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A proof of a Shilnikov theorem for C^1-smooth dynamical systems

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Abstract

Dynamical systems with a homoclinic loop to a saddle equilibrium state are considered. Andronov and Leontovich have shown (see [1939], [1959]) that a generic bifurcation of a two-dimensional C^1-smooth dynamical system with a homoclinic loop leads to appearance of a unique periodic orbit. This result holds true in the multi-dimensional setting if some additional conditions are satisfied, which was proved by Shilnikov [1962, 1963, 1968] for the case of dynamical systems of sufficiently high smoothness. In the present paper we reprove the Shilnikov theorem for dynamical systems in C^1.

1 Main theorem

Let us consider a family of C^1-smooth vector fields X_μ on an $(n+1)$-dimensional manifold. We assume that the vector field X_μ depends on μ continuously, along with the first derivatives. Let the following hold.

(A) *The system X_μ has a saddle equilibrium state O, and the roots $\lambda_n, \dots, \lambda_1, \gamma$ of the characteristic equation of the linearized system at the point O at $\mu = 0$ satisfy the following inequalities* $\operatorname{Re}(\lambda_n) \leq \ldots \leq \operatorname{Re}(\lambda_1) < 0 < \gamma$.

Thus, we can introduce local coordinates (x, y) ($x \in R^n$, $y \in R^1$) in a small neighborhood of O such that the system X_μ takes the following form near O at $\mu = 0$

$$\left\{ \begin{array}{rcl} \dot{x} & = & \Lambda x + \dots \,, \\ \dot{y} & = & \gamma y + \dots \,. \end{array} \right. \tag{1.1}$$

Here Λ is an $(n \times n)$-matrix with the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$; the dots stand for nonlinearities.

The unstable manifold W^u of O is one-dimensional (it is tangent to y-axis at O) and consists of three orbits: the point O itself and two *separatrices* leaving O in the opposite directions. The stable manifold W^s is n-dimensional; it divides a small neighborhood of the equilibrium into two parts: U^+ and U^- (see Fig.1). Assume that

(B) *at $\mu = 0$ one of the separatrices Γ is homoclinic to O; i.e.,* $\Gamma \subset (W^s \cap W^u)$.

Without loss of generality we assume that the separatrix Γ leaves the point O towards the region U^+ (i.e. towards positive y; see Fig.1).

We consider behavior of orbits in a small neighborhood U of the homoclinic loop $\mathcal{L} = O \cup \Gamma$.

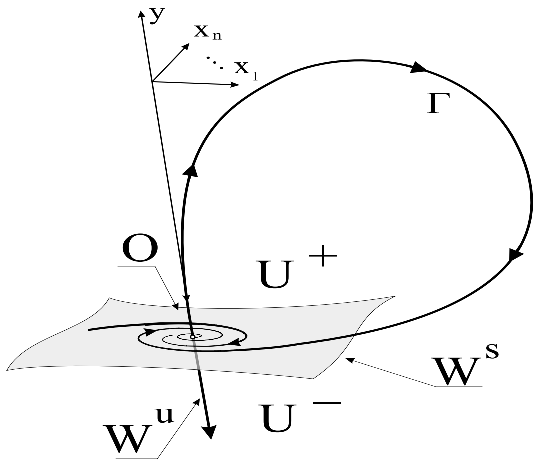

Figure 1: The system X_0 has a homoclinic orbit Γ to the saddle equilibrium O. The stable manifold W^s divides a small neighborhood of O into two regions: U^+ and U^-.

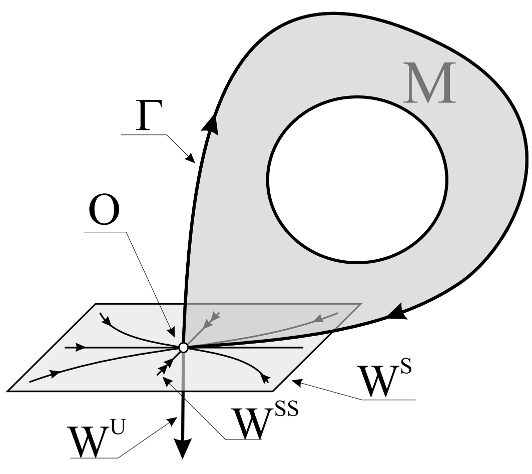

Figure 2: A two-dimensional invariant manifold M exists near the homoclinic loop $\mathcal{L} = O \cup \Gamma$ if and only if the leading eigenvalue λ_1 is real and simple, the loop does not lie in the strong stable manifold W^{ss} and some additional transversality conditions are fulfilled.

For systems on the plane ($n = 1$) this problem was completely solved by Andronov and Leontovich [1938], [1951], [1959] (see also [Andronov *et al.* 1967]). In particular, it was shown that if the saddle value $\sigma = \lambda_1 + \gamma$ is non-zero, the bifurcations of the homoclinic loop produce only one periodic orbit. Thus, the bifurcation of such homoclinic loop was proven to be one of the four main bifurcations of the birth of a limit cycle on a plane.

The analogous multidimensional problem was considered by Shilnikov [1963]. From the modern point of view, one should immediately get the result similar to the two-dimensional one, if a smooth, normally-hyperbolic two-dimensional invariant manifold exists near the homoclinic loop (see Fig.2). However, the existence of such a manifold requires some extra conditions. First, the nearest to the imaginary axis negative eigenvalue λ_1 has to be real and simple. The orbit Γ should not lie in the strong stable submanifold W^{ss} which corresponds to the eigenvalues $\lambda_n, \ldots, \lambda_2$. Moreover, some transversality conditions must be satisfied by the flow map near Γ (see [Turaev, 1984; 1996; Shashkov, 1991; 1994; Homburg, 1996; Sandstede, 1994 - this also includes the PDE case; Shashkov and Turaev, 1999]).

In fact, the existence of a two-dimensional invariant manifold is not so much relevant for the dynamics near a homoclinic loop. It was a remarkable discovery of Shilnikov [1965; 1970] that if the characteristic exponents at the point O satisfy a condition which reads in our case as $\mathrm{Im}\,(\lambda_1) \neq 0$, $-\mathrm{Re}\,(\lambda_1) < \gamma$, then generically there exist non-trivial hyperbolic sets in a small neighborhood of the loop. In other words, the dynamics near a homoclinic loop to a saddle-focus with positive saddle value is quite opposite to that in dimension two. By now, the Shilnikov homoclinic loop is one of the most simple in the setting and the most complicated in dynamics models of chaotic behavior.

