# On an equilibrium problem for a cracked body with electrothermoconductivity

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**Abstract**

We consider a problem related to resistance spot welding. The mathematical model describes the equilibrium state of an elastic, cracked body subjected to heat transfer and electroconductivity and can be viewed as an extension to the classical thermistor problem.

We prove existence of a solution in Sobolev spaces.

**Key words:** *crack, thermistor, thermoelastic contact, spot welding*

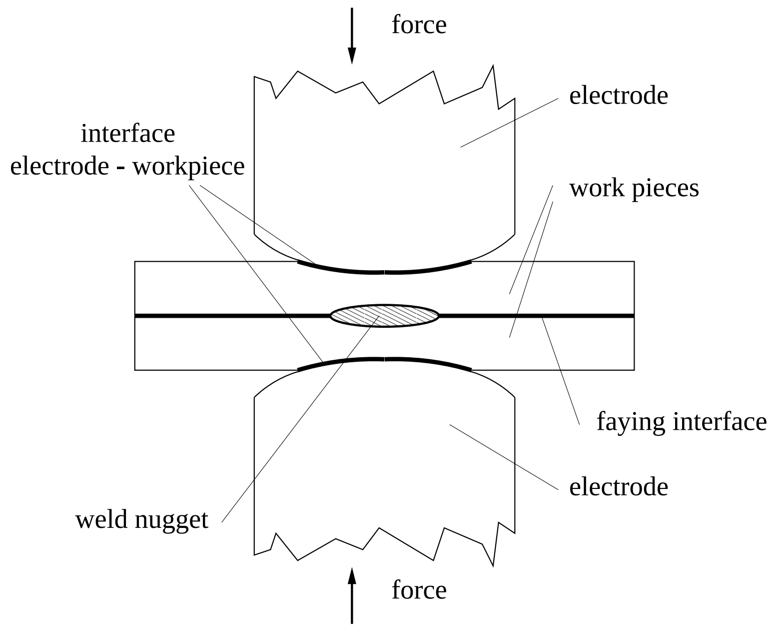


Figure 1: Schematic of the resistance spot welding process.

# 1 Introduction

In resistance spot welding two workpieces are pressed together by electrodes. Owing to the Joule effect and the high resistivity in the contact area between the workpieces, the welding current leads to an increase in temperature, until finally a weld nugget is formed (cf. Fig. 1).

For a complete description of the process, one has to take into account mechanical, thermal and electrical effects, as well as the free boundary between liquid metal and solid. To the knowledge of the authors mathematical models up to now have only considered the thermal and electrical effects, neglecting mechanics (cf. e.g. [5]).

Obviously, the most important control parameters for the process are the force, applied to join the work-pieces and the shape of the electrode. To achieve a uniform current density between the electrodes, flat electrodes would be desirable. On the other hand, to reduce wear, a domed electrode is more favourable. Hence, the area of contact between electrode and workpiece is very important to control the quality of the weld joint.

The aim of the present paper is to initiate the investigation of this contact problem. Owing to the quadratic Joule heating term in the energy balance a crucial point for the analysis will be the regularity of solutions for the electric potential equation. To avoid the additional difficulties, which arise from the geometric singularity at the boundary of the contact between electrode and workpiece, we will focus on the simplified problem of a cracked thermoelastic body.

In the next section we give a precise formulation of the model. An existence result is proved in Section 3.

## 2 Mathematical model

Let $\Omega \subset R^2$ be a bounded domain with smooth boundary $\Gamma$, and $\Xi \subset \Omega$ be a smooth curve without selfintersections. Denote $\Omega_c = \Omega \setminus \overline{\Xi}, Q_c = \Omega_c \times (0,T), Q = \Omega \times (0,T), T > 0$. Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ , $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\text{meas}\Gamma_1 > 0$.

