n-1 \quad .$$

The interface condition reads for i such that the angle φ_i coincides with a part of the interface

$$\begin{aligned} b_i \cos(\lambda(\varphi_{i+1} - c_i)) &= b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \\ k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) &= k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})) \quad , \end{aligned}$$

and for singular points $x \in \partial\Omega$ the boundary conditions read

$$\begin{aligned} &\text{either } b_0 \cos(\lambda(\varphi_0 - c_0)) = 0 \text{ or } -b_0 \sin(\lambda(\varphi_0 - c_0)) = 0 \\ &\text{either } b_{n-1} \cos(\lambda(\varphi_n - c_{n-1})) = 0 \text{ or } -b_{n-1} \sin(\lambda(\varphi_n - c_{n-1})) = 0 \quad . \end{aligned}$$

3 The quasi-monontone case

3.1 Quasi-monotonicity bounds eigenvalues from below

In this section we show that if the weight function k is quasi-monotone, the eigenvalues of the Sturm-Liouville eigenvalue problem are bounded from below. We precede the proof of this fact by two technical lemmata.

Lemma 3.1 *Let functions $t_i(\varphi) = b_i \cos(\varphi - b_i), i = 1, 2$ be given which fulfill conditions*

$$\begin{aligned} t_1(\varphi_1) &= t_2(\varphi_1) \\ k_1 t_1'(\varphi_1) &= k_2 t_2'(\varphi_1) \quad , \end{aligned} \tag{6}$$

for some $\varphi_1, k_i > 0, b_i > 0, i = 1, 2$.

Let

- $t_1'(\varphi_1) < t_2'(\varphi_1)$

- *or* $k_1 < k_2$ *and* $t_1'(\varphi_1) \leq 0$ *or* $t_2'(\varphi_1) \leq 0$

Then $t_1(\varphi) \leq t_2(\varphi), \varphi_1 \leq \varphi \leq \varphi_1 + \pi$ *and* $t_2(\varphi) \leq t_1(\varphi), \varphi_1 - \pi \leq \varphi \leq \varphi_1$

PROOF Observe that $t_2 - t_1 = b_3 \cos(\varphi - c_3)$ for some b_3, c_3. It is not hard to see that $c_3 \in \{\varphi_1 - \pi/2, \varphi_1 + \pi/2\}$ and $b_3 = (t_2 - t_1)'(\varphi_1)$. Let choose $c_3 = \varphi_1 - \pi/2$. It remains to show $0 < b_3 = (t_2 - t_1)'(\varphi_1)$.

If $k_1 < k_2$ this follows from equation (24)

$$\frac{t_1'(\varphi_1)}{t_2'(\varphi_1)} = \frac{k_2}{k_1} > 1$$

and $t_i'(\varphi_1) < 0, i = 1, 2$. ■

Lemma 3.2 *Let numbers* $0 = \varphi_0 < \varphi_1 < \ldots < \varphi_n < \pi/2$ *and* $k_i, i = 1, .., n$ *with* $0 < k_0 \leq k_1 \leq \ldots \leq k_{n-1}$ *be given. Further let numbers* $c_i \in [0, 2\pi), b_i, i = 0, \ldots, n-1$ *be given which define a function*

$$s(\varphi) = \sum_{i=0\ldots n-1} b_i \cos(\varphi - c_i)\, \chi_{[\varphi_i, \varphi_{i+1})} \quad , \tag{7}$$

where $\chi_{[\varphi_i,\varphi_{i+1})}$ *denotes the characteristical function of the interval* $[\varphi_i, \varphi_{i+1})$.

Let the function $s(\varphi)$ *be continuous and let the derivatives weighted with* k_i *be also continuous:*

$$b_i \cos(\varphi_{i+1} - c_i) = b_{i+1} \cos(\varphi_{i+1} - c_{i+1}) \quad , \quad i = 0, .., n-2 \tag{8}$$
$$k_i b_i \sin(\varphi_{i+1} - c_i) = k_{i+1} b_{i+1} \sin(\varphi_{i+1} - c_{i+1}) \quad , \quad i = 0, .., n-2 \tag{9}$$

Let $c_0 = 0$ *and let* $b_0 > 0$. *Then* $s(\varphi) > 0, 0 \leq \varphi \leq \varphi_n$.

PROOF Define auxilary functions $t_i(\varphi) := b_i \cos(\varphi - c_i)$. These functions are illustrated in figure 4. Multiplying the function $s(\varphi)$ by a constant we can assure $b_0 = 1$. We want to prove

$$\begin{aligned} & 0 < \cos(\varphi) = t_0(\varphi) \leq .. \leq t_j(\varphi)\, , \varphi_j \leq \varphi \leq \varphi_n < \pi/2 \\ & t_j'(\varphi_j) \leq 0 \end{aligned} \tag{10}$$

with help of lemma 3.1 through induction over $j = 0, .., n-1$.

For $j = 0$ inequality (10) is clearly fulfilled.

Suppose $i > 0$ and inequality (10) is fulfilled for $j = i - 1$. Observe that $t_{i-1}'(\varphi_{i-1}) \leq 0$ and $t_{i-1}(\varphi) > 0, \varphi_{i-1} \leq \varphi \leq \varphi_i$ implies $t_{i-1}'(\varphi_i) < 0$. Condition (9) gives then $t_i'(\varphi_i) < 0$. Thus the assumptions of lemma 3.1 are fulfilled for t_{i-1}, t_i with φ_i and we can show inequality (10). ■

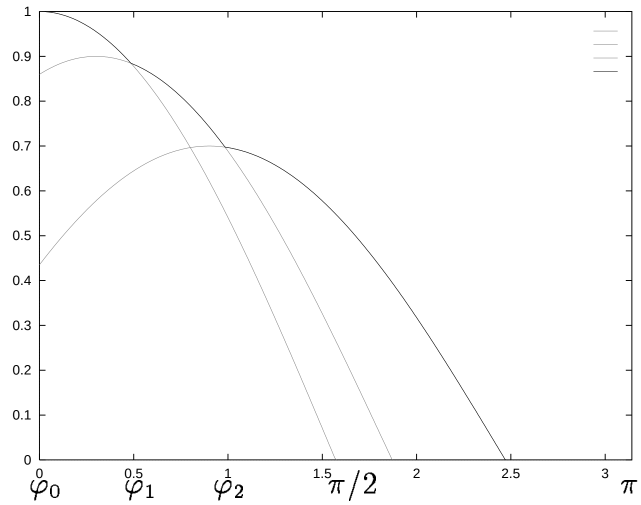

Figure 4: function s from equation (19) is the upper envelope and colored dark in case of decreasing k_i, functions t_i are colored light dark

