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Numerical construction of a hedging strategy against the multi-asset European claim

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ABSTRACT. For evaluating a hedging strategy we have to know at every instant the solution of the Cauchy problem for a parabolic equation (the value of the hedging portfolio) and its derivatives (the deltas). We suggest to find these magnitudes by Monte Carlo simulation of the corresponding system of stochastic differential equations using weak solution schemes. It turns out that with one and the same control function a variance reduction can be achieved simultaneously for the claim value as well as for the deltas. We consider asset models with an instantaneous saving bond and the Jamshidian LIBOR rate model.

1. Introduction

Let us consider a model for the financial market consisting of a cash bond (riskless asset) with price $B(t)$ and m stocks (risky assets) with prices per share $X^i(t)$, $i = 1, ..., m$, satisfying the equations

$$dB = r(t)Bdt, \ B(t_0) = 1, \tag{1.1}$$

$$dX^i = X^i(\mu^i(t, X)dt + \sum_{j=1}^{m} \nu_{ij}(t, X)dW^j(t)), \ t \geq t_0, \ i = 1, ..., m.$$

Here, for the time being, $r(t)$ is a deterministic interest rate, $X = (X^1, ..., X^m)^\top$, $W = (W^1, ..., W^m)^\top$ is an m-dimensional standard Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. We denote by $\{\mathcal{F}_t\}$ the P-augmentation of the filtration generated by W. It is assumed that $r(t)$, the vector $(\mu^1(t, x), ..., \mu^m(t, x))^\top$, and the matrix $\nu(t, x) = \{\nu_{ij}(t, x)\}$, $t \in [t_0, T]$, $x \in R^m_+ := \{x : x^1 > 0, ..., x^m > 0\}$, are sufficiently smooth and such that there exists a unique process $X(t) \in R^m_+$, $t \in [t_0, T]$, with $X(t_0) \in R^m_+$ satisfying (1.1) (for example, all the μ^i, ν_{ij} are smooth and bounded). Moreover, we assume that the volatility matrix $\sigma(t, x) = \{\sigma_{ij}(t, x)\} = \{x^i \nu_{ij}(t, x)\}$ has full rank for every (t, x), $t \in [t_0, T]$, $x \in R^m_+$. From now on we shall not always state explicitly the properties of the originating functions which we regard as sufficiently good in analytical sense.

We consider a model where the stocks pay dividends to the share holders at a rate $r^i(t, X(t))$ for the i-th stock and a consumption process C is assumed and defined by a consumption rate $c(t, X(t))$, $t_0 \leq t \leq T$,

$$dC = c(t, X(t))dt, \ C(t_0) = 0. \tag{1.2}$$

The portfolio value $V(t)$ of a trading strategy $(\varphi_t, \psi_t) = (\varphi_t, \psi^1_t, ..., \psi^m_t)$, i.e. the positions in bond $B(t)$ and stocks $X^j(t)$ respectively, is given by

$$V(t) = \varphi_t B(t) + \sum_{i=1}^{m} \psi^i_t X^i(t). \tag{1.3}$$

A portfolio (φ_t, ψ_t) is called (generalized) self-financing, if its value $V(t)$ satisfies

$$dV = \varphi_t dB + \sum_{i=1}^{m} \psi^i_t dX^i + \sum_{i=1}^{m} r^i(t, X(t))\psi^i_t X^i(t)dt - c(t, X(t))dt \tag{1.4}$$

$$= \varphi_t r(t) B dt + \sum_{i=1}^{m} \psi_t^i X^i \cdot (\mu^i(t,X)dt + \sum_{j=1}^{m} \nu_{ij}(t,X) dW^j(t))$$

$$+ \sum_{i=1}^{m} r^i(t, X(t)) \psi_t^i X^i(t) dt - c(t, X(t)) dt,$$

which is equivalent with

$$B d\varphi_t + \sum_{i=1}^{m} X^i d\psi_t^i + \sum_{i=1}^{m} d\psi_t^i dX^i = \sum_{i=1}^{m} r^i(t, X(t)) \psi_t^i X^i(t) dt - c(t, X(t)) dt. \tag{1.5}$$

Let a European claim at maturity time T be specified by a payoff function f which depends on $X(T)$ only and let $V(t)$ be the present value of the claim. Since the model is Markovian we have

$$V(t) = \varphi_t B(t) + \sum_{i=1}^{m} \psi_t^i X^i(t) = v(t, X(t)),\ V(T) = v(T, X(T)) = f(X(T)), \tag{1.6}$$

where v is a function of the variables $t, x^1, ..., x^m$.

Just as in the one dimensional case we may derive a parabolic pde for the function $v(t,x)$ (see, e.g., [9]). Due to Itô's formula we have

$$dv(t, X(t)) = \frac{\partial v}{\partial t} dt + \sum_{i=1}^{m} \frac{\partial v}{\partial x^i} dX^i + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 v}{\partial x^i \partial x^j} dX^i dX^j \tag{1.7}$$

$$= \frac{\partial v}{\partial t} dt + \sum_{i=1}^{m} \frac{\partial v}{\partial x^i} X^i \mu^i dt + \sum_{i=1}^{m} \frac{\partial v}{\partial x^i} \sum_{j=1}^{m} \sigma_{ij} dw^j(t) + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 v}{\partial x^i \partial x^j} dt,$$

where

$$a_{ij}(t,x) = \sum_{k=1}^{m} \sigma_{ik}\sigma_{jk} = x^i x^j \sum_{k=1}^{m} \nu_{ik}\nu_{jk},$$

i.e., the matrix $a = \{a_{ij}\}$ is equal to $a = \sigma\sigma^\top$.

Comparing (1.4) with (1.7), we obtain

$$\psi_t^i = \psi^i(t, X(t)) = \frac{\partial v}{\partial x^i}(t, X(t)),\ \psi^i(t,x) = \frac{\partial v}{\partial x^i}(t,x), \tag{1.8}$$

and

$$\frac{\partial v}{\partial t}(t, X(t)) + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(t, X(t)) \frac{\partial^2 v}{\partial x^i \partial x^j}(t, X(t)) \tag{1.9}$$

$$= \varphi_t r(t) B(t) + \sum_{i=1}^{m} r^i(t, X(t)) \psi_t^i X^i(t) - c(t, X(t)).$$

Substituting (see(1.6) and (1.8))

$$\varphi_t B(t) = v(t, X(t)) - \sum_{i=1}^{m} \psi_t^i X^i(t) = v(t, X(t)) - \sum_{i=1}^{m} \frac{\partial v}{\partial x^i}(t, X(t)) X^i(t),$$

in (1.9) and taking into account (1.8), we get the following Cauchy problem for the function $v(t,x)$:

$$Lv(t,x)+c(t,x):=\frac{\partial v}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}(t,x)\frac{\partial^2 v}{\partial x^i\partial x^j} \tag{1.10}$$

$$+\sum_{i=1}^{m}b^i(t,x)\frac{\partial v}{\partial x^i}-r(t)v+c(t,x)=0,$$

$$v(T,x)=f(x), \tag{1.11}$$

where we introduced the notation $b^i=(r-r^i)x^i,\ b=(b^1,...,b^m)$.

Let $v(t,x)$ be the solution of the problem (1.10)-(1.11). Then the required hedging strategy $(\varphi_t,\psi_t^1,...,\psi_t^m)$ as a function of $(t,X(t))$ is given by

$$\varphi_t=\frac{1}{B(t)}(v(t,X(t))-\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}(t,X(t))X^i(t)),\ \psi_t^i=\frac{\partial v}{\partial x^i}(t,X(t)),\ i=1,...,m. \tag{1.12}$$

The relation (1.5) for this strategy can be checked directly. Indeed

$$Bd\varphi_t=-r(v-\sum_{i=1}^{m}X^i\frac{\partial v}{\partial x^i})dt+d(v-\sum_{i=1}^{m}X^i\frac{\partial v}{\partial x^i})$$

$$=-r(v-\sum_{i=1}^{m}X^i\frac{\partial v}{\partial x^i})dt+dv-\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}dX^i-\sum_{i=1}^{m}X^id(\frac{\partial v}{\partial x^i})-\sum_{i=1}^{m}d(\frac{\partial v}{\partial x^i})dX^i,$$

$$\sum_{i=1}^{m}X^id\psi_t^i=\sum_{i=1}^{m}X^id(\frac{\partial v}{\partial x^i}),$$

$$\sum_{i=1}^{m}d\psi_t^idX^i=\sum_{i=1}^{m}d(\frac{\partial v}{\partial x^i})dX^i.$$

Therefore the left part of (1.5) is equal to

$$Bd\varphi_t+\sum_{i=1}^{m}X^id\psi_t^i+\sum_{i=1}^{m}d\psi_t^idX^i=-r(v-\sum_{i=1}^{m}X^i\frac{\partial v}{\partial x^i})dt+dv-\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}dX^i. \tag{1.13}$$

Further, see(1.7),

$$dv-\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}dX^i=\frac{\partial v}{\partial t}dt+\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}\frac{\partial^2 v}{\partial x^i\partial x^j}dt$$

and according to (1.10)

$$dv-\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}dX^i=(-c+rv-\sum_{i=1}^{m}(r-r^i)x^i\frac{\partial v}{\partial x^i})dt,$$

which combined with (1.13) gives (1.5).

Remark 1.1. Consider the model (1.1) with now r depending on t and X, i.e., (1.1) with the first equation

$$dB = r(t,X)Bdt, \; B(t_0) = 1.$$

Then in general $V(t)$ depends on t, $X(t)$, $B(t)$, i.e., $V(t) = v(t, X(t), B(t))$. Arguing as above, we obtain that v satisfies the following equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(t,x)\frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_{i=1}^{m} b^i(t,x)\frac{\partial v}{\partial x^i} + r(t,x)B\frac{\partial v}{\partial B} - r(t,x)v + c(t,x) = 0.$$

But, if the claim depends as before on $X(T)$ only the solution of the above equation satisfying condition (1.11) is independent of B. So $\partial v/\partial B = 0$ and we obtain Cauchy problem (1.10)-(1.11) where $r = r(t,x)$. The formulas for the required hedging strategy, (1.12), remain the same.

