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The attractor for a nonlinear reaction-diffusion system in an unbounded domain

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ABSTRACT. In this paper the quasilinear second order parabolic systems of a reaction-diffusion type in an unbounded domain are considered. Our aim in this article is to study the long-time behaviour of parabolic systems for which the nonlinearity depends explicitely on the gradient of the unknown functions. To this end we give a systematic study of given parabolic systems and their attractors in weighted Sobolev spaces. Dependence of the Hausdorff dimension of attractors from weight of the Sobolev spaces are considered.

CONTENTS

INTRODUCTION

In this paper the quasilinear second order parabolic equations and systems of a reaction-diffusion type

$$\begin{cases} \partial_t u - \Delta_x u + f(u, \nabla_x u) - \lambda_0 u = g;\ x \in \Omega \\ u\big|_{t=0} = u_0\,, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}$$

are considered.

Here $\Omega \subset \mathbb{R}^3$ is an unbounded domain in $\mathbb{R}^3$ with a sufficiently smooth boundary (see §1), $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to $x = (x_1, x_2, x_3)$, f and g are given functions and λ_0 is a fixed positive constant.

It is assumed also that the nonlinear term $f(u, \nabla_x u)$ satisfies the conditions

$$\begin{cases} 1.\ f \in C(\mathbb{R}^k \times \mathbb{R}^{3k}, \mathbb{R}^k) \\ 2.\ f(u, \nabla_x u).u \geq 0 \\ 3.\ |f(u, \nabla_x u)| \leq C|u|(1 + |\nabla_x u|^r)(1 + |u|^p),\ p - \text{is arbitrary},\ 0 \leq r < 2 \end{cases} \tag{0.2}$$

Here and below we denote by $u.v$ the inner product in the space R^k.

It is well known that in many cases the long-time behaviour of dynamical systems generated by evolutionary equations of mathematical physics can be naturally described in terms of attractors of the corresponding semigroups (see [2], [19], [32] and references

therein). In bounded domains the existence of the attractor is established for a large class of equations such as reaction-diffusion equations, nonlinear wave equations, 2D Navier–Stokes system, etc. Under some natural assumptions it is proved that for all equations mentioned above the attractor has a finite Hausdorff and fractal dimension (see [2], [19], [32]).

For unbounded domains Ω the behaviour of solutions for (0.1) becomes much more complicated mainly due to the noncompactness of the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$. Nevertheless some progress in studying these equations in unbounded domains has been obtained by using appropriate weighted Sobolev spaces. (see [1], [2], [6], [17], [27], [28], [26])

Indeed, for the case where $f(u, \nabla_x u) \equiv f(u)$ the problems of the type (0.1) were studied (using the scale $W^{s,p}_{\langle\alpha\rangle}$ of weighted Sobolev spaces with power weights $\phi_{\langle\alpha\rangle}(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$) in [2]. Under some natural conditions on the nonlinear term f the existence of the attractor $\mathcal{A}_\alpha$ for $\alpha \in \mathbb{R}$, but for $\alpha \leq 0$ only in a weak topology of the space $L^2_{\langle\alpha\rangle}$, was obtained. Moreover, they proved that in the case where $\alpha > 0$ the attractor $\mathcal{A}_\alpha$ has a finite Hausdorff dimension. An example of an infinite dimensional attractor in the case $\alpha < -3/2$ was also constructed.

The compact attractor in a strong topology of the space $L^2_{\langle\alpha\rangle}$ for $\alpha < -3/2$ was considered in [26].

The case with the explicit dependence of the nonlinear term on $\nabla_x u$ ($f = f(u, \nabla_x u)$) under essentially more restrictive conditions on the nonlinear term and $\alpha > 0$ was considered in [16].

The Kolmogorov's ε-entropy of the attractor of (0.1) for the case where it has infinite fractal and Hausdorff dimension was studied in [38], [13].

In this paper we give a systematic study of the equations of type (0.1) and their attractors in weighted Sobolev spaces $W^{s,p}_\phi$. We restrict ourselves by considering only weight functions $\phi(x)$ which satisfy the condition

$$C_1 e^{-\varepsilon|x|} \leq \phi(x) \leq C_2 e^{\varepsilon|x|} \tag{0.3}$$

where ε is a sufficiently small positive number which depends on the equation, and consequently we consider only solutions of (0.1) whose rate of growth with respect to $|x| \to \infty$ does not exceed the exponent $e^{\varepsilon|x|}$.

In fact, most of our analytic results (such as a priori estimates, existence of solutions, smoothness, uniqueness, etc.) will be obtained first with the scale $W^{s,p}_{\{\varepsilon\}}$ of weighted Sobolev spaces with exponential weights $\phi_{\{\varepsilon\}}(x) = e^{-\varepsilon|x|}$ with a sufficiently small positive ε. After that, using the technique developed in Section 1 we extend straightforwardly these results to the weighted Sobolev spaces $W^{s,p}_{\langle\alpha\rangle}$ with power weights, which are traditional in the attractor industry (see [1], [2], [26]). Notice that our approach is applicable to a more general classes of weights, for instance for the anisotropic weights $\phi(x) = (1 + |x_1|^{p_1} + |x_2|^{p_2} + |x_3|^{p_3})^\alpha$, $\alpha \in \mathbb{R}$.

It is worthwhile to emphasize that the explicit dependence of the nonlinear term on the gradient ($f = f(u, \nabla_x u)$) leads to the new difficulties especially in the case $\alpha < 0$. In this case for instance we are faced with the problem of proving the uniqueness of solutions of the problem (0.1) (see §5 for an explanation). To avoid these problems we use the concept of the trajectory attractor developed in [7-10], [34], [37].

For the convenience of the reader we recall some basic results from the theory of attractors using equation (0.1) as a model example. Indeed, assume first that the problem (0.1) has a *unique* solution for every u_0 from a certain phase space $\Phi_{\langle\alpha\rangle}$. (It

is proved in §5 that is true under natural assumptions when $g \in L^2_{\langle\alpha\rangle}(\Omega)$ with $\alpha \geq 0$). Then equation (0.1) generates a semigroup in the phase space $\Phi_{\langle\alpha\rangle}$

$$S_t : \Phi_{\langle\alpha\rangle} \to \Phi_{\langle\alpha\rangle},\ S_t u(0) = u(t) \tag{0.4}$$

The attractor $\mathcal{A}$ of the semigroup (0.4) is called the (global) atractor of the equation (0.1). This means that

1. $\mathcal{A} \subset \Phi_{\langle\alpha\rangle}$ is a compact subset of $\Phi_{\langle\alpha\rangle}$.
2. The set $\mathcal{A}$ is strictly invariant with respect to S_t, i.e. $S_t\mathcal{A} = \mathcal{A}$.
3. The set $\mathcal{A}$ is an attracting set for the semigroup S_t, i.e. for any bounded subset $B \subset \Phi_{\langle\alpha\rangle}$ and any neighbourhood $\mathcal{O}(\mathcal{A})$ there is a number $T = T(\mathcal{O}, B)$ such that

$$S_t B \subset \mathcal{O}(\mathcal{A}) \text{ for every } t \geq T \tag{0.5}$$

There exists however a number of important examples of partial differential equations, such as 3D Navier-Stokes system, nonlinear wave equations with a strong nonlinearity, elliptic equations, etc., for which we do not have uniqueness or at least it is not yet proved. At present there are several approaches which can handle these equations from the dynamical point of view.

The first approach is based on the concept of multivalued semigroups and their attractors (see [3], [5]).

The alternative approach which we will use below involves the concept of trajectory attractor. The applications of this concept to evolutionary equations (such as 3D Navier-Stokes system and nonlinear wave equations) in bounded domains and for elliptic boundary value problems in unbounded domains can be found in [7-10] and [18], [31], [34], [37] respectively.

We explain now the main idea of the trajectory attractor approach using our equation as a model example. Indeed let us consider the case when we do not have uniqueness ($g \in L^2_{\langle\alpha\rangle}(\Omega)$ with $\alpha < 0$). Denote by $K^+_{\langle\alpha\rangle}$ the set of all solutions of the problem (0.1) for all initial values $u_0 \in \Phi_{\langle\alpha\rangle}$ such that $u(t) \in \Phi_{\langle\alpha\rangle}$ for every $t \geq 0$ (it is proved in Section 5 that this set is not empty for the appropriate 'phase space' $\Phi_{\langle\alpha\rangle}$). Since our equation does not depend explicitly on t then the semigroup $\{T_h, h \geq 0\}$ of positive shifts along the t-axis acts on $K^+_{\langle\alpha\rangle}$:

$$T_h K^+_{\langle\alpha\rangle} \subset K^+_{\langle\alpha\rangle},\ h \geq 0,\ (T_h u)(t) \equiv u(t+h) \tag{0.6}$$

We endow the space $K^+_{\langle\alpha\rangle}$ with the appropriate topology (roughly speaking, this topology is induced by the embedding $K^+_{\langle\alpha\rangle} \subset C^{loc}(\mathbb{R}_+, \Phi_{\langle\alpha\rangle})$). By definition the attractor $\mathcal{A}^{tr}$ of the semigroup T_h acting in the space $K^+_{\langle\alpha\rangle}$ is called the trajectory attractor of the equation (0.1). Note that the choice of the topology of local convergence with respect to $t \in \mathbb{R}_+$ in $K^+_{\langle\alpha\rangle}$ guarantees the equivalence of the trajectory attractor ($\mathcal{A}^{tr}$) and the global one ($\mathcal{A}^{gl}$) in the case of uniqueness. Indeed, let $\alpha \geq 0$ then as proved below the trace operator Π_0 ($\Pi_0 u \equiv u(0)$) realizes a C^1-diffeomorphism between the spaces $K^+_{\langle\alpha\rangle}$ and $\Phi_{\langle\alpha\rangle}$. Thus, the semigroup S_t, defined by (0.4) and the semigroup T_t, defined by (0.6) are conjugated by this diffeomorphism

$$T_t = (\Pi_0)^{-1} S_t \Pi_0, \quad t \geq 0$$

Notice that although we formulate our main results about the attractors (see Section 5) in power weighted spaces $W^{s,p}_{\langle\alpha\rangle}$ (following tradition), it can be easily extended to other weighted spaces as well.

The rest of our paper is devoted to the study of the Hausdorf dimension of the attractor $\mathcal{A}$. This problem is essentially different for the case $\alpha \geq 0$ and for the case $\alpha < 0$.

In the case when $\alpha \geq 0$, we prove that under natural additional assumptions on the nonlinear term f the attractor $\mathcal{A}$ has finite dimension. It is worthwile to emphasize that we obtain finite dimensionality of the attractor in the ordinary (unweighted) Sobolev spaces $W^{s,p}(\Omega)$ also. To the best of our knowledge, this result has not been proved before.

Studying the attractor in the case when $\alpha < 0$, we restrict ourselves to the gradient independent case $f(u, \nabla_x u) \equiv f(u)$ and $\Omega = \mathbb{R}^3$. In this case we prove that for every nonlinear term f from our class such that the function $f(u)+\lambda_0 u$ is nonmonotonic there exists the right-hand side $g \in L^2_{\langle\alpha\rangle}$ such that the dimension of the attractor is infinite. This Theorem is based on our construction of the infinite dimensional unstable manifold associated with the equilibrium point z_0 of the equation (0.1) (see Section 7). Notice that in contrast to the case of bounded domains, in our situation the equilibrium point z_0 is not hyperbolic in general, hence the usual theorems on the unstable manifolds do not work.

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Part 1. The existence of solutions, uniqueness, differentiability.

This part is devoted to study the analytical properties of solutions (such as a priori estimates, existence, uniqueness, etc.) of (0.1) in unbounded domains.

In Section 1 we introduce a wide class of weights and the corresponding weighted Sobolev spaces and formulate a number of auxiliary results which will be essentially used throughout the paper.

The linear equation (with $f(u, \nabla_x u) \equiv 0$) in weighted Sobolev spaces is considered in Section 2.

The a priori estimates for the solutions of the nonlinear equation (0.1) are obtained in Section 3.

Using these estimates we prove in Section 4 the existence of solutions for (3.1).

Section 5 is devoted to study the problems connected with the uniqueness of solutions and its differentiability with respect to the initial value u_0.

§1 Weight functions and weighted spaces.

In this Section we introduce and study the family of weight functions and the corresponding weighted Sobolev spaces which will be used throughout of the paper.

Definition 1.1. *A function $\phi \in L^\infty_{loc}(\mathbb{R}^n)$ is called a weight function with the rate of growth $\mu \geq 0$ if the condition*

$$\phi(x+y) \leq C_\phi e^{\mu|x|}\phi(y), \ \phi(x) > 0 \tag{1.1}$$

is satisfied for every $x, y \in \mathbb{R}^n$.

Remark 1.1. *It is not difficult to deduce from* (1.1) *that*

$$\phi(x+y) \geq C_\phi^{-1} e^{-\mu|x|}\phi(y) \tag{1.2}$$

are also satisfied for every $x, y \in \mathbb{R}^n$.

Proposition 1.1. *Let* ϕ_1 *and* ϕ_2 *be weight functions with the rates of growth* μ_1 *and* μ_2 *respectively. Then,*

1. $\alpha\phi_1 + \beta\phi_2$, $\max\{\phi_1, \phi_2\}$, *and* $\min\{\phi_1, \phi_2\}$ *are weight functions with the rate of growth* $\max\{\mu_1, \mu_2\}$ *for every* $\alpha, \beta > 0$.

2. $\phi_1 \cdot \phi_2$ *and* $\phi_1 \cdot (\phi_2)^{-1}$ *are weight functions with the rate of growth* $\mu_1 + \mu_2$.