Remark 3.1.1 *Lemma 3.2 could be sharpend to hold also for* $\varphi_n \leq \pi/2$ *if* $n > 1$*. To show this use* $k_0 < k_1$ *and show* $0 < c_1$*.*

Theorem 3.3 *Let an interior heterogeneous singular point* $x_l \in \overset{\circ}{\Omega}$ *be given and let the distribution of the weights* $k_{l,i}$ *be quasi-monotone with respect to* x_l*. Then the smallest non vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem is greater then* $(1/4)^2$*. This bound is sharp.*

PROOF

We choose a eigenfunction of the associated Sturm-Liouville eigenvalue problem with eigenvalue λ^2. The eigenfunction has the representation

$$s(\varphi) = \sum_{i=0...n-1} b_i \cos(\lambda(\varphi - c_i))\ \chi_{[\varphi_i, \varphi_{i+1})} \quad , \tag{11}$$

where $\chi_{[\varphi_i, \varphi_{i+1})}$ denotes the characteristical function of the interval $[\varphi_i, \varphi_{i+1})$ and $b_i, c_i \in [0, 2\pi), i = 0, .., n-1$ are real numbers. Possibly substituting c_i with $c_i + \pi$ or $c_i - \pi$ we can assume $b_i \geq 0$. The eigenfunction $s(\varphi)$ has to fulfill the following interface conditions

$$b_i \cos(\lambda(\varphi_{i+1} - c_i)) \ = b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \tag{12}$$
$$k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) \ = k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})) \tag{13}$$

Let us have a closer look onto $s(\varphi)$. This function is continuous and achieves therefore a minimum at a point φ_{min} and a maximum at φ_{max}.

Choose j such that $\varphi_{max} \in [\varphi_j, \varphi_{j+1})$. If φ_{max} lies in the interior of an interval $[\varphi_j, \varphi_{j+1}]$ we see $c_j = \varphi_{max}$ and hence $s(\varphi_{max}) > 0$

If $\varphi_{max} = \varphi_j$ for some j proceed as follows. Since φ_{max} is a maximum it is clear that $s(\varphi_j - 0)' \geq 0$ and $s(\varphi_j + 0)' \leq 0$. Condition (13) implies on the other hand that $s(\varphi_j - 0)'$ and $s(\varphi_j + 0)'$ can not have different signs. Hence $s(\varphi_j - 0)' = s(\varphi_j + 0)' = 0$ and also in this case holds $c_j = \varphi_{max}$. From this follows $s(\varphi_{max}) > 0$.

Similary $s(\varphi_{min}) < 0$ and we conclude that there are two points $\varphi_{zero,1}$ and $\varphi_{zero,2}$ with $s(\varphi_{zero,1}) = s(\varphi_{zero,2}) = 0$. Without loss of generality let us assume

$$\varphi_{zero,1} < \varphi_{max} < \varphi_{zero,2} < \varphi_{min}$$

Now we exploit the quasi-monotonicity condition. We want to show that there is a extremum φ_{ex} from $\{\varphi_{min}, \varphi_{max}\}$ and a point φ_{zero} from $\{\varphi_{zero,1}, \varphi_{zero,2}\}$ such that $k_x(\varphi)$ does not decrease when going from φ_{ex} to φ_{zero}. This means we want to show that $k_x(\varphi)$ is increasing on $[\varphi_{ex}, \varphi_{zero}]$ or decreasing on $[\varphi_{zero}, \varphi_{ex}]$.

To do so denote with I_{min}, I_{max} the intervals where $k_x(\varphi)$ reaches the minimum and maximum. The quasi-monotonicity condition implies that $k_x(\varphi)$ is monotone on intervals $[\delta_{min}, \delta_{max}]$ and $[\delta_{max}, \delta_{min}]$ with $\delta_{min} \in I_{min}, \delta_{max} \in I_{max}$. Without loss of generality assume that $I_{min} \cap [\varphi_{zero,2}, \varphi_{min}]$ and choose $\delta_{min} \in I_{min} \cap [\varphi_{zero,2}, \varphi_{min}]$.

Then there are two cases. In the first case $I_{max} \cap [\varphi_{zero,1}, \varphi_{zero,2}] \neq \emptyset$. Then $k_x(\varphi)$ increases on $[\varphi_{min}, \varphi_{zero,1}]$. In the second case $k_x(\varphi)$ is monotone in $[\varphi_{zero,1}, \varphi_{zero,2}]$ and either $k_x(\varphi)$ increases $[\varphi_{max}, \varphi_{zero,2}]$ or decreases on $[\varphi_{zero,1}, \varphi_{max}]$.

Multiplying with -1 in (11), rotatating the polar coordinate system and possible reflecting it on the x-axis we can assure

$$0 = \varphi_{ex} = \varphi_{max} < \varphi_{zero} < 2\pi \quad . \tag{14}$$

Remember that $k_x(\varphi)$ increases on $[\varphi_{ex}, \varphi_{zero}]$.

Choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$. We show as before $c_j = \varphi_{ex} = 0$. Further since φ_{ex} is a maximum $b_j > 0$.

Choose the largest m such that $\varphi_{j+m-1} < \varphi_{zero}$. We introduce an homogeneous transformation

$$F : [\varphi_{ex}, \varphi_{zero}] \to [0, \lambda\varphi_{zero}] \text{ with } F(\varphi) = \lambda\varphi \tag{15}$$

and define $s_F(F(\varphi)) = s(\varphi), \varphi \in [\varphi_{ex}, \varphi_{zero}]$.

Under this transformation we obtain a sequence $\widehat{\varphi}_0 < \widehat{\varphi}_1 < \ldots < \widehat{\varphi}_m$ where $\widehat{\varphi}_0 = 0, \widehat{\varphi}_i = F(\varphi_{i+j}), 0 < i < m$ and $\widehat{\varphi}_m = F(\varphi_{zero})$.

It follows that s_F fulfills

$$s_F(\varphi) = \sum_{i=0 \ldots n-1} \widehat{b}_i \cos(\varphi - \widehat{c}_i)\, \chi_{[\widehat{\varphi}_i, \widehat{\varphi}_{i+1})} \quad ,$$

and

$$\begin{aligned} \widehat{b}_i \cos(\widehat{\varphi}_{i+1} - \widehat{c}_i) &= \widehat{b}_{i+1} \cos(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \\ \widehat{k}_i \widehat{b}_i \sin(\widehat{\varphi}_{i+1} - \widehat{c}_i) &= \widehat{k}_{i+1} \widehat{b}_{i+1} \sin(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \quad , \end{aligned}$$

for some $\widehat{c}_i, \widehat{b}_i = b_{i+j}, \widehat{k}_i = k_{i+j}$ with $i = 0, .., m-1$. Due to the choice of $\varphi_{ex}, \varphi_{zero}$ we have $\widehat{c}_0 = 0$ and $s_F(\widehat{\varphi}_m) = 0$ with $\widehat{\varphi}_m = \lambda\varphi_{zero} < \lambda 2\pi$. Further $\widehat{k}_i \leq \widehat{k}_{i+1}, i = 0, .., m-1$.