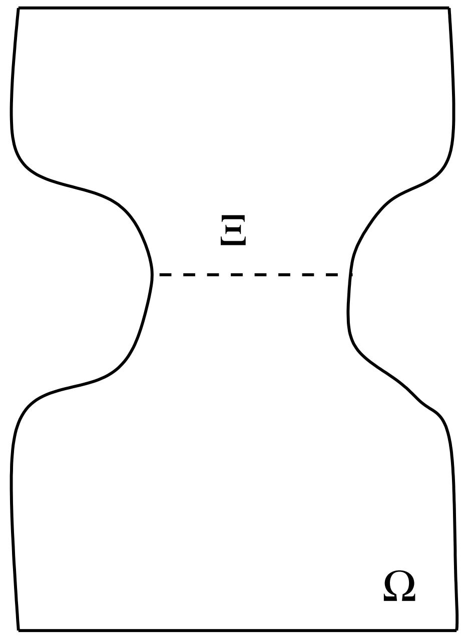


Figure 2: The domain $\Omega_c$.

In the domain $Q_c$, we want to find a solution $u = (u_1\,, u_2), \theta, \varphi$ of the following boundary value problem

$$-\sigma_{ij,j} + \delta^2\theta_{,i} = 0, \tag{1}$$

$$\theta_t - \Delta\theta + \delta^2 \frac{\partial}{\partial t}\text{div}u = \gamma(\theta)|\nabla\varphi|^2, \tag{2}$$

$$\text{div}(\gamma(\theta)\nabla\varphi) = 0, \tag{3}$$

$$\theta = \theta_0 \text{ for } t = 0, \tag{4}$$

$$\varphi = \varphi_0, \quad \theta = 0 \text{ on } \Gamma \times (0,T), \tag{5}$$

$$\sigma_{ij}n_j = g_i \text{ on } \Gamma_2 \times (0,T), i = 1,2, \tag{6}$$

$$[\varphi] = \left[\gamma(\theta)\frac{\partial\varphi}{\partial\nu}\right] = 0, \quad [\theta] = \left[\frac{\partial\theta}{\partial\nu}\right] = 0 \text{ on } \Xi \times (0,T), \tag{7}$$

$$u = 0 \text{ on } \Gamma_1 \times (0,T); \quad [u]\cdot\nu \geq 0 \text{ on } \Xi \times (0,T), \tag{8}$$

$$\sigma_\nu \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\tau = 0, \quad \sigma_\nu \cdot [u]\cdot\nu = 0 \text{ on } \Xi \times (0,T). \tag{9}$$

Here $\delta$ is a positive constant describing the thermal expansion, $\gamma$ is a given $C^1-$function, $\gamma_1 \leq \gamma(s) \leq \gamma_2, \;\; s \in R$; $\gamma_1, \gamma_2$ are positive constants, $\sigma_{ij} = \sigma_{ij}(u)$ denote the stress tensor components, $i,j = 1,2$, $\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(u)$ is the Hooke's law, $\varepsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k})$ are strain tensor components, elastic coefficients $a_{ijkl}$ are smooth and satisfy the usual assumptions of symmetry and positive definiteness. We select a unit normal vector $\nu = (\nu_1, \nu_2)$ to $\Xi$, and $n = (n_1, n_2)$ is a unit normal vector to $\Gamma$,

$$\{\sigma_{ij}\nu_j\} = \sigma_\tau + \sigma_\nu \cdot \nu \ , \quad i = 1,2, \quad \tau = (-\nu_2, \nu_1) \ .$$

The mathematical model (1)-(9) describes the equilibrium state of an elastic body subjected to the heat transfer and electroconductivity. The function $u = (u_1, u_2)$ describes the displacement field in the body, $\theta$ is the temperature, $\varphi$ stands for the electric potential, the brackets $[v] = v^+ - v^-$ mean the jump of $v$ across $\Xi$, $v^+, v^-$ stands for the values of $v$ on $\Xi^+, \Xi^-$, respectively, where $\Xi^+, \Xi^-$ are defined for given choice of positive and negative directions of $\nu$ on $\Xi$.