3. $(\phi_1)^\alpha$ *is weight function with the rate of growth* $|\alpha|\mu_1$.

The assertions of this proposition are immediate corollaries of (1.1) and (1.2).

The following two examples of weight functions are of fundamental significance for our purposes:

$$1.\ \phi_{\langle\alpha\rangle}(x) = (1+|x|^2)^{\alpha/2} \quad 2.\ \phi_{\{\varepsilon\}}(x) = e^{-\varepsilon|x|},\ \alpha, \varepsilon \in \mathbb{R}$$

(Evidently the second weight has the rate of growth $|\varepsilon|$ and the first one satisfies (1.1) for any $\mu > 0$).

Definition 1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be some (unbounded) domain in* $\mathbb{R}^n$ *and let* ϕ *be a weight function with the rate of growth* μ. *Define the space*

$$L^p_\phi(\Omega) = \left\{ u \in D'(\Omega) \ :\ \|u, \Omega, \|_{\phi(x),0,p} \equiv \int_\Omega \phi|u(x)|^p\,dx < \infty \right\}$$

Analogously we define the weighted Sobolev space $H^{l,p}_\phi(\Omega)$, $l \in \mathbb{N}$ *as the space of distributions whose derivatives up to the order* l *inclusively belong to* $L^p_\phi(\Omega)$ *For the simlicity of notations we will right troughout of the paper* $W^{s,p}_{\langle\alpha\rangle}$ *instead of* $W^{s,p}_{(1+|x|^2)^{\alpha/2}}$ *and* $W^{s,p}_{\{\varepsilon\}}$ *instead of* $W^{s,p}_{e^{-\varepsilon|x|}}$.

We define also the Sobolev spaces of functions bounded with respect to $|x| \to \infty$

$$W^{l,p}_b(\Omega) = \{u \in D'(\Omega)\ : \|u, \Omega\|_{b,l,p} = \sup_{x_0 \in \mathbb{R}^n} \|u, \Omega \cap B^1_{x_0}\|_{l,p} < \infty\}$$

Here and below we denote by $B^R_{x_0}$ *the ball in* R^n *of radius* R, *centred in* x_0, *and* $\|u, V\|_{l,p}$ *means* $\|u\|_{W^{l,p}(V)}$.

Theorem 1.1. *Let* $u \in L^p_\phi(\Omega)$ *where* ϕ *is a weight function with the rate of growth* μ. *Then for any* $1 \leq q \leq \infty$ *the following estimate is valid*

$$\left(\int_\Omega \phi(x_0)^q \left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx \right)^q dx_0 \right)^{1/q} \leq C \int_\Omega \phi(x)|u(x)|^p\,dx \tag{1.3}$$

for every $\varepsilon > \mu$, *where the constant* C *depends only on* ε, μ *and* C_ϕ *from* (1.1) *(and is independent of* Ω*)*

Proof. Let $q = 1$. Then due to (1.1)

$$\int_\Omega \int_\Omega \phi(x_0) e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\,dx_0 \leq$$

$$\leq C_\phi \int_{\Omega^2} e^{\mu|x-x_0|} e^{-\varepsilon|x-x_0|}\phi(x)|u(x)|^p\,dx\,dx_0 \leq$$

$$\leq C_\phi \left(\int_{\mathbb{R}^n} e^{-(\varepsilon-\mu)|y|}\,dy \right) \left(\int_\Omega \phi(x)|u(x)|^p\,dx \right) \leq C_1 \int_\Omega \phi(x)|u(x)|^p\,dx$$

Let now $q = \infty$ then applying (1.1) again we obtain

$$\sup_{x_0\in\Omega}\left\{\phi(x_0)\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right\} \le$$
$$\le C_\phi \int_{x\in\Omega} \sup_{x_0\in\Omega}\left\{e^{\mu|x-x_0|}e^{-\varepsilon|x-x_0|}\right\}\phi(x)|u(x)|^p\,dx \le$$
$$\le C_\phi \int_{x\in\Omega}\phi(x)|u(x)|^p\,dx$$

Thus we proved the inequality (1.3) for $q = 1$ and $q = \infty$. For $1 < q < \infty$ it follows then from the interpolation inequality (see [33])

$$\|\cdot\|_{L^q} \le \|\cdot\|^{\theta}_{L^1}\|\cdot\|^{1-\theta}_{L^\infty},\ \theta = 1/q$$

Theorem 1.1 is proved. □

Corollary 1.1. *Let ϕ be a weight function with the rate of growth $\mu < \frac{\varepsilon}{4}$, $u \in L^p_\phi(\Omega)$ and $R \in \mathbb{R}_+$. Then*

$$\left(\int_{\Omega\cap\{|x_0|>R\}}\phi(x_0)^q\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le \tag{1.4}$$
$$\le C\int_{\Omega\cap\{|x|>R/2\}}\phi(x)|u(x)|^p\,dx +$$
$$+ Ke^{-\beta R}\int_\Omega \phi(x)|u(x)|^p\,dx$$

for some $\beta > 0$, $C > 0$, $K > 0$ which depends only on ε, μ and C_ϕ from (1.1) *(and are independent of R).*

Proof. Indeed

$$\int_{\Omega\cap\{|x_0|>R\}}\phi(x_0)^q\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0 \le \tag{1.5}$$
$$\le C^q\int_{\Omega\cap\{|x_0|>R/2\}}\phi(x_0)^q\left(\int_{\Omega\cap\{|x|>R/2\}} e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0 +$$
$$+ C^q\int_{\Omega\cap\{|x_0|>R\}}\phi(x_0)^q\left(\int_{\Omega\cap\{|x|<R/2\}} e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0$$

The first integral in the right–hand side of (1.5) can be estimated by using (1.3) with $\Omega\cap\{|x| > R/2\}$ instead of Ω. So it remains to estimate the second one. If $|x_0| > R$ and $|x| < R/2$ then

$$|x - x_0| \ge 1/4|x - x_0| + 3/4|x - x_0| \ge 1/4(|x_0| - R/2) + 3R/8 \ge 1/4(|x_0| + R)$$

and due to (1.2) $\phi_\alpha(x) \geq C_\phi^{-1} e^{-\mu R/2}\phi(0)$. Hence

$$\int_{\Omega\cap\{|x_0|>R\}} \phi(x_0)^q \left(\int_{\Omega\cap\{|x|<R/2\}} e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0 \leq$$

$$\leq Ce^{-q\varepsilon R/4}\int_{\Omega\cap\{|x_0|>R\}} \phi^q(x_0)e^{-q\varepsilon|x_0|/4}dx_0\left(\int_{\Omega\cap\{|x|<R/2\}}|u(x)|^p\,dx\right)^q \leq$$

$$\leq C_1 e^{-q\varepsilon R/4}e^{-q\mu R/2}\left(\int_{\mathbb{R}^n} e^{-q(\varepsilon/4-\mu)|x_0|}\,dx_0\right)\left(\int_{\Omega\cap\{|x|<R/2\}}\phi(x)|u(x)|^p\,dx\right)^q \leq$$

$$\leq C_2 e^{-\beta R}\left(\int_\Omega \phi(x)|u(x)|^p\,dx\right)^q$$

Corollary 1.1 is proved.

Remark 1.2. *The assertion of Corollary 1.1 can be extended to the case $\mu < \varepsilon$ (instead of $4\mu < \varepsilon$) by replacing $R/2$ in the estimate* (1.4) *by R/N for a sufficiently large $N = N(\mu/\varepsilon)$.*

Corollary 1.2. *Let the assumption of the previous Corollary hold. Then the following analogues of estimates* (1.3) *and* (1.4) *are valid;*

$$\sup_{x_0\in\Omega}\left\{\phi(x_0)\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|^p\}\right\} \leq C\sup_{x\in\Omega}\{\phi(x)|u(x)|^p\} \tag{1.6}$$

and

$$\sup_{x_0\in\Omega\cap\{|x_0|>R\}}\left\{\phi(x_0)\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|^p\}\right\} \leq \tag{1.7}$$

$$\leq C\sup\nolimits_{x\in\Omega\cap\{|x|>R/2\}}\{\phi(x)|u(x)|^p\}+$$

$$Ce^{-\beta R}\sup\nolimits_{x\in\Omega}\{\phi(x)|u(x)|^p\}$$

The proof of this Corollary is the same as the proof of Theorem 1.1 for $q = \infty$ and Corollary 1.1.

For a more detailed study of the weighted Sobolev spaces defined above we need some regularity assumptions on the domain $\Omega \subset \mathbb{R}^n$ which are assumed to be valid throughout the paper.

We suppose that there exists a positive number $R_0 > 0$ such that for every point $x \in \Omega$ there exists a smooth domain $V_x \subset \Omega$ such that

$$B_{x_0}^{R_0}\cap\Omega \subset V_x \subset B_{x_0}^{R_0+1}\cap\Omega \tag{1.8}$$

where we denote by $B_{x_0}^R$ the ball of radius R, centred in x_0.

Moreover it is assumed also that there exists a diffeomorphism $\theta_x : B_0^1 \to V_x$ such that

$$\|\theta_x\|_{C^N} + \|\theta_x^{-1}\|_{C^N} \leq K \tag{1.9}$$

uniformly with respect to $x \in \Omega$ for sufficiently large N. (For simplicity we suppose below that (1.8) and (1.9) hold for $R_0 = 2$.) Notice that in the case when Ω is bounded the conditions (1.8) and (1.9) are equivalent to the condition: the boundary $\partial\Omega$ is a smooth manifold, but for unbounded domains the smoothness of the boundary is not sufficient to obtain the regular structure of Ω when $|x| \to \infty$ since some conditions on uniformity with respect to $x \in \Omega$ smoothness are required. It is most convenient for us to formulate these conditions in the form (1.8) and (1.9).

Theorem 1.2. *Let the domain Ω satisfy the conditions* (1.8) *and* (1.9), *the weight function – the condition* (1.1) *and let R be some positive number. Then the following estimates are valid*

$$C_2 \int_\Omega \phi(x)|u(x)|^p\,dx \le \int_\Omega \phi(x_0) \int_{\Omega\cap B^R_{x_0}} |u(x)|^p\,dx\,dx_0 \le \tag{1.10}$$
$$\le C_1 \int_\Omega \phi(x)|u(x)|^p\,dx$$

Proof. Let us change the order of integration in (1.10);

$$\int_\Omega \phi(x_0) \int_{\Omega\cap B^R_{x_0}} |u(x)|^p\,dx\,dx_0 = \tag{1.11}$$
$$= \int_\Omega |u(x)|^p \left(\int_\Omega \chi_{\Omega\cap B^R_x}(x_0)\phi(x_0)\,dx_0 \right) dx$$

Here $\chi_{\Omega\cap B^R_x}$ is the characteristic function of the set $\Omega\cap B^R_x$.

It follows from the inequalities (1.1) and (1.2) that

$$C_1\phi(x) \le \inf_{x_0\in B^R_x} \phi(x_0) \le \sup_{x_0\in B^R_x} \phi(x_0) \le C_2\phi(x) \tag{1.12}$$

and the assumptions (1.8) and (1.9) imply that

$$0 < C_1 \le \mu(x\in\Omega\cap B^R_{x_0}) \le C_2 \tag{1.13}$$

uniformly with respect to $x_0\in\Omega$.

Estimate (1.10) is an immediate corollary of the estimates (1.11)–(1.13). Theorem 1.2 is proved. □

Corollary 1.3. *Let* (1.8) *and* (1.9) *be valid. Then an equivalent norm in the weighted Sobolev space $H^{l,p}_\phi(\Omega)$ is given by the following expression:*

$$\|u,\Omega\|_{\phi,l,p} = \left(\int_\Omega \phi(x_0) \|u,\Omega\cap B^R_{x_0}\|^p_{l,p}\,dx_0 \right)^{1/p} \tag{1.14}$$

Here and below $\|u,V\|_{l,p}$ means $\|u\|_{W^{l,p}(V)}$. In particular we obtain also that the norms (1.14) *are equivalent for different $R\in\mathbb{R}_+$.*

To study the equation (0.1) we need also the weighted Sobolev spaces with fractional derivatives $s\in\mathbb{R}_+$ (not only $s\in\mathbb{Z}$). For the first we recall (see [33] for details) that if V is a bounded domain the norm in the space $W^{s,p}(V)$, $s=[s]+l$, $0<l<1$, $[s]\in\mathbb{Z}_+$ can be given by the following expression

$$\|u,V\|^p_{s,p} = \|u,V\|^p_{[s],p} + \sum_{|\alpha|=[s]} \int_{x\in V}\int_{y\in V} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+lp}}\,dx\,dy \tag{1.15}$$

It is not difficult to prove, arguing as in Theorem 1.2 and using this representation, that for any bounded domain V with a sufficiently smooth boundary

$$\|u,V\|^p_{s,p} \le C_1 \int_{x_0\in V} \|u,V\cap B^R_{x_0}\|^p_{s,p}\,dx_0 \le C_2\|u,V\|^p_{s,p} \tag{1.16}$$

This justifies the following definition.

Definition 1.3. *Let us define the space* $W_\phi^{s,p}$ *for any* $s \in \mathbb{R}_+$ *by the norm* (1.14).

It is not difficult to check that these norms are equivalent for different $R > 0$.

In the sequel we will use the $W_\phi^{s,p}$-valued functions, so we will formulate a simple continuity criteria for such functions. To this end we formulate below the continuity criteria for such functions.