Suppose $\lambda \leq 1/4$. Thus $\widehat{\varphi}_m < \lambda 2\pi \leq \pi/2$ and the partition and s_F defined on $[\widehat{\varphi}_0, \widehat{\varphi}_m]$ with the sequence $0 < \widehat{\varphi}_1 < \ldots < \widehat{\varphi}_m < \pi/2$ fulfills the assumption of lemma 3.2. We conclude from lemma 3.2 that s_F does not vanish on $[0, \widehat{\varphi}_m]$. But this is an contradiction with $s_F(\widehat{\varphi}_m) = 0$ and hence $1/4 < \lambda$.

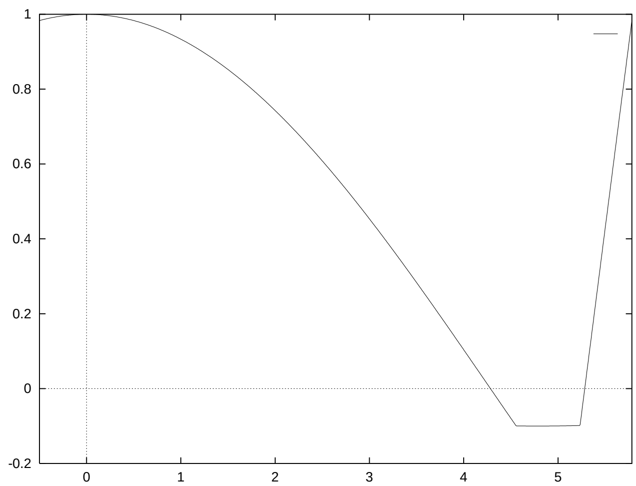

Figure 5: $s_1(\varphi)$ for $\varepsilon = 0.5$

From the above proof it is not hard to see how to construct such an eigenfunction $s_1(\varphi)$, that the bound $1/4 < \lambda$ is sharp. Choose $\varepsilon > 0$ and set $\lambda = \frac{\pi/2}{2\pi - 4\varepsilon}$. Set $\varphi_0 = -\varepsilon, c_0 = 0, b_0 = 1$. Define $c_2 = 2\pi - 2\varepsilon - \frac{3\pi}{2\lambda}$ and define $b_2 = cos(\lambda(\varphi_0 - c_0))/cos(\lambda(2\pi + \varphi_0 - c_2))$.

Further define $\varphi_1 = 2\pi - 3\varepsilon$ and choose $\varphi_2 > \varphi_1$ such that $cos(\lambda(\varphi_2 - c_2)) = cos(\lambda(\varphi_1 - c_0))$. Set $c_1 = 0.5(\varphi_1 + \varphi_2)$ and $b_1 = cos(\lambda(\varphi_1))/cos(\lambda(\varphi_1 - c_1))$.

Now one sees that $s_1(\varphi)$ achieves a maximum at $\varphi = 0$ and vanishes at $\varphi = 2\pi - 4\varepsilon$ and $\varphi = 2\pi - 2\varepsilon$. Furthermore a minimum is attained in $\varphi = c_1$. The function $s_1(\varphi)$ is plotted in figure 5.

Set $k_0 = 1$ and choose k_1, k_2 in such a way that equations (13) are fulfilled. We see that the smallest λ is obtained when the interior angle of a subdomain tends to 2π. ■

Theorem 3.4 *Let a heterogeneous singular point $x_l \in \partial\Omega$ on the boundary be given and let the distribution of the weights $k_{l,i}$ be quasi-monotone distributed with respect to x_l.*

Then the smallest non vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem λ^2 fulfills $(\frac{1}{4})^2 < \lambda^2$.

These bounds are sharp.

PROOF The proof runs similar to that of theorem 3.3. The eigenfunction of the associated Sturm-Liouville eigenvalue problem with eigenvalue λ^2 has the representation

$$s(\varphi) = \sum_{i=0 \dots n-1} b_i \cos(\lambda(\varphi - c_i)) \, \chi_{[\varphi_i, \varphi_{i+1})} \quad ,$$

where $\chi_{[\varphi_i,\varphi_{i+1})}$ denotes the characteristical function of the interval $[\varphi_i, \varphi_{i+1})$ and $b_i, c_i \in [0, 2\pi), i = 0, .., n-1$ are real numbers. The eigenfunction $s(\varphi)$ has to fulfill the interface conditions for $i = 0, .., n-2$

$$b_i \cos(\lambda(\varphi_{i+1} - c_i)) = b_{i+1} \cos(\lambda(\varphi_{i+1} - c_{i+1})) \tag{16}$$

$$k_i b_i \sin(\lambda(\varphi_{i+1} - c_i)) = k_{i+1} b_{i+1} \sin(\lambda(\varphi_{i+1} - c_{i+1})) \tag{17}$$

and some boundary conditions which will be specified later.

Since we deal with two different boundary conditions there are three possibilities how to combine them. We will treat each case separately. In any case $s(\varphi)$ is not a constant function. Denote with F_1, F_2 parts of the boundary on both sides of $x \in \partial\Omega \cap B_x(r)$.

Case I. $F_1 \subset \Gamma_D, F_2 \subset \Gamma_D$

The quasi-monotonicity condition means that the local weight function $k_x(\varphi)$ has not more then one local maximum $[\varphi_i, \varphi_{i+1}]$ and this local maximum is achieved in $[\varphi_0, \varphi_1]$ or $[\varphi_{n-1}, \varphi_n]$. We may suppose without loss of generality that the maximum of $k_x(\varphi)$ is obtained on $[\varphi_{n-1}, \varphi_n]$ and set $\varphi_{zero} = \varphi_n$. From the quasi-monotonicity condition we conclude that $k_x(\varphi)$ achieves its minimum on the interval $[\varphi_0, \varphi_1]$ and that $k_x(\varphi)$ is decreasing on $[\varphi_0, \varphi_n]$.

The function $s(\varphi)$ vanishes at φ_0 and φ_n. Since $\lambda > 0$ and since $s(\varphi)$ is continuous it achieves therefore in a point φ_{ex} an extremum $s(\varphi_{ex})$ different from 0.

We choose j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and show as in the proof of theorem 3.3 that $c_j = \varphi_{ex}$.