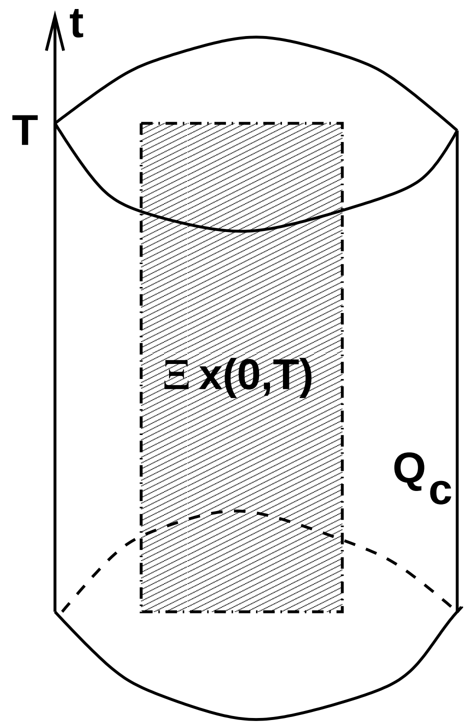


Figure 3: The cylinder $Q_c$.

The curve $\Xi$ presents the crack in the body, and the second inequality of (8) corresponds to the mutual nonpenetration condition between the crack faces. In the following we assume that

$$\theta_0 \in H_0^1(\Omega); \quad g_i \in H^1(0,T;L^2(\Gamma_2)), \quad i = 1,2; \quad \varphi_0 \in L^\infty(0,T;H^{\frac{3}{2}}(\Gamma)) \ .$$

Here

$$H_0^1(\Omega) = \{v \in H^1(\Omega) |\ v = 0 \text{ on } \ \Gamma\}.$$

The space $H^{\frac{3}{2}}(\Gamma)$ can be defined as the space of traces on $\Gamma$ of all functions from $H^2(\Omega)$.
Our aim is to prove an existence theorem for the problem (1)-(9).

Note that the so-called thermistor problem for finding the temperature and electrical potential was considered in [1], [2], [8], [9]. The Stefan problem with Joule's heating was analysed in [5]. On the other hand, there are many papers related to equilibrium of elastic bodies with cracks and nonpenetration conditions imposed on the crack faces (see [14],[13], [12]), and to thermoelastic bodies with linear and nonlinear boundary conditions of Signorini's type (see [3], [4], [10],[11]). Thermoelastic problems are formulated in terms of the displacement vector and the temperature.

# 3 Existence theorem and proof

To prove the existence of a solution to (1)-(9) we substitute the function $\theta = \overline{\theta}$ in (3) and determine the function $\varphi$ from (3) and the first conditions of (5),(7), respectively. Then we consider $\gamma(\theta)|\nabla\varphi|^2$ as a given function in the right-hand side of (2) and solve the equations (1), (2) along with all boundary and initial conditions. In such a way we find the functions $u, \theta$. Next step of the proof is to show that the mapping

$$\overline{\theta} \longrightarrow \theta$$

admits a fixed point in an appropriate functional space. To this end we use the Schauder fixed point theorem.
Let $\overline{\theta} \in L^2(0,T;H^{\frac{3}{2}}(\Omega_c))$ be any fixed function. Consider the following problem

$$\mathrm{div}(\gamma\,(\overline{\theta})\nabla\varphi) = 0 \quad \text{in} \quad Q_c, \tag{10}$$

$$\varphi = \varphi_0 \quad \text{on} \quad \Gamma\times(0,T), \tag{11}$$

$$[\varphi] = 0, \ \left[\gamma\,(\overline{\theta})\,\frac{\partial\varphi}{\partial\nu}\right] = 0 \quad \text{on} \quad \Xi\times(0,T). \tag{12}$$