Theorem 1.3. *Let* $u : [0,T] \to W_\phi^{s,p}(\Omega)$ *– be some* $W_\phi^{s,p}$*–valued function,* $1 \le p < \infty$. *Then* $u \in C([0,T], W_\phi^{s,p})$ *if and only if*

$$u\big|_{\Omega \cap B_0^R} \in C([0,T], W^{s,p}(\Omega \cap B_0^R)) \tag{1.17}$$

and uniformly with respect to $t \in [0,T]$

$$\lim_{R\to\infty} \|u(t), \Omega \cap \{|x| > R\}\|_{\phi,s,p} = 0 \tag{1.18}$$

Proof. Indeed let (1.17) and (1.18) be valid. Then

$$\|u(t_1) - u(t_2), \Omega\|_{\phi,s,p} \le C\|u(t_1) - u(t_2), \Omega \cap \{|x| < R\}\|_{s,p} + \\ + C \sup_{t\in[0,T]} \|u(t), \Omega \cap \{|x| > R\}\|_{\phi,s,p}$$

The second term in the right–hand of this inequality can be choosen arbitrary small by taking R large enough (due to condition (1.18)) and for the fixed R the first one can be choosen arbitrary small by taking $|t_1 - t_2|$ small enough (due to the condition (1.17)). Thus, $u \in C([0,T], W_\phi^{s,p}(\Omega))$.

Let us suppose now that $u \in C([0,T], W_\phi^{s,p}(\Omega))$. Then (1.17) is evidently holds. It remains only to verify (1.18). Indeed since u is continious then the set $\{u(t), t \in [0,T]\}$ is compact in $W_\phi^{s,p}$. The estimate (1.18) is an immediate corollary of this compactness. Theorem 1.3 is proved.

Remark 1.3. *Note that if* $p = \infty$ *then Theorem 1.3 gives only a sufficient condition for the continuity which evidently not necessary.*

Note also, that as a rule we will check the condition (1.18) *by using the estimate* (1.4).

We consider now the other class of weighted functional spaces which we significantly use to obtain adequate a priori estimates of solutions of the equation (0.1). Note that they are of independent interest.

Definition 1.4. *Let* $\Omega \subset \mathbb{R}^n$ *satisfy the conditions* (1.8) *and* (1.9) *and let* ϕ *be the weight function with the rate of growth* μ. *For every* $1 \le p < \infty$, *and* $R > 0$ *we define the following spaces*

$$L_\phi^{(p,\infty)}(\Omega) = \left\{ u \in D'(\Omega) : \|u, \Omega\|_{\phi,(p,\infty)}^p = \int_{x\in\Omega} \phi(x) \|u, \Omega \cap B_x^R\|_{0,\infty}^p dx < \infty \right\}$$

For the simplicity of notation we will write below $L_{\langle\alpha\rangle}^{(p,\infty)}$ *and* $L_{\{\varepsilon\}}^{(p,\infty)}$ *instead of* $L_{(1+|x|^2)^{\alpha/2}}^{(p,\infty)}$ *and* $L_{e^{-\varepsilon|x|}}^{(p,\infty)}$ *correspondingly*

It can be shown that in fact these spaces are independent of the choice of $R > 0$.

Proposition 1.2. *Let the conditions of the previous definition be valid. Then* $L_{\phi}^{(p,\infty)}(\Omega) \subset L_{\phi/p}^{\infty}(\Omega)$, *i.e.*

$$\sup_{x\in\Omega}\{\phi(x)|u(x)|^p\} \le C\int_{x\in\Omega}\phi(x)\|u,\Omega\cap B_x^R\|_{0,\infty}^p dx \tag{1.19}$$

Proof. Let us estimate the left-hand side of (1.19) using the estimates (1.12) and the inequality $\sup z_1,\cdot,z_k,\cdots \le \sum_{l=1}^{\infty} z_l$ for nonnegative z_l;

$$\sup_{x\in\Omega}\{\phi(x)|u(x)|^p\} = \sup_{l\in\mathbb{Z}^n}\{\sup_{x\in B_l^R\cap\Omega}\phi(x)|u(x)|^p\} \le C\sum_{l\in\mathbb{Z}^n}\phi(l)\|u,\Omega\cap B_l^R\|_{0,\infty}^p$$

$$\le C_1\sum_{l\in\mathbb{Z}^n}\phi(l)\int_{x\in B_l^1}\|u,\Omega\cap B_x^{R+1}\|_{0,\infty}^p dx \le$$

$$\le C_2\int_{x\in\Omega}\phi(x)\|u,\Omega\cap B_x^{R+2}\|_{0,\infty}^p dx \le C_3\int_{x\in\Omega}\phi(x)\|u,\Omega\cap B_x^R\|_{0,\infty}^p dx$$

Here we assumed, evidently, that $R > n^{1/2}$. Proposition 1.2 is proved.

Let us note that the weight function

$$\phi_{x_0,\varepsilon} = e^{-\varepsilon|x-x_0|} \tag{1.20}$$

satisfy the conditions (1.1) *uniformly* with respect to $x_0 \in \mathbb{R}^n$. Consequently all estimates obtained above for the arbitrary weights will be valid for the family (1.20) with the constants, independent of $x_0 \in \mathbb{R}^n$ as well. Since these estimates are of fundamental significance for us we write it explicitly by a number of corollaries formulated below.

Corollary 1.4. *Let* Ω *be the same as in Proposition 1.2. Then for any* $\varepsilon \ge 0$ *the following estimate is valid uniformly with respect* $x_0 \in \mathbb{R}^n$

$$\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|^p\} \le C\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u,\Omega\cap B_x^R\|_{0,\infty}^p dx \tag{1.21}$$

Corollary 1.5. *Let* $u \in L_{\{\delta\}}^p(\Omega)$ *for* $0 < \delta < \varepsilon$, $\varepsilon > 0$. *Then the following estimate holds uniformly with respect to* $y \in \mathbb{R}^n$

$$\left(\int_\Omega e^{-q\delta|x_0-y|}\left(\int_\Omega e^{-\varepsilon|x-x_0|}|u(x)|^p\,dx\right)^q dx_0\right)^{1/q} \le \tag{1.22}$$

$$\le C_{\varepsilon,q}\int_\Omega e^{-\delta|x-y|}|u(x)|^p\,dx$$

Moreover if $u \in L_{\{\delta\}}^{\infty}(\Omega)$, $\delta < \varepsilon$ *then*

$$\sup_{x_0\in\Omega}\left\{e^{-\delta|x_0-y|}\sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(x)|\}\right\} \le C_{\varepsilon,\delta}\sup_{x\in\Omega}\{e^{-\delta|x-y|}|u(x)|\} \tag{1.23}$$

and if $4\delta < \varepsilon$ *the appropriate analogues of the estimates* (1.4), *and* (1.7) *are also valid uniformly with respect to* $y \in \mathbb{R}^n$.

Corollary 1.6. *Let $u \in L^p_b(\Omega)$ and $\varepsilon > 0$. Then*

$$\sup_{x_0 \in \Omega} \left\{ \int_{x \in \Omega} e^{-\varepsilon |x - x_0|} |u(x)|^p \, dx \right\} \le C \|u, \Omega\|^p_{b,l,p} \tag{1.24}$$

Indeed,

$$\sup_{x_0 \in \Omega} \left\{ \int_{x \in \Omega} e^{-\varepsilon |x - x_0|} |u(x)|^p \, dx \right\} \le C \sup_{x_0 \in \Omega} \left\{ \int_{x \in \Omega} e^{-\varepsilon |x - x_0|} \|u, \Omega \cap B^1_x|^p_{0,p} \, dx \right\} \le$$

$$\le C \sup_{x \in \Omega} \left\{ \|u, \Omega \cap B^1_x\|^p_{0,p} \right\} \sup_{x_0 \in \Omega} \left\{ \int_{x \in \mathbb{R}^n} e^{-\varepsilon |x - x_0|} \, dx \right\} \le C_1 \|u, \Omega\|^p_{b,0,p}$$

§2 The linear equation

This Section is devoted to the study of the linear problem of the type (0.1)

$$\begin{cases} \partial_t u - \Delta_x u + \lambda_0 u = g(t) \\ u\big|_{t=0} = u_0 \ ; \ u\big|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

in an unbounded domain Ω which is assumed to satisfy the conditions (1.8) and (1.9) formulated in the previous Section. To this end we will use weighted Sobolev spaces introduced in Section 1.

Theorem 2.1. *Let $g \in L^2([0,T], L^2_{\{\varepsilon_1\}}(\Omega))$ for some $\varepsilon_1 \ge 0$ and let $u_0 \in L^2_{\{\varepsilon_1\}}(\Omega)$. Then there exists the unique solution of the problem such that*

$$u \in L_2([0,T], W^{1,2}_{\{\varepsilon_1\}}(\Omega)) \cap W^{1,2}([0,T], W^{-1,2}_{\{\varepsilon_1\}}(\Omega)) \tag{2.2}$$

and for any $\varepsilon > \varepsilon_1$, the following estimate is valid uniformly with respect to $x_0 \in \mathbb{R}^3$:

$$\|u(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_{T-1}^{T} \|u(t), \Omega \cap B^1_{x_0}\|^2_{1,2} + \|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{-1,2} \, dt \le \tag{2.3}$$

$$\le C(|v_0|^2, e^{-\varepsilon |x - x_0|}) e^{-(\lambda_0 - \varepsilon^2)T} + \int_0^T e^{(\lambda_0 - \varepsilon^2)(t-T)} (|g(t)|^2, e^{-\varepsilon |x - x_0|}) \, dt$$

Here and below $\int_{T-1}^T$ means $\int_0^T$ if $T < 1$.

Proof. We restrict ourselves to obtain only the a priori estimate (2.3) for the solutions of the problem (2.1). The existence of solutions can be obtained from this estimate in a standard way.

Let us multiply the equation (2.1) by $u(t) e^{-\varepsilon |x - x_0|}$ and integrate over $x \in \Omega$

$$\partial_t (|u(t)|^2, e^{-\varepsilon |x - x_0|}) + 2\lambda_0 (|u(t)|^2, e^{-\varepsilon |x - x_0|}) + 2(|\nabla_x u(t)|^2, e^{-\varepsilon |x - x_0|}) =$$

$$= 2(g, u e^{-\varepsilon |x - x_0|}) - 2(\nabla_x u, u \nabla_x e^{-\varepsilon |x - x_0|})$$

Applying Holder's inequality to the right–hand side of the last formula and using the obvious estimate $|\nabla_x e^{-\varepsilon |x - x_0|}| \le \varepsilon e^{-\varepsilon |x - x_0|}$ we obtain

$$\partial_t (|u(t)|^2, e^{-\varepsilon |x - x_0|}) + (\lambda_0 - \varepsilon^2)(|u(t)|^2, e^{-\varepsilon |x - x_0|}) + \tag{2.4}$$

$$+ (|\nabla_x u(t)|^2, e^{-\varepsilon |x - x_0|}) \le C(|g|^2, e^{-\varepsilon |x - x_0|})$$

Applying Gronwall's inequality to the estimate (2.4) we obtain the estimate

$$(2.5)\quad (|u(T)|^2, e^{-\varepsilon|x-x_0|}) + \int_{T-1}^{T} (|\nabla_x u(t)|^2, e^{-\varepsilon|x-x_0|})\,dt \le$$
$$\le (|u(0)|^2, e^{-\varepsilon|x-x_0|}) e^{-(\lambda_0-\varepsilon^2)T} + C\int_0^T e^{(\lambda_0-\varepsilon^2)(t-T)} (|g(t)|^2, e^{-\varepsilon|x-x_0|})\,dt$$

Taking into account that $e^{-\varepsilon|x-x_0|} \ge C$ if $x \in B^1_{x_0}$ uniformly with respect to $x_0 \in \mathbb{R}^3$ we obtain from (2.5)

$$(2.6)\quad \|u(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_{T-1}^{T} \|\nabla_x u(t), \Omega \cap B^1_{x_0}\|^2_{1,2}\,dt \le$$
$$\le C(|u(0)|^2, e^{-\varepsilon|x-x_0|}) e^{-(\lambda_0-\varepsilon^2)T} + C\int_0^T e^{(\lambda_0-\varepsilon^2)(t-T)} (|g(t)|^2, e^{-\varepsilon|x-x_0|})\,dt$$

The estimate of $\partial_t u$ follows now from (2.6) and from the equation (2.1).The estimate (2.3) is proved. It remains to prove (2.2). Let us take in (2.3) $\varepsilon > \varepsilon_1$, multiply it by $e^{-\varepsilon_1|x_0|}$ and integrate over $x_0 \in \Omega$. Then after using the estimates (1.3) and (1.10) we obtain that

$$\|u(T), \Omega\|^2_{\{\varepsilon_1\},0,2} + \int_{T-1}^{T} \|u(t), \Omega\|^2_{\{\varepsilon_1\},1,2} + \|\partial_t u(t), \Omega\|^2_{\{\varepsilon_1\},-1,2}\,dt \le$$
$$\le \|u(0), \Omega\|^2_{\{\varepsilon_1\},0,2} e^{-(2\lambda_0-\varepsilon^2)T} + \int_0^T e^{(2\lambda_0-\varepsilon^2)(t-T)} \|g(t), \Omega\|^2_{\{\varepsilon_1\},0,2}\,dt$$

Theorem 2.1 is proved. □

Theorem 2.2. *Let u be a solution of (2.1) satisfying (2.2). Let $\varepsilon > \varepsilon_1 \ge 0$ be the same as in previous theorem, and let $u_0 \in W^{1,2}_{\{\varepsilon_1\}}(\Omega)$ and g be the same as in the previous Theorem. Then*

$$(2.7)\qquad u \in L^2([0,T], W^{2,2}_{\{\varepsilon_1\}}(\Omega)) \cap W^{1,2}([0,T], L^2_{\{\varepsilon_1\}}(\Omega))$$

and the following estimate is valid uniformly with respect to $x_0 \in \mathbb{R}^3$;

$$(2.8)\quad \|u(T), \Omega \cap B^1_{x_0}\|^2_{1,2} + \int_{T-1}^{T} \|u(t), \Omega \cap B^1_{x_0}\|^2_{2,2} + \|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{0,2}\,dt \le$$
$$\le C\left(\|\nabla_x u(0), \Omega \cap B^3_{x_0}\|^2 + (|u_0|^2, e^{-\varepsilon|x-x_0|})\right) e^{-\gamma T} +$$
$$+ C\int_0^T e^{\gamma(t-T)} (|g(t)|^2, e^{-\varepsilon|x-x_0|})\,dt$$

for some positive $\gamma = \gamma(\varepsilon) > 0$.