The quasi-monotonicity condition implies now that $k_x(\varphi)$ is monotonically increasing on $[\varphi_{ex}, \varphi_{zero}]$.

By rotation of the coordinate system, possible reflection on the x-axis and multiplication of $s(\varphi)$ with -1 we can assume $s(\varphi_{ex}) > 0$ and

$$0 = \varphi_{ex} < \varphi_{zero} \le \theta < 2\pi \quad , \tag{18}$$

for some θ.

Choose m such that $\varphi_{j+m-1} < \varphi_{zero}$.

We transform the sequence $\varphi_{ex} < \varphi_{j+1} < \dots < \varphi_{j+m-1} < \varphi_{zero} < \theta$ with the affine transformation defined in (15) and obtain a new sequence $\widehat{\varphi}_0 < \widehat{\varphi}_1 < \dots < \widehat{\varphi}_m$ where $\widehat{\varphi}_0 = 0, \widehat{\varphi}_i = F(\varphi_{i+j}) = \lambda\varphi_{i+j}, 1 < i < m-1$ and $\widehat{\varphi}_m = F(\varphi_{zero}) = \lambda\varphi_{zero}$.

Suppose $\lambda \leq \frac{1}{4}$. Defining $s_F(F(\varphi)) := s(\varphi)$ we obtain a scaled function which fulfills the modified conditions (16) and (17) that is

$$\begin{aligned} \widehat{b}_i \cos(\widehat{\varphi}_{i+1} - \widehat{c}_i) &= \widehat{b}_{i+1} \cos(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \\ \widehat{k}_i \widehat{b}_i \sin(\widehat{\varphi}_{i+1} - \widehat{c}_i) &= \widehat{k}_{i+1} \widehat{b}_{i+1} \sin(\widehat{\varphi}_{i+1} - \widehat{c}_{i+1}) \ , \end{aligned}$$

for some $\widehat{c}_i, \widehat{b}_i = b_{i+j}$ and $\widehat{k}_i = k_{i+j}$ with $i = 0, .., m-1$. Further $\widehat{c}_0 = 0$, $s_F(0) > 0$ and $s_F(\widehat{\varphi}_m) = 0$ with $\widehat{\varphi}_m < \lambda\theta \leq \frac{1}{4}\theta \leq \pi/2$ and $\widehat{k}_i \leq \widehat{k}_{i+1}, i = 0, .., m-1$.

Hence s_F fulfills the assumption of lemma 3.2 and it follows that s_F does not vanish on $[0, \widehat{\varphi}_m]$. But this is an contradiction since s_F vanishes at $\widehat{\varphi}_m$.

Case II. $F_1 \subset \Gamma_N, F_2 \subset \Gamma_D$

Suppose that the Dirichlet conditions are set on the angle φ_n. The quasi-monotonicity condition implies that the local weight function $k_x(\varphi)$ has not more then one local maximum $[\varphi_i, \varphi_{i+1}]$ and this local maximum is achieved for $i = n-1$.

Hence $k_x(\varphi)$ is monotone increasing on $[\varphi_{ex}, \varphi_{zero}]$.

Suppose $s(\varphi_{ex}) > 0$. Choose j in such a way that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$. If $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$ and $\varphi_{ex} \neq \varphi_0$ we conclude as before $c_j = \varphi_{ex}$. If $\varphi_{ex} = \varphi_0$ and hence $j = 0$ it follows from $\frac{\partial u}{\partial n} = 0$ that $s(\varphi)$ has vanishing right hand side derivative and thus $c_j = \varphi_{ex}$. We show as in the case I that $\frac{1}{4} \leq \lambda$. The case $\lambda = \frac{1}{4}$ occurs with $n_l = 1$ in a slit domain. To show that $\lambda < \frac{1}{4}$ for $n_l > 1$ use remark 3.1.1.

Case III. $F_1 \subset \Gamma_N, F_2 \subset \Gamma_N$

Since $s(\varphi) \not\equiv const$ and since this function is continuous on $[\varphi_0, \varphi_n]$ it attains a minimum and a maximum. As in the proof of theorem 3.3 we conclude that the extrema have different sign and that there is a point φ_{zero} where $s(\varphi)$ vanishes. The quasi-monotonicity condition implies that the local weight function $k_x(\varphi)$ has not more then one local maximum $[\varphi_j, \varphi_{j+1}]$.

We show that there is an extremum φ_{ex} of $s(\varphi)$ such that $k_x(\varphi)$ increases monotone, when going from φ_{ex} to φ_{zero} with positive or with negativ orientation. Suppose that $\varphi_{zero} < \varphi_j$. Then $k_x(\varphi)$ is monotone increasing on $[\varphi_0, \varphi_{zero}]$ and φ_{ex} is choosen from $[\varphi_0, \varphi_{zero}]$. Otherwise $k_x(\varphi)$ is monotone decreasing on $[\varphi_{zero}, \varphi_n]$ and φ_{ex} is from $[\varphi_{zero}, \varphi_n]$.

By reflection of the coordinate system on the x-axis we can assume that $k_x(\varphi)$ is monotone increasing on $[\varphi_0, \varphi_{zero}]$. Choose an extremum φ_{ex} of $s(\varphi)$ such that $s(\varphi)$ does not vanish on $[\varphi_{ex}, \varphi_{zero})$ and j such that $\varphi_{ex} \in [\varphi_j, \varphi_{j+1})$. We show as before $c_j = \varphi_{ex}$ and $\lambda > \frac{1}{4}$.

To prove sharpness we use the example from theorem 3.3. Denote with $[\varphi_0, \varphi_2]$ the closure of the support $\max\{0, s_1(\varphi)\}$. We define the eigenfunction $s_2(\varphi) := s_1(\varphi)$ on $[\varphi_0, \varphi_2]$. This eigenfunction has the eigenvalue $\lambda = \frac{\pi/2}{2\pi - 4\varepsilon}$. ■

Using the bound $\theta < 2\pi$ in inequality (18) it is not hard to show with the assumptions of the theorem 3.4 the improved bound $\left(\frac{2\pi}{4\theta}\right)^2 < \lambda^2$. Similary one could derive better estimates for the lowest nonvanishing eigenvalue in theorem 3.3 if one substitutes in equation (14) 2π by θ, where θ is the lenght of the largest angle, where $k_x(\varphi)$ is monotone.

3.2 Special cases

One can use the above techniques to derive in special situations sharper bounds on the minimal eigenvalue of the Sturm-Liouville problem. The idea is to prove that the distance $\varphi_{zero} - \varphi_{ex}$, where φ_{ex} and φ_{zero} are from equation (14), is bounded by an angle θ and to use the bound $\theta < 2\pi$. This yields improved estimates for the eigenvalues of the associated Sturm-Liouville eigenvalue problem. We illustrate this in the case of three subdomains sharing an interior heterogeneous singular point.