Here, $t$ plays the role of a parameter. Note first that the conditions $\xi \in H^1(\Omega_c)$, $[\xi] = 0$ on $\Xi$ provide the inclusion $\xi \in H^1(\Omega)$.
Consider the problem (10), (11) with the first condition of (12). The solution of this problem can be defined as follows

$$\begin{gathered} \varphi \in L^\infty(0,T;H^1(\Omega)), \\ \int_Q \gamma(\overline{\theta})\nabla\varphi\cdot\nabla\psi = 0 \quad \forall\psi \in L^2(0,T;H_0^1(\Omega)) \end{gathered} \tag{13}$$

with the condition (11). It is easy to obtain the estimate for the function $\varphi$ by choosing $\psi = \varphi - \Phi_0$. Here we take $\Phi_0$ as an element of the space $L^\infty(0,T;H^2(\Omega))$ such that $\Phi_0 = \varphi_0$ on $\Gamma\times(0,T)$. From the condition imposed on $\varphi_0$ it follows that such an extension of $\varphi_0$ in the domain $Q$ exists. As a result of the substitution we have the equality

$$\int_Q \gamma(\overline{\theta})\nabla\varphi\cdot(\nabla\varphi - \nabla\Phi_0) = 0$$

which provides the estimate

$$\gamma_1 \int_Q |\nabla\varphi|^2 \le \gamma_2 \int_Q |\nabla\varphi \cdot \nabla\Phi_0|.$$

Hence the above inclusion $\varphi \in L^\infty(0,T;H^1(\Omega))$ follows. Existence of the solution is proved by the standard variational method. Moreover, the second condition of (12) is fulfilled since the equation (10) holds in $Q$ (compare [13], [14]). Indeed, in the domain $Q$, consider the zeroth distribution $\mathrm{div}(\gamma(\overline{\theta})\nabla\varphi)$. Denote by $\langle\cdot,\xi\rangle$ the value of a distribution at the point $\xi$. We divide $\Omega_c$ into two subdomains $\Omega_1,\Omega_2$ by extending the curve $\Xi$. In so doing we assume that the extended curve crosses the boundary $\Gamma$ at two points, and the boundaries $\partial\Omega_i,\ i=1,2$, with unit external normals $\nu^1,\nu^2$, respectively, to possess the Lipschitz property. We have in $Q$,

$$\langle \mathrm{div}(\gamma(\overline{\theta})\nabla\varphi),\xi\rangle = 0, \quad \xi \in C_0^\infty(Q).$$

Consequently,

$$\langle \mathrm{div}(\gamma(\overline{\theta})\nabla\varphi),\xi\rangle = -\int_{\Omega_1\times(0,T)} \gamma(\overline{\theta})\nabla\varphi\cdot\nabla\xi - \int_{\Omega_2\times(0,T)} \gamma(\overline{\theta})\nabla\varphi\cdot\nabla\xi =$$

$$= \langle \mathrm{div}(\gamma(\overline{\theta})\nabla\varphi),\xi\rangle_{\Omega_1\times(0,T)} + \langle \mathrm{div}(\gamma(\overline{\theta})\nabla\varphi),\xi\rangle_{\Omega_2\times(0,T)} +$$
$$+ \int_0^T \langle[\gamma(\overline{\theta})\frac{\partial\varphi}{\partial\nu}],\xi\rangle_{\Xi,1/2}dt = 0.$$

Here we use the following well-known fact. Let $D \subset R^2$ be a bounded domain with a Lipschitz boundary $\partial D$. Then the conditions $u \in H^1(D),\ \mathrm{div}(a\nabla u) \in L^2(D), a \in L^\infty(D)$ imply $a\frac{\partial u}{\partial\nu} \in H^{-1/2}(\partial D)$, and the Green formula holds

$$\int_D \mathrm{div}(a\nabla u)\xi = \langle a\frac{\partial u}{\partial n},\xi\rangle_{1/2} - \int_D a\nabla u\cdot\nabla\xi \quad \forall\xi \in H^1(D),$$

where $\langle\cdot,\cdot\rangle_{1/2}$ is the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$. This implies the second condition of (12),

$$\int_0^T \langle[\gamma(\overline{\theta})\frac{\partial\varphi}{\partial\nu}],\xi\rangle_{\Xi,1/2}dt = 0,$$

which holds in the sense

$$\int_0^T \langle\gamma(\overline{\theta})\frac{\partial\varphi}{\partial\nu^1},\xi\rangle_{\partial\Omega_1,1/2}dt + \int_0^T \langle\gamma(\overline{\theta})\frac{\partial\varphi}{\partial\nu^2},\xi\rangle_{\partial\Omega_2,1/2}dt = 0, \quad \xi \in C_0^\infty(Q).$$