Proof. Recall firstly that we assume that the domain Ω satisfies the conditions (1.8) and (1.9) and the constant $R_0 = 2$.

Let us consider the cut–off function $\psi_{x_0}(x) \in C^\infty_0(\mathbb{R}^3)$ such that $\psi_{x_0} = 1$ if $x \in B^1_{x_0}$ and $\psi_{x_0} = 0$ if $x \notin B^2_{x_0}$ and let $v_{x_0} = \psi_{x_0} v$. It follows from the equation (2.1) and from the condition (1.8) that v_{x_0} is the solution of the following equation

$$(2.9)\qquad \begin{cases} \partial_t v_{x_0} - \Delta_x v_{x_0} + \lambda_0 v_{x_0} = \psi_{x_0} g - 2\nabla_x \psi_{x_0} \nabla_x v_{x_0} - \Delta_x \psi_{x_0} v_{x_0} \equiv h_{x_0}(t) \\ v_{x_0}\big|_{\partial V_{x_0}} = 0\ ;\ v\big|_{t=0} = \psi_{x_0} u(0) \end{cases}$$

where the domains V_{x_0} were defined in (1.8) and (1.9).

Multiplying the equality (2.9) by $\Delta_x v_{x_0}$ and integrating over $x \in V_{x_0}$ we obtain after simple computation involving integration by parts and Gronwall's inequality that the following estimate holds uniformly with respect to $x_0 \in \Omega$:

$$\|v_{x_0}(T), V_{x_0}\|^2_{1,2} + \int_{T-1}^{T} \|\Delta_x v_{x_0}(t), V_{x_0}\|^2_{0,2} + \|\partial_t v_{x_0}(t), V_{x_0}\|^2_{0,2}\, dt \le \tag{2.10}$$

$$\le C_1 \|v_{x_0}(0), V_{x_0}\|^2_{1,2} e^{-\lambda_0 T} + C_1 \int_0^T e^{\lambda_0 (t-T)} \|h_{x_0}, V_{x_0}\|^2_{0,2}\, dt \le$$

$$\le C_2 \|u(0), \Omega \cap B^3_{x_0}\|^2_{1,2} e^{-\lambda_0 T} + C_2 \int_0^T e^{\lambda_0 (t-T)} \|h_{x_0}, \Omega \cap B^3_{x_0}\|^2_{0,2}\, dt$$

Taking into the account the assumptions (1.8) and (1.9) for the domains V_{x_0} we obtain from the elliptic regularity theorem (see [33]) that

$$\|u(t), \Omega \cap B^1_{x_0}\|_{2,2} \le C \|v_{x_0}, V_{x_0}\|_{2,2} \le C_1 \|\Delta_x v_{x_0}, V_{x_0}\|_{0,2}$$

Estimating

$$\|h_{x_0}(t), \Omega \cap B^3_{x_0}\|^2_{0,2} \le C(\|g, \Omega \cap B^3_{x_0}\|^2_{0,2} + \|u(t), \Omega \cap B^3_{x_0}\|^2_{1,2}) \tag{2.11}$$

and using the estimate (2.3) we obtain now the estimate (2.8). The assertion (2.7) can be deduced from (2.8) in the same way as (2.2) from (2.3). Theorem 2.2 is proved. □

Corollary 2.1. *Let u – be a solution of the problem (2.1) satisfying (2.2). Let us suppose also that $g \in L^2([0,T], L^2_\phi(\Omega))$ and $u_0 \in W^{1,2}_\phi(\Omega)$ for a some weight ϕ satisfying (1.1) with a sufficiently small rate of growth μ ($\mu < \varepsilon$) and ε was introduced in Theorem 2.2). Then*

$$u \in W^{1,2}([0,T], L^2_\phi(\Omega)) \cap L^2([0,T], W^{2,2}_\phi(\Omega)) \cap C([0,T], W^{1,2}_\phi(\Omega)) \tag{2.12}$$

and the following estimate is valid

$$\|u(T), \Omega\|^2_{\phi,1,2} + \int_{T-1}^{T} \|u(t), \Omega\|^2_{\phi,2,2} + \|\partial_t u(t), \Omega\|^2_{\phi,0,2}\, dt \le \tag{2.13}$$

$$\le C \|u(0), \Omega\|^2_{\phi,1,2} e^{-\gamma T} + C \int_0^T e^{\gamma(t-T)} \|g(t), \Omega\|^2_{\phi,0,2}\, dt$$

for some positive γ.

Proof. Indeed let us multiply the estimate (2.8) by $\phi(x_0)$ and integrate over Ω. Then using the estimates (1.3) and (1.10) we obtain (2.13). Thus, we have proved that $u \in L^\infty([0,T], W^{1,2}_\phi(\Omega))$ and it remains to prove continuity. To this end we use Theorem 1.3. Multiplying (2.8) by $\phi(x_0)$ and integrating over $\Omega \cap \{|x_0| > R\}$ we obtain using (1.4) that

$$\|u(t), \Omega \cap \{|x| > R\}\|^2_{\phi,1,2} \to 0$$

when $R \to \infty$ uniformly with respect to $t \in [0,T]$. So we should prove only that $u\big|_{\Omega \cap \{|x| < R\}} \in C([0,T], W^{1,2}(\Omega \cap \{|x| < R\}))$. But it follows from a well known interpolation theorem for unweighted Sobolev spaces (see [25]) that

$$C([0,T], W^{1,2}) \subset L^2([0,T], W^{2,2}) \cap W^{1,2}([0,T], L^2)$$

Corollary 2.1 is proved. □

Theorem 2.3. *Let u be a solution of the problem (2.1) which satisfies (2.2), $\varepsilon > \varepsilon_1 \geq 0$ such as in Theorem 2.2, $u_0 \in W^{2-\delta,2}_{\{\varepsilon_1\}}(\Omega)$ and $g \in C([0,T], L^2_{\{\varepsilon_1\}}(\Omega))$ for some $0 < \delta \leq 1$. Then*

$$u \in C([0,T], W^{2-\delta,2}_{\{\varepsilon_1\}}(\Omega)) \cap C^{1-\delta/2}([0,T], L^2_{\{\varepsilon_1\}}(\Omega)) \tag{2.14}$$

and the following estimate is valid uniformly with respect to $x_0 \in \mathbb{R}^3$:

$$\begin{aligned} \|u(T), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} + \|u\|^2_{C^{1-\delta/2}([T-1,T], L^2(\Omega \cap B^1_{x_0}))} \leq \\ \leq C \left(\|u(0), \Omega \cap B^3_{x_0}\|^2_{2-\delta,2} + (|u(0)|^2, e^{-\varepsilon|x-x_0|}) \right) e^{-\gamma T} + \\ + \int_0^T e^{\gamma(t-T)} |t-T|^{-1+\delta/2} (|g(t)|^2, e^{-\varepsilon|x-x_0|})\, dt \end{aligned} \tag{2.15}$$

for some positive constant γ

The proof of this Theorem is based on the following result for the auxiliary problem (2.9)

Lemma 2.1. *Let v_{x_0} be the solution of the problem (2.9) and let $v_{x_0}(0) \in W^{2-\delta,2} \cap W^{1,2}_0(V_{x_0})$ and $h_{x_0} \in C([0,T], L^2(V_{x_0})$. Then*

$$v_{x_0} \in C([0,T], W^{2-\delta,2}(V_{x_0})) \cap C^{1-\delta/2}([0,T], L^2(V_{x_0})) \tag{2.16}$$

and the following estimate is valid uniformly with respect to $x_0 \in \Omega$

$$\begin{aligned} \|v_{x_0}(T), V_{x_0}\|^2_{2-\delta,2} + \|u\|^2_{C^{1-\delta/2}([T-1,T], L^2(V_{x_0}))} \leq \\ \leq C \|v_{x_0}(0), V_{x_0}\|^2_{2-\delta,2} e^{-\lambda_0 T} + C_1 \int_0^T e^{\lambda_0(t-T)} |t-T|^{-1+\delta/2} \|h_{x_0}, V_{x_0}\|^2_{0,2}\, dt \end{aligned} \tag{2.17}$$

Proof. Let $A = A_{x_0} = -\Delta_x + \lambda_0$. Then from the variation of constants formula we obtain that

$$v_{x_0}(T) = e^{-AT} v_{x_0} + \int_0^T e^{A(t-T)} h_{x_0}(t) \tag{2.18}$$

Let us derive the estimate of $W^{2-\delta,2}$ norm. The Holder continuity for v_{x_0} can be obtained analogously.

It is well known that the operator A generates a holomorphic semigroup in $L^2(V_{x_0})$. Then, for $\beta \geq 0$, $s > 0$ the following estimates are valid, (see [33], [11], [20])

$$\|e^{-As}\|_{W^{2-\delta,2} \to W^{2-\delta,2}} \leq C e^{-\lambda_0 s}\ ;\ \|A^\beta e^{-As}\|_{L^2 \to L^2} \leq C_1 s^{-\beta} e^{-\lambda_0 s} \tag{2.19}$$

and due to the regularity conditions (1.9), the constants C and C_1 do not depend on $x_0 \in \Omega$.

Thus, $\|e^{-AT}v_{x_0}(0), V_{x_0}\|_{2-\delta,2} \le Ce^{-\lambda_0 T}\|v_{x_0}(0), V_{x_0}\|_{2-\delta,2}$ and using Holder's inequality we obtain after simple transformations

$$\Big\| \int_0^T e^{A(t-T)}h_{x_0}(t)\,dt\Big\|_{W^{2-\delta,2}} \le \int_0^T \|A^{1-\delta/2}e^{A(t-T)}\|_{L^2\to L^2}\|h_{x_0}(t), V_{x_0}\|_{0,2}\,dt \le$$

$$C\int_0^T e^{\lambda_0(t-T)}|t-T|^{-1+\delta/2}\|h_{x_0}, V_{x_0}\|_{0,2}\,dt \le C\left(\int_0^T e^{\lambda_0(t-T)}|t-T|^{-1+\delta/2}\,dt\right)^{1/2}$$

$$\times\left(\int_0^T e^{\lambda_0(t-T)}|t-T|^{-1+\delta/2}\|h_{x_0}, V_{x_0}\|_{0,2}^2\,dt\right)^{1/2} \le$$

$$\le C_1\left(\int_0^T e^{\lambda_0(t-T)}|t-T|^{-1+\delta/2}\|h_{x_0}, V_{x_0}\|_{0,2}^2\,dt\right)^{1/2}$$

Lemma 2.1 is proved.

The proof of Theorem 2.3. It follows from Lemma 2.1 that

$$\|u\|^2_{C^{1-\delta/2}([T-1,T],L^2(\Omega\cap B^1_{x_0}))} + \|u(T), \Omega\cap B^1\|^2_{2-\delta,2} \le \tag{2.20}$$

$$\le C\left(\|v_{x_0}, V_{x_0}\|^2_{2-\delta,2} + \|v_{x_0}\|^2_{C^{1-\delta/2}([T-1,T],L^2(V_{x_0}))}\right) \le$$

$$\le C_1\left(\|v_{x_0}(0), V_{x_0}\|^2_{2-\delta,2}e^{-\lambda_0 T} + \int_0^T e^{\lambda_0(t-T)}|t-T|^{-1+\delta/2}\|h_{x_0}(t), V_{x_0}\|^2_{0,2}\}\right)$$

We used here the notations of Theorem 2.2.

Estimating the last integral in (2.20) by the estimates (2.11) and (2.8) we obtain the inequality (2.15) after some evident calculations. Hence it remains to obtain only that $u \in C([0,T], W^{2-\delta,2}_{\{\varepsilon\}}(\Omega))$. This continuity can be proved using the 'tail estimates' (1.4) and Theorem 1.3 as in Theorem 2.2 and Corollary 2.1. Theorem 2.3 is proved. □

Corollary 2.2. *Let u – be a solution of the problem* (2.1) *satisfying* (2.2). *Let us suppose also that $g \in C([0,T], L^2_\phi(\Omega))$ and $u_0 \in W^{2-\delta,2}_\phi(\Omega)$, $0 \le \delta < 1$ and ϕ is the same as in Corollary 2.1. Then*

$$u \in C([0,T], W^{2-\delta,2}_\phi(\Omega)) \cap C^{1-\delta/2}([0,T], L^2_\phi(\Omega)) \tag{2.21}$$

and the following estimate is valid

$$\|u(T), \Omega\|^2_{\phi,2-\delta,2} + \|u\|^2_{C^{1-\delta/2}([T-1,T],L^2_\phi(\Omega))} \le C\|u(0), \Omega\|^2_{\phi,2-\delta,2}e^{-\gamma T} + \tag{2.22}$$

$$+ C\sup_{t\in[0,T]}\{e^{\gamma(t-T)}\|g(t), \Omega\|^2_{\phi,0,2}\}$$

The proof of this Corollary is the same as Corollary 2.1, only instead of the estimate (2.8) we should use the estimate (2.15).