Lemma 3.5 *Let an heterogeneous singular point $x \in \bar{\Omega}$ be given such that if $x \in \mathring{\Omega}$, there are only three subdomains $x \in \partial\Omega_i$. In case of $x \in \partial\Omega$ there are only two subdomains $x \in \partial\Omega_i$ and the boundary conditions do not change in x . Let the maximal interior angle of these subdomains be smaller then θ.*

Then the smallest non vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem is greater then $(\pi/(2\theta))^2$. This bound is sharp.

PROOF The proof is a special case of these of theorem 3.3 and theorem 3.4. Let us first consider the case $x \in \mathring{\Omega}$. Since there are four points

$$\varphi_{zero,1} < \varphi_{max} < \varphi_{zero,2} < \varphi_{min}$$

and only three subdomains we conclude that there is an interval $[\varphi_j, \varphi_{j+1}]$ which contains an extremum $\varphi_{ex} \in \{\varphi_{min}, \varphi_{max}\}$ and a point $\varphi_{zero} \in \{\varphi_{zero,1}, \varphi_{zero,1}\}$.

As before we can assume $0 = \varphi_{ex} = \varphi_{max} < \varphi_{zero} < \theta$. Further $c_j = \varphi_{ex} = 0$.

Thus $s(\varphi) = b_j \cos(\lambda\varphi), \varphi \in [\varphi_{ex}, \varphi_{zero})$ and $s(\varphi_{zero}) = 0$. It is clear that $\lambda\varphi_{zero} = \pi/2$. This together with $\varphi_{zero} < \theta$ implies $\pi/(2\theta) < \lambda$.

The case $x \in \partial\Omega$ is done similary.

Sharpness of the bound follows from functions $s_1(\varphi)$, $s_2(\varphi)$ defined in the proof of theorem 3.3 and 3.4 respectively. ■

Another interesting situation is the special case of an interior heterogeneous singular point, where the interface consists of two lines intersecting with angle ψ. This situation was considered also in [6]. Let the weights k_i be distributed quasi-monotone with respect to this singular point.

We know that there are two extrema $\varphi_{max}, \varphi_{min}$ of $k_x(\varphi)$ and points $\varphi_{zero,1}, \varphi_{zero,2}$ at which $k_x(\varphi)$ vanishes. The points can be ordered like

$$\varphi_{zero,1} < \varphi_{max} < \varphi_{zero,2} < \varphi_{min} \quad .$$

The idea is to show, that there are always a points φ_{ex} and φ_{zero} such that either $k_x(\varphi)$ is monotone increasing on $[\varphi_{ex}, \varphi_{zero}]$ and the lenght of the interval $[\varphi_{ex}, \varphi_{zero}]$ is not greater then θ and $\theta < \pi$ or $k_x(\varphi)$ is monotone decreasing on $[\varphi_{zero}, \varphi_{ex}]$ and the lenght of the interval $[\varphi_{zero}, \varphi_{ex}]$ does not exceed $\theta < \pi$.

Consider the case that $\varphi_{ex,1}$, $\varphi_{ex,2}$ lie in intervals which are neighbors (theire closures intersect). Then there is a point $\varphi_{zero,j}$ contained in the same interval as $\varphi_{ex,1}$ or $\varphi_{ex,2}$ which proves $\theta \leq \psi < \pi$.

If $\varphi_{ex,1}$, $\varphi_{ex,2}$ lie in intervals which are not neighbors, then there a two possibilities. We check that there is $\varphi_{zero} \in \{\varphi_{zero,1}, \varphi_{zero,2}\}$ and an extremum $\varphi_{ex} \in \{\varphi_{ex,1}, \varphi_{ex,2}\}$ such that $k_x(\varphi)$ is monotone increasing, when going from φ_{ex} to φ_{zero}. Either φ_{zero} lies in an interval together with an extremum and then $\theta \leq \psi < \pi$ or it lies in a halfplane together with φ_{ex}. In the later case $\theta \leq \pi$.

Argumenting similary on checks that the case $\theta = \pi$ does not occur. We proved

Lemma 3.6 *Let an heterogeneous singular point $x_l \in \Omega$ be given such that the interface consists of two intersecting lines. Let the weights $k_{l,i}, i = 0, .., 3$ be distributed quasi-monotone with respect to x_l.*

Then the smallest non vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem is greater then $(1/2)^2$.

This bound is sharp.

PROOF The proof follows from the above considerations. To see that the bound is sharp, regard the special case $k_1 = k_2 = k_3$, that means the case of two subdomains only. Define an eigenfunction as done in equation (3.17) in [6]. ■

A special case of this lemma is the case of two subdomains sharing a singular point. Note that in this case we get the bounds as [6].

3.3 Regularity results in the quasi-montone case

Here we present our main results.

Theorem 3.7 *Let the distribution of weights $k_{l,i}, i = 0, .., n_l - 1$ be quasi-monotone with respect to all singular points x_l. The solution of (1) fulfills $u \in H^{1+1/4-\varepsilon}(\Omega), \varepsilon > 0$ if Ω contains slits and $u \in H^{1+1/4}(\Omega)$ if not . This is the maximal regularity independent of the bounds of k.*

Let a heterogeneous singular point $x_l \in \Omega$ be given and let U be a neighborhood containing no other singular points. Then $u \in H^{1+1/4}(\Omega \cap U)$.

These regularity results are optimal.

PROOF The assertion follows with lemma 2.6 from theorem 3.3,3.4, lemma 2.2 and lemma 2.1 . ■

Note that we get in principle the same regularity as in case of $k = 1$ if Ω contains no slits.

If one does not want to impose restrictions on k one has to restrict the number of subdomains n_l to which boundary the singular point x_l belongs.

Additionally in some special cases sharper bounds are possible, if one introduces further parameters depending on the geometry.

Theorem 3.8 *Let a singular point $x_l \in \Omega$ be given. Denote with n_l the number of subdomains $x \in \partial\Omega_i$ and let U be a neighborhood containing no other singular points.*

If x is an interior singular point let $n_l \leq 3$. If $x \in \partial\Omega$ then let $n_l \leq 2$ and additionally the boundary conditions do not change in x_l.

Then the solution of (1) fulfills $u \in H^{1+1/4}(\Omega \cap U)$.

This is the maximal regularity independent of k and the restrictions on n_l are sharp.

Denote with θ the largest interior angle of all subdomains $x \in \partial\Omega_i, i = 0, .., n_l - 1$.