Hence we obtain the following boundary value problem for $\varphi$,

$$\mathrm{div}(\gamma\left(\overline{\theta}\right)\nabla\varphi) = 0 \quad \text{in} \quad Q, \tag{14}$$
$$\varphi = \varphi_0 \quad \text{on} \quad \Gamma\times(0,T). \tag{15}$$

Now we aim to show an additional regularity for $\varphi$. First note that for any given $g \in W_p^1(\Omega), p > 1$, the solution $w$ of the problem

$$\operatorname{div}(\nabla w + g) = 0 \quad \text{in} \quad \Omega,$$
$$w = 0 \quad \text{on} \quad \Gamma$$

exists and the following estimate holds,

$$\|\nabla w\|_{L^p(\Omega)} \leq \Lambda_p \|\nabla g\|_{L^p(\Omega)} \tag{16}$$

with the positive constant $\Lambda_p$ depending on $p$.
We can rewrite the problem (14), (15) in the form

$$\operatorname{div}\left(\frac{\gamma_1+\gamma_2}{2}\nabla v + \tilde{\gamma}(\overline{\theta})\nabla v + \gamma(\overline{\theta})\nabla\Phi_0\right) = 0 \quad \text{in} \quad Q, \tag{17}$$

$$v = 0 \quad \text{on} \quad \Gamma \times (0,T). \tag{18}$$

Here $v = \varphi - \Phi_0$ is unknown function, $\tilde{\gamma}(s) = \gamma(s) - \frac{\gamma_1+\gamma_2}{2}$. Note that $|\tilde{\gamma}(s)| \leq \frac{\gamma_2-\gamma_1}{2}$, $s \in R$.
Take any function $v^0 \in L^\infty(0,T; W_4^1(\Omega))$ and apply the iteration method for solving the problem (17), (18):

$$\operatorname{div}\left(\frac{\gamma_1+\gamma_2}{2}\nabla v^{n+1} + \tilde{\gamma}(\overline{\theta})\nabla v^n + \gamma(\overline{\theta})\nabla\Phi_0\right) = 0 \quad \text{in} \quad Q, \tag{19}$$

$$v^{n+1} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{20}$$

where $n = 0, 1, 2, ...$ Assume that the oscillation of the function $\gamma$ is small enough so that $\Lambda < 1$, $\Lambda = \frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}\Lambda_4$. According to (16) for almost all $t \in (0,T)$ we have the estimate

$$\|\nabla v^{n+1}\|_{L^4(\Omega)} \leq \Lambda\|\nabla v^n\|_{L^4(\Omega)} + \lambda\|\nabla\Phi_0\|_{L^4(\Omega)}, \ \lambda = \frac{2\gamma_2\Lambda_2}{\gamma_1+\gamma_2}.$$

Consequently, for almost all $t \in (0,T)$ this implies

$$\|\nabla v^{n+1}\|_{L^4(\Omega)} \leq \|\nabla v^n\|_{L^4(\Omega)} + \frac{\lambda}{1-\Lambda}\|\nabla\Phi_0\|_{L^4(\Omega)}, \ n = 0, 1, 2, ...$$

Whence the following estimate is obtained,

$$\|\nabla v^n\|_{L^\infty(0,T;L^4(\Omega)} \leq c \tag{21}$$

being uniform in $n$. Taking the difference $v^n - v^l, n > l$, we easily derive that

$$\|\nabla(v^n - v^l)\|_{L^\infty(0,T;L^4(\Omega)} \leq \Lambda^l\|\nabla(v^{n-l} - v^0)\|_{L^\infty(0,T;L^4(\Omega)}.$$