Corollary 2.3. *Let the condition of the previous Theorem be valid, $\delta < 1/2$ and $u_0 = 0$. Assume also that* $\dim\Omega = n \le 3$. *Then*

$$|u(t,x_0)|^2 \le \sup_{t\in[0,T]}\{e^{\gamma(t-T)}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\} \tag{2.23}$$

for some positive γ

Indeed, the estimate (2.23) follows immediately from (2.15), Sobolev's embedding theorem $W^{2-\delta,2} \subset C$ for $\delta < 1/2$ and the following simple inequality

$$\int_0^T e^{\gamma(t-T)}|t-T|^{-1+\delta/2}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\,dt \leq \tag{2.24}$$
$$\leq C \sup_{t\in[0,T]}\{e^{\gamma(t-T)/2}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\}$$

Theorem 2.4. *Let the conditions of Theorem 2.3 be valid,* $\delta < 1/2$ *and* $\dim\Omega = n \leq 3$. *Then*

$$|u(T,x_0)|^2 \leq \sup_{x\in\Omega}\left\{e^{-\varepsilon|x-x_0|}|u(0,x)|^2\right\}e^{-\gamma T} + \tag{2.25}$$
$$+ \sup_{t\in[0,T]}\{e^{\gamma(t-T)}(|g(t)|^2, e^{-\varepsilon|x-x_0|})\}$$

for some $\gamma > 0$

Proof. Due to Corollary 2.3 it is sufficient to prove (2.25) only for $g \equiv 0$.

Let us consider the function $\psi(x) \in C^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} |\nabla_x\psi(x)| \leq C\varepsilon\psi(x)\ ;\ |\Delta_x\psi(x)| \leq C\varepsilon^2\psi(x) \\ \psi(x) > 0\ ;\ \psi(x) = e^{-\varepsilon|x|} \text{ for } |x| \geq 1 \end{cases} \tag{2.26}$$

It is not difficult to prove that such a function exists.

Let us consider also the function $w_{x_0}(t,x) = \psi(x-x_0)u(t,x)$. It follows from (2.1) that this function satisfies the equation

$$\partial_t w_{x_0} - \Delta_x w_{x_0} + \lambda_0 w_{x_0} - K_1(x)w_{x_0} - K_2(x)\nabla_x w_{x_0} = 0 \tag{2.27}$$

and due to the condition (2.26) $\sup_{x\in\mathbb{R}^3}|K_i(x)| \leq C\varepsilon$, $i = 1,2$. Hence for sufficiently small $\varepsilon > 0$ the maximum principle is valid for the equation (2.27) (see, [23]). Thus

$$|w_{x_0}(t,x)| \leq \sup_{x\in\Omega}|w_{x_0}(0,x)|e^{-\gamma t} \tag{2.28}$$

for some $\gamma > 0$. Taking $x = x_0$ in (2.28) we obtain (2.25) for $g = 0$. Theorem 2.4 is proved. □

We finish this Section with a version of the comparison principle for parabolic equations in weighted Sobolev spaces.

Theorem 2.5. *Let a function* u *satisfy* (2.7) *for a certain* $\varepsilon_1 > 0$, $u(0) = 0$ *and let the following inequality be valid almost everywhere in* $[0,T]\times\Omega$*:*

$$\partial_t u - \Delta_x u + \lambda_0 u \geq 0 \tag{2.29}$$

Then almost everywhere in $[0,T]\times\Omega$ $\quad u(t,x) \geq 0$.

Proof. Let us consider the functions $u_+(t,x) = \max\{0, u(t,x)\}$, $u_-(t,x) = u_+(t,x) - u(t,x)$. By using the technique of [3] it is not difficult to prove that

$$u_-, u_+ \in W^{1,2}([0,T], L^2_{\{\varepsilon\}}(\Omega)) \cap L^2([0,T], W^{1,2}_{\{\varepsilon\}}(\Omega)) \tag{2.30}$$

for $\varepsilon \geq \varepsilon_1$ and the following equalities are valid almost everywhere

$$(\partial_t u_+(t), u_-(t))_{\{\varepsilon\}} = (\partial_t u_-(t), u_+(t))_{\{\varepsilon\}} = (\nabla_x u_+(t), \nabla_x u_-(t))_{\{\varepsilon\}} = 0 \tag{2.31}$$

Let us multiply (2.29) by u_- and integrate over Ω. Then due to (2.31) we obtain

$$-1/2\partial_t\|u_-(t),\Omega\|^2_{\{\varepsilon\},0,2} - \lambda_0\|u_-(t),\Omega\|^2_{\{\varepsilon\},0,2} - \varepsilon^2\|u_-(t),\Omega\|^2_{\{\varepsilon\},0,2} \geq 0 \tag{2.32}$$

Applying Gronwall's inequality to the estimate (2.32) and taking into account that $u_-(0) = 0$ we obtain that $\|u_-(t),\Omega\|_{\{\varepsilon\},0,2} = 0$ almost everywhere, i.e. $u \geq 0$ almost everywhere. Theorem 2.5 is proved.

§3 The nonlinear equation. A priori estimates.

In this Section we consider the parabolic boundary problem:

$$\begin{cases} \partial_t u - \Delta_x u + f(u, \nabla_x u) + \lambda_0 u = g(x) \\ u\big|_{t=0} = u_0; \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{3.1}$$

in the unbounded domain Ω which is assumed to be 3-dimensional ($\Omega \subset \mathbb{R}^3$) and to satisfy the conditions (1.8) and (1.9) formulated in Section 1.

Recall that $u = (u^1, \cdots, u^k)$, $\lambda_0 > 0$ – is some positive number, $f = (f^1, \cdots, f^k)$, $g = (g^1, \cdots, g^k)$ and the nonlinear term f satisfies the following conditions

$$\begin{cases} 1. & f(u, \nabla_x u).u \geq 0 \\ 2. & |f(u, \nabla_x u)| \leq C|u|(1 + |\nabla_x u|^r)(1 + |u|^p),\ p > 0 \text{ – is arbitrary} \\ 3. & f \in C(\mathbb{R}^k \times \mathbb{R}^{3k}\,;\, \mathbb{R}^k) \text{ and } r < 2 \end{cases} \tag{3.2}$$

We suppose in this Section that the right-hand side $g = g(x)$ is from the space $\cap_{\varepsilon>0} L^2_{\{\varepsilon\}}(\Omega)$ and that the initial date u_0 is from the space $\cap_{\varepsilon>0} H^{2-\delta,2}_{\{\varepsilon\}}(\Omega)$ (where the exponent $0 < \delta = \delta(r) < 1/2$ will be defined below).

A solution of the equation (3.1) is defined to be a function u from the space $\cap_{\varepsilon>0} C([0,T], W^{2-\delta,2}_{\{\varepsilon\}}(\Omega))$ which satisfies the equation (3.1) in a distribution sense.

The main aim of this Section is to prove a number of a priori estimates for the solutions of (3.1) which will be used below in order to prove the existence of solutions, their uniqueness, etc.

Note that all results formulated below remain valid for $g \in L^2_{\{\varepsilon\}}(\Omega)$, $u_0 \in W^{2-\delta,2}_{\{\varepsilon\}}(\Omega)$ and $u \in C([0,T], W^{2-\delta,2}_{\{\varepsilon\}}(\Omega))$ with a sufficiently small positive ε which depends on the equation (3.1).

Theorem 3.1. *Let u be the solution of* (3.1). *Then the following estimate is valid uniformly with respect to $x_0 \in \Omega$*

$$\|u(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_{T-1}^{T} \|u(t), \Omega \cap B^1_{x_0}\|^2_{1,2} + \|\partial_t u(t), \Omega \cap B^1_{x_0}\|^2_{-1,2}\, dt \leq \\ \leq C(|u_0|^2, e^{-\varepsilon|x-x_0|})e^{-\gamma T} + C(|g|^2, e^{-\varepsilon|x-x_0|}) \tag{3.3}$$

for some positive $\gamma > 0$ and sufficiently small $\varepsilon > 0$.

The proof of this Theorem is the same as the proof of Theorem 2.1 (the nonlinear term disappears in the estimates due to the first condition of (3.2)).

Theorem 3.2. *Let u be the solution of* (3.2). *Then the following estimate is valid uniformly with respect to $x_0 \in \Omega$*

$$|u(T, x_0)|^2 \leq C \sup_{x\in\Omega}\{e^{-\varepsilon|x-x_0|}|u(0,x)|^2\}e^{-\gamma T} + C(|g|^2, e^{-\varepsilon|x-x_0|}) \tag{3.4}$$

for some positive $\gamma > 0$ and sufficiently small $\varepsilon > 0$.

Proof. Let us consider the function $w(t,x) = u(t,x).u(t,x)$. Then due to the equation (3.1)

$$\partial_t w - \Delta_x w + 2\lambda_0 w = -2\nabla_x u.\nabla_x u - 2f(u, \nabla_x u).u + 2g.u \leq 2g.u \equiv h_u(t) \tag{3.5}$$

We consider also the auxiliary linear problem

$$\left\{\begin{array}{l} \partial_t v - \Delta_x v + 2\lambda_0 v = h_u(t) \\ v\big|_{t=0} = w\big|_{t=0} = u_0.u_0 \end{array}\right. \tag{3.6}$$

Due to the comparison principle (Theorem 2.5),

$$w(t,x) \le v(t,x), \quad (t,x) \in [0,T]\times\Omega \tag{3.7}$$

Applying Theorem 2.4 to the linear equation (3.6) and using (1.23) we obtain

$$\begin{aligned} \sup_{x\in\Omega}\{e^{-2\varepsilon|x-x_0|}|w(T,x)|^2\} \le \sup_{x\in\Omega}\{e^{-2\varepsilon|x-x_0|}|v(T,x)|^2\} \le \\ \le C\sup_{x\in\Omega}\{e^{-2\varepsilon|x-x_0|}|u_0(x)|^4\}e^{-2\gamma T} + \\ + C\sup_{t\in[0,T]}\{e^{2\gamma(t-T)}(|h_u(t)|^2, e^{-2\varepsilon|x-x_0|})\} \end{aligned} \tag{3.8}$$

Denote $Z_{x_0}(T) = \sup_{x\in\Omega}\{e^{-2\varepsilon|x-x_0|}|w(T,x)|^2\}$ and estimate the last term in the right–hand side of (3.8)

$$\begin{aligned} (|h_u(t)|^2, e^{-2\varepsilon|x-x_0|}) \le C(|u(t)|^2|g|^2, e^{-2\varepsilon|x-x_0|}) \le \\ \le C\sup_{x_0\in\Omega}\{e^{-\varepsilon|x-x_0|}|w(t,x)|\}(|g|^2, e^{-\varepsilon|x-x_0|}) \le \mu Z_{x_0}(t) + C_\mu(|g|^2, e^{-\varepsilon|x-x_0|})^2 \end{aligned} \tag{3.9}$$

and the estimate (3.9) is valid for every $\mu > 0$.

It follows from the inequalities (3.8) and (3.9) that

$$\begin{aligned} Z_{x_0}(T) \le C\sup_{x\in\Omega}\{e^{-2\varepsilon|x-x_0|}|u_0(x)|^4\}e^{-2\gamma T} + C_\mu(|g|^2, e^{-\varepsilon|x-x_0|})^2 + \\ + \mu\sup_{t\in[0,T]}\{e^{2\gamma(t-T)}Z_{x_0}(t)\} \end{aligned} \tag{3.10}$$

To complete the proof of Theorem 3.2 we need the following Lemma

Lemma 3.1. *Let the function $Z_{x_0}(t)$ be a solution of the following inequality*

$$Z_{x_0}(T) \le C_1 e^{-\beta T} + C_2 + \mu \sup_{t\in[0,T]}\{e^{\beta(t-T)}Z_{x_0}(t)\} \tag{3.11}$$

and let $\mu \le 1/2$ and $\beta > 0$. Then

$$Z_{x_0}(T) \le 2C_1 e^{-\beta T} + 2C_2 \tag{3.12}$$

Proof. Multiplying the inequality (3.11) by $e^{\beta T}$ and taking the supremum $\sup_{T\in[0,s]}$ of the both sides of the inequality we get after simple calculations

$$\sup_{T\in[0,s]}\{e^{\beta T}Z_{x_0}(T)\} \le C_1 + C_2 e^{\beta s} + \mu\sup_{T\in[0,s]}\{e^{\beta T}Z_{x_0}(T)\}$$

Taking into account that $\mu < 1/2$ we get

$$\sup_{t\in[0,T]}\{e^{\beta t}Z_{x_0}(t)\} \le 2C_1 + 2C_2 e^{\beta T}$$

Replacing the last term in (3.11) by this estimate we obtain (3.12).

The end of the proof of Theorem 3.2. Applying the result of Lemma 3.1 to the estimate (3.10) we will get for sufficiently small $\mu > 0$

$$Z_{x_0}(T) \le C\sup_{x\in\Omega}\{e^{-2\varepsilon|x-x_0|}|u_0(x)|^4\}e^{-2\gamma T} + C_\mu(|g|^2, e^{-\varepsilon|x-x_0|})^2$$

Taking the square root of the both sides of the last inequality and taking $x = x_0$ in the left side of it we obtain the inequality (3.4). Theorem 3.2 is proved. □

Corollary 3.1. *Under the assumptions of the previous Theorem the following estimate is valid uniformly with respect to* $x_0 \in \Omega$

$$|u(T,x_0)|^2 \le Ce^{-\gamma T}\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^2_{0,\infty}\,dx + C(|g|^2, e^{-\varepsilon|x-x_0|}) \tag{3.13}$$

Indeed the estimate (3.13) is an immediate corollary of (3.4) and (1.21).