From the other hand, in the case of negative saddle value, i.e., if

(C) $\sigma = \mathrm{Re}\,(\lambda_1) + \gamma < 0$,

the bifurcation of the homoclinic loop leads to the appearance of only one stable periodic orbit, exactly as for the systems on a plane, no matter is the equilibrium state O a saddle or a saddle-focus [Shilnikov, 1963].

In the present paper we reprove the corresponding result for C^1-smooth systems. In order to describe bifurcations of X_μ, we introduce the small parameter μ as described below. Namely, we suppose that

(D) *the separatrix* Γ *does not belong to* W^s *if* $\mu \neq 0$.

It follows from the continuity with respect to μ that after leaving a small neighborhood of O the separatrix Γ at $\mu \neq 0$ stays close to the locus of the homoclinic loop $\mathcal{L}$, until it enters the small neighborhood of O once again. Without loss of generality we assume that Γ enters U^+ at $\mu > 0$ and U^- at $\mu < 0$.

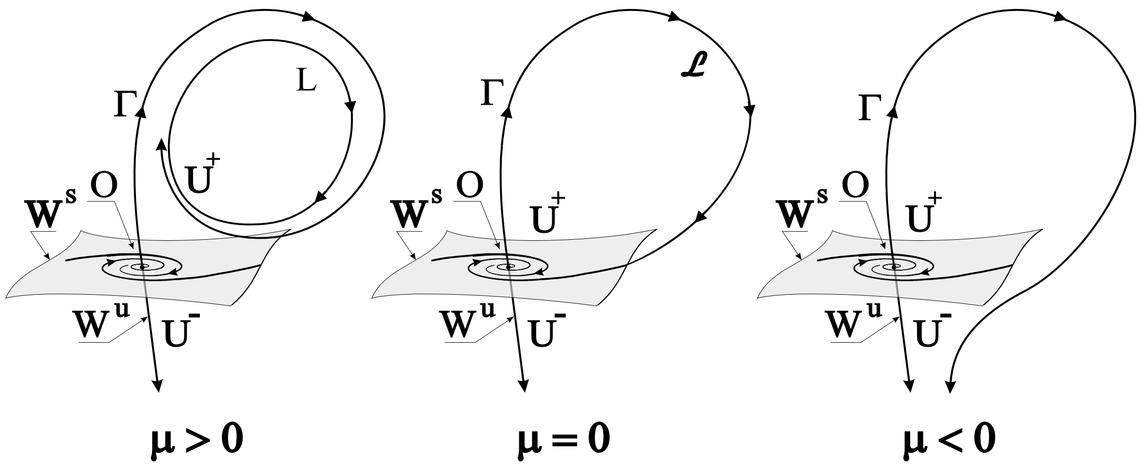

Figure 3: At $\mu > 0$, a stable periodic orbit L is born from the loop $\mathcal{L}$ ($\mu = 0$). All the orbits (except for those tending to O) leave at $\mu < 0$.

Theorem 1.1 (see Fig.3). *If conditions* **(A)** – **(D)** *are fulfilled, then there exists a small neighborhood U of the homoclinic loop such that at all small $\mu > 0$ the system has a unique periodic orbit L, which is stable and, in particular, the separatrix Γ tends to L as $t \to +\infty$. The other orbits in U which do not lie in W^s either tend to L or leave U in a finite time. At $\mu = 0$ the periodic orbit becomes a homoclinic loop (which may attract some orbits of $U \backslash W^s$, the other orbits leave U). At all small $\mu < 0$ all orbits of $U \backslash W^s$ leave U in a finite time.*

Proof. We will follow the lines of the original proof in Shilnikov [1963]. Take a small cross-section S^0 to the stable manifold W^s so that to intersect the homoclinic loop at $\mu = 0$. The stable manifold of O divides S^0 into two regions: $S^0_+ = S^0 \cap U_+$ and $S^0_- = S^0 \cap U_-$ (i.e. S^0_+ lies above W^s; see Fig.4). Let P^*_μ be the intersection point $\Gamma \cap S^0$. At $\mu = 0$ the separatrix Γ forms a homoclinic loop, so $P^*_0 \in \{S^0 \cap W^s\}$. Thus, the intersection point exists for all small μ. Let $d(\mu)$ be the distance from P^*_μ to $W^s \cap S^0$, taken with the sign, positive when $P^*_\mu \in S^0_+$ and negative when $P^*_\mu \in S^0_-$. By virtue of assumption **(D)** , the sign of $d(\mu)$ coincides with the sign of μ (Fig.4).

An orbit which starts with a point $P \in S^0_+$ goes close to the stable manifold in a small neighborhood of O and then leaves the neighborhood, close to the separatrix Γ. If μ is sufficiently small, then moving along Γ, such orbit intersects S^0 again at some point $\bar{P}$ close to the point P^*_μ. Thus, the Poincaré map $T : P \mapsto \bar{P}$ is defined on S^0_+ close to W^s. On $W^s \cap S^0$ the map T is defined by continuity: $T(W^s \cap S^0) = P^*_\mu$. The orbits which start on S^0_- leave a small neighborhood of O close to the other separatrix and, therefore, they leave the neighbourhood U of the homoclinic loop under consideration. Thus, the Poincaré map T is not defined on S^0_-.

Shilnikov proves in [1963] that if the saddle value σ (see **(C)**) is negative, then the map T is *strongly contracting* at small μ (i.e. $\text{dist}(TP_1, TP_2) \leq K\text{dist}(P_1, P_2)$ where the contraction factor K tends uniformly to zero as both P_1, P_2 tend to $W^s \cap S^0$). Then, he artificially defines the map T on S^0_-. We will do the same, assuming, say,

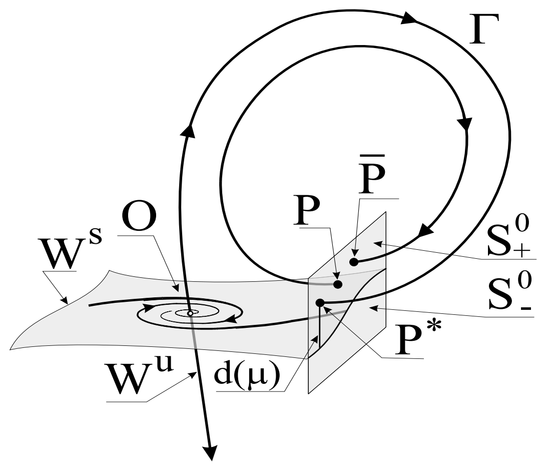

Figure 4: The Poincaré map $T : S^0_+ \to S^0$ is defined near $S^0 \cap W^s$. The image $T(W^s \cap S^0) = P^*_\mu$ is defined by continuity. The point $P^*_\mu = \Gamma \cap S^0$ lies on the distance $|d(\mu)|$ from $S^0 \cap W^s$.