By (21), it follows that the sequence $v^n$ is fundamental, and we can assume that as $n \to \infty$

$$v^n \to v \quad \text{in} \quad L^\infty(0,T;L^4(\Omega)).$$

This allows us to pass to the limit in (19), (20) as $n \to \infty$. Hence the problem (17), (18) (or, what is the same, the problem (14), (15)) has the solution which satisfies the inclusion $v \in L^\infty(0,T;W_4^1(\Omega))$. Consequently,

$$\nabla\varphi \in L^\infty(0,T;L^4(\Omega)). \tag{22}$$

This implies that

$$\gamma\left(\overline{\theta}\right)|\nabla\varphi|^2 \in L^2(Q_c),$$

and we can consider the following initial-boundary value problem in $Q_c$ for unknown functions $u = (u_1\ ,u_2), \theta$:

$$-\sigma_{ij,j} + \delta^2\theta_{,i} = 0, \tag{23}$$

$$\theta_t - \Delta\theta + \delta^2\frac{\partial}{\partial t}\text{div}u = \gamma\left(\overline{\theta}\right)|\nabla\varphi|^2, \tag{24}$$

$$u = 0 \text{ on } \Gamma_1 \times (0,T); \sigma_{ij}n_j = g_i \text{ on } \Gamma_2 \times (0,T), i = 1,2, \tag{25}$$

$$[u]\cdot\nu \geq 0, \quad \sigma_\nu \leq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\tau = 0, \quad \sigma_\nu \cdot [u]\cdot\nu = 0 \text{ on } \Xi \times (0,T), \tag{26}$$

$$\theta = 0 \text{ on } \Gamma \times (0,T), \tag{27}$$

$$[\theta] = \left[\frac{\partial\theta}{\partial\nu}\right] = 0 \text{ on } \Xi \times (0,T), \tag{28}$$

$$\theta = \theta_0 \text{ for } t = 0. \tag{29}$$

The problem (23)-(29) with the given right-hand side $h = \gamma\left(\overline{\theta}\right)|\nabla\varphi|^2 \in L^2(Q_c)$ can be solved for small $\delta$ (see [3], [7]), with the following estimates

$$\|\theta_t\|_{L^2(Q_c)} + \|\theta\|_{L^2(0,T;H^1(\Omega_c))} \leq c_1\delta\|u\|_{H^1(0,T;H^1(\Omega_c))} + c_2\|h\|_{L^2(Q_c)}\ , \tag{30}$$

$$\|u\|_{H^1(0,T;H^1(\Omega_c))} \leq c_3\delta\|\theta\|_{H^1(Q_c)} + c_4\|g\|_{H^1(0,T;L^2(\Gamma_2))} + c_5\|\theta_0\|_{H^1(\Omega)}, \tag{31}$$

and the constants $c_i$ are independent of $\delta$, $\overline{\theta}$ . This solution $(u,\theta)$ satisfies the variational inequality

$$\int_{Q_c}\sigma_{ij}\varepsilon_{ij}(\overline{u}-u) - \delta^2\int_{Q_c}\theta\text{div}(\overline{u}-u) \geq \int_{\Gamma_2\times(0,T)} g_i(\overline{u}_i - u_i) \quad \forall\overline{u} \in K\ , \tag{32}$$

and the identity

$$\int_{Q_c}\left(\theta_t+\delta^2\frac{\partial}{\partial t}\mathrm{div}u-\gamma\left(\overline{\theta}\right)|\nabla\varphi|^2\right)\eta=-\int_{Q_c}\nabla\theta\cdot\nabla\eta \quad \forall\eta\in L^2(0,T;H_0^1(\Omega)). \tag{33}$$

Here

$$K=\{v\in L^2(0,T;H^1(\Omega_c)|v=0 \text{ on } \Gamma_1\times(0,T), \quad [v]\cdot\nu\geq 0 \quad \text{on } \Xi\times(0,T)\}.$$