Moreover it is possible to consider the model in which all the coefficients depend on t, X and B and the claim is a function $f(X(T), B(T))$. In this case we derive in a similar way the following degenerate problem

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(t,x,B)\frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_{i=1}^{m} b^i(t,x,B)\frac{\partial v}{\partial x^i} \tag{1.14}$$

$$+r(t,x,B)B\frac{\partial v}{\partial B} - r(t,x,B)v + c(t,x,B) = 0,$$

$$v(T,x,B) = f(x,B). \tag{1.15}$$

If this problem has a solution $v = v(t,x,B)$, then a hedging strategy is given by

$$\varphi_t = \frac{1}{B(t)}(v(t,X(t),B(t)) - \sum_{i=1}^{m} \frac{\partial v}{\partial x^i}(t,X(t),B(t))X^i(t)),$$

$$\psi_t^i = \frac{\partial v}{\partial x^i}(t,X(t),B(t)), \; i = 1, ..., m.$$

Remark 1.2. We note that a Cauchy problem is considered in spite of the fact that the variable x belongs to $R_+^m = \{x : \; x^1 > 0, ..., x^m > 0\}$. This is possible because every solution $X(t)$, $X(t_0) \in R_+^m$, of system (1.1) evolves in R_+^m during the whole time interval $[t_0,T]$. Consider a stock model with prices evolving in an open parallelepiped $\Pi = \{x : \; 0 \le \pi_1^1 < x^1 < \pi_2^1, ..., 0 \le \pi_1^m < x^m < \pi_2^m\}$, where π_1^k, π_2^k, $k = 1, ..., m$, are constants (it is possible to consider cases when some of π_2 are equal to ∞). For example,

$$dX^i = (X^i - \pi_1^i)(\pi_2^i - X^i)(\mu^i(t,X)dt + \sum_{j=1}^{m} \nu_{ij}(t,X)dW^j(t)), \; t \ge t_0, \; i = 1, ..., m,$$

with suitable coefficients μ^i and ν_{ij}.

For such a model the construction of a hedging strategy leads to a corresponding Cauchy problem as well (not to a boundary value problem).

Remark 1.3. Let us consider a model consisting of a cash bond $B(s)$ and a stock $X(s)$ (we take only one stock for notational simplicity), where the price of the stock satisfies the equation

$$dX = \mu(s, X)ds + \sigma(s, X)dW(s). \tag{1.16}$$

Let $0 \le \pi_1 < \pi_2,\ \pi_1 < x < \pi_2,\ \tau = \tau_{t,x} = T \wedge \inf\{s : X_{t,x}(s) \notin [\pi_1, \pi_2],\ t \le s \le T\}$ (we put inf to be equal ∞ for an empty set). We now consider an example of a barrier option. The option is specified by a payoff equal to zero if $\tau < T$ and equal to $f(X_{t,x}(T))$ if $\tau = T$, where $f(x)$ is a function defined on $[\pi_1, \pi_2]$. We note that a more rigorous notation for (1.16) would be

$$dX = 1_{\{\tau > s\}}\mu(s, X)ds + 1_{\{\tau > s\}}\sigma(s, X)dW(s),$$

but we use the simplified notation as long as it doesn't lead to any confusion. In addition, we assume that $f(x)$ is equal to zero in some neighborhood of π_1 and π_2 respectively. Then, it is not difficult to show that the portfolio value $V(t)$ of the hedging strategy is equal to $v(t, X(t))$ where $v(t, x)$ satisfies the following boundary value problem

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 v}{\partial x^2} + r(t)x\frac{\partial v}{\partial x} - r(t)v = 0,\ t_0 \le t < T,\ \pi_1 < x < \pi_2, \tag{1.17}$$

$$v(T, x) = f(x),\ v(t, \pi_1) = v(t, \pi_2) = 0 \tag{1.18}$$

and as before we have

$$V(t) = v(t, X(t)) = \varphi_t B(t) + \psi_t X(t),$$

with

$$\varphi_t = \frac{1}{B(t)}(v(t, X(t)) - \frac{\partial v}{\partial x}(t, X(t))X(t)),\ \psi_t = \frac{\partial v}{\partial x}(t, X(t)).$$

Note that for this example we did not use the multipliers X^i (see (1.1)) in the stock model.

2. Evaluation of a hedging strategy

Frequently, works in numerics for finance (see, e.g., [13] and references therein) are devoted to the evaluation of a portfolio value $v(t, x)$. Of course, in case $v(t, x)$ is known, it is possible to find $\partial v(t, x)/\partial x^i$ approximately as $[v(t, x^1, ..., x^i + \Delta x^i, ..., x^m) - v(t, x^1, ..., x^i, ..., x^m)]/\Delta x^i$ (or as $[v(t, x^1, ..., x^i + \Delta x^i, ..., x^m) - v(t, x^1, ..., x^i - \Delta x^i, ..., x^m)]/2\Delta x^i$) but such an approach requires very accurate calculations for v. In this sequel we give special attention to the probabilistic evaluation of the deltas $\partial v(t, x)/\partial x^i$ and other Greeks.

Usually, in many-dimensional cases (in reality for $m \ge 3$) it is impossible to find $v(t, x)$ for all (t, x) because of the complexity of problem (1.10)-(1.11). However, for constructing the hedging strategy we only have to find at any instant t the individual values $v(t, X(t))$ and $\partial v(t, X(t))/\partial x^i,\ i = 1, ..., m$, where $X(t)$ is the known state of the market.

The probabilistic approach for the evaluation of a particular value $v(t, x)$ is well known. It turns out that for specific (t, x) the values $\partial v(t, x)/\partial x^i,\ i = 1, ..., m$, can be found effectively by a probabilistic approach as well. Let us recall the probabilistic representation for the solution of the Cauchy problem (1.10)-(1.11), where now we take $r(t, x)$ in (1.10) instead of $r(t)$.

In fact, the solution to problem (1.10)-(1.11) has various probabilistic representations:

$$v(t,x)=E[f(X_{t,x}(T))\cdot Y_{t,x,1}(T)+Z_{t,x,1,0}(T)],\ t\le T,\ x\in R_{+}^{m}, \tag{2.1}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s\ge t$, is the solution of the following system of stochastic differential equations:

$$dX=(b(s,X)-\sigma(s,X)h(s,X))ds+\sigma(s,X)dW(s),\ X(t)=x, \tag{2.2}$$

$$dY=-r(s,X)Yds+h^{\top}(s,X)YdW(s),\ Y(t)=y, \tag{2.3}$$

$$dZ=c(s,X)Yds,\ Z(t)=z. \tag{2.4}$$

Here $h(t,x)=(h^1(t,x),...,h^m(t,x))^{\top}$, h^i are fairly arbitrary functions, Y and Z are scalars. In what follows we assume that all the coefficients in (1.10)-(1.11) and in (2.2)-(2.4) and the solution of (1.10)-(1.11) are sufficiently smooth and satisfy necessary growth conditions for large $|x|$, so that we may apply the theory of weak methods for numerical integration of SDEs. The usual probabilistic representation (see, e.g., [1], [2]) follows from (2.1)-(2.4) for $h=0$. The representation for $h\neq 0$ is a consequence of Girsanov's theorem.

We introduce the notation

$$u_k(t,x)=\frac{\partial v}{\partial x^k}(t,x),\ k=1,...,m. \tag{2.5}$$

The functions v and u_k, $k=1,...,m$, satisfy the Cauchy problem for the following system of $m+1$ linear parabolic equations consisting of (1.10)-(1.11) and

$$\frac{\partial u_k}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}(t,x)\frac{\partial^2 u_k}{\partial x^i\partial x^j}+\sum_{i=1}^{m}b^i(t,x)\frac{\partial u_k}{\partial x^i}-r(t,x)\cdot u_k \tag{2.6}$$

$$+\frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial a_{ij}}{\partial x^k}(t,x)\frac{\partial u_j}{\partial x^i}+\sum_{i=1}^{m}\frac{\partial b_i}{\partial x^k}(t,x)\frac{\partial v}{\partial x^i}-\frac{\partial r}{\partial x^k}(t,x)\cdot v+\frac{\partial c}{\partial x^k}(t,x)=0,$$

$$u_k(T,x)=\frac{\partial f}{\partial x^k}(x),\ k=1,...,m. \tag{2.7}$$

The Cauchy problem (1.10)-(1.11), (2.6)-(2.7) belongs to the class of problems, which solutions has probabilistic representations given in [8] . However, we obtain a representation from (2.1)-(2.4) directly by differentiating (2.1) with respect to x_k. We get

$$u_k(t,x)=\frac{\partial v}{\partial x^k}(t,x) \tag{2.8}$$

$$=E\left[\sum_{i=1}^{m}\frac{\partial f}{\partial x^i}(X_{t,x}(T))\cdot\delta_k X^i(T)\cdot Y_{t,x,1}(T)+f(X_{t,x}(T))\cdot\delta_k Y(T)+\delta_k Z(T)\right],$$

where

$$\delta_k X^i(s):=\delta_k X^i_{t,x}(s):=\frac{\partial X^i_{t,x}(s)}{\partial x^k},\ \delta_k Y(s):=\delta_k Y_{t,x,1}(s):=\frac{\partial Y_{t,x,1}(s)}{\partial x^k}, \tag{2.9}$$

$$\delta_k Z(s):=\delta_k Z_{t,x,1,0}(s):=\frac{\partial Z_{t,x,1,0}(s)}{\partial x^k},\ t\le s\le T.$$

Let $\delta_k X = (\delta_k X^1, ..., \delta_k X^m)^\top$. The functions $\delta_k X(s)$, $\delta_k Y(s)$ and $\delta_k Z(s)$ satisfy the following system of first order variation associated with (2.2)-(2.4) (we remind that we keep k fixed),

$$d\delta_k X = \sum_{l=1}^m \frac{\partial(b(s,X) - \sigma(s,X)h(s,X))}{\partial x^l} \cdot \delta_k X^l ds \tag{2.10}$$

$$+ \sum_{l=1}^m \frac{\partial \sigma(s,X)}{\partial x^l} \cdot \delta_k X^l dW(s),\ \delta_k X^l(t) = 0,\ \text{if } l \neq k,\ \text{and } \delta_k X^k(t) = 1,$$

$$d\delta_k Y = -\sum_{l=1}^m \frac{\partial r(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y ds - r(s,X)\delta_k Y ds \tag{2.11}$$

$$+ \sum_{l=1}^m \frac{\partial h^\top(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y dW(s) + h^\top(s,X)\delta_k Y dW(s),\ \delta_k Y(t) = 0,$$

$$d\delta_k Z = \sum_{l=1}^m \frac{\partial c(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y ds + c(s,X)\delta_k Y ds,\ \delta_k Z(t) = 0. \tag{2.12}$$

We underline here that there is an opportunity of parallelizing: one can consider m problems (2.8), (2.2)-(2.4), (2.10)-(2.12) for every fixed $k = 1, ..., m$ separately.