$TP \equiv P^*_\mu$ at $P \in S^0_-$. This extended map is contracting too (with the same factor K). In particular, at $\mu = 0$, this map takes a small neighborhood of the point P^*_0 into itself. The same, obviously, holds true for all small μ. Thus, the Banach principle can be applied which gives the existence of a unique fixed point; moreover, this point attracts iterations (by the map T extended onto all S^0) of every initial point on S^0.

At $\mu \le 0$ the fixed point is, by definition, the point P^*_μ. Since it lies in the region $S^0_- \cup (W^s \cap S^0)$ where the Poincaré map is not defined, no periodic orbit corresponds to this point: it is a homoclinic loop at $\mu = 0$ or just a fake at $\mu < 0$.

At $\mu > 0$, the fixed point is the limit of the iterations of the point P^*_μ. This point is the image of the line $W^s \cap S^0$ and it lies on the distance $d(\mu)$ from this line. Therefore, due to the contraction, all the iterations of this point (and their limit - the fixed point) lie in the ball of radius $\dfrac{K}{1-K} d(\mu)$ with the center at P^*_μ. If μ is sufficiently small, one can assume $K < \dfrac{1}{2}$ and in this case the radius is less than $d(\mu)$. Thus, at $\mu > 0$ the fixed point of the extended map belongs to the region S^0_+. Hence, it is the fixed point of the true Poincaré map, which a periodic orbit of the system corresponds to.

All this is in a complete correspondence with the statement of the theorem. The key point in the proof is to show that the Poincaré map is strongly contracting. To this aim, computations involving explicitly second derivatives of the right-hand sides of the system were used in Shilnikov [1963]. Below (Sections 2 and 3) we prove the contraction in the case of minimal smoothness (C^1), by means of the method of a boundary value problem of Shilnikov [1967]. End of the proof.

At the first glance, the transition from, say, C^2 to C^1 is an insignificant step. However, the dynamical systems of low smoothness appear naturally when studying high-dimensional systems reduced onto a normally hyperbolic invariant manifold (say, to inertial manifold, or to a non-local center manifold as in the example below). The smoothness of such a manifold - and therefore the smoothness of the reduced system - does not correlate with the smoothness of the original system. In particular, the conditions for the existence of a C^2-smooth invariant manifold are much more restrictive than for a C^1 one. Thus, the study of the bifurcational problems in as less smoothness as possible may be crucial for a rigorous description of the high-dimensional dynamics.

As an example, consider a C^1-version of the result of Shilnikov [1968]: a generalization of Theorem 1.1 onto the case where the dimension of the unstable manifold of O is larger than one. Namely, let X_μ be a continuous family of C^1-smooth dynamical systems on an $(n+m)$-dimensional manifold. Let us modify conditions **(A)** , **(B)** in the following way.

(A′) *The system X_μ has a hyperbolic equilibrium state O, and the characteristic exponents $\lambda_n, \cdots, \lambda_1, \gamma, \gamma_2, \cdots, \gamma_m$ at the point O at $\mu = 0$ satisfy the following condition:* $\mathrm{Re}\,(\lambda_n) \leq \cdots \leq \mathrm{Re}\,(\lambda_1) < 0 < \gamma < \mathrm{Re}\,(\gamma_2) \leq \cdots \leq \mathrm{Re}\,(\gamma_m)$.

(B') *At $\mu = 0$ there exists a homoclinic orbit Γ; i.e., $\Gamma \subset (W^s \cap W^u)$.*

The conditions **(C)** , **(D)** remain unchanged.

In this case the dimension of the unstable manifold W^u is equal to m and, moreover, there exists an $(m-1)$-dimensional strong unstable invariant submanifold $W^{uu} \subset W^u$. The characterizing feature of W^{uu} is that all orbits in it are tangent to the linear subspace which corresponds to the eigenvalues $\gamma_2, \ldots \gamma_m$ whereas all orbits of $W^u \backslash W^{uu}$ are tangent to the eigendirection corresponding to the *leading* eigenvalue γ. Assume that

(E) *the homoclinic orbit Γ does not belong to W^{uu} (see Fig.5).*

The next assumption is necessary [Turaev, 1996] for the presence of an $(n+1)$-dimensional global invariant manifold (as well as condition **(E)**). Denote by $E^{s+} \subset R^{n+1}$ the invariant subspace of the system X_0 linearized at the point O, corresponding to the eigenvalues $\lambda_n, \ldots, \lambda_1, \gamma$. It is well known (see for instance [Hirsch *et al.*, 1977]) that there exists an invariant C^1-smooth manifold M^{s+} tangent to E^{s+} at O (see Fig.5). The manifold M^{s+} contains W^s. It is not uniquely defined but any two of them have the same tangent at each point of W^s. We require the following condition to be fulfilled.

(F) *The manifold M^{s+} is transverse to the manifold W^u at each point of Γ (see Fig.5).*

Note that the transversality must be verified only at one point on Γ because the manifolds M^{s+} and W^u are invariant with respect to the flow defined by the system

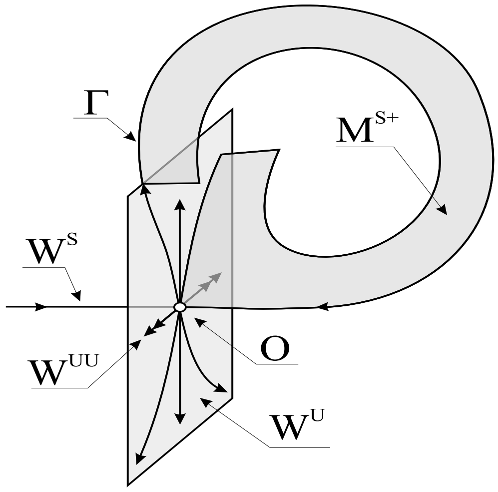

Figure 5: The orbit Γ does not lie in the strong unstable submanifold W^{uu}. The extended stable manifold M^{s+} is transverse to the unstable manifold W^u.

X_0. One can check, that condition **(F)** is equivalent to the requirement of Shilnikov [1968] of non-vanishing of some specific determinant. Note also that conditions **(E)** and **(F)** are not so much restrictive because they are fulfilled in general position.