In fact, the presence of the estimates (30)-(31) allows us to find $\delta_0$ such that for all $\delta\leq\delta_0$ the problem (23)-(29) is solvable. In what follows we fix any $\delta\leq\delta_0$ which provides the solvability of (23)-(29). The solution of (23)-(29) satisfies the following inclusions

$$\theta_t\in L^2(Q_c),\ \theta\in L^2(0,T;H^1(\Omega_c)),\ u\in H^1(0,T;H^1(\Omega_c)).$$

Actually, the function $\theta$ has a higher regularity. To see this we write the equation (33) in $Q_c$ in the following form

$$-\Delta\theta=-\theta_t-\delta^2\frac{\partial}{\partial t}\mathrm{div}u+\gamma\left(\overline{\theta}\right)|\nabla\varphi|^2 \tag{34}$$

with the right-hand side $-\theta_t-\delta^2\frac{\partial}{\partial t}\mathrm{div}u+\gamma\left(\overline{\theta}\right)|\nabla\varphi|^2$ belonging to $L^2(Q)$. Of course, the derivative $\frac{\partial}{\partial t}\mathrm{div}u$ is defined with respect to the domain $Q_c$. Conditions (28) provide that the equation (34) holds in $Q$. In this case we can argue as in the case of the boundary value problem (10)- (12) which, in fact, removes the singularity surface $\Xi\times(0,T)$. Consequently, by (27),

$$\theta\in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\ .$$

Note that the estimate

$$\|\theta\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))}\leq c_6$$

for the solution to the system (32)-(33) is also independent of the norm $\|\overline{\theta}\|_{L^2(0,T;H^{\frac{3}{2}}(\Omega_c))}$. The constant $c_6$ depends on $Q$ and the $L^2-$norm of the right-hand side of (34).

Divide next the domain $\Omega_c$ into two subdomains $\Omega_1,\Omega_2$ with Lipschitz boundaries as before, and notice that the space

$$\theta_t\in L^2(0,T;L^2(\Omega_i)), \quad \theta\in L^2(0,T;H^2(\Omega_i))$$

is compactly imbedded in the space [15]

$$\theta\in L^2(0,T;H^{\frac{3}{2}}(\Omega_i))\ , i=1,2.$$

Consequently, the space

$$\theta_t\in L^2(0,T;L^2(\Omega_c)), \quad \theta\in L^2(0,T;H^2(\Omega_c))$$

has a compact imbeddeding in the space

$$\theta \in L^2(0,T;H^{\frac{3}{2}}(\Omega_c)) \ .$$

This means that if $\overline{\theta}$ belongs to any ball in the space $L^2(0,T;H^{\frac{3}{2}}(\Omega_c))$, i.e.

$$\|\overline{\theta}\|_{L^2(0,T;H^{\frac{3}{2}}(\Omega_c))} \leq R$$

for $R$ sufficiently large, the solution $\theta$ belongs to the same ball, and the mapping $L^2(0,T;H^{\frac{3}{2}}(\Omega_c)) \ni \overline{\theta} \longrightarrow \theta \in L^2(0,T;H^{\frac{3}{2}}(\Omega_c))$ is compact. So we can apply the Schauder fixed point theorem to assure the existence of a solution to the problem (1)-(9). As a result we have the following existence theorem.

**Theorem 3.1** *Assume that all assumptions concerning $g_i, \gamma, \varphi_0, \theta_0, a_{ijkl}$ are satisfied. Then for small $\delta$ there exists a solution to the problem (1)-(9) such that*

$$\theta_t \in L^2(Q_c), \quad \theta \in L^2(0,T;H^1(\Omega_c)), \ \ u \in K, \ u \in H^1(0,T;H^1(\Omega_c)), \tag{35}$$

$$\varphi \in L^\infty(0,T;H^1(\Omega_c)), \tag{36}$$

$$\int_{Q_c} \sigma_{ij}\varepsilon_{ij}(\overline{u}-u) - \delta^2 \int_{Q_c} \theta \, div(\overline{u}-u) \geq \int_{\Gamma_2\times(0,T)} g_i(\overline{u}_i - u_i) \quad \forall \overline{u} \in K, \tag{37}$$

$$\int_{Q_c} \left(\theta_t + \delta^2 \frac{\partial}{\partial t} div u - \gamma(\theta)\left|\nabla\varphi\right|^2\right)\eta = -\int_{Q_c} \nabla\theta\cdot\nabla\eta \quad \forall \eta \in L^2(0,T;H_0^1(\Omega)), \tag{38}$$

$$\int_Q \gamma(\theta)\nabla\varphi\cdot\nabla\psi = 0 \quad \forall\psi \in L^2(0,T;H_0^1(\Omega)). \tag{39}$$