Remark 2.1. The solution of the boundary value problem (1.17)-(1.18) for the barrier option has the following probabilistic representation

$$v(t,x) = E1_{\{\tau_{t,x}=T\}}[f(X_{t,x}(T)) \cdot Y_{t,x,1}(T)], \tag{2.13}$$

where

$$dX = (r(t)X - \sigma(s,X)h(s,X))ds + \sigma(s,X)dW(s),\ X(t) = x, \tag{2.14}$$

$$dY = -r(t)Y ds + h(s,X)Y dW(s),\ Y(t) = 1,$$

and

$$\frac{\partial v}{\partial x}(t,x) = E1_{\{\tau_{t,x}=T\}}\left[\frac{\partial f}{\partial x}(X_{t,x}(T)) \cdot \delta X(T) \cdot Y_{t,x,1}(T) + f(X_{t,x}(T)) \cdot \delta Y(T)\right], \tag{2.15}$$

where the equations for $\delta X(T)$ and $\delta Y(T)$ are analogous to (2.10), (2.11).

The option under consideration is known as nullified barrier option [6]. For more general barrier options the boundary value conditions are nonzero and instead of (1.18) we have

$$v(T,x) = f(x),\ v(t,\pi_1) = v_1(t),\ v(t,\pi_2) = v_2(t). \tag{2.16}$$

Let Γ denote the set where the condition (2.16) is specified. Then (2.16) can be written as

$$v\,|_\Gamma = g, \tag{2.17}$$

where $g(T,x) = f(x),\ g(t,\pi_1) = v_1(t),\ g(t,\pi_2) = v_2(t)$.

Instead of (2.13) we may write

$$v(t,x) = E[g(\tau_{t,x}, X_{t,x}(\tau_{t,x})) \cdot Y_{t,x,1}(\tau_{t,x})]. \tag{2.18}$$

We note that in this case there is no expression for $\partial v(t,x)/\partial x$ such as (2.15) because the dependence on x is more complicated now due to the presence of $\tau_{t,x}$ and the problem of effective numerical construction of a hedging strategy requires special examination.

Thus, to find $v(t,x)$ and $\partial v/\partial x^k(t,x)$ we need to evaluate the expectations (2.1) and (2.8). Let us consider (2.1). Usually it is impossible to simulate the random variables $X_{t,x}(T))$, $Y_{t,x,1}(T)$, $Z_{t,x,1,0}(T)$ directly and we are forced to simulate some approximate random variables $\overline{X}_{t,x}(T))$, $\overline{Y}_{t,x,1}(T)$, $\overline{Z}_{t,x,1,0}(T)$. To this aim we may use weak methods for numerical integration of SDEs (see [5], [7]). The error of such a weak approximation is of order of $O(h^p)$ where p is the order of weak convergence, depending on the specific method, and h is a time discretization step. For simplicity we consider equidistant partitions of the time interval $[t,T]: t=\overline{t}_0<t_1<...<t_L=T$ with step size $h=(T-t)/L$. For example, the Euler method with simplified simulation of Wiener processes applied to system (2.2)-(2.4) gives

$$\overline{X}(t)=x,\ \overline{X}(t_{l+1})=\overline{X}(t_l)+(b_l-\sigma_l h_l)\cdot h+\sigma_l\cdot\zeta_l\sqrt{h}, \tag{2.19}$$

$$\overline{Y}(t)=1,\ \overline{Y}(t_{l+1})=\overline{Y}(t_l)\ -r_l\overline{Y}(t_l)\cdot h+h_l^{\top}\overline{Y}(t_l)\cdot\zeta_l\sqrt{h},$$

$$\overline{Z}(t)=0,\ \overline{Z}(t_{l+1})=\overline{Z}(t_l)\ +c_l\overline{Y}(t_l)\cdot h,\ l=0,...,L-1,$$

where b_l, σ_l, h_l, r_l, and c_l are values of the corresponding functions (scalar, vector or matrix) at $(t_l,\overline{X}(t_l))$ and $\zeta_l=(\zeta_l^1,...,\zeta_l^m)^{\top}$ is a vector of two-point random variables ζ_l^j distributed by the law $P(\zeta_l^j=\pm1)=1/2$ and independent in $j=1,...,m;\ l=0,...,L-1$. We obtain the usual Euler method if ζ_l^j are simulated as $N(0,1)$-distributed random variables. In either case the order of weak convergence is equal to 1, i.e., the following relation

$$|v(t,x)-E[f(\overline{X}(T))\cdot\overline{Y}(T)+\overline{Z}(T)]|=O(h)$$

is fulfilled for a sufficiently large class of functions f.

Among methods with a higher order of weak convergence let us consider the Talay-Tubaro extrapolation method [16]. We denote an approximation (2.19) with step size h by $\overline{X}^h$, $\overline{Y}^h$, $\overline{Z}^h$. According to the Talay-Tubaro method we have in particular,

$$|v(t,x)-2E[f(\overline{X}^{h/2}(T))\cdot\overline{Y}^{h/2}(T)+\overline{Z}^{h/2}(T)]+E[f(\overline{X}^{h}(T))\cdot\overline{Y}^{h}(T)+\overline{Z}^{h}(T)]|=O(h^2).$$

The value $E[f(\overline{X}(T))\cdot\overline{Y}(T)+\overline{Z}(T)]$ can be evaluated by the Monte-Carlo method

$$E[f(\overline{X}(T))\cdot\overline{Y}(T)+\overline{Z}(T)]\simeq\frac{1}{N}\sum_{n=1}^{N}[f(\overline{X}^{(n)}(T))\cdot\overline{Y}^{(n)}(T)+\overline{Z}^{(n)}(T)], \tag{2.20}$$

where $\overline{X}^{(n)}(t_l)$, $\overline{Y}^{(n)}(t_l)$, $\overline{Z}^{(n)}(t_l)$, $n=1,...,N$, are independent approximate trajectories (generally in weak sense) of the solution of system (2.2)-(2.4).

The statistical error in (2.20) is usually defined by $(D\overline{\xi}(T)/N)^{1/2}$, where $\overline{\xi}(T)=f(\overline{X}(T))\cdot\overline{Y}(T)+\overline{Z}(T)$.

Thus, we have

$$(2.21)\quad v(t,x) \simeq E[f(\overline{X}(T)) \cdot \overline{Y}(T) + \overline{Z}(T)] \simeq \frac{1}{N}\sum_{n=1}^{N}[f(\overline{X}^{(n)}(T)) \cdot \overline{Y}^{(n)}(T) + \overline{Z}^{(n)}(T)].$$

The first approximate equality in (2.21) involves an error due to the approximate integration, whereas the second approximate equality involves a statistical error due to the Monte-Carlo method.

Of course the same consideration holds with respect to the evaluation of $\partial v/\partial x^k(t,x)$.

3. Variance reduction

This section is concerned with two methods of variance reduction in connection with the Monte Carlo approach for the linear parabolic Cauchy problem: with the method of importance sampling [3], [7], [10], [11], [17], and with the method of control variates [10], [11] (for the initial-boundary value problem see [7], [12]). We consider variance reduction for the evaluation of the portfolio as well as for the evaluation of the deltas.

We introduce the variables

$$(3.1)\qquad \xi(s) := v(s, X_{t,x}(s)) \cdot Y_{t,x,1}(s) + Z_{t,x,1,0}(s),$$

$$(3.2)\qquad \eta_k(s) := \sum_{i=1}^{m}\frac{\partial v}{\partial x^i}(s, X_{t,x}(s)) \cdot \delta_k X^i(s) \cdot Y_{t,x,1}(s) + v(s, X_{t,x}(s)) \cdot \delta_k Y(s) + \delta_k Z(s).$$

Clearly

$$(3.3)\qquad \xi := \xi(T) = f(X_{t,x}(T)) \cdot Y_{t,x,1}(T) + Z_{t,x,1,0}(T),$$

(3.4)

$$\eta_k := \eta_k(T) = \sum_{i=1}^{m}\frac{\partial f}{\partial x^i}(X_{t,x}(T)) \cdot \delta_k X^i(T) \cdot Y_{t,x,1}(T) + f(X_{t,x}(T)) \cdot \delta_k Y(T) + \delta_k Z(T).$$

Because $D\xi$ $(D\eta_k)$ is close to $D\overline{\xi}$ $(D\overline{\eta}_k)$, the error of a Monte Carlo evaluation of $v(t,x)$ depends on the variance of the random variable ξ, see (2.1) whereas the Monte Carlo error of an evaluation of $u_k(t,x) = \partial v(t,x)/\partial x^k$ depends on the variance of η_k, see (2.8).

The method of evaluating $v(t,x)$ by importance sampling coincides with the method described in [7]: it is clear that $E\xi$ does not depend on the choice of h. At the same time, the variance $D\xi = E\xi^2 - (E\xi)^2$ does depend on h. Therefore it is natural to regard $h^1, ..., h^m$ as controls and to choose them such that the variance $D\xi$ is minimal. This problem is solved in [7]. It turns out that the variance can be reduced to zero.

Proposition 3.1. *Let the solution $v(t,x)$ of the problem* (1.10)-(1.11) *be positive. Let*

$$(3.5)\qquad h^j = -\frac{1}{v}\sum_{i=1}^{m}\sigma_{ij}\frac{\partial v}{\partial x^i}\ .$$

Suppose that for any (t,x), $t_0 \le t \le T$, $x \in R^m_+$, *there is a solution of the system* (2.2)-(2.4), *with* h^j *as in* (3.5), *for* $t \le s \le T$. *Then,* ξ *in* (3.3), *computed according to* (2.2)-(2.4) *with* h *as in* (3.5), *is deterministic, i.e.,* $D\xi = 0$.