It is shown in Shashkov & Turaev [1999] (in the case of higher smoothness in Turaev [1991], Shashkov [1994] or Homburg [1993; 1996]) that when conditions **(A')** , **(B')** , **(E)** and **(F)** are fulfilled, then
there exists a small neighborhood U of the homoclinic orbit Γ such that, for all μ small enough, the system X_μ has an $(n+1)$-dimensional repelling invariant C^1-manifold $\mathcal{M}_\mu$ depending continuously on μ and such that any orbit not lying in $\mathcal{M}_\mu$ leaves U as $t \to +\infty$. The manifold $\mathcal{M}_\mu$ is tangent at the point O to the linear subspace corresponding to the eigenvalues $(\lambda_n, \ldots, \lambda_1, \gamma)$.

Due to this result, the study of the $(n+m)$-dimensional system is reduced to the study of the $(n+1)$-dimensional system on the invariant manifold $\mathcal{M}_\mu$. Evidently, for the reduced system conditions **(A)** – **(D)** hold, therefore, Theorem 1.1 is immediately transferred to the multidimensional case. Note that the periodic orbit L born from the loop is now stable only on the invariant manifold $\mathcal{M}_\mu$ and since the manifold is repelling, the orbit L is unstable in the normal directions. Thus, in this case, L is a saddle periodic orbit with m-dimensional unstable and $(n+1)$-dimensional stable manifolds.

2 The Shilnikov Boundary Value Problem

In order to prove Theorem 1.1 we need appropriate estimates (strong contraction) for the Poincaré map near the homoclinic loop $\mathcal{L}$. The study of the solutions near

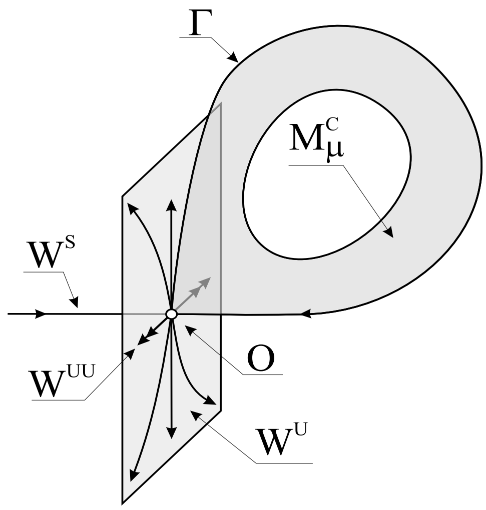

Figure 6: There exists an $(n+1)$-dimensional C^1-smooth center invariant manifold M^c_μ if conditions **(A')** , **(B')** , **(E)** and **(F)** are fulfilled.

the equilibrium state is the most complicated point here because the flight time near O is unbounded and, therefore, we need the estimates which hold true for the unboundedly large times. The question on the local estimates does not appear if the system can be linearized in the neighborhood of the equilibrium point. However, the smooth linearization requires a lot of resonance restrictions plus extra-smoothness. Therefore, to find suitable estimates near the equilibrium we use the method which is based on the consideration of some boundary value problem (see [Shilnikov, 1967; 1970], [Ovsyannikov & Shilnikov, 1986; 1991], [Shilnikov *et al.*, 1998]). In this section we investigate solutions of the Shilnikov boundary value problem for the case where the smoothness of the system is C^1 only.

Let us introduce local coordinates (x, y) ($x \in R^n$, $y \in R^1$) in a neighborhood of the saddle O such that the system X_μ takes the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & \Lambda x + f(x,y,\mu)\ , \\ \dot{y} & = & \gamma y + g(x,y,\mu)\ . \end{array} \right. \tag{2.1}$$

Here Λ is a matrix $(n \times n)$ such that $\mathrm{Spectr}\,(\Lambda) = \{\lambda_1 \ldots \lambda_n\}$. The functions f and g are smooth with respect to (x, y) and depend continuously on μ along with the derivatives. Moreover,

$$f(0,0,\mu) = 0,\ \ g(0,0,\mu) = 0,\quad \left.\frac{\partial(f,g)}{\partial(x,y)}\right|_{(x,y,\mu)=0} = 0\ . \tag{2.2}$$

According to Shilnikov [1967], for any $\tau > 0$ and x^0 and y^1 small enough, in a small neighborhood of O there exists a unique orbit $\{x(t), y(t)\}_{t\in[0,\tau]}$ of system (2.1) which satisfies the following boundary conditions

$$x(0) = x^0, \quad y(\tau) = y^1. \tag{2.3}$$

Let us denote the solution of this boundary value problem as

$$x(t) = x(t; x^0, y^1, \tau, \mu) , \quad y(t) = y(t; x^0, y^1, \tau, \mu). \tag{2.4}$$

The theorem below follows from Shilnikov [1967] (we give here an adjusted proof for the sake of completeness; $\|(x,y)\|$ denotes $\max(\|x\|, \|y\|)$).

Theorem 2.1 *For any small $\varepsilon > 0$, if $\|(x^0, y^1)\| \le \varepsilon$, then $\|(x(t), y(t))\| \le \varepsilon$ in (2.4) for all $t \in [0, \tau]$ and all small μ. The solution (2.4) depends smoothly on $(t; x^0, y^1, \tau)$ and, along with the derivatives, depends continuously on μ. The following estimates hold for the derivatives:*

$$\left\| \frac{\partial(x,y)}{\partial x^0} \right\| \le C e^{-\alpha t} , \quad \left\| \frac{\partial(x,y)}{\partial y^1} \right\| \le C e^{-\beta(\tau - t)} , \tag{2.5}$$

where C, α and β are some constants such that

$$C > 0 , \quad \mathrm{Re}\,(\lambda_n) \le \cdots \le \mathrm{Re}\,(\lambda_1) < -\alpha < 0 < \beta < \gamma . \tag{2.6}$$

Moreover, as ε diminishes, the constants α and β can be made arbitrarily close to $|\mathrm{Re}\,(\lambda_1)\,|$ and γ, respectively.