Theorem 3.3. *Let* u *be the solution of the problem* (3.1) *and let* $0<\delta<2-r$. *Then the following estimate is valid uniformly with respect to* $x_0\in\Omega$

$$\|u(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta,2} \le Ce^{-\gamma T}\left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{2-\delta,2}\,dx + \right. \tag{3.14}$$

$$\left. + \int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{0,\infty}\,dx + \left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{0,\infty}\,dx\right\}^K\right) +$$

$$+ C\int_{x\in\Omega}|g(x)|^2e^{-\varepsilon|x-x_0|}\,dx + C\left(\int_{x\in\Omega}|g(x)|^2e^{-\varepsilon|x-x_0|}\right)^K$$

Where $K=K(r,p,\delta)=\frac{2(p+2)}{2-r-\delta}$ *and* $\gamma>0$, $\varepsilon>0$ *are sufficiently small.*

Proof. Let us multiply the equation (3.1) by $\psi_{x_0}(x)$, the same as in the proof of Theorem 2.2 and denote $v_{x_0}=\psi_{x_0}u$

$$\partial_t v_{x_0} - \Delta_x v_{x_0} + a v_{x_0} = -\psi_{x_0} f(u,\nabla_x u) - 2\nabla_x\psi_{x_0}\nabla_x u - \Delta_x\psi_{x_0}u + g \equiv h_{x_0}(t) \tag{3.15}$$

Then due to the estimate (2.17)

$$\|u(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta,2} \le C\|v_{x_0},V_{x_0}\|^2_{2-\delta,2} \le \tag{3.16}$$

$$\le C_1(\|v_{x_0}(0),V_{x_0}\|^2_{2-\delta,2}e^{-\gamma T} + C_2\int_0^T e^{\gamma(t-T)}|t-T|^{-1+\delta/2}\|h_{x_0}(t),V_{x_0}\|^2_{0,2}\,dt$$

Let us estimate the right–hand side of (3.16). It follows from the definition of h_{x_0} that

$$\|h_{x_0},V_{x_0}\|^2_{0,2} \le C\|f(u,\nabla_x u),V_{x_0}\|^2_{0,2} + C\|u,V_{x_0}\|^2_{1,2} + \|g,V_{x_0}\|^2_{0,2} \tag{3.17}$$

Due to the second condition of (3.2)

$$\|f(u,\nabla_x u),V_{x_0}\|^2_{0,2} \le C\left(\|u,V_{x_0}\|^2_{0,\infty} + \|u,V_{x_0}\|^{2(p+1)}_{0,\infty}\right)\left(\|u,V_{x_0}\|^{2r}_{1,2r}+1\right) \tag{3.18}$$

Using the interpolation inequality (see [33]) we obtain that

$$\|u,V_{x_0}\|_{1,2r} \le C\|u,V_{x_0}\|^{1-\theta}_{0,\infty}\|u,V_{x_0}\|^{\theta}_{2-\delta,2} \tag{3.19}$$

where $\theta=\frac{1}{2-\delta}\in(0,1)$. It follows from the condition $\delta<2-r$ that $2r\theta<2$. Hence

$$\|f(u,\nabla_x u),V_{x_0}\|^2_{0,2} \le \tag{3.20}$$

$$\le C_\mu\|u,V_{x_0}\|^2_{0,\infty} + C_\mu\left(\|u,V_{x_0}\|^{2(p+1)}_{0,\infty}\|u,V_{x_0}\|^{2r(1-\theta)}_{0,\infty}\right)^{\frac{1}{1-r\theta}} + \mu\|u,V_{x_0}\|^2_{2-\delta,2} \le$$

$$\le C_\mu\left(\|u,V_{x_0}\|^2_{0,\infty} + \|u,V_{x_0}\|^{2K}_{0,\infty}\right) + \mu\|u,V_{x_0}\|^2_{2-\delta,2}$$

Here $\mu > 0$ is an arbitrary positive number. Indeed

$$\frac{2(p+1+r(1-\theta)}{1-r\theta} = \frac{2r(p+1+r(1-\theta))}{2-r-\delta} < 2K$$

Using the estimates (3.17)–(3.20) we obtain from (3.16) that

$$\begin{aligned}(3.21)\quad \|u(T), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} \le C_1 \|v_{x_0}(0), V_{x_0}\|^2_{2-\delta,2} e^{-\gamma T} + \\ + C_\mu \int_0^T e^{\gamma(t-T)} |t-T|^{-1+\delta/2} \left(\|u, V_{x_0}\|^2_{0,\infty} + \|u, V_{x_0}\|^{2K}_{0,\infty} \right) dt + \\ + \mu \int_0^T e^{\gamma(t-T)} |t-T|^{-1+\delta/2} \|u(t), V_{x_0}\|^2_{2-\delta,2} \, dt\end{aligned}$$

Applying the estimate (3.13) to the right-hand side of the inequality (3.21) and using the evident inequality

$$\int_0^T e^{\gamma(t-T)} |t-T|^{-1+\delta/2} e^{-\gamma_1 t} \, dt \le C e^{-\gamma_1 T}, \quad \gamma_1 < \gamma$$

we will have

$$\begin{aligned}\|u(T), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} \le C_\mu e^{-\gamma_1 T} \Bigg(\|u(0), V_{x_0}\|^2_{2-\delta,2} + \\ \int_{x \in \Omega} e^{-\varepsilon_1 |x-x_0|} \|u(0), \Omega \cap B^1_x\|^2_{0,\infty} \, dx + \left\{ \int_{x \in \Omega} e^{-\varepsilon_1 |x-x_0|} \|u(0), \Omega \cap B^1_x\|^2_{0,\infty} \, dx \right\}^K \Bigg) \\ + C_\mu \left(\int_{x \in \Omega} e^{-\varepsilon_1 |x-x_0|} |g(x)|^2 dx + \left\{ \int_{x \in \Omega} e^{-\varepsilon_1 |x-x_0|} |g(x)|^2 dx \right\}^K \right) + \\ + \mu \int_0^T e^{\gamma(t-T)} |t-T|^{-1+\delta/2} \|u(t), V_{x_0}\|^2_{2-\delta,2} \, dt\end{aligned}$$

for $\gamma_1 < \gamma$ and ε_1 small enough. Multiplying this estimate by $e^{-K\varepsilon|x_0-y|}$ with $K\varepsilon < \varepsilon_1$, integrating over $x_0 \in \Omega$ and using the inequalities (1.22) we obtain after simple calculations that

$$\begin{aligned}(3.22)\quad \int_{x \in \Omega} e^{-K\varepsilon|y-x|} \|u(T), \Omega \cap B^1_x\|^2_{2-\delta,2} \, dx \le \\ \le C_\mu e^{-\gamma_1 T} \left(\int_\Omega e^{-K\varepsilon|x-y|} \|u(0), V_x\|^2_{2-\delta,2} \, dx + \right. \\ \int_{x \in \Omega} e^{-\varepsilon|x-y|} \|u(0), \Omega \cap B^1_x\|^2_{0,\infty} \, dx + \left\{ \int_{x \in \Omega} e^{-\varepsilon|x-y|} \|u(0), \Omega \cap B^1_x\|^2_{0,\infty} \, dx \right\}^K \Bigg) \\ + C_\mu \left(\int_{x \in \Omega} e^{-\varepsilon|x-y|} |g(x)|^2 dx + \left\{ \int_{x \in \Omega} e^{-\varepsilon|x-y|} |g(x)|^2 dx \right\}^K \right) + \\ + \mu \int_0^T e^{\gamma(t-T)} |t-T|^{-1+\delta/2} \int_{x \in \Omega} e^{-K\varepsilon|x-y|} \|u(t), V_x\|^2_{2-\delta,2} \, dx \, dt\end{aligned}$$

Let us estimate the last integral into the right–hand side of inequality (3.22). It follows from the inequality (1.16) that uniformly with respect to $x \in \Omega$,

$$\|u, V_x\|^2_{2-\delta,2} \le C \int_{z \in V_x} \|u, \Omega \cap B^1_z\|^2_{2-\delta,2}\, dz \tag{3.23}$$

Multiplying the estimate (3.23) by $e^{-K\varepsilon|y-x|}$ and integrating over $x \in \Omega$ we obtain due to Theorem 1.2 (with $u(z) := \|u, \Omega \cap B^1_z\|_{2-\delta,2}$, $p = 2$ and $\phi = e^{-K\varepsilon|x-y|}$), that

$$\int_{x\in\Omega} e^{-K\varepsilon|y-x|}\|u, V_x\|^2_{2-\delta,2}\, dx \le C \int_{x\in\Omega} e^{-K\varepsilon|y-x|}\|u, \Omega \cap B^1_x\|^2_{2-\delta,2}\, dx \tag{3.24}$$

Replacing the last integral in the right–hand side of (3.22) by the estimates (3.24) and (2.24) and using Lemma 3.1 with

$$Z_y(T) = \int_{x\in\Omega} e^{-K\varepsilon|y-x|}\|u(T), \Omega \cap B^1_x\|^2_{2-\delta,2}\, dx$$

we obtain that for sufficiently small $\mu > 0$, $\varepsilon > 0$ and $\gamma > 0$

$$\begin{aligned}
&\int_\Omega e^{-K\varepsilon|y-x|}\|u(T), \Omega \cap B^1_x\|^2_{2-\delta,2}\, dx \le \\
&\le C_1 e^{-\gamma T}\left(\int_\Omega e^{-K\varepsilon|x-y|}\|u(0), V_x\|^2_{2-\delta,2}\, dx + \right. \\
&\left. \int_{x\in\Omega} e^{-\varepsilon|x-y|}\|u(0), \Omega \cap B^1_x\|^2_{0,\infty}\, dx + \left\{\int_{x\in\Omega} e^{-\varepsilon|x-y|}\|u(0), \Omega \cap B^1_x\|^2_{0,\infty}\, dx\right\}^K\right) \\
&+ C_2\left(\int_{x\in\Omega} e^{-\varepsilon|x-y|}|g(x)|^2 dx + \left\{\int_{x\in\Omega} e^{-\varepsilon|x-y|}|g(x)|^2 dx\right\}^K\right)
\end{aligned} \tag{3.25}$$

uniformly with respect to $y \in \Omega$.

It is not difficult to prove using (3.23) that

$$\|u(T), \Omega \cap B^1_y\|^2_{2-\delta,2} \le C \int_\Omega e^{-K\varepsilon|y-x|}\|u(T), \Omega \cap B^1_x\|^2_{2-\delta,2}\, dx \tag{3.26}$$

uniformly with respect to $y \in \Omega$. The estimate (3.14) follows from the estimates (3.25) and (3.26). Theorem 3.3 is proved. $\square$

Corollary 3.2. *Let u be the solution of* (3.1) *and let the assumptions of previous Theorem be valid. Then*

$$\begin{aligned}
&\|f(u(T), \nabla_x u(T)), \Omega \cap B^1_{x_0}\|^2_{0,2} \le \\
&\le Ce^{-\gamma T}\left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u_0, \Omega \cap B^1_x\|^2_{2-\delta,2}\, dx + \right. \\
&\left. \int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0), \Omega \cap B^1_x\|^2_{0,\infty}\, dx + \left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0), \Omega \cap B^1_x\|^2_{0,\infty}\, dx\right\}^K\right) \\
&+ C_2\left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(x)|^2 dx + \left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(x)|^2 dx\right\}^K\right)
\end{aligned} \tag{3.27}$$

Indeed (3.27) follows from the estimates (3.20), (3.14) and (3.13).

Corollary 3.3. *Let u be the solution of* (3.1) *and let the assumptions of previous Theorem be valid. Then $u \in C^{1-\delta/2}(\mathbb{R}, L^2_{\{\varepsilon\}}(\Omega))$ and*

$$\|u\|^2_{C^{1-\delta/2}([T,T+1],L^2(\Omega\cap B^1_{x_0}))} \le \tag{3.28}$$

$$\le Ce^{-\gamma T}\left(\int_{x\in\Omega} e^{\varepsilon|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{2-\delta,2}\,dx+\right.$$

$$\left.\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^2_{0,\infty}\,dx+\left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^2_{0,\infty}\,dx\right\}^K\right)$$

$$+C_2\left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(x)|^2dx+\left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(x)|^2dx\right\}^K\right)$$

Indeed, let us rewrite the equation (3.1) in the form

$$\partial_t u-\Delta_x u+\lambda_0 u=-f(u,\nabla_x u)+g \tag{3.29}$$

The equation (3.29) has the view (2.1) with the right–hand side $f(u,\nabla_x u)+g$ Thus, applying Theorem 2.3 (the assertion(2.14) and the estimate (2.15)) to the equation (3.29) and taking into account the estimate (3.27) we easily obtain all of the assertions of Corollary 3.3. □

Corollary 3.4. *Let u be a solution of* (3.1) *and suppose that the assumptions of Theorem 3.3 hold. Then for every fixed $0<\delta_1<\delta$, $\varepsilon>0$ and arbitrary $T\ge 1$*

$$u\in C([T,T+1],W^{2-\delta_1,2}_{\{\varepsilon\}}(\Omega))\cap C^{1-\delta_1/2}([T,T+1],L^2_{\{\varepsilon\}}(\Omega)) \tag{3.30}$$

and the following estimate is valid

$$\|u\|^2_{C([T,T+1],W^{2-\delta_1,2}(\Omega\cap B^1_{x_0}))}+\|u\|^2_{C^{1-\delta_1/2}([T,T+1],L^2(\Omega\cap B^1_{x_0}))}\le \tag{3.31}$$

$$\le C_1e^{-\gamma T}\left(\int_{x\in\Omega} e^{\varepsilon|x-x_0|}\|u_0,\Omega\cap B^1_x\|^2_{2-\delta,2}\,dx+\right.$$

$$\left.\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^2_{0,\infty}\,dx+\left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|u(0),\Omega\cap B^1_x\|^2_{0,\infty}\,dx\right\}^K\right)$$

$$+C_2\left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(x)|^2dx+\left\{\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}|g(x)|^2dx\right\}^K\right)$$

where the constants $C_1=C_1(\delta_1)$ and $C_2(\delta_1)$ are independent of x_0 and $T\ge 1$.