Proof. By using Itô's formula and taking into account $Lv + c = 0$ we derive

$$d[v(s,X_{t,x}(s))\cdot Y_{t,x,1}(s) + Z_{t,x,1,0}(s)] = (Lv+c)\cdot Y ds - \sum_{i=1}^{m}\frac{\partial v}{\partial x^i}(\sigma h)^i \cdot Y ds$$

$$+\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}\cdot Y(\sigma dW(s))^i + v\cdot Yh^\top dW(s) + \sum_{i=1}^{m}\frac{\partial v}{\partial x^i}(\sigma dW(s))^i\cdot Yh^\top dW(s)$$

$$= Y\cdot(\sum_{i=1}^{m}\frac{\partial v}{\partial x^i}(\sigma dW(s))^i + vh^\top dW(s)) = Y\cdot\sum_{j=1}^{m}(\sum_{i=1}^{m}\sigma_{ij}\frac{\partial v}{\partial x^i} + vh^j)dW^j(s),$$

whence

$$v(s,X_{t,x}(s))\cdot Y_{t,x,1}(s) + Z_{t,x,1,0}(s) = v(t,x) + \int_t^s Y\cdot\sum_{j=1}^{m}(\sum_{i=1}^{m}\sigma_{ij}\frac{\partial v}{\partial x^i} + vh^j)dW^j. \tag{3.6}$$

For h in (3.5), equation (3.6) becomes an identity with respect to s, x, ω ($\omega \in \Omega$),

$$\xi(s) := v(s,X_{t,x}(s))\cdot Y_{t,x,1}(s) + Z_{t,x,1,0}(s) \equiv v(t,x), \tag{3.7}$$

i.e., $\xi(s)$ is deterministic. Moreover, $\xi(s)$ is independent of $t \le s \le T$. In particular, by (1.11), we get for $s = T$,

$$\xi(T) = \xi = f(X_{t,x}(T))\cdot Y_{t,x,1}(T) + Z_{t,x,1,0}(T) = v(t,x). \tag{3.8}$$

The proposition is proved.

>From the proof of the proposition above we obtain the following corollary.

Corollary 3.1.For an arbitrary h (of course, the usual conditions of smoothness and boundedness are supposed) the variance $D\xi(T)$ is equal to

$$D\xi(T) = E\int_t^T Y^2_{t,x,1}(s)\cdot\sum_{j=1}^{m}(\sum_{i=1}^{m}\sigma_{ij}\frac{\partial v}{\partial x^i} + vh^j)^2 ds,$$

where the functions σ_{ij}, $\partial v/\partial x^i$, v, h^j have $s, X_{t,x}(s)$ as their arguments.

Remark 3.1. Of course, the h^j, $j = 1, ..., m$, cannot be constructed without knowing the function v. Nevertheless, the obtained result establishes that, in principle, it is possible to reduce the variance $D\xi$ arbitrarily by conveniently choosing the functions h^j. The results can be used, e.g., in the following situation. Let all the parameters of the considered problem be close to those one for which the solution is known and equal to v_0. By taking h^j as in (3.5) equal to

$$h^j = -\frac{1}{v_0}\sum_{i=1}^{m}\sigma_{ij}\frac{\partial v_0}{\partial x^i}, \tag{3.9}$$

the variance $D\xi$, although not zero, will be small. Also, it is shown in [15] that in certain situations it is optimal to precompute a rough approximation for the solution of the Cauchy problem by some finite difference method and next to proceed with variance reduced Monte Carlo where the controls h^j are computed from the rough approximation.

Remark 3.2. If the condition $v > 0$ in Proposition 3.1 is not satisfied, but e.g., if $v > -K,\ K > 0$, then we consider $\widetilde{v} = v + K$ as a solution of the problem

$$L\widetilde{v} + Kr + c = 0,\ \widetilde{v}(T, x) = f(x) + K$$

and consider instead of (2.4),

$$d\widetilde{Z} = (Kr(s, X) + c(s, X))Y ds,\ \widetilde{Z}(t) = z.$$

Next, taking

$$\widetilde{h}^j = -\frac{1}{v+K}\sum_{i=1}^{m}\sigma_{ij}\frac{\partial v}{\partial x^i} = -\frac{1}{\widetilde{v}}\sum_{i=1}^{m}\sigma_{ij}\frac{\partial \widetilde{v}}{\partial x^i}$$

in (2.2)-(2.3) leads to $\widetilde{\xi} = (f(X_{t,x}(T)) + K)\cdot Y_{t,x,1}(T) + \widetilde{Z}_{t,x,1,0}(T)$, as being a deterministic variable.

A remarkable fact now is that the variables $\eta_k,\ k = 1, ..., m$, for h^j as in (3.5) are deterministic as well.

Proposition 3.2. *Under the hypotheses of Proposition 3.1 the variables* $\eta_k = \eta_k(T),\ k = 1, ..., m$, *in* (3.4), *computed according to* (2.2)-(2.4) *and* (2.10)-(2.12) *with* h *as in* (3.5) *are deterministic.*

Proof. By differentiating (3.7) with respect to x^k we get

$$\frac{\partial v}{\partial x^k}(t,x) = \sum_{i=1}^{m}\frac{\partial v}{\partial x^i}(s, X_{t,x}(s))\cdot\delta_k X^i_{t,x}(s)\cdot Y_{t,x,1}(s) + v(s, X_{t,x}(s))\cdot\delta_k Y_{t,x,1}(s) + \delta_k Z_{t,x,1,0}(s).$$

Thus, we have proved that the variables $\eta_k(s)$ (see (3.2)) are deterministic (moreover they do not depend on $s,\ t \le s \le T$). Therefore all $\eta_k(T)$ are deterministic. Proposition 3.2 is proved.

We now proceed to the method of control variates. In (2.2)-(2.4), we consider h to be fixed and introduce the new random variable

$$\xi_F(T) = \xi(T) + \int_t^T Y_{t,x,1}(s)\cdot\sum_{j=1}^{m}F_j(s, X_{t,x}(s))dW^j(s), \tag{3.10}$$

where $F_j(s,x)$ are rather arbitrary functions depending on (s,x).

Clearly, the expectation $E\xi_F(T)$ is equal to $E\xi(T)$ and does not depend on the choice of F. At the same time, the variance $D\xi_F(T)$ does depend on F. Also in this situation it turns out that the variance can be reduced to zero.

Proposition 3.3. *Let* h *in* (2.2)-(2.4) *be a fixed function. Then for*

$$F_j(s,x) = -(\sum_{i=1}^{m}\sigma_{ij}(s,x)\frac{\partial v}{\partial x^i}(s,x) + v(s,x)h^j(s,x)),\ j = 1, ..., m, \tag{3.11}$$

the variable $\xi_F(T)$ *is deterministic, i.e.,* $D\xi_F(T) = 0$.

Proof. The proposition is a consequence of the following equality (see (3.6))

$$\xi_F(T) = f(X_{t,x}(T)) \cdot Y_{t,x,1}(T) + Z_{t,x,1,0}(T) + \int_t^T Y_{t,x,1}(s) \cdot \sum_{j=1}^m F_j(s, X_{t,x}(s)) dW^j(s)$$

$$= v(t,x) + \int_t^T Y_{t,x,1}(s) \cdot \sum_{j=1}^m (\sum_{i=1}^m \sigma_{ij} \frac{\partial v}{\partial x^i} + vh^j) dW^j(s) + \int_t^T Y_{t,x,1}(s) \cdot \sum_{j=1}^m F_j dW^j(s),$$

where the functions σ_{ij}, $\partial v / \partial x^i$, v, h^j, F^j have $s, X_{t,x}(s)$ as their arguments.

Clearly,

$$D\xi_F(T) = E \int_t^T Y_{t,x,1}^2(s) \cdot \sum_{j=1}^m (\sum_{i=1}^m \sigma_{ij} \frac{\partial v}{\partial x^i} + vh^j + F_j)^2 ds \tag{3.12}$$

which is equal to zero for F_j according to (3.11). Proposition 3.3 is proved.

Of course, a remark similar to Remark 3.1. applies here as well.

The method of control variates in the case $h = 0$ was first considered by N.J. Newton [10]. Following [10], let us look for $F = (F_1, ..., F_m)$ of the form

$$F_j(s,x) = \sum_{i=1}^m \sigma_{ij}(s,x) \sum_{r=1}^l c_r \gamma_r^i(s,x), \tag{3.13}$$

where $\gamma_r = (\gamma_r^1, ..., \gamma_r^m)$, $r = 1, ..., l$, are known row vectors and c_r are constants. According to (3.12) we have

$$D\xi_F(T) = E \int_t^T Y_{t,x,1}^2(s) \cdot \sum_{j=1}^m (\sum_{i=1}^m \sigma_{ij} [\frac{\partial v}{\partial x^i} + \sum_{r=1}^l c_r \gamma_r^i] + vh^j)^2 ds.$$

However, determination of c_r directly by minimization of the right-hand-side of this relation is impossible because the functions v and $\partial v / \partial x^i$ are unknown. But by using

$$v(s, X_{t,x}(s)) = E(\xi(T; s, X_{t,x}(s)) \mid X_{t,x}(s)),$$

$$\frac{\partial v}{\partial x^i}(s, X_{t,x}(s)) = E(\eta_i(T; s, X_{t,x}(s)) \mid X_{t,x}(s)),$$

where

$$\xi(T; s, X_{t,x}(s)) = f(X_{s,X_{t,x}(s)}(T)) \cdot Y_{s,X_{t,x}(s),1}(T) + Z_{s,X_{t,x}(s),1,0}(T),$$

$$\eta_i(T; s, X_{t,x}(s)) = \sum_{k=1}^m \frac{\partial f}{\partial x^k}(X_{s,X_{t,x}(s)}(T)) \cdot \delta_i X^k_{s,X_{t,x}(s)}(T) \cdot Y_{s,X_{t,x}(s),1}(T)$$

$$+ f(X_{s,X_{t,x}(s)}(T)) \cdot \delta_i Y_{s,X_{t,x}(s),1}(T) + \delta_i Z_{s,X_{t,x}(s),1,0}(T),$$

it is not difficult to show that the mentioned minimization problem is equivalent to the following one

$$E\int_t^T Y_{t,x,1}^2(s)\cdot\sum_{j=1}^m\Big(\sum_{i=1}^m \sigma_{ij}[\eta_i(T;\cdot)+\sum_{r=1}^l c_r\gamma_r^i]+\xi(T;\cdot)h^j\Big)^2 ds \to \min_{c_1,\dots,c_l}, \tag{3.14}$$

where the functions $\sigma_{ij},\ \gamma_r^i,\ \xi(T;\cdot),\ \eta_i(T;\cdot)$ have $s, X_{t,x}(s)$ as their arguments.