Note that, in fact, we have some additional regularity for the solution of (35)-(39), in particular,

$$\theta \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \ \ . \tag{40}$$

The inclusion (40) follows from the equation

$$-\Delta\theta = -\theta_t - \delta^2\frac{\partial}{\partial t}\mathrm{div} u + \gamma(\theta)\left|\nabla\varphi\right|^2$$

and the given bounary conditions for $\theta$ on $\Gamma\times(0,T)$ and $\Xi\times(0,T)$. Recall that the boundary conditions on $\Xi\times(0,T)$ remove the singularity surface $\Xi\times(0,T)$.

Boundary conditions (6),(9) are included in the variational inequality (37). It can be shown (see [7]) that the displacement $u$ also has an additional regularity, in particular, for any $x \in \Xi$ there exists a neighbourhood $V$ such that

$$u \in L^2(0,T;H^2(V \cap \Omega_c)).$$

Consequently, from the variational inequality (37) it follows that boundary conditions (9) hold for almost all $(x,t) \in \Xi \times (0,T)$.
We can state an additional smoothness of $\varphi$ provided that $|\gamma'(s)| < \gamma_3,\ \gamma_3 = const$. Namely,

$$\varphi \in L^q(0,T;H^2(\Omega))\ , q < 4. \tag{41}$$

Indeed, the equation (39) reads

$$\Delta\varphi = -\frac{\gamma'(\theta)}{\gamma(\theta)}\nabla\theta \cdot \nabla\varphi \quad \text{in} \quad Q. \tag{42}$$

According to [15] the space

$$\theta_t \in L^2(0,T;L^2(\Omega)), \quad \theta \in L^2(0,T;H^2(\Omega))$$

has (compact) imbeddeding in the space

$$\theta \in L^q(0,T;H^{\frac{3}{2}}(\Omega)), q < 4.$$

Consider the right-hand side of the equation (42). Since the imbedding $H^{1/2}(\Omega) \subset L^4(\Omega)$ is continuous for the two dimensional case we have

$$\nabla\theta \in L^q(0,T;L^4(\Omega))\ . \tag{43}$$

Consequently, by (22), (43),

$$\nabla\varphi \cdot \nabla\theta \in L^q(0,T;L^2(\Omega))\ .$$

The right-hand side of the equation (42) belongs to $L^q(0,T;L^2(\Omega))$, and $\Phi_0 \in L^\infty(0,T;H^2(\Omega))$, hence the inclusion (41) follows.
It is clear that to prove the existence theorem we can choose the function $\overline{\theta} \in L^2(0,T;H^{\frac{3}{2}}(\Omega))$ satisfying the additional conditions $[\overline{\theta}] = \left[\frac{\partial\overline{\theta}}{\partial\nu}\right] = 0$ on $\Xi \times (0,T)$. In this case the proposed scheme of the proof also works since the solution $\theta$ of the problem (32), (33) is smooth , i.e. $\theta_t \in L^2(Q)$, $\theta \in L^2(0,T;H^2(\Omega))$, and consequently, we obtain a compact imbedding of the space $L^2(0,T;H^{\frac{3}{2}}(\Omega)$ into itself.
Also, note that to prove the theorem it suffices to require a weaker regulariry assumptions on $\varphi_0$. We can assume that $\varphi_0$ is a trace on $\Gamma \times (0,T)$ of a function $\Phi_0 \in L^\infty(0,T;W^1_4(\Omega))$.

# References


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