Then solution of problem (1) fulfills $u \in H^{1+\pi/(2\theta)}(\Omega_i \cap U), i = 0, .., n_l - 1$.

PROOF One checks that under the above restrictions on n_l the weight function k is quasi-monotone. The first part follows from theorem 3.7. The second part follows from lemma 3.5 together with lemma 2.6. To see that the restrictions on n_l are sharp we refer to the examples from section 4.1. ■

Thus if three subdomains meet in an interior point regularity is $H^{5/4}$ in a vicinity of this point. Such a result seems to be new.

For heterogeneous singular points on the boundary with quasi-monotone distributed weights k_i the results could be sharpend for interior angles $\theta < 2\pi$ to hold local regularity $H^{1+max(1,\pi/(2\theta))}$. But nevertheless there are examples that if the quasi-monotonicity assumption is violated, the lowest non-vanishing eigenvalue will go to 0 even for arbitrary small interior angles θ and thus the maximal regularity is any case H^1 only.

The special case, where the interface consists locally of two intersecting lines, was already considered in [6]. We give an regularity result for the quasi-monotone case.

Theorem 3.9 *Let an interior heterogeneous singular point $x_l \in \Omega$ be given and let U be a neighborhood containing no other singular points. The interface consists a neighborhood of x of two intersecting lines . Let the distribution of weights $k_{l,i}, i = 0,..,3$ be quasi-monotone with respect to x_l.*

Then the solution of problem (1) fulfills $u \in H^{1+1/2}(\Omega_i \cap U), i = 1,..,4$. This bound is sharp.

PROOF The assertion follows from lemma 2.6 and lemma 3.6, lemma 2.2. To prove sharpness define a singular function similar to s_1 defined in the proof of theorem 3.3. ■

A special case of the last two theorems is the case of two subdomains sharing a singular point. In this sense results concerning two different weights from lemma 2.4 or lemma 2.3 that means from [6], [9], [11], [8] are a special case of theorem 3.8 or theorem 3.9.

Remark 3.3.1 *One notices that lemma 3.2 is the key ingredient for deriving lower bounds for the eigenfunctions of the Sturm-Liouville problem. It uses explicitely that the eigenfunctions of the Sturm-Liouville problem are piecewise scaled and shifted cosines. One could prove a similar result by using only concavity of the positive part of the eigenfunctions. In such a way extensions to other problems are possible.*

4 The general case

4.1 Example with detoriating regularity

It was shown that quasi-monotonicity of k is sufficient to prove regularity independent of k and better then H^1. Quasi-monotonicity is necessary in the sense, that there are no better spaces $H^s, 1 \leq s$ independent of the global bounds on k such that the solution is contained in H^s, then $H^s = H^1$, if the quasi-monotonicity condition is violated. In other words: without the quasi-monotonicity condition being fulfilled, the regularity will depend on the bounds on k.

We want to discuss such an example. This example is taken from [6]. Let the interface be the intersection of two lines. We define $s_3(\varphi)$ with eigenvalue λ^2

$$s_3(\varphi) := \begin{cases} \cos(\lambda(\pi - \theta - c))\, cos(\lambda(\varphi - \theta + b)) & \text{for } 0 \leq \varphi \leq \theta \\ \cos(\lambda b)\, cos(\lambda(\varphi - \pi + c)) & \text{for } \theta \leq \varphi \leq \pi \\ \cos(\lambda c)\, cos(\lambda(\varphi - \pi - b)) & \text{for } \pi \leq \varphi \leq \pi + \theta \\ \cos(\lambda(\theta - b))\, cos(\lambda(\varphi - \theta - \pi - c)) & \text{for } \pi + \theta \leq \varphi \leq 2\pi \end{cases}$$

The parameter $\theta \in (0, \pi/2]$ is the intersection angle between the two lines of the interface that means $\varphi_0 = 0, \varphi_1 = \theta, \varphi_2 = \pi, \varphi_3 = 2\pi - \theta$.

One can vary λ between $(0,1]$ to get different regularity of the singular function $u_3(r,\varphi) = r^\lambda s_3(\varphi)$ (without cut off function). We choose $\theta = \pi/2$, $b = 0.5\theta, c = \pi/2(1+\frac{1}{\lambda}) - b$.The corresponding values for the weight function k are $k_0 = k_2 = -\tan(\lambda c)$ and $k_1 = k_3 = \tan(\lambda b)$. Here the maxima of $k_x(\varphi)$ are achieved at $[\varphi_0, \varphi_1]$ and $[\varphi_2, \varphi_3]$ and hence the quasi-montonicity condition is violated. One checks that $k_1 \to \left(\lambda\frac{\pi}{4}\right)^{-1}$ and $k_2 \to \lambda\frac{\pi}{4}$ with $\lambda \to 0$.

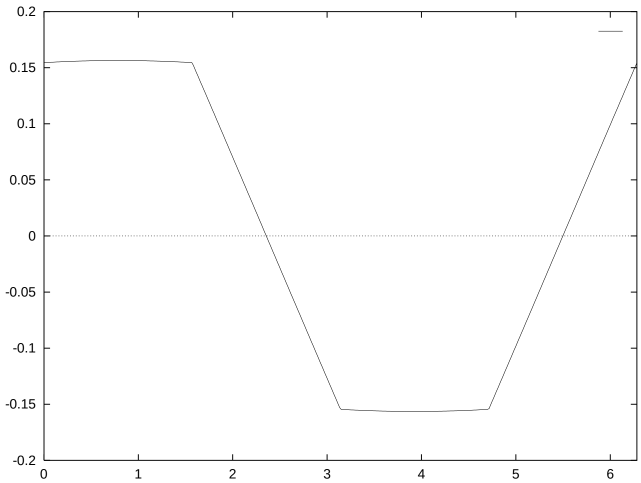

Figure 6: $s_3(\varphi)$ for $\lambda = 0.2$

In figure 6 we show a plot of $s_3(\varphi)$ for $\lambda = 0.2$, $\theta = \pi/2$ and $k_1 \approx 6.31, k_2 \approx 0.16$. The the singular function $u_3(r,\varphi)$ for $\lambda = 0.1$ (without cut off function) is depicted in figure 2.

One can use this example to show that quasi-monotonicity is also necessary for better regularity then H^1 for heterogeneous singular points on the boundary. In case of of Dirichlet boundary conditions take $u_3(r,\varphi)$ defined on the cone given by $[-\pi/4, 3\pi/4]$. If Neumann boundary conditions are imposed, take the sector defined by $[\pi/4, 5\pi/4]$. If the boundary conditions change, take $u_3(r,\varphi)$ defined on the cone given by $[-\pi/4, \pi/4]$.