The solution of the problem (3.14) provides optimal values for c and leads to reduced variance.

To conclude we consider the following system

$$dX=(b(s,X)-\sigma(s,X)h(s,X))ds+\sigma(s,X)dW(s),\ X(t)=x, \tag{3.15}$$

$$dY=-r(s,X)Yds+h^\top(s,X)YdW(s),\ Y(t)=1, \tag{3.16}$$

$$dZ=c(s,X)Yds+F^\top(s,X)YdW(s),\ Z(t)=0, \tag{3.17}$$

and the random variables $\xi(s),\ \eta_k(s)$ according to (3.1), (3.2). Of course, the equation for $\delta_k Z$ becomes of the following form (instead of (2.12))

$$d\delta_k Z=\sum_{l=1}^m \frac{\partial c(s,X)}{\partial x^l}\cdot\delta_k X^l\cdot Yds+c(s,X)\delta_k Y ds \tag{3.18}$$

$$+\sum_{l=1}^m \frac{\partial F^\top(s,X)}{\partial x^l}\cdot\delta_k X^l\cdot YdW(s)+F^\top(s,X)\delta_k YdW(s),\ \delta_k Z(t)=0.$$

Note that the variables $\xi(s)$ and $\eta_k(s)$ depend on h and F and a more correct notation would be, for example, $\xi_{h,F}(s)$ instead of $\xi(s)$ but the accepted notation does not lead to any confusion.

The following proposition can be proved analogously to the previous ones.

Proposition 3.4. *Let h and F be such that*

$$\sum_{i=1}^m \sigma_{ij}\frac{\partial v}{\partial x^i}+vh^j+F_j=0,\ j=1,\dots,m. \tag{3.19}$$

Then $\xi(T)$ from (3.3) *computed according to* (3.15)-(3.17) *and*$\eta_k(T)$ *in* (3.4) *computed according to* (2.10)-(2.11), (3.18) *are deterministic.*

Example 3.1. Let all the parameters $r,\ \mu^i,\ \nu_{ij},\ c,\ r^i$ be independent of x, i.e., they are known deterministic functions of t, and let the payoff function be a sum

$$f(X(T))=f_1(X^1(T))+\dots+f_m(X^m(T)).$$

Then the system (2.2)-(2.4) becomes of the following form (we put $h=0$):

$$dX^i=X^i\cdot(r(s)-r^i(s))ds+X^i\cdot\sum_{j=1}^m \nu_{ij}(s)dW^j(s),\ X^i(t)=x^i,\ i=1,\dots,m,$$

$$dY=-r(s)Yds,\ Y(t)=1,$$

$$dZ = c(s)Y ds,\ Z(t) = 0,\ t \le s \le T.$$

We derive explicitly

$$X^i_{t,x}(T) = x^i \cdot k^i(t) \cdot \exp(\int_t^T \sum_{j=1}^m \nu_{ij}(s)dW^j(s)) = x^i \cdot k^i(t) \cdot \exp(\alpha^i \lambda_i(t)),$$

where

$$k^i(t) = \exp(\int_t^T (r(s) - r^i(s))ds - \frac{1}{2}\int_t^T \sum_{j=1}^m \nu_{ij}^2(s)ds),$$

$$\lambda_i(t) = (\int_t^T \sum_{j=1}^m \nu_{ij}^2(s)ds)^{1/2},$$

and α^i is a normal random variable with zero mean and variance 1.

>From (2.1) we obtain

$$v^0(t, x^1, ..., x^m) = \sum_{i=1}^m E\left[f_i(X^i_{t,x}(T)) \cdot Y_{t,x,1}(T) + Z_{t,x,1,0}(T)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{i=1}^m \int_{-\infty}^{\infty} f_i(x^i k^i(t) \exp(\alpha\lambda_i(t))) \cdot \exp(-\alpha^2/2)d\alpha \cdot \exp(-\int_t^T r(s)ds)$$

$$+ \int_t^T c(s) \exp(-\int_s^T r(s')ds')ds,$$

whence the derivatives $\partial v^0/\partial x^i,\ i = 1, ..., m$, can be found explicitly as well.

In case the parameters of an original problem do not differ too much from the considered ones above, we can use the recommendation of Remark 3.1 and, for example, take h^j according to (3.9) with $F_j = 0$ or $h^j = 0$ with $F_j = -\sum_{i=1}^m \sigma_{ij}(s,x)\partial v^0(s,x)/\partial x^i$.

4. **Gamma, vega, theta**

Clearly, differentiation with respect to x^j in (2.8) gives the probabilistic representation for the gammas $\partial^2 v(t,x)/\partial x^k \partial x^j,\ i,k = 1, ..., m$. The representation involves along with the first variations $\delta_k X^i,\ \delta_k Y,\ \delta_k Z$ the second ones

$$\delta_{kj} X^i(s) := \frac{\partial^2 X^i_{t,x}(s)}{\partial x^k \partial x^j},\ \delta_{kj} Y(s) := \frac{\partial^2 Y_{t,x,1}(s)}{\partial x^k \partial x^j},\ \delta_{kj} Z(s) := \frac{\partial^2 Z_{t,x,1,0}(s)}{\partial x^k \partial x^j},\ t \le s \le T.$$

Let us write down the system for these variables. For notational simplicity we restrict ourselves to the case $m = 1$. In this case $X,\ b,\ h,\ \sigma$ and W in (2.1)-(2.4) are scalars. We have for the delta

$$u(t,x) = \frac{\partial v}{\partial x}(t,x) \tag{4.1}$$

$$= E\left[\frac{df}{dx}(X_{t,x}(T)) \cdot \delta X(T) \cdot Y_{t,x,1}(T) + f(X_{t,x}(T)) \cdot \delta Y(T) + \delta Z(T)\right],$$

where (together with (2.2)-(2.4))

$$d\delta X = \frac{\partial(b(s,X) - \sigma(s,X)h(s,X))}{\partial x} \cdot \delta X ds + \frac{\partial \sigma(s,X)}{\partial x} \cdot \delta X dW(s),\ \delta X(t) = 1, \tag{4.2}$$

$$d\delta Y = -\frac{\partial r(s,X)}{\partial x} \cdot \delta X \cdot Y ds - r(s,X) \cdot \delta Y ds \tag{4.3}$$

$$+\frac{\partial h(s,X)}{\partial x} \cdot \delta X \cdot Y dW(s) + h(s,X) \cdot \delta Y dW(s),\ \delta Y(t) = 0,$$

$$d\delta Z = \frac{\partial c(s,X)}{\partial x} \cdot \delta X \cdot Y ds + c(s,X) \cdot \delta Y ds,\ \delta Z(t) = 0. \tag{4.4}$$

We introduce the notation

$$\gamma X(s) := \frac{\partial^2 X_{t,x}(s)}{\partial x^2},\ \gamma Y(s) := \frac{\partial^2 Y_{t,x,1}(s)}{\partial x^2},\ \gamma Z(s) := \frac{\partial^2 Z_{t,x,1,0}(s)}{\partial x^2},$$

and obtain for the gamma

$$u^\gamma(t,x) := \frac{\partial^2 v}{\partial x^2}(t,x) = E\left[\frac{d^2 f}{dx^2}(X_{t,x}(T)) \cdot [\delta X(T)]^2 \cdot Y_{t,x,1}(T)\right] \tag{4.5}$$

$$+E\left[\frac{df}{dx}(X_{t,x}(T)) \cdot [\gamma X(T) \cdot Y_{t,x,1}(T) + 2\delta X(T) \cdot \delta Y(T)] + f(X_{t,x}(T)) \cdot \gamma Y(T) + \gamma Z(T)\right],$$

where

$$d\gamma X = \frac{\partial(b(s,X) - \sigma(s,X)h(s,X))}{\partial x} \cdot \gamma X ds + \frac{\partial \sigma(s,X)}{\partial x} \cdot \gamma X dW(s) \tag{4.6}$$

$$+\frac{\partial^2(b(s,X) - \sigma(s,X)h(s,X))}{\partial x^2} \cdot [\delta X]^2 ds + \frac{\partial^2 \sigma(s,X)}{\partial x^2} \cdot [\delta X]^2 dW(s),\ \gamma X(t) = 0,$$

$$d\gamma Y = -\frac{\partial r(s,X)}{\partial x} \cdot \gamma X \cdot Y ds - r(s,X) \cdot \gamma Y ds + \frac{\partial h(s,X)}{\partial x} \cdot \gamma X \cdot Y dW(s) \tag{4.7}$$

$$+h(s,X) \cdot \gamma Y dW(s) - \frac{\partial^2 r(s,X)}{\partial x^2}[\delta X]^2 \cdot Y ds - 2\frac{\partial r(s,X)}{\partial x}\delta X \cdot \delta Y ds$$

$$+\frac{\partial^2 h(s,X)}{\partial x^2} \cdot [\delta X]^2 \cdot Y dW(s) + 2\frac{\partial h(s,X)}{\partial x} \cdot \delta X \cdot \delta Y dW(s),\ \gamma Y(t) = 0,$$

$$d\gamma Z = \frac{\partial c(s,X)}{\partial x} \cdot \gamma X \cdot Y ds + c(s,X) \cdot \gamma Y ds \tag{4.8}$$

$$+\frac{\partial^2 c(s,X)}{\partial x^2} \cdot [\delta X]^2 \cdot Y ds + 2\frac{\partial c(s,X)}{\partial x} \cdot \delta X \cdot \delta Y ds,\ \delta Z(t) = 0.$$

Thus, to calculate the gamma one needs to evaluate the expectation (4.5) by virtue of the system consisting of equations (2.2)-(2.4), (4.2)-(4.4)), and (4.6)-(4.8).