Proof. It follows from (2.2), that for any small $\xi > 0$ there exists small $\varepsilon > 0$ such that at $\|(x,y)\| \le \varepsilon$ and small μ

$$\|(f,g)\| \le \xi\varepsilon , \quad \left\| \frac{\partial(f,g)}{\partial(x,y)} \right\| < \xi . \tag{2.7}$$

Note also that for any λ such that

$$\max_{i=1,\ldots,n} \mathrm{Re}\,(\lambda_i) < -\lambda, \tag{2.8}$$

the norm of $x \in R^n$ may be defined such that

$$\left\| e^{\Lambda\, s} \right\| \le e^{-\lambda s} \quad \text{at} \quad s \ge 0. \tag{2.9}$$

Consider the Banach space H of the continuous functions $(x(t), y(t))$ which are defined for $t \in [0, \tau]$, with the uniform norm

$$\|(x(t), y(t))\|_H = \sup_{t \in [0,\tau]} \|(x(t), y(t))\|. \tag{2.10}$$

Let H_ε be the ε-neighborhood of zero in H (i.e. H_ε is the set of continuous functions with the norm not greater than ε). Let us take a small $\varepsilon > 0$ and introduce an integral operator $T : H_\varepsilon \to H$, which maps a function $(x(t), y(t))$ into the function $(\overline{x}(t), \overline{y}(t))$ defined by the following rule:

$$\begin{cases} \overline{x}(t) = e^{\Lambda t} x^0 + \displaystyle\int_0^t e^{\Lambda(t-s)} f(x(s), y(s), \mu) ds , \\ \\ \overline{y}(t) = e^{\gamma(t-\tau)} y^1 + \displaystyle\int_\tau^t e^{\gamma(t-s)} g(x(s), y(s), \mu) ds . \end{cases} \tag{2.11}$$

It is easy to see that any solution of the boundary value problem (2.3) is a fixed point of the integral operator T as well as any fixed point of the operator (2.11) is a solution of the boundary value problem. Therefore, the existence and uniqueness of the solution of the boundary value problem follows from the fact that T is *a contraction operator which maps H_ε into itself.* Indeed, take any function (x, y) from H_ε. Using (2.7) – (2.9), its image $(\bar{x}, \bar{y})$ by T is estimated as follows:

$$\|\overline{x}(t)\| \leq e^{-\lambda t}\|x^0\| + \int_0^t e^{-\alpha(t-s)}\xi\|(x(s), y(s))\|ds \leq e^{-\lambda t}\|x^0\| + \frac{\xi}{\lambda}\left(1 - e^{-\lambda t}\right)\|(x,y)\|_H \,,$$

$$\|\overline{y}(t)\| \leq e^{-\gamma(\tau-t)}\|y^1\| + \int_t^\tau e^{-\gamma(s-t)}\xi\|(x(s), y(s))\|ds \leq e^{-\gamma(\tau-t)}\|y^1\| + \frac{\xi}{\gamma}\left(1 - e^{-\gamma(\tau-t)}\right)\|(x,y)\|_H. \tag{2.12}$$

On the interval $0 \leq t \leq \tau$ the factors $e^{-\lambda t}$ and $e^{-\gamma(\tau-t)}$ are bounded in $[0, 1]$. Therefore, if $\|(x, y)\|_H \leq \varepsilon$ and $\|(x^0, y^1)\| \leq \varepsilon$, then assuming

$$\xi \max(\lambda^{-1}, \gamma^{-1}) < 1 \tag{2.13}$$

we get $\|(\bar{x}, \bar{y})\|_H \leq \varepsilon$; i.e., the ε-neighborhood of zero in H is T-invariant indeed.

To show contraction, take any functions (x_1, y_1) and (x_2, y_2) from H_ε. As above, we have the following estimates

$$\begin{aligned} \|\overline{x}_1(t) - \overline{x}_2(t)\| &\leq \frac{\xi}{\lambda}\left(1 - e^{-\lambda t}\right)\|(x_1 - x_2, y_1 - y_2)\|_H \,, \\ \|\overline{y}_1(t) - \overline{y}_2(t)\| &\leq \frac{\xi}{\gamma}\left(1 - e^{-\gamma(\tau-t)}\right)\|(x_1 - x_2, y_1 - y_2)\|_H \,. \end{aligned} \tag{2.14}$$

Thus, if ε is so small that (2.13) holds, the contraction follows (i.e. $\|(\bar{x}_1 - \bar{x}_2, \bar{y}_1 - \bar{y}_2)\|_H < q\|(x_1 - x_2, y_1 - y_2)\|_H$ with $q < 1$).

According to the Banach principle of contraction mapping, the operator T has a unique fixed point in H_ε for all (x^0, y^1, τ, μ), i.e. the boundary value problem (2.3) has a unique solution. It depends smoothly on the boundary data (x^0, y^1) because the integral operator T is smooth on H_ε (i.e. its Freche derivative is uniformly continuous) and it depends smoothly on (x^0, y^1) so the latter holds true for its fixed point as well.

Since T is a smooth contracting operator smoothly depending on parameters (x^0, y^1), the iterations of any initial function in H_ε converge to the fixed point, along with the derivatives with respect to (x^0, y^1). Thus the sequence of functions $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$, obtained by the iterations

$$(x_{n+1}(t), y_{n+1}(t)) = T\left(x_n(t), y_n(t)\right) \tag{2.15}$$

with $(x_0(t), y_0(t)) = 0$, converges to the solution of the boundary value problem and the derivatives $\partial(x_n, y_n)/\partial(x^0, y^1)$ converge to the corresponding derivative of the solution.

Thus, to prove estimates (2.5) it is sufficient to check that, for appropriately chosen constants C, α and β (see (2.6)), if some function (x, y) satisfies (2.5), then its image by T satisfies (2.5) too, with the same values of the constants.

By differentiation of (2.11) we get

$$\begin{cases} \dfrac{\partial \overline{x}(t)}{\partial x^0} = e^{\Lambda t} + \displaystyle\int_0^t e^{\Lambda(t-s)} \frac{\partial f}{\partial(x,y)} \frac{\partial(x(s), y(s))}{\partial x^0}\, ds \;, \\[2ex] \dfrac{\partial \overline{y}(t)}{\partial x^0} = \displaystyle\int_\tau^t e^{\gamma(t-s)} \frac{\partial g}{\partial(x,y)} \frac{\partial(x(s), y(s))}{\partial x^0}\, ds \;. \end{cases}$$

If the first inequality of (2.5) holds for $(x(s), y(s))$, then the equations above give (using (2.7)-(2.9)):

$$\left\| \frac{\partial(\overline{x}(t), \bar{y}(t))}{\partial x^0} \right\| \le e^{-\lambda t} + C\xi \max \left\{ \int_0^t e^{-\lambda(t-s)} e^{-\alpha s} ds \;, \int_t^\tau e^{-\gamma(s-t)} e^{-\alpha s} ds \right\},$$

or if α is close to λ and less than it,

$$\left\| \frac{\partial(\overline{x}(t), \bar{y}(t))}{\partial x^0} \right\| \le e^{-\alpha t} \left(1 + \xi \frac{C}{\lambda - \alpha} \right). \tag{2.16}$$

Analogously, if $(x(s), y(s))$ satisfies the second inequality of (2.5), then

$$\left\| \frac{\partial(\overline{x}(t), \bar{y}(t))}{\partial y^1} \right\| \le e^{-\beta(\tau - t)} \left(1 + \xi \frac{C}{\gamma - \beta} \right) \tag{2.17}$$

for β close to and less than γ.