4.2 Regularity results depending on global bounds

We saw in section 4.1 that in the case of a non quasi-monotone weigth function, the regularity may go down to H^1. This may happen if M_k gets large.

In this section we derive explicit bounds on the regularity depending on M_k. We show that $u \in H^{1+cM_k^{-1}}$, where c is a constant not depending on the problem. From the preceding section we know that this is the maximal (assymptotic) regularity.

Lemma 4.1 *Let a number* $0 < k_{min} < 1$ *and numbers* $0 = \varphi_0 < \varphi_1 < \ldots < \varphi_n = \arctan(k_{min}^{-1/2})$ *be given. Further let* $k_i, i = 1,..,n$ *with* $k_{min} \le k_i \le 1$ *be given. Let numbers* $c_i \in [-\pi/2, 3/2\pi)\,, b_i, i = 0, ..., n-1$ *be given which define a function*

$$s(\varphi) = \sum_{i=0\dots n-1} b_i \cos(\varphi - c_i)\, \chi_{[\varphi_i,\varphi_{i+1})} \quad , \tag{19}$$

where $\chi_{[\varphi_i,\varphi_{i+1})}$ denotes the characteristical function of the interval $[\varphi_i, \varphi_{i+1})$.

Let the function $s(\varphi)$ be continuous and let the derivatives weighted with k_i be also continuous:

$$b_i \cos(\varphi_{i+1} - c_i) \ = b_{i+1} \cos(\varphi_{i+1} - c_{i+1}) \ \ , \ \ i = 0,..,n-2 \tag{20}$$

$$k_i b_i \sin(\varphi_{i+1} - c_i) \ = k_{i+1} b_{i+1} \sin(\varphi_{i+1} - c_{i+1}) \ \ , \ \ i = 0,..,n-2 \tag{21}$$

Let $c_0 = 0, b_0 = 1$. Then $s(\varphi) > 0, 0 \le \varphi < \varphi_n$.

PROOF We define $t_i(\varphi) := b_i \cos(\varphi - c_i)$. We may suppose $k_0 > k_1$. Otherwise regard the discussion in the end of the proof.

The proof is done in three steps.

The idea is to bound function t_i from below by functions t_{j_i}. The we show that the function t_{j_i} is greater then a function u_{j_i}. In the last step we discuss the functions u_{j_i}.

In the first step our goal is to show that for $i = 0,..,n-1$ there is a index $0 \le j \le n-1$ and a number φ_j^- fulfilling

$$\begin{aligned} t_j(\varphi_j^-) = t_0(\varphi_j^-), 0 < \varphi_j^- \le \varphi_i \ \text{ and } t_j(\varphi) \le t_i(\varphi), \varphi_i \le \varphi \le \varphi_n \\ t_j(\varphi) \le t_0(\varphi), \varphi_j^- \le \varphi \le \varphi_n \quad . \end{aligned} \tag{22}$$

To denote the dependence of j from i we write j_i. In a second step we define functions u_{j_i} such that $u_{j_i}(\varphi) \le t_{j_i}(\varphi), \varphi_i \le \varphi \le \varphi_{i+1}$. In the third step we show that $0 < u_{j_i}(\varphi), \varphi \in [0, \varphi_n), i = 0,..,n-1$.

First Step. The proof of the first step is somewhat technical. We show equation (22) through induction with respect to $i = 1,..,n-1$.

Initial step $i = 1$**.** Simply define $\varphi_{j_1}^- := \varphi_1$ and $j_1 = 1$. As $k_0 > k_1$ lemma 3.1 implies $t_{j_1}(\varphi) \le t_0(\varphi), \varphi_{j_1}^- \le \varphi \le \varphi_n$. We showed equation (22) for $i = 1$.

Induction with $i > 1$**.** Set $J = j_{i-1}$. There are two cases.

In the first case $t_J(\varphi) \le t_i(\varphi), \varphi_i \le \varphi \le \varphi_n$. We define $j_i := J$ and proved (22).

In the second case we define $j_i := i$. This case is illustrated in figure 7. There is a $\varphi^+ \in (\varphi_i, \varphi_n]$ with

$$t_J(\varphi^+) = t_i(\varphi^+) \quad .$$

Further due to equations (20), (22) $t_J(\varphi_i) \le t_{i-1}(\varphi_i) = t_i(\varphi_i)$. The last equations imply $0 \le (t_J - t_i)'(\varphi^+)$. We may use lemma 3.1 to show $t_J(\varphi) \le t_i(\varphi), 0 \le \varphi \le \varphi^+$.

From equation (22) and from $\varphi_J^- < \varphi_i < \varphi^+$ follows

$$t_0(\varphi_J^-) = t_J(\varphi_J^-) \le t_i(\varphi_J^-) \quad .$$

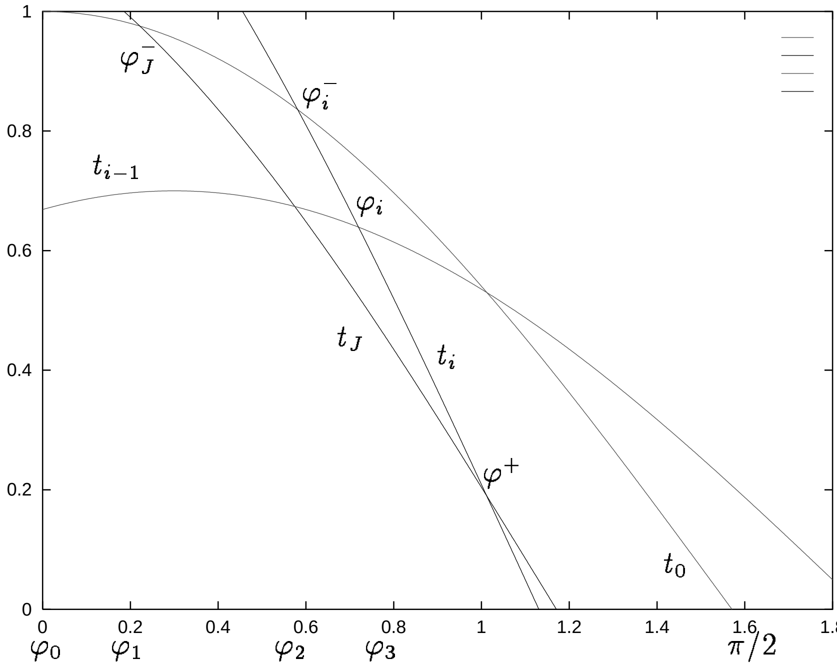

Figure 7: step $i = 3$ is illustrated, here $J = 1$, note $\varphi_J \leq \varphi \leq \varphi_i \leq \varphi^+$

We conclude that there is a φ fulfilling $\varphi_J^- \leq \varphi \leq \varphi_i$ with $t_0(\varphi) = t_i(\varphi)$. We define $\varphi_i^- := \varphi$. It is not hard to see that $t_i(\varphi) \leq t_0(\varphi), \varphi_i^- \leq \varphi_i \leq \varphi \leq \varphi_n$ and hence we proved (22).