One can prove that the gammas for h^j as in (3.5) are deterministic as well.

Clearly, if the problem under consideration depends on some parameter α, then $v = v(t,x;\alpha)$ and it is possible to find $\partial v(t,x;\alpha)/\partial\alpha$ in the same way as above. Let us find, for example, the vega $\partial v(t,x;\alpha)/\partial\alpha$ in the case of the one-dimensional model (1.1) ($m=1$), where instead of $\sigma(t,x) = x\nu(t,x)$ we have $\sigma(t,x;\alpha) = \alpha x\nu(t,x)$. We have

$$v(t,x;\alpha) = E[f(X_{t,x}(T;\alpha)) \cdot Y_{t,x,1}(T;\alpha) + Z_{t,x,1,0}(T;\alpha)], \tag{4.9}$$

where

$$dX = (b(s,X) - \sigma(s,X;\alpha)h(s,X;\alpha))ds + \sigma(s,X;\alpha)dW(s),\ X(t) = x, \tag{4.10}$$

$$dY = -r(s,X)Yds + h(s,X;\alpha)YdW(s),\ Y(t) = y, \tag{4.11}$$

$$dZ = c(s,X)Yds,\ Z(t) = z. \tag{4.12}$$

Therefore

$$\frac{\partial v}{\partial\alpha}(t,x;\alpha) = E\left[\frac{df}{dx}(X_{t,x}(T;\alpha)) \cdot \delta_\alpha X(T;\alpha) \cdot Y_{t,x,1}(T;\alpha)\right] \tag{4.13}$$

$$+E[f(X_{t,x}(T;\alpha)) \cdot \delta_\alpha Y(T;\alpha) + \delta_\alpha Z(T;\alpha)],$$

where

$$\delta_\alpha X(s;\alpha) = \frac{\partial X_{t,x}(s;\alpha)}{\partial\alpha},\ \delta_\alpha Y(s;\alpha) = \frac{\partial Y_{t,x,1}(s;\alpha)}{\partial\alpha},\ \delta_\alpha Z(s;\alpha) = \frac{\partial Z_{t,x,1,0}(s;\alpha)}{\partial\alpha}$$

satisfy the following system

(4.14)

$$d\delta_\alpha X = \frac{\partial(b-\sigma h)}{\partial x} \cdot \delta_\alpha X ds + \frac{\partial\sigma}{\partial x} \cdot \delta_\alpha X dW(s) - \frac{\partial(\sigma h)}{\partial\alpha}ds + \frac{\partial\sigma}{\partial\alpha}dW(s),\ \delta_\alpha X(t) = 0,$$

$$d\delta_\alpha Y = -\frac{\partial r}{\partial x} \cdot \delta_\alpha X \cdot Yds - r \cdot \delta_\alpha Y ds \tag{4.15}$$

$$+\frac{\partial h}{\partial x} \cdot \delta_\alpha X \cdot YdW(s) + h\delta_\alpha YdW(s) + \frac{\partial h}{\partial\alpha}YdW(s),\ \delta_\alpha Y(t) = 0,$$

$$d\delta_\alpha Z = \frac{\partial c}{\partial x} \cdot \delta_\alpha X \cdot Yds + c\delta_\alpha Yds,\ \delta_\alpha Z(t) = 0. \tag{4.16}$$

Let us now point out how to find theta; $u_{m+1}(t,x) := \partial v(t,x)/\partial t$. The above way of differentiation under the expectation sign is now impossible because of the nondifferentiability of $X_{t,x}(s)$ with respect to t (e.g., the problem $dX = dW(s),\ X(t) = x,\ s \geq t$, has the solution $X_{t,x}(s) = x + W(s) - W(t)$ which is evidently nondifferentiable with respect to t). Of course, one can evaluate theta by to the initial equation (1.10) after finding the deltas and the gammas. But if we do not need the gammas, this way is irrational. It is better to consider the system of $m+2$ parabolic equations consisting of (1.10)-(1.11), (2.6)-(2.7) and

$$\frac{\partial u_{m+1}}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(t,x)\frac{\partial^2 u_{m+1}}{\partial x^i \partial x^j} + \sum_{i=1}^{m} b^i(t,x)\frac{\partial u_{m+1}}{\partial x^i} - r(t,x) \cdot u_{m+1} \tag{4.17}$$

$$+\frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial a_{ij}}{\partial t}(t,x)\frac{\partial u_j}{\partial x^i}+\sum_{i=1}^{m}\frac{\partial b_i}{\partial t}(t,x)\frac{\partial v}{\partial x^i}-\frac{\partial r}{\partial t}(t,x)\cdot v+\frac{\partial c}{\partial t}(t,x)=0,$$

$$u_{m+1}(T,x)=-\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}(T,x)\frac{\partial^2 f}{\partial x^i\partial x^j}(x)-\sum_{i=1}^{m}b^i(T,x)\frac{\partial f}{\partial x^i}(x) \tag{4.18}$$

$$+r(T,x)\cdot f(x)-c(T,x):=g(x),$$

and to use the probabilistic representations from [8] consequently.

Let us consider a model in which the coefficients ν_{ij} (and consequently a_{ij}) do not depend on t. In such a case we have a system of parabolic equations consisting of two equations for v and u_{m+1} only. Namely, (1.10)-(1.11) and the following equation

$$\frac{\partial u_{m+1}}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{m}a_{ij}(t,x)\frac{\partial^2 u_{m+1}}{\partial x^i\partial x^j}+\sum_{i=1}^{m}b^i(t,x)\frac{\partial u_{m+1}}{\partial x^i}-r(t,x)\cdot u_{m+1} \tag{4.19}$$

$$+\sum_{i=1}^{m}\frac{\partial b_i}{\partial t}(t,x)\frac{\partial v}{\partial x^i}-\frac{\partial r}{\partial t}(t,x)\cdot v+\frac{\partial c}{\partial t}(t,x)=0,$$

$$u_{m+1}(T,x)=g(x), \tag{4.20}$$

with $g(x)$ as in (4.18) and $a_{ij}(T,x)=a_{ij}(x)$.

The probabilistic representation for the solution of the Cauchy problem (1.10)-(1.11), (4.19)-(4.20) has the following simple form (see [8]). Introduce the system of stochastic differential equations

$$dX=(b(s,X)-\sigma(s,X)h(s,X))ds+\sigma(s,X)dW(s),\ X(t)=x, \tag{4.21}$$

$$dY^1=-r(s,X)Y^1ds-\frac{\partial r(s,X)}{\partial s}Y^2ds+h^\top(s,X)Y^1dW(s),\ Y^1(t)=y^1, \tag{4.22}$$

$$dY^2=-r(s,X)Y^2ds+h^\top(s,X)Y^2dW(s) \tag{4.23}$$

$$+(\sigma^{-1}(s,X)\frac{\partial b(s,X)}{\partial s})^\top Y^2dW(s),\ Y^2(t)=y^2,$$

$$dZ=c(s,X)Y^1ds+\frac{\partial c(s,X)}{\partial s}Y^2ds,\ Z(t)=0, \tag{4.24}$$

and the random variable

$$\xi_{t,x,y^1,y^2}=f(X_{t,x}(T))\cdot Y^1_{t,x,y^1,y^2}(T)+g(X_{t,x}(T))\cdot Y^2_{t,x,y^1,y^2}(T)+Z_{t,x,y^1,y^2,0}(T), \tag{4.25}$$

where Y^1 and Y^2 are scalars.

Then the required solution $v(t,x)$, $u_{m+1}(t,x)$ can be found from the relations

$$v(t,x)=E\xi_{t,x,1,0}\ ,\ u_{m+1}(t,x)=E\xi_{t,x,0,1}\ . \tag{4.26}$$

This fact can be verified in the following way. We show by Itô's formula that

$$v(s,X_{t,x}(s))\cdot Y^1_{t,x,y^1,y^2}(s)+u_{m+1}(s,X_{t,x}(s))\cdot Y^2_{t,x,y^1,y^2}(s)+Z_{t,x,y^1,y^2,0}(s) \tag{4.27}$$

$$-v(t,x)\cdot y^1 - u_{m+1}(t,x)\cdot y^2$$

$$= \int_t^s \left[F_1^\top(\vartheta, X_{t,x}(\vartheta)\cdot Y^1_{t,x,y^1,y^2}(\vartheta) + F_2^\top(\vartheta, X_{t,x}(\vartheta)\cdot Y^2_{t,x,y^1,y^2}(\vartheta)\right] dW(\vartheta),$$

where F_1 and F_2 are some known vector-functions. From this the relations (4.26) follow immediately.

5. **The Jamshidian LIBOR rate model**

In this section we drop the assumption of an instantaneous saving bond numeraire and consider a system of assets (we now use the notation from [4] which differs a little from the one used in the previous sections)

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i \circ dW = \mu_i dt + \sum_{k=1}^m \sigma_{ik} dW_k,\ t_0 \le t \le T,\ i = 1, ..., m, \tag{5.1}$$

under the arbitrage free condition [4]:

$$\mu_i = r + \sigma_i \circ \varphi, \quad i = 1, ..., m,$$

for some processes r and φ. We assume that the system is nondegenerate, i.e., $rank(\sigma_i \circ \sigma_j) = m$ almost surely.