Indeed, introduce the function $w(t)=(t-T+1)u(t)$. Then

$$\begin{cases}\partial_t w-\Delta_x w+\lambda_0 w=(t-T+1)g-(t-T+1)f(u,\nabla_x u)+u\equiv h(t)\\ w\big|_{t=T-1}=0;\ w\big|_{\partial\Omega}=0\end{cases} \tag{3.32}$$

Note, that the equation (3.32) has the form of (2.1) and due to (3.14) and (3.27) the right-hand side $h\in C([T-1,T+1],L^2_{\{\varepsilon\}}(\Omega))$. Applying Theorem 2.3 (with $\delta=\delta_1$) and using the estimates (3.14) and (3.27) we obtain the estimate (3.31) and the assertion (3.30). □

§4 The nonlinear equation. Existence of solutions.

In this Section we prove the existence of solutions for the problem (3.1). To this end we prove first the existence of solutions for the problem (3.1) in a smooth *bounded* domain Ω and after that by approximating the initial unbounded domain by bounded ones, we extend this result to every domain which satisfies (1.8) and (1.9).

Theorem 4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and let $u_0 \in W^{2-\delta,2}(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}$, $\delta < \min\{1/2, 2-r\}$, where r was defined by (0.2). Then for any $T > 0$ there exists at least one solution*

$$u \in W \equiv C^{1-\delta/2}([0,T], L^2(\Omega)) \cap C([0,T], W^{2-\delta,2}(\Omega) \cap W_0^{1,2}(\Omega)) \tag{4.1}$$

of the problem (3.1)

Proof. Let P be the inverse operator of the parabolic operator

$$Lu = \partial_t u - \Delta_x u + \lambda_0 u\,; \quad u\big|_{t=0} = 0;\ u\big|_{\partial\Omega} = 0 \tag{4.2}$$

Then, due to Lemma 2.1,

$$P = L^{-1} : C([0,T], L^2(\Omega)) \to W \text{ for every } \delta > 0 \tag{4.2}$$

Let us introduce the function $w(t) = e^{t(\Delta_x - \lambda_0)}u_0$ as the solution of linear homogeneous parabolic problem. Then Lemma 2.1 implies that $w \in W$ and

$$\sup_{t\in[0,T]} \|w(t), \Omega\|_{2-\delta,2} + \|w\|_{C^{1-\delta/2}([0,T],L^2(\Omega))} \le C\|u_0, \Omega\|_{2-\delta,2}$$

We rewrite the equation (3.1) with respect to a new variable $v(t) = u(t) - w(t)$ and apply the operator P to both sides of the obtained equation. After these transformations we get the following equation in the space W

$$v + F(v) = h\ ,\ v \in W \tag{4.3}$$

Here $h = Pg \in W$ and the nonlinear operator F is given by the following expression

$$F(v) = -Pf(v + w, \nabla_x v + \nabla_x w) : W \to W. \tag{4.4}$$

We are going to apply the Leray–Schauder fixed–point principle (see [29], [36]) to the equation (4.3). First we check that the operator F is compact. Indeed, let $\xi_k \in W$ be some bounded sequence in W. Then due to the compactness of embedding

$$W \subset\subset C([0,T], W^{2-\delta-\alpha,2}(\Omega)) \text{ for every } \alpha > 0 \tag{4.5}$$

and due to the Sobolev embedding theorem $W^{\mu,2} \subset C$ if $\mu > 3/2$ and $W^{2-\delta-\alpha,2} \subset W^{1,2r}$ for sufficiently small $\alpha > 0$ (here r the same as in the condition (3.2)), we can suppose without loss of generality that the sequence $(\xi_k, \nabla_x \xi_k) \to (\xi, \nabla_x \xi)$ in the space $C([0,T], C(\Omega) \times L^{2r}(\Omega))$. But due to the conditions (3.2) the Nemitskij operator

$$f(u,v) : C([0,T], C(\Omega) \times L^{2r}(\Omega)) \to C([0,T], L^2(\Omega))$$

is continuous (see [22]). Hence the sequence

$$f(\xi_k + w, \nabla_x \xi_k + \nabla_x w) \to f(\xi + w, \nabla_x \xi + \nabla_x w)$$

in the space $C([0,T], L^2(\Omega))$. Thus $F(\xi_k) \to F(\xi)$ in W and consequently F is compact.

Due to the Leray–Schauder principle it is sufficient to prove now that for a sufficiently large ball B in the space W

$$v + sF(v) \neq h \text{ for } s \in [0,1],\ v \in \partial B. \tag{4.6}$$

Let us prove (4.6). Indeed let us suppose that (4.6) is not valid for some $s = s_0 \in [0,1]$. Then denoting $u = v + w$ we obtain that

$$\begin{cases} \partial_t u - \Delta_x u + \lambda_0 u + s_0 f(u, \nabla_x u) = g \\ u\big|_{t=0} = u_0\ ,\ u\big|_{\partial\Omega} = 0 \end{cases} \tag{4.7}$$

The equation (4.7) has the form of (3.1). Then due to Theorem 3.3 and Corollary 3.3, $\|u\|_W \leq C(u_0)$ uniformly with respect to $s_0 \in [0,1]$. Hence for a sufficiently large ball in W the inequality (4.6) is valid.

Thus due to the Leray–Schauder fixed–point principle, the equation (4.3) has at least one solution $v \in W$. Theorem 4.1 is proved. □

Theorem 4.2. *Let Ω satisfy the conditions* (1.8)– (1.9) *and the nonlinear term $f(u, \nabla_x u)$ – conditions* (3.2). *Let us suppose also that $u_0 \in \cap_{\varepsilon>0} \in W^{2-\delta,2}_{\{\varepsilon\}}(\Omega)$ for some $\delta < \min\{1/2, 2-r\}$ and $g \in \cap_{\varepsilon>0} L^2_{\{\varepsilon\}}(\Omega)$. Then the problem* (3.1) *has at least one solution*

$$u \in \cap_{\varepsilon>0} \left\{ C([0,T], W^{2-\delta,2}_{\{\varepsilon\}}(\Omega)) \cap C^{1-\delta/2}([0,T], L^2_{\{\varepsilon\}}(\Omega)) \right\} \tag{4.8}$$

Proof. Let Ω_N, $N = 1, 2, \cdots$ be a sequence of smooth bounded domains which satisfies the conditions (1.8) and (1.9) uniformly with respect to $N \in \mathbb{N}$ and such that

$$\begin{cases} \Omega_N \subset \Omega_{N+1} \subset \Omega\ ;\ \Omega = \cup_{N=1}^{\infty} \Omega_N \\ \Omega \cap B_0^N \subset \Omega_N \subset \Omega \cap B_0^{N+1} \end{cases} \tag{4.9}$$

It is not difficult to check that such a sequence exists.

Let us introduce the sequence of cut–off functions $\psi_N(x) \in C_0^\infty(\mathbb{R}^3)$ such that

$$\psi_N(x) = 1 \text{ if } x \in B_0^{N-1} \text{ and } \psi_N(x) = 0 \text{ if } x \notin B_0^N \text{ and } \|\psi_N\|_{C^2} \leq C$$

Let u_N be a solution of the following problem

$$\begin{cases} \partial_t u_N - \Delta_x u_N + \lambda_0 u_N + f(u_N, \nabla_x u_N) = g \\ u_N\big|_{\partial\Omega_N} = 0\ ;\ u_N\big|_{t=0} = \psi_N u_0 \end{cases} \tag{4.10}$$

Since the conditions (1.8) and (1.9) holds for Ω_N *uniformly* with respect to $N \in \mathbb{N}$ then (due to Theorem 3.3 and Corollary 3.3) the estimates (3.14) and (3.28) with u replaced by u_N are also valid uniformly with respect to $N \in \mathbb{N}$. Thus, for every $M \in \mathbb{N}$ the sequence $u_N\big|_{\Omega \cap B_0^M}$, $N \geq M$ is bounded in the space

$$C^{1-\delta/2}([0,T], L^2(\Omega \cap B_0^M)) \cap C([0,T], W^{2-\delta,2}(\Omega \cap B_0^M)) \tag{4.11}$$

Let us extract from the sequence u_N a subsequence (which we denote by u_N also for simplicity) converging $*$-weakly to u in $L^\infty([0,T], W^{2-\delta,2}(\Omega \cap B_0^M))$ for every $M \in \mathbb{N}$ (it is possible to do using the Cantor diagonal procedure). Hence due to the embedding (4.5) we obtain as in the previous Theorem that

$$(4.12) \qquad \begin{cases} u_N \to u \text{ strongly in } C([0,T] \times \Omega \cap B_0^M) \cap C([0,T], W^{1,2}(\Omega \cap B_0^M) \\ f(u_N, \nabla_x u_N) \to f(u, \nabla_x u) \text{ strongly in } C([0,T], L^2(\Omega \cap B_0^M)) \end{cases}$$

The assertions (4.12) imply immediately that the function u satisfies the equation (3.1). So it remains to prove that (4.8) holds. To this end we prove that $u_N \to u$ strongly in the space (4.11). Indeed

$$(4.13) \quad \partial_t(\psi_M u_N) - \Delta_x(\psi_M u_N) = h_M(u_N) \equiv -\psi_M f(u_N, \Delta_x u_N) - \\ - 2\nabla_x \psi_M \nabla_x u_N - \Delta_x \psi_M u_N - \lambda_0 \psi_M u_N + \psi_M g, \quad \psi_M u_N\big|_{\partial\Omega_M} = 0$$

It follows from (4.12) that $h_M(u_N) \to h_M(u)$ in $C([0,T], L^2(\Omega_M))$ consequently due to the regularity theorem, applied to the linear problem (4.13), $u_N \to u$ strongly in the space (4.11).

Passing to the limit $N \to \infty$ in the estimates (3.14) and (3.28) (with u replaced by u_N) we obtain that the limit function u also satisfies these estimates. Moreover we obtain also that u belongs to the space (4.11) for every $M > 0$. Let us fix a sufficiently small $\mu > 0$, multiply the estimates (3.14) and (3.28) by $e^{-\mu|x_0|}$ and integrate over $x_0 \in \Omega$. Then using our assumptions on u_0 and g and the estimate (1.3) we obtain as in the proof of Corollaries 2.1 and 2.2 that

$$u \in L^\infty([0,T], W^{2-\delta,2}_{\{\mu\}}(\Omega)) \cap C^{1-\delta/2}([0,T], L^2_{\{\mu\}}(\Omega))$$

The continuity of u ($u \in C([0,T], W^{2-\delta,2}_{\{\mu\}})$) can be proved using the 'tail estimates' (1.4) and Theorem 1.3. Thus, (4.8) holds and consequently u is a solution of the problem (3.1). Theorem 4.2 is proved. $\square$

§5 The nonlinear equation. Uniqueness of solutions. Differentiability with respect to u_0.

In this Section we study the uniqueness problem for equation (0.1). We require the nonlinear term to satisfy (3.2) and the assumptions

$$(5.1) \qquad \begin{cases} 1. & f \in C^1(\mathbb{R}^k \times \mathbb{R}^{3k}; \mathbb{R}^k) \\ 2. & |f'_u(u, \nabla_x u)| \le C(1 + |u|^p)(1 + |\nabla_x u|^r), \text{ where } r < 2 \\ 3. & |f'_{\nabla_x u}(u, \nabla_x u)| \le C(1 + |u|^{p+1})(1 + |\nabla_x u|^{r-1}) \end{cases}$$

Note that under these assumptions we can prove the uniqueness only in the case where the right-hand side g and the initial value u_0 are bounded with respect to $|x| \to \infty$ (see Remark 5.1).

Theorem 5.1. *Let the nonlinear term satisfy the assumptions* (3.2) *and* (5.15) *and let the initial data* $u_0 \in W_b^{2-\delta,2}(\Omega)$ *for some* $\delta < \min\{\frac{1}{r} - \frac{1}{2}, \frac{1}{2}\}$ *(see Definition 1.2).*

Suppose also that the right–hand side $g \in L^2_b(\Omega)$. *Then the problem* (3.1) *has a unique solution in the class* $\cap_{\{\varepsilon>0\}} C(\mathbb{R}_+, W^{2-\delta,2}_{\{\varepsilon\}}(\Omega))$.

Proof. Let u_1, u_2 be two solutions of (3.1). Then applying $\sup_{x_0\in\Omega}$ to both sides of (3.14) and using the boundedness of the initial condition u_0 and the right–hand side g and the estimate (1.24) we obtain that

$$\|u_i(T),\Omega\|^2_{b,2-\delta,2} \le C(\|u_0,\Omega\|^2_{b,2-\delta,2} + \|u_0,\Omega\|^{2K}_{b,2-\delta,2})e^{-\alpha T} + \\ + C(\|g,\Omega\|^2_{b,0,2} + \|g,\Omega\|^{2K}_{b,0,2}) \tag{5.2}$$

for i=1,2. Thus, under our assumptions, all solutions are also bounded with respect to $|x| \to \infty$.