Thus, the image $(\bar{x}, \bar{y})$ satisfies estimates (2.5) with the new constant factor

$$C_{new} = 1 + Cq$$

where $q = \xi \max\left((\lambda - \alpha)^{-1}, (\gamma - \beta)^{-1}\right)$. Given α and β, assume ξ is so small that $q < 1$. In this case, if $C \ge (1 - q)^{-1}$, then $C_{new} \le C$ which completes the proof of estimates (2.5).

It remains to prove the smoothness of the solution of the boundary value (2.1), (2.3) with respect to t and τ. Since $(x(t; x^0, y^1, \tau, \mu), y(t; x^0, y^1, \tau, \mu))$ is an orbit of the system X_μ, the smoothness with respect to t follows immediately. Let us now fix any initial point (x^0, y^0) and let $y^1(\tau)$ be the y-coordinate of its time τ shift by the flow X_μ. By the definition of $(x(t; x^0, y^1, \tau, \mu), y(t; x^0, y^1, \tau, \mu))$ as the unique solution of the boundary value problem (2.1),(2.3) we have that $(x(t; x^0, y^1(\tau), \tau, \mu), y(t; x^0, y^1(\tau), \tau, \mu))$ is the time t shift of (x^0, y^0), independently of the value of τ. Thus,

$$\frac{d}{d\tau}(x(t; x^0, y^1(\tau), \tau, \mu), y(t; x^0, y^1(\tau), \tau, \mu)) \equiv 0. \tag{2.18}$$

Now, since $y^1(\tau)$ depends smoothly on τ and since $(x(t;x^0,y^1,\tau,\mu),y(t;x^0,y^1(\tau),\tau,\mu))$ depends smoothly on y^1 (as we just have proved), the smoothness of $(x(t;x^0,y^1,\tau,\mu),y(t;x^0,y^1(\tau),\tau,\mu))$ with respect to τ follows from (2.18) immediately (usefull expressions for the derivatives are given by (3.13),(3.14) in the next Section).

3 Poincaré map

Let us now prove that the Poincaré map near the homoclinic loop $\mathcal{L}$ is strongly contracting. This map is represented as a superposition of two maps: T_{loc} and T_{glo} where T_{loc} is defined by the flow near the equilibrium point and T_{glo} is defined by the flow near the global piece of the homoclinic orbit Γ. These maps are defined on the small cross-sections S^0 and S^1 (which we construct below): $T_{loc}: S^0 \mapsto S^1$ and $T_{glo}: S^1 \mapsto S^0$.

The n-dimensional stable manifold W^s of the point O is tangent to the plane $y=0$ at the point $O=(0,0)$ at $\mu=0$. Thus, W^s is locally the graph of a smooth function

$$y = y^s(x,\mu)\ ,\quad y^s(0,\mu)=0\ ,\quad \left.\frac{\partial y^s(x,\mu)}{\partial x}\right|_{(x,\mu)=0} = 0\ . \tag{3.1}$$

The unstable manifold W^u of O is locally the graph of a smooth function

$$x = x^u(y,\mu)\ ,\quad x^u(0,\mu)=0\ ,\quad \left.\frac{\partial x^u(y,\mu)}{\partial y}\right|_{(y,\mu)=0} = 0\ . \tag{3.2}$$

At $\mu=0$, the orbit Γ tends to O as $t\to+\infty$. Therefore, the surface

$$S^0 = \Big\{(x,y)\mid \|x\|=\xi,\ \|x-x^+,y-y^+\|\le\delta\Big\} \tag{3.3}$$

is a cross-section for the orbits close to Γ if μ is small enough. Here (x^+,y^+) are the coordinates of the first intersection of Γ with the surface $\|x\|=\xi$ at $\mu=0$ (see Fig.7), and ξ and δ are small positive constants.

The manifold W^u consists of three orbits: the equilibrium point O and two separatrices one of which is the orbit Γ which forms the homoclinic loop at $\mu=0$. Without loss of generality we assume that the orbit Γ leaves O towards the positive y. So, for small positive δ and y^- and for small μ, the surface

$$S^1 = \Big\{(x,y)\mid y=y^-,\ \|(x-x^-)\|\le\delta\Big\} \tag{3.4}$$

is a cross-section for the orbits close to Γ. Here (x^-,y^-) are the coordinates of the first intersection of Γ with the plane $y=y^-$ at $\mu=0$.

Both the cross-sections are n-dimensional. Without loss of generality, we take $(x_1,\dots,x_n)$ as the coordinates on the cross-section S^1 and $(x_1,\dots,x_{n-1},y)$ as the coordinates on S^0. Below we use the following notations (see Fig.7): S^0_0 for the

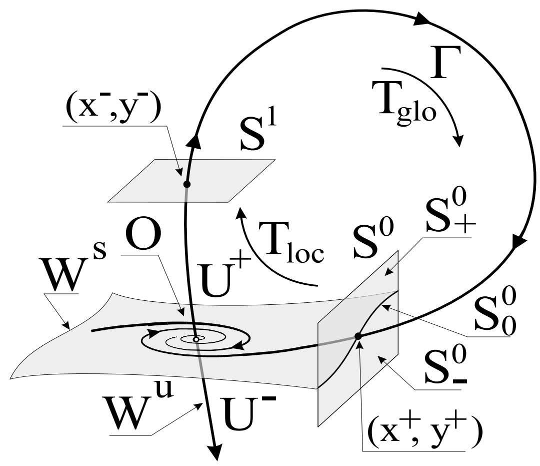

Figure 7: Two cross-sections can be constructed in neighborhood of O: S^0 near the point (x^+, y^+) and S^1 near (x^-, y^-). The flow defines the maps: $T_{loc} : S^0_+ \to S^1$ and $T_{glo} : S^1 \to S^0$.

intersection of W^s with S^0; S^0_+ for $S^0 \cap U^+$ and S^0_- for $S^0 \cap U^-$; also (x^0, y^0) for the coordinates on the cross-section S^0 and x^1 for the coordinates on S^1.