A portfolio (ψ, B) is said to be a self-financing trading strategy when

$$V(t) := \sum_{k=1}^m \psi_k(t) B_k(t) = V(0) + \int_0^t \psi_k(s) dB_k(s). \tag{5.2}$$

We consider self-financing trading strategies (ψ, B) where ψ has the form $\psi = \psi(t, B)$. So the corresponding portfolio value is of the form $V = V(t, B)$ as well and we have

$$\sum_{i=1}^m B_i \frac{\partial V}{\partial B_i} = V, \tag{5.3}$$

$$\frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 V}{\partial B_i \partial B_j} dB_i dB_j = 0. \tag{5.4}$$

Indeed, the self-financing property implies $\sum_k B_k d\psi_k = 0$ and consequently

$$0 = (\sum_{k=1}^m B_k d\psi_k)(\sum_{k'=1}^m B_{k'} d\psi_{k'}) = (\sum_{k=1}^m \sum_{l=1}^m B_k \frac{\partial \psi_k}{\partial B_l} dB_l)(\sum_{k'=1}^m \sum_{l'=1}^m B_{k'} \frac{\partial \psi_{k'}}{\partial B_{l'}} dB_{l'})$$

$$= \sum_{k,k'=1}^m \sum_{l,l'=1}^m B_k \frac{\partial \psi_k}{\partial B_l} B_{k'} \frac{\partial \psi_{k'}}{\partial B_{l'}} dB_l dB_{l'} = \sum_{l,l'=1}^m \sigma_l \circ \sigma_{l'} \sum_{k=1}^m B_k \frac{\partial \psi_k}{\partial B_l} \sum_{k'=1}^m B_{k'} \frac{\partial \psi_{k'}}{\partial B_{l'}} dt.$$

Then by non-degeneracy it follows that $\sum_k B_k \partial\psi_k / \partial B_l = 0$, or,

$$\psi_l = \sum_{k=1}^m \frac{\partial(\psi_k B_k)}{\partial B_l} = \frac{\partial V}{\partial B_l},\ l = 1, ..., m, \tag{5.5}$$

which gives (5.3). Next, by expanding dV by Itô's formula, by the self-financing property and (5.5) we get (5.4). From (5.3) we conclude that the value of this self-financing portfolio is homogeneous of degree 1 in B, i.e.

$$V(t, \alpha B) = \alpha V(t, B) \qquad \alpha > 0.$$

Therefore it is clear that any path-independent self-financing portfolio must satisfy this homogeneity condition [4].

We now assume in addition that the process B is an Ito diffusion, i.e. μ and σ are functions of (t, B), hence

$$dB_i dB_j = B_i B_j \Gamma_{ij}(t, B) dt,$$

where $\Gamma_{ij}(t, x) := (\sigma_i \cdot \sigma_j)(t, x)$. Then (5.3)-(5.4) can be rewritten in the form

$$\sum_{i=1}^{m} x_i \frac{\partial V}{\partial x_i} = V, \tag{5.6}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{m} \Gamma_{ij} x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j} = 0, \tag{5.7}$$

We assume also that

$$\sigma_i(\alpha x) = \sigma_i(x), \tag{5.8}$$

for all $\alpha > 0$, $x > 0$, $i = 1, ..., m$.

The (abstract) LIBOR process is defined as the $m - 1$ dimensional process given by

$$L_i := \delta_i^{-1}\left(\frac{B_i}{B_{i+1}} - 1\right), \quad i = 1, ..., m-1,$$

where the constants δ_i are so called "daycount fractions". For the LIBOR dynamics we can derive straightforwardly (an empty sum is defined to be 0)

$$dL_i = -\sum_{j=i+1}^{m-1} \frac{\delta_j L_i L_j \gamma_i \circ \gamma_j}{(1 + \delta_j L_j)} dt + L_i \gamma_i \circ (\varphi - \sigma_m) dt + L_i \gamma_i \circ dW, \tag{5.9}$$

where

$$\gamma_i := \delta_i^{-1} L_i^{-1} (1 + \delta_i L_i)(\sigma_i - \sigma_{i+1}). \tag{5.10}$$

It is possible to write

$$dL_i = (L_i + \delta_i^{-1})(\sigma_i - \sigma_{i+1}) \circ (\varphi - \sigma_{i+1}) dt + (L_i + \delta_i^{-1})(\sigma_i - \sigma_{i+1}) \circ dW$$

but we prefer the representation (5.9) since the class of LIBOR market models, specified by deterministic or even constant γ_i is of high practical importance.

Since the value $V(t, B)$ of a path-independent portfolio is homogeneous of degree 1 in B, we assume a payoff function $g(B(T))$ to be homogeneous of degree 1 as well. The function $V(t, B)$ can be expressed as

$$V(t, B) = B_m(t) v(t, L)$$

and therefore

$$g(B(T)) = B_m(t) f(L(T))$$

for some functions $v(\cdot,\cdot): \mathbb{R}\times\mathbb{R}^{m-1} \to \mathbb{R}$ and $f(\cdot) : \mathbb{R}^{m-1} \to \mathbb{R}$.

Hence,

$$V(t,x) = V(t, x_1, ..., x_m) = x_m v(t, \delta_1^{-1}(\frac{x_1}{x_2} - 1), ..., \delta_{m-1}^{-1}(\frac{x_{m-1}}{x_m} - 1))$$

$$= x_m v(t, y_1, .., y_{m-1}) = x_m v(t, y),$$

where (y_k corresponds to L_k)

$$y_k = \delta_k^{-1}(\frac{x_k}{x_{k+1}} - 1), \quad k = 1, ..., m-1,$$

and

$$v(T, y) = f(y).$$

Let us derive a partial differential equation for v.

Using (5.6), we deduce from (5.7) straightforwardly

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,,j=1}^{m-1} \Phi_{ij}(x) x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j} = 0, \tag{5.11}$$

where

$$\Phi_{ij} = \Gamma_{ij} - \Gamma_{im} - \Gamma_{jm} + \Gamma_{mm} = (\sigma_i - \sigma_m) \circ (\sigma_j - \sigma_m).$$

For $i, j < m$ we have

$$\frac{\partial V}{\partial x_i} = x_m \sum_{k=1}^{m-1} \frac{\partial v}{\partial y_k} \frac{\partial y_k}{\partial x_i}$$

and

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = x_m \sum_{k,l=1}^{m-1} \frac{\partial^2 v}{\partial y_k y_l} \frac{\partial y_l}{\partial x_j} \frac{\partial y_k}{\partial x_i} + x_m \sum_{k=1}^{m-1} \frac{\partial v}{\partial y_k} \frac{\partial^2 y_k}{\partial x_i \partial x_j}.$$

Hence (5.11) yields

$$\frac{\partial v}{\partial t} + \frac{1}{2} \sum_{i,,j=1}^{m-1} \Phi_{ij}(x) x_i x_j \left\{ \sum_{k,l=1}^{m-1} \frac{\partial^2 v}{\partial y_k y_l} \frac{\partial y_l}{\partial x_j} \frac{\partial y_k}{\partial x_i} + \sum_{k=1}^{m-1} \frac{\partial v}{\partial y_k} \frac{\partial^2 y_k}{\partial x_i \partial x_j} \right\} = 0$$

or

$$\frac{\partial v}{\partial t} + \frac{1}{2} \sum_{k,l=1}^{m-1} \frac{\partial^2 v}{\partial y_k y_l} \sum_{i,,j=1}^{m-1} \Phi_{ij}(x) x_i x_j \frac{\partial y_l}{\partial x_j} \frac{\partial y_k}{\partial x_i} + \frac{1}{2} \sum_{k=1}^{m-1} \frac{\partial v}{\partial y_k} \sum_{i,,j=1}^{m-1} \Phi_{ij}(x) x_i x_j \frac{\partial^2 y_k}{\partial x_i \partial x_j} = 0.$$

Next observe that

$$\sum_{i,,j=1}^{m-1} \Phi_{ij}(x)x_ix_j \frac{\partial y_l}{\partial x_j}\frac{\partial y_k}{\partial x_i} = \Phi_{lk}(x)x_lx_k\frac{\partial y_l}{\partial x_l}\frac{\partial y_k}{\partial x_k} + \Phi_{l+1,k}(x)x_{l+1}x_k\frac{\partial y_l}{\partial x_{l+1}}\frac{\partial y_k}{\partial x_k} +$$

$$\Phi_{l,k+1}(x)x_lx_{k+1}\frac{\partial y_l}{\partial x_l}\frac{\partial y_k}{\partial x_{k+1}} + \Phi_{l+1,k+1}(x)x_{l+1}x_{k+1}\frac{\partial y_l}{\partial x_{l+1}}\frac{\partial y_k}{\partial x_{k+1}} =$$

$$[\Phi_{lk}(x) - \Phi_{l+1,k}(x) - \Phi_{l,k+1}(x) + \Phi_{l+1,k+1}(x)]\frac{\delta_k^{-1}\delta_l^{-1}x_kx_l}{x_{k+1}x_{l+1}} =$$

$$(\sigma_l - \sigma_{l+1})\circ(\sigma_k - \sigma_{k+1})\frac{(1+\delta_ky_k)(1+\delta_ly_l)}{\delta_k\delta_l} = y_ky_l\,\gamma_k\circ\gamma_l$$

and

$$\frac{1}{2}\sum_{i,,j=1}^{m-1}\Phi_{ij}(x)x_ix_j\frac{\partial^2 y_k}{\partial x_i\partial x_j} = \sum_{i,,j=1;i<j}^{m-1}\Phi_{ij}(x)x_ix_j\frac{\partial^2 y_k}{\partial x_i\partial x_j} + \frac{1}{2}\sum_{i=1}^{m-1}\Phi_{ii}(x)x_i^2\frac{\partial^2 y_k}{\partial x_i^2}$$

$$= \Phi_{k,k+1}(x)x_kx_{k+1}\frac{\partial^2 y_k}{\partial x_k\partial x_{k+1}} + \frac{1}{2}\Phi_{k+1,k+1}(x)x_{k+1}^2\frac{\partial^2 y_k}{\partial x_{k+1}^2}$$

$$= [-\Phi_{k,k+1}(x) + \Phi_{k+1,k+1}(x)]\delta_k^{-1}\frac{x_k}{x_{k+1}} = -\delta_k^{-1}(1+\delta_ky_k)(\sigma_k - \sigma_{k+1})\circ(\sigma_{k+1} - \sigma_m)$$

$$= -y_k\,\gamma_k\circ(\sigma_{k+1} - \sigma_m) = -\sum_{p=k+1}^{m-1} y_k\,\gamma_k\circ(\sigma_p - \sigma_{p+1}) = -\sum_{p=k+1}^{m-1}\frac{\delta_py_py_k}{1+\delta_py_p}\,\gamma_k\circ\gamma_p.$$

We thus find

$$\frac{\partial v}{\partial t} - \sum_{k=1}^{m-1}\sum_{p=k+1}^{m-1}\frac{\partial v}{\partial y_k}\frac{\delta_py_py_k}{1+\delta_py_p}\,\gamma_k\circ\gamma_p + \frac{1}{2}\sum_{k,l=1}^{m-1}\frac{\partial^2 v}{\partial y_k\partial y_l}y_ky_l\,\gamma_k\circ\gamma_l \;= 0.$$

Note that because of assumption (5.8) γ_i are functions of y indeed.