Second Step. Set $j = j_i$ and define u_j by

$$u_j = a_j \cos(\varphi - d_j) \quad , \tag{23}$$

where a_j, d_j are choosen in such a way that the following interface conditions are fulfilled

$$t_0(\varphi_j^-) \ = u_j(\varphi_j^-) \tag{24}$$
$$k_0 t_0'(\varphi_j^-) \ = k_{min} u_j^-(\varphi_j) \quad , \tag{25}$$

for $\varphi_j^- \in [0, \varphi_n]$. Since $u_j(\varphi_j^-) = t_0(\varphi_j^-) = t_j(\varphi_j^-)$ and $k_{\min} < k_j$ we conclude with help of lemma 3.1 that $u_j(\varphi) \leq t_j(\varphi), \varphi_j^- \leq \varphi_j \leq \varphi \leq \varphi_n \leq \pi/2$. This yields together with equation (22)

$$u_j(\varphi) \leq t_j(\varphi) \leq t_i(\varphi) \quad , \quad \varphi_i \leq \varphi \leq \varphi_n \quad . \tag{26}$$

Third Step. We want to show $0 < u_j(\varphi), \varphi \in [0, \varphi_n), i = 0, .., n-1$ by showing that $\varphi_n - \pi/2 < d_j$.

Therefore we choose $\varphi = \varphi_j^-, d := d_j$ and rewrite (24)

$$a_j \cos(\varphi - d) = \cos(\varphi)$$
$$k_{min} a_j \sin(\varphi - d) = k_0 \sin(\varphi) \quad .$$

We now look for the minimal value of d depending on φ. Set $k = k_{min}/k_0$. Clearly $k \geq k_{min}$ and divide the two equations be each other to obtain

$$d(\varphi) = \varphi - \arctan(k^{-1}\tan(\varphi)) \quad . \tag{27}$$

Differentiating with respect to φ reveals that minimum is attained for $\tan(\varphi) = k^{1/2}$. Insertion of the minimum leads to $d(\varphi) = \arctan(k^{1/2}) - \arctan(k^{-1/2}) > \arctan(k^{1/2}) - \pi/2 \geq \arctan(k_{min}^{1/2}) - \pi/2$.

Now we collect the results from the previous three steps to obtain from inequality (26)

$$0 < u_j(\varphi) \leq t_j(\varphi) \leq t_i(\varphi) \quad , \quad \varphi_i \leq \varphi \leq \varphi_{i+1}, i = 1, .., n-1$$

In the case that $k_0 \leq k_1$ denote with j the first index $j = i$ such that $k_i > k_{i+1}$. If there is no such index then $k_0 \leq k_1 \leq ... \leq k_{n-1}$ and we use lemma 3.2 to prove the assertion. If $j < n-1$ calculation shows that $c_0 \leq c_1 \leq ... \leq c_j \leq \pi/2$. This implies that t_i does not vanish on $[\varphi_0, \varphi_n], 0 \leq i \leq j$ and we are let to prove the assertion for functions $t_i, i > j$. We have to show $t_i > 0, i = j, .., n-1$ on $[0, c_j]$ since we already showed $t_i > 0, i = j, .., n-1$ and on $[c_j, \varphi_n]$. Note that from the last fact and from the fact that $t_i'(\varphi_i) \leq 0$ follows $t_i(\varphi) > 0, i = j, .., n-1$ on $[0, c_j]$. ■

Theorem 4.2 *Let an heterogeneous singular point* $x_l \in \overset{\circ}{\Omega}$ *be given and let* $c_1 M^{-1} \leq k_{l,i} \leq c_1 M, i = 0, .., n_l - 1$ *for some constants* c_1. *Then the smallest non vanishing eigenvalue of the associated Sturm-Liouville eigenvalue problem is greater then* cM^{-1}, *where c is a constant independent of the problem. This bound is sharp.*

PROOF Multiplying k with a constant we may assume $k_{min} := M^{-2} \leq k_{l,i} \leq 1$. As in the proof of theorems 3.3,3.4 we conclude that there are points $\varphi_{max}, \varphi_{zero}$ such that $s(\varphi)$ achieves a maximum in φ_{max} and vanishes in φ_{zero}. We choose j such that $\varphi_{max} \in [\varphi_j, \varphi_{j+1})$ and show as before $c_j = 0$. Further we choose the maximal n such that $\varphi_{j+1} < ... < \varphi_n \leq \varphi_{zero}$. Changing the coordinate system we may set $\varphi_{max} = 0 < \varphi_{zero} < 2\pi$.

We introduce the homogenous scaling $F : [0, \varphi_{zero}] \to [0, \widehat{\varphi}_{zero}]$ with $F(\varphi) = \widehat{\varphi} = \lambda\varphi$. Define $s_F(F(\varphi)) := s(\varphi), \varphi \in [\varphi_{ex}, \varphi_{zero}]$. We have $\widehat{\varphi}_{zero} \leq \lambda 2\pi$.

Observe that $s_F(\widehat{\varphi})$ fulfills the assumption of lemma 4.1. We conclude from lemma 4.1 that since s_F vanishes in $\widehat{\varphi}_{zero}$ that $M^{-1} = k_{min}^{1/2} \approx \arctan(k_{min}^{1/2}) < \widehat{\varphi}_{zero} \leq \lambda 2\pi$.

Function s_3 defined in section 4.1 shows the sharpness of the bound. ■

Theorem 4.3 *The solution of problem (1) fulfills* $u \in H^{1+\min\{1, cM_k^{-1}\}}(\Omega), \varepsilon > 0$ *where c is a constant not depending on the problem.*

Let an heterogeneous singular point $x_l \in \overset{\circ}{\Omega}$ *be given and let* $c_1 M^{-1} \leq k_{l,i} \leq c_1 M, i = 0,..,n_l - 1$ *for some constants* c_1, M. *Let* $x \in U$ *be a neighborhood containing no other singular points. Then* $u \in H^{1+\min\{1,cM^{-1}\}}(U \cap \Omega), where c is a constant independent of the problem.$

This is the maximal (assymptotic) regularity independent k.

PROOF The assertion follows with lemma 2.6 from theorem 4.2 and lemma 2.1, lemma 2.2. Sharpness follows from the function u_3 defined in section 4.1. ■

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