As we mentioned, the Poincaré map T near the homoclinic loop is a superposition of the two maps T_{loc} and T_{glo}. The local map is defined on S^0_+, it takes the region which corresponds to small y^0 greater than $y^s(x^0, \mu)$ to a small neighborhood of the point $x^1 = x^-$ on S^1 (the orbits starting with $y^0 < y^s(x^0, \mu)$, i.e. below the stable manifold, go close to the other separatrix and do not reach S^1). By continuity, the map T_{loc} may be defined at $y^0 = y^s(x^0, \mu)$:

$$T_{loc} S^0_0 = x^-.$$

The global map takes a small neighbourhood of the point $x^1 = x^-$ on S^1 into S^0 (see Fig.7). The flight time from S^1 to S^0 is bounded, therefore, the map T_{glo} is a diffeomorphism. In particular, its derivatives are bounded. Thus, to show the required contraction of the Poincaré map $T = T_{glo} \circ T_{loc}$ it is sufficient to prove the following lemma which, basically, shows that the local map is arbitrarily strong contracting in a sufficiently small neighbourhood of S^0_0.

Lemma 3.1 *The map T_{loc} is written as*

$$x^1 = \varphi(x^0, y^0; \mu)$$

where φ is a C^1 function of (x^0, y^0) defined on $S^0_+ \cup S^0_0$ and its first derivatives vanish at S^0_0.

Proof. According to Section 2, given $\tau > 0$ and small x^0, y^1, there exists a unique orbit $(x(t), y(t)) = (x(t; x^0, y^1, \tau, \mu), y(t; x^0, y^1, \tau, \mu))$ which, at $t = 0$, starts with the point $(x^0, y(0))$ and reaches the point $(x(\tau), y^1)$ at $t = \tau$. Thus, fixing $y^1 = y^-$ and

$\|x^0\| = \xi$, we get that the orbit of a point $(x^0, y^0) \in S^0_+$ reaches the cross-section S^1 at a point x^1 at the time $\tau(x^0, y^0, \mu)$ if and only if

$$y^0 = y(0; x^0, y^-, \tau(x^0, y^0, \mu), \mu) \tag{3.5}$$

and

$$x^1 = \varphi(x^0, y^0, \mu) \equiv x(\tau(x^0, y^0, \mu); x^0, y^-, \tau(x^0, y^0, \mu), \mu). \tag{3.6}$$

It follows from (3.5) that

$$\frac{\partial \tau}{\partial x^0} = -\left(\left.\frac{\partial y}{\partial \tau}\right|_{t=0} \right)^{-1} \left.\frac{\partial y}{\partial x^0}\right|_{t=0}, \quad \frac{\partial \tau}{\partial y^0} = \left(\left.\frac{\partial y}{\partial \tau}\right|_{t=0} \right)^{-1}. \tag{3.7}$$

By (3.6) and (3.7)

$$\begin{aligned} \frac{\partial \varphi}{\partial x^0} &= \left.\frac{\partial x}{\partial x^0}\right|_{t=\tau} + \left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} \right) \frac{\partial \tau}{\partial x^0} = \\ & \left.\frac{\partial x}{\partial x^0}\right|_{t=\tau} - \left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} \right) \left(\left.\frac{\partial y}{\partial \tau}\right|_{t=0} \right)^{-1} \left.\frac{\partial y}{\partial x^0}\right|_{t=0}, \\ \frac{\partial \varphi}{\partial y^0} &= \left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} \right) \frac{\partial \tau}{\partial y^0} = \\ & \left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} \right) \left(\left.\frac{\partial y}{\partial \tau}\right|_{t=0} \right)^{-1}. \end{aligned} \tag{3.8}$$

To prove Lemma 3.1 we must show that

$$\lim_{\tau \to +\infty} \frac{\partial \varphi}{\partial (x^0, y^0)} = 0 \tag{3.9}$$

(because, the limit $\tau = +\infty$ corresponds to the starting point on the stable manifold W^s, or, what is the same, to $(x^0, y^0) \in S^0_0$). According to theorem 2.1,

$$\begin{aligned} & \left\| \left.\frac{\partial (x,y)}{\partial x^0}\right|_{t=0} \right\| \le C, \quad \left\| \left.\frac{\partial (x,y)}{\partial y^1}\right|_{t=0} \right\| \le Ce^{-\beta\tau}, \\ & \left\| \left.\frac{\partial (x,y)}{\partial x^0}\right|_{t=\tau} \right\| \le Ce^{-\alpha\tau}, \quad \left\| \left.\frac{\partial (x,y)}{\partial y^1}\right|_{t=\tau} \right\| \le C. \end{aligned} \tag{3.10}$$

Thus, by virtue of (3.8),(3.10), it is sufficient to show

$$\lim_{\tau \to +\infty} \left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} \right) \left(\left.\frac{\partial y}{\partial \tau}\right|_{t=0} \right)^{-1} = 0. \tag{3.11}$$

To find estimates for the derivatives of x, y with respect to τ, note that by the definition of the function (x, y) as the unique solution of the boundary value problem (2.1),(2.3) we have the identities:

$$\begin{aligned} & y(t; x^0, y^1, \tau, \mu) \equiv y(t; x^0, y(\tau + \Delta\tau; x^0, y^1, \tau, \mu), \tau + \Delta\tau, \mu) \;, \\ & x(t; x^0, y^1, \tau, \mu) \equiv x(t + \Delta\tau; x(-\Delta\tau; x^0, y^1, \tau, \mu), y^1, \tau + \Delta\tau, \mu) \;. \end{aligned} \tag{3.12}$$

The differentiation of (3.12) with respect to $\Delta\tau$ at $\Delta\tau = 0$ gives

$$\frac{\partial y}{\partial \tau} = -\frac{\partial y}{\partial y^1} \, \dot{y}|_{t=\tau} \tag{3.13}$$

and

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial \tau} = \frac{\partial x}{\partial x^0} \, \dot{x}|_{t=0} \quad . \tag{3.14}$$

Now, by (3.10) and (3.14) we have

$$\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} = O(e^{\alpha\tau}) \quad \text{as} \quad \tau \to +\infty \;. \tag{3.15}$$

Since $\dot{y}|_{t=\tau}$ is bounded away from zero (this is the value of $\dot{y}$ on the cross-section S^1), it remains to estimate $\dfrac{\partial y}{\partial y^1}$ *from below.* To this aim, let us consider the orbit $\left(x^*(t; x^0, y^0, \mu), y^*(t; x^0, y^0, \mu)\right)$ which starts with the point (x^0, y^0) at $t = 0$, i.e. the solution of the initial value problem.