We introduce

$$\lambda_k(t,y) = -\sum_{p=k+1}^{m-1}\frac{\delta_py_py_k}{1+\delta_py_p}\,\gamma_k\circ\gamma_p,$$

$$\eta_{kl}(t,y) = y_ky_l\,\gamma_k\circ\gamma_l.$$

So the Cauchy problem for v reads,

$$\frac{\partial v}{\partial t} + \sum_{k=1}^{m-1}\lambda_k(t,y)\frac{\partial v}{\partial y_k} + \frac{1}{2}\sum_{k,l=1}^{m-1}\eta_{kl}(t,y)\frac{\partial^2 v}{\partial y_k\partial y_l} \;= 0, \tag{5.12}$$

$$v(T,y) = f(y). \tag{5.13}$$

Next, we will derive a representation for the hedge quantities ψ_i. For $i < m$ we have

$$(5.14) \qquad \frac{\partial V}{\partial x_i} = x_m \sum_{k=1}^{m-1} \frac{\partial v}{\partial y_k}\frac{\partial y_k}{\partial x_i} = -x_m \frac{\partial v}{\partial y_{i-1}} \delta_{i-1}^{-1} \frac{x_{i-1}}{x_i^2} + x_m \frac{\partial v}{\partial y_i} \delta_i^{-1} \frac{1}{x_{i+1}}$$

$$= -\frac{\partial v}{\partial y_{i-1}} \delta_{i-1}^{-1} \cdot (1+\delta_{i-1}y_{i-1}) \prod_{k=i}^{m-1} \frac{1}{1+\delta_k y_k} + \frac{\partial v}{\partial y_i} \delta_i^{-1} \cdot \prod_{k=i+1}^{m-1} \frac{1}{1+\delta_k y_k} := \varphi_i(t,y)$$

and

$$(5.15) \quad \frac{\partial V}{\partial x_m} = v(t,y) - \frac{1}{x_m} \sum_{i=1}^{m-1} x_i \frac{\partial V}{\partial x_i} = v(t,y) - \sum_{i=1}^{m-1} \prod_{k=i}^{m-1} (1+\delta_k y_k) \varphi_i(t,y) := \varphi_m(t,y).$$

Therefore the hedging strategy is constructed by

$$(5.16) \qquad V(t) = \sum_{i=1}^{m} \varphi_i(t, L(t)) B_i(t).$$

According to (5.14) and (5.15), for calculating $\varphi_i(t, L(t)),\ i = 1, ..., m$, we have to find v and $\partial v / \partial y_i,\ i = 1, ..., m-1$. Clearly they can be found by solving the Cauchy problem (5.12)-(5.13) in the same manner as it was done in the previous sections.

Set

$$u_i := \frac{\partial v}{\partial y_i},$$

then differentiating with respect to y_i yields

(5.17)

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{m-1} \frac{\partial \lambda_k}{\partial y_i} u_k + \sum_{k=1}^{m-1} \lambda_k(t,y) \frac{\partial u_i}{\partial y_k} + \frac{1}{2} \sum_{k,l=1}^{m-1} \frac{\partial \eta_{kl}}{\partial y_i} \frac{\partial u_k}{\partial y_l} + \frac{1}{2} \sum_{k,l=1}^{m-1} \eta_{kl}(t,y) \frac{\partial^2 u_i}{\partial y_k y_l} = 0.$$

If we have a Cauchy problem (5.12)-(5.13) for v, we have also a Cauchy problem for the u_i :

$$(5.18) \qquad u_i(T,y) = \frac{\partial f}{\partial y_i}.$$

6. Applications to European LIBOR derivative claims

Now, in practice we are given a fixed time tenor structure $T_1 < T_2 < ... < T_m$, where $\delta_i = T_i - T_{i-1}$ and a system of zero-coupon bonds B_i which mature at T_i with $B_i(T_i) = 1$. In [4] it is shown that when $\gamma_k(t, L)$ are measurable, bounded and locally Lipschitz in L, such a zero-coupon bond system always exists. However, although this system is not uniquely determined it turns out that the price and hedge of LIBOR derivatives does not depend on a particular choice of the bond system. Moreover, it is not difficult to see that it is possible to identify a system of bonds which is an Ito diffusion and thus Markovian. Indeed, for given $B_m(t_0) > 0$, define B_m as follows (an empty product is defined to be 1):

$$B_m(t) = \alpha_0(t) B_m(t_0) + \alpha_1(t) \frac{1}{\prod_{j=1}^{m-1}(1+\delta_j L_j(t))}, \qquad t \le T_1,$$

$$B_m(t) = \alpha_1(t)B_m(T_1) + \alpha_2(t)\frac{1}{\prod_{j=2}^{m-1}(1+\delta_j L_j(t))}, \qquad T_1 < t \le T_2,$$

....

$$B_m(t) = \alpha_{m-1}(t)B_m(T_{m-1}) + \alpha_m(t), \qquad T_{m-1} < t \le T_m,$$

where the functions $\alpha_j(t),\ 0 \le j \le m,$ are smooth and such that for any $t_0 \le t \le T_m$

$$\alpha_j(t) \ge 0, \qquad j = 0, ..., m, \qquad \sum_{j=0}^{m} \alpha_j(t) = 1,$$

$$\alpha_j(t) + \alpha_{j+1}(t) = 1, \qquad j = 0, ..., m-1,$$

$$\alpha_0(t_0) = \alpha_j(T_j) = 1, \qquad j = 1, ..., m.$$

Then the system $B = (B_1, ..., B_m)$ where

$$B_i = B_m \prod_{j=i}^{m-1}(1+\delta_j L_j(t)), \qquad t_0 \le t \le T_i,\ i = 1, ..., m-1,$$

is arbitrage free and satisfies $B_i(T_i) = 1$ (see [4]). In addition it is easily seen that the system B thus constructed is an Ito diffusion on the probability space given by the L process.

The developed general probabilistic method for the price and hedge of a European claim can be applied to certain European LIBOR derivatives. We discuss two examples: the "swaption" and the "callable" reverse floater. For a LIBOR market model, in [14], one factor analytical approximation formulas are derived both for the swaption and for the callable reverse floater. Clearly these analytical approximation can be used for variance reduction in the Monte Carlo method presented in this sequel.

6.1. **The European swaption.** A swap contract with maturity T_1 and strike κ on a loan of \$1 over the period $[T_1, T_m]$ obliges to pay a fixed coupon κ and receive spot LIBOR at the settlement dates $T_2, ..., T_m$. From a standard portfolio argument it is obvious that the present value of this contract is equal to

$$Swap(t) = B_1(t) - B_m(t) - \kappa \sum_{j=1}^{m-1} \delta_j B_{j+1}(t),\ t_0 \le t \le T_1.$$

The swap rate $S(t)$ is now defined as that fixed coupon which sets this contract value to zero:

$$S(t) := \frac{B_1(t) - B_m(t)}{\sum_{j=1}^{m-1} \delta_j B_{j+1}(t)}.$$

A swaption contract with maturity T_1, strike κ and principal \$1 gives the right to contract at T_1 to pay a fixed coupon κ and receive spot LIBOR at the settlement dates $T_2, ..., T_m$.

Equivalently, one can contract for receiving the T_1-swaprate and one can show that the payoff of the swaption is equivalent to a T_1 cashflow of

$$Swpn(T_1) = \sum_{j=1}^{m-1} 1_A B_{j+1}(T_1)(L_j(T_1) - \kappa)\delta_j, \tag{6.1}$$

where A denotes the $\mathcal{F}_{T_1}$ measurable event $\{S(T_1) > \kappa\}$ and the swaprate $S(T_1)$ is given by (see [14])

$$S(T_1) := \frac{B_1(T_1) - B_m(T_1)}{\sum_{j=1}^{m-1} \delta_j B_{j+1}(T_1)} = \frac{-1 + \prod_{i=1}^{m-1}(1 + \delta_i L_i(T_1))}{\sum_{j=1}^{m-1} \delta_j \prod_{i=j+1}^{m-1}(1 + \delta_i L_i(T_1))}.$$

>From (6.1) we see that the swaption cashflow is homogeneous of degree one. Therefore we may compute the swaption price and the corresponding hedge by Monte Carlo simulation of the probabilistic representations for (5.12), (5.17) with final value conditions (5.13), (5.18), with f given by

$$f(y) := \sum_{j=1}^{m-1} 1_A(y)(y_j - \kappa)\delta_j \prod_{k=j+1}^{m-1} (1 + \delta_k y_k), \tag{6.2}$$

where

$$A = \left\{ y : \quad \frac{-1 + \prod_{k=1}^{m-1}(1 + \delta_k y_k)}{\sum_{k=1}^{m-1} \delta_k \prod_{i=k+1}^{m-1}(1 + \delta_i y_i)} > \kappa \right\}$$

and, for instance, use variance reduction from a one factor approximation formula derived in [14].

6.2. **The callable reverse floater.** Let $K, K' > 0$. A reverse floater (RF) contracts for receiving $L_i(T_i)$ while paying $\max(K - L_i(T_i), K')$ at time T_{i+1} for $i = 1, .., m-1$, with respect to a unit principal. A callable reverse floater (CRF) is an option to enter into a reverse floater at T_1. In [14] it is shown that in a LIBOR market model the reverse floater can be evaluated analytically and for $K' = 0$ the reverse floater contract is equivalent with a T_1-cashflow of

$$RF(T_1) = B_1(T_1) - B_m(T_1) - \sum_{i=1}^{m-1} B_{i+1}(T_1) F_i(T_1, K), \tag{6.3}$$

where $F_i(T_1, K)$ is known explicitly as a Black-type formula, only involving T_1, K and the deterministic $\gamma_i,\ i = 1, ..., m-1$, [14]. So the payoff of the CRF, being

$$CRF(T_1) = \max(RF(T_1), 0),$$

is clearly homogeneous of degree one and the reverse floater price and hedge may be computed by Monte Carlo simulation of the probabilistic representations for the system (5.12), (5.17) with final value conditions (5.13), (5.18) and f given by an expression similar to (6.2). Moreover, in [14] a one factor approximation formula is derived which can be used for variance reduction.

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