All the time that the orbit $(x^*(t), y^*(t))$ belongs to a small neighborhood of the equilibrium state O, the following estimate holds for any fixed $\gamma^* > \gamma$:

$$\frac{d}{dt} \left\| \frac{\partial \left(x^*(t), y^*(t)\right)}{\partial y^0} \right\| \leq \gamma^* \left\| \frac{\partial \left(x^*(t), y^*(t)\right)}{\partial y^0} \right\| \tag{3.16}$$

(this is true because the spectrum of the linearization matrix of the system (2.1) at the point O lies to the left of the straight line $\mathrm{Re}\,(\cdot) = \gamma$ on the complex plane). Since γ^* may be chosen arbitrary close to γ and α arbitrary close to $-\mathrm{Re}\,(\lambda_1)$ (see Theorem 2.1), we may assume by the condition **(C)**

$$\alpha + \gamma^* < 0 \;. \tag{3.17}$$

Inequality (3.16) implies that

$$\left\| \frac{\partial y^*(t)}{\partial y^0} \right\| \leq c\, e^{\gamma^* t} \tag{3.18}$$

for some positive constant c.

By definition,

$$y^1 \equiv y^* \left(\tau; x^0, y\left(0; x^0, y^1, \tau, \mu\right), \tau, \mu\right) \tag{3.19}$$

(recall that the star indicates the solution of the initial value problem, whereas y without the star corresponds to the solution of the boundary value problem). Identity (3.19) implies

$$\left.\frac{\partial y^*}{\partial y^0}\right|_{t=\tau} \left.\frac{\partial y}{\partial y^1}\right|_{t=0} \equiv 1 \,. \tag{3.20}$$

By (3.18) and (3.20)

$$\left\| \left.\frac{\partial y}{\partial y^1}\right|_{t=0} \right\| \geq \frac{1}{c}\, e^{-\gamma^* \tau} \,. \tag{3.21}$$

Now, by (3.15),

$$\left(\left.\frac{\partial x}{\partial t}\right|_{t=\tau} + \left.\frac{\partial x}{\partial \tau}\right|_{t=\tau} \right) \left(\left.\frac{\partial y}{\partial \tau}\right|_{t=0} \right)^{-1} = O(e^{(\alpha+\gamma^*)\tau}) \,, \tag{3.22}$$

which, along with (3.17), gives the lemma.

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References

[1] Andronov, A.A. & Leontovich, E.A. [1938] "To the theory of changing of qualitative structure of trajectories on the plane," *Dokl. Acad. Nauk USSR* **21**, No.9.

[2] Leontovich, E.A. [1951] "On appearance of limit cycles from a separatrix," *Dokl. Acad. Nauk USSR* **28**, No.4.

[3] Andronov, A.A. & Leontovich, E.A. [1959] "On appearance of limit cycles from a loop of separatrix to a saddle and from a loop of separatrix to a saddle-node," *Mat. Sb.* **48**, 335–376.

[4] Andronov, A.A., Leontovich, E.A., Gordon, I.I. & Mayer, A.G. [1967] "Theory of Bifurcations of Dynamical Systems on a plane," Israel Program of Scientific Translations, Jerusalem.

[5] Hirsch, M., Pugh, C. & Shub, M. [1977] "Invariant manifolds," *Lect. Notes Math.* **583**, Springer-Verlag.

[6] Homburg, A.J. [1993] "Some global aspects of homoclinic bifurcations of vector fields," PhD thessis, University of Groeningen.

[7] Homburg, A.J. [1996] "Global aspects of homoclinic bifurcations of vector fields," *Memoirs of the A.M.S.* **578**.

[8] Ovsyannikov, I.M. & Shilnikov, L.P. [1986] "On systems with a saddle-focus homoclinic curve," *Math. USSR Sb.* **58**, 557–574.

[9] Ovsyannikov, I.M. & Shilnikov, L.P. [1991] "Systems with a homoclinic curve of multidimensional saddle-focus type, and spiral chaos," *Math. USSR Sb.* **73**, 415–443.

[10] Sandstede, B. [1994] "Center manifolds for homoclinic solutions," preprint WIAS

[11] Shashkov, M.V. [1991] "On existence of a smooth invariant two-dimensional attractive manifold for systems with a separatrix contour," *Methods of qualitative theory and theory of bifurcations* (Nizhny Novgorod State University), 61–73.

[12] Shashkov, M.V. [1994] "Bifurcations of separatrix contours," PhD Thesis, Nizhny Novgorod University.

[13] Shashkov M.V. & Turaev D.V. [1999] "An existence theorem of smooth nonlocal center invariant manifolds for systems close to a system with a homoclinic loop," *Journal of Nonlinear Science,* Vol.9, No.5 pp.525-573.

[14] Shilnikov, L.P. [1962] "Some cases of generation of periodic motions in the n-dimensional space," *DAN SSSR* **143**, 289–292.

[15] Shilnikov, L.P. [1963] "Some cases of generation of periodic motion from singular trajectories," *Mat. Sb.* **61**, 443–466.

[16] Shilnikov, L.P. [1965] "A case of the existence of a denumerable set of periodic motions," *Sov. Math. Dokl.* **6**, 163–166.

[17] Shilnikov, L.P. [1967] "On a Poincaré-Birkhoff problem," *Math. USSR Sb.* **3**, 415–443.

[18] Shilnikov, L.P. [1968] "On the generation of a periodic motion from a trajectory doubly asymptotic to an equilibrium state of saddle type," *Math. USSR Sb.* **6**, 415–443.

[19] Shilnikov, L.P. [1970] "A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type," *Math. USSR Sb.* **10**, 91–102.

[20] Shilnikov, L., Shilnikov, A., Turaev, D. & Chua, L. [1998] "Methods of qualitative theory in nonlinear dynamics. Part I" (World Scientific, Singapore, 1998).

[21] Turaev, D.V. [1984] "On a case of bifurcations of a contour composed by two homoclinic curves of a saddle," *Methods of qualitative theory of differential equations* (Gorky State University), 162–175.

[22] Turaev, D.V. [1991] *On Bifurcations of Dynamical Systems with Two Homoclinic Curves of a Saddle,* PhD Thesis, Nizhny Novgorod University.

[23] Turaev, D.V. [1996] "On dimension of non-local bifurcational problems," *Int. J. of Bifurcation and Chaos* **6**, 911–948.