Let $v(t) = u_2(t) - u_1(t)$. Then

$$\begin{cases} \partial_t v - \Delta_x v + \lambda_0 v = -\widehat{L}_1(t,x)v - \widehat{L}_2(t,x)\nabla_x v \\ v\big|_{t=0} = u_2(0) - u_1(0) \end{cases} \tag{5.3}$$

Here

$$\begin{cases} \widehat{L}_1(t,x) = \int_0^1 f'_u(u_1 + \theta v, \nabla_x u_1 + \theta\nabla_x v)\, d\theta \\ \widehat{L}_2(t,x) = \int_0^1 f'_{\nabla_x u}(u_1 + \theta v, \nabla_x u_1 + \theta\nabla_x v)\, d\theta \end{cases} \tag{5.4}$$

It follows from the condition (5.1) that

$$\begin{cases} |\widehat{L}_1(t)| \le C(1 + |u(t)|^p + |\nabla_x u(t)|^r + |u(t)|^p|\nabla_x u(t)|^r) \\ |\widehat{L}_2(t)| \le C(1 + |u(t)|^{p+1} + |\nabla_x u(t)|^{r-1} + |u(t)|^{p+1}|\nabla_x u(t)|^{r-1}) \end{cases} \tag{5.5}$$

We denote here by $|u(t)|^p = |u_1(t)|^p + |u_2(t)|^p$, $|\nabla_x u_1(t)|^r = |\nabla_x u_1(t)|^r + |\nabla_x u_2(t)|^r$.

After multiplying the equation (5.3) by v in the space $L^2_{\{\varepsilon\}}(\Omega)^k$ for sufficiently small $\varepsilon > 0$ we obtain after simple transformations

$$\partial_t\|v(t)\|^2_{\{\varepsilon\},0,2} + \|v(t)\|^2_{\{\varepsilon\},1,2} + \lambda_0\|v(t)\|^2_{\{\varepsilon\},0,2} + \\ + \left\langle \widehat{L}_1 v, v\right\rangle_{\{\varepsilon\}} + \left\langle \widehat{L}_2\nabla_x v, v\right\rangle_{\{\varepsilon\}} \le 0 \tag{5.6}$$

Let's estimate the two nonlinear terms in (5.6) separately.

It follows from the estimate (5.2) and Sobolev's embedding theorem that $|u_i(t,x)| \le C$ for all $t \in [0,T]$, $x \in \Omega$. Hence

$$I_1(t) \equiv |\langle L_1 v, v\rangle_{\{\varepsilon\}}| \le C(\langle v, v\rangle_{\{\varepsilon\}} + \langle |u|^p v, v\rangle_{\{\varepsilon\}} + \\ + \langle |\nabla_x u|^r v, v\rangle_{\{\varepsilon\}} + \langle |u|^p|\nabla_x u|^r v, v\rangle_{\{\varepsilon\}}) \le \\ \le C_1(\|v(t),\Omega\|^2_{\{\varepsilon\},0,2} + \langle |\nabla_x u|^r v, v\rangle_{\{\varepsilon\}}) \tag{5.7}$$

Let us estimate the last integral at the right–hand side of (5.7). To this end we use a trick based on (1.10), Holder inequality and embedding Theorem $W^{1,2} \subset L^6$. Indeed,

$$\langle |\nabla_x u|^r v, v\rangle_{\{\varepsilon\}} \le C\int_\Omega e^{-\varepsilon|x_0|}\|v\cdot v\cdot|\nabla_x u|^r, V_{x_0}\|_{0,1}\, dx_0 \le \\ \le C_1\|u,\Omega\|^r_{b,1,3r}\int_\Omega e^{-\varepsilon|x_0|}\|v, V_{x_0}\|_{0,2}\|v, V_{x_0}\|_{0,6}\, dx_0 \le \\ \le C_\mu\|u,\Omega\|^r_{b,1,3r}\int_\Omega e^{-\varepsilon|x_0|}\|v, V_{x_0}\|^2_{0,2}\, dx_0 + \\ + \mu\|u,\Omega\|^r_{b,1,3r}\int_\Omega e^{-\varepsilon|x_0|}\|v, V_{x_0}\|^2_{1,2}\, dx_0 \le \\ \le C_\mu\|u,\Omega\|^r_{b,1,3r}\|v(t),\Omega\|^2_{\{\varepsilon\},0,2} + \mu\|u,\Omega\|^r_{b,1,3r}\|v(t),\Omega\|^2_{\{\varepsilon\},1,2} \tag{5.8}$$

Taking into account the condition $\delta < \frac{1}{r} - \frac{1}{2}$ and using the Sobolev embedding theorem and inequality (5.2) we obtain that

$$\|u(t), \Omega\|_{b,1,3r} \le C \max\{\|u_i(t), \Omega\|_{b,2-\delta,2},\ i = 1, 2\} \le C_1 \text{ for } t \in \mathbb{R}_+ \tag{5.9}$$

Hence

$$I_1(t) \le C_\mu \|v(t), \Omega\|^2_{\{\varepsilon\},0,2} + \mu \|v(t)\|^2_{\{\varepsilon\},1,2} \tag{5.10}$$

Analogously

$$\begin{aligned} I_2(t) \equiv \left\langle \widehat{L}_2 \nabla_x v, v \right\rangle_{\{\varepsilon\}} &\le C(\langle v, \nabla_x v\rangle_{\{\varepsilon\}} + \left\langle |u|^{p+1} \nabla_x v, v \right\rangle_{\{\varepsilon\}} + \\ &+ \left\langle |\nabla_x u|^{r-1} \nabla_x v, v \right\rangle_{\{\varepsilon\}} + \left\langle |u|^{p+1} |\nabla_x u|^{r-1} \nabla_x v, v \right\rangle_{\{\varepsilon\}}) \le \\ &\le C_\mu \|v, \Omega\|^2_{\{\varepsilon\},0,2} + \mu \|v, \Omega\|^2_{\{\varepsilon\},1,2} + C \left\langle |\nabla_x u(t)|^{r-1} \nabla_x v, v \right\rangle_{\{\varepsilon\}} \end{aligned} \tag{5.11}$$

Arguing as in (5.8) and using the interpolation inequality (see [33])

$$\|v, V_{x_0}\|_{0,3} \le C \|v, V_{x_0}\|_{1/2,2} \le C \|v, \Omega\|^{1/2}_{0,2} \|v, \Omega\|^{1/2}_{1,2} \tag{5.12}$$

we obtain using that $6(r-1) < 3r$

$$\begin{aligned} \left\langle |\nabla_x u|^{r-1} v, \nabla_x v \right\rangle_{\{\varepsilon\}} &\le C \int_\Omega e^{-\varepsilon|x_0|} \| \, |v| \cdot |\nabla_x v| \cdot |\nabla_x u|^{r-1}, V_{x_0}\|_{0,1} \, dx_0 \le \\ &\le C_1 \|u, \Omega\|^{r-1}_{b,1,6(r-1)} \int_\Omega e^{-\varepsilon|x_0|} \|v, V_{x_0}\|_{1,2} \|v, V_{x_0}\|_{0,3} \, dx_0 \le \\ &\le C_1 \|u, \Omega\|^{r-1}_{b,1,3r} \int_\Omega e^{-\varepsilon|x_0|} \|v, V_{x_0}\|^{3/2}_{1,2} \|v, V_{x_0}\|^{1/2}_{0,2} \, dx_0 \le \\ &\le C_\mu \int_\Omega e^{-\varepsilon|x_0|} \|v, V_{x_0}\|^2_{0,2} \, dx_0 + \mu \int_\Omega e^{-\varepsilon|x_0|} \|v, V_{x_0}\|^2_{1,2} \, dx_0 \le \\ &\le C_\mu \|v(t), \Omega\|^2_{\{\varepsilon\},0,2} + \mu \|v(t), \Omega\|^2_{\{\varepsilon\},1,2} \end{aligned} \tag{5.13}$$

Hence

$$I_2(t) \le C_\mu \|v(t), \Omega\|^2_{\{\varepsilon\},0,2} + \mu \|v(t), \Omega\|^2_{\{\varepsilon\},1,2} \tag{5.14}$$

Replace the integrals I_1 and I_2 in (5.6) by their estimates (5.10) and (5.14).

$$\partial_t \|v(t), \Omega\|^2_{\{\varepsilon\},0,2} + \beta \|v(t), \Omega\|^2_{\{\varepsilon\},1,2} \le C \|v(t), \Omega\|^2_{\{\varepsilon\},0,2} \tag{5.15}$$

Applying the Gronwall inequality to (5.15) we obtain that $v(t) \equiv 0$. Theorem 5.1 is proved. □

Corollary 5.1. *Let the conditions of Theorem 5.1 be valid and let* $u_1(0), u_2(0) \in W^{2-\delta,2}_b(\Omega)$. *Then the following estimate is valid uniformly with respect to* $x_0 \in \Omega$

$$\begin{aligned} \|u_1(T) - u_2(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_0^T \|u_1(T) - u_2(T), \Omega \cap B^1_{x_0}\|^2_{1,2} \le \\ \le C e^{C_1 T} (e^{-\varepsilon|x-x_0|}, |u_1(0) - u_2(0)|^2) \end{aligned} \tag{5.16}$$

Here the constants C *and* C_1 *depend on* $\|u_i(0), \Omega\|_{b,2-\delta,2}$

The proof of the estimate is the same as the prove of Theorem 5.1 but instead of multiplying by $ve^{-\varepsilon|x|}$ we should multiply the equation (5.3) by $ve^{-\varepsilon|x-x_0|}$.

Our main task now (after proving the uniqueness) is to establish some regularity properties for the corresponding semigroup. They are formulated in the following propositions.

Proposition 5.1. *Under the assumptions of previous Theorem the following estimate is valid:*

$$\|u_1(T)-u_2(T),\Omega\cap B^1_{x_0}\|^2_{1,2}+\int_0^T\|u_1(t)-u_2(t),\Omega\cap B^1_{x_0}\|^2_{2,2}\,dt\le \tag{5.17}$$
$$\le Ce^{C_1T}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{1,2}\,dx$$

Here the constants C and C_1 depends on $\|u_i(0),\Omega\|_{b,2-\delta,2}$ and $\varepsilon>0$ is sufficiently small.

Proof. Applying the estimate (2.8) to the equation (5.3) we get

$$\|v(T),\Omega\cap B^1_{x_0}\|^2_{1,2}+\int_0^T\|v(t),\Omega\cap B^1_{x_0}\|^2_{2,2}\,dt\le \tag{5.18}$$
$$\le C\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(0),\Omega\cap B^1_x\|^2_{1,2}\,dx+$$
$$+C\int_0^T(|\widehat{L}_1(t)v(t)|^2+|\widehat{L}_2(t)\nabla_x v(t)|^2,e^{-\varepsilon|x-x_0|})\,dt$$

It can be shown analogously with (5.8) and (5.13) using the Sobolev embedding theorems $W^{3/2,2}\subset W^{1,3}$ and $W^{2-\delta_1,2}\subset C$ for $\delta_1<1/2$ and the appropriate Holder inequality that there exists $1/2>\delta_1>\delta$ such that

$$(|\widehat{L}_1(t)v(t)|^2+|\widehat{L}_2(t)\nabla_x v(t)|^2,e^{-\varepsilon|x-x_0|})\le \tag{5.19}$$
$$\le C\|u(t),\Omega\|^{2r}_{b,2-\delta,2}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(t),\Omega\cap B^1_{x_0}\|^2_{2-\delta_1,2}\,dx$$

Indeed, let us estimate only the most complicated term in the left-hand side of the inequality (5.19). The rest of the terms can be esimated analogously

$$\int_\Omega e^{-\varepsilon|x-x_0|}|\nabla_x u|^{2(r-1)}|\nabla_x v|^2\,dx\le$$
$$\le C\int_\Omega e^{-\varepsilon|x-x_0|}\|\,|\nabla_x u|^{2(r-1)}\cdot|\nabla_x v|^2,V_x\|_{0,1}\,dx\le$$
$$\le C_1\int_\Omega e^{-\varepsilon|x-x_0|}\|u,V_x\|^{2(r-1)}_{1,6(r-1)}\cdot\|v,V_x\|^2_{1,3}\,dx\le$$
$$\le C_2\|u,\Omega\|^{2(r-1)}_{b,2-\delta,2}\int_\Omega e^{-\varepsilon|x-x_0|}\|v,V_x\|^2_{3/2,2}\,dx$$

According to the interpolation inequality

$$\|\cdot\|_{2-\delta_1,2}\le C\|\cdot\|^\theta_{2,2}\cdot\|\cdot\|^{1-\theta}_{0,2}\le C_\mu\|\cdot\|^2_{0,2}+\mu\|\cdot\|^2_{2,2}$$

we obtain that

$$\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v,V_x\|^2_{2-\delta_1,2}\,dx\le \tag{5.20}$$
$$\le C_\mu\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v,\Omega\cap B^1_x\|^2_{0,2}\,dx+\mu\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v,\Omega\cap B^1_x\|^2_{2,2}\,dx$$

Replacing the last integral on the right–hand side of the estimate (5.18) by the estimates (5.19) and (5.20) and using (5.2) and (5.16) we obtain after some calculations that

$$\|v(T),\Omega\cap B^1_{x_0}\|^2_{1,2}+\int_0^T\|v(t),\Omega\cap B^1_{x_0}\|^2_{2,2}\,dt\le \tag{5.21}$$
$$\le C_\mu e^{C_1T}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(0),\Omega\cap B^1_x\|^2_{1,2}\,dx+$$
$$+\mu\int_0^T\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|v(t),\Omega\cap B^1_x\|^2_{2,2}\,dx\,dt$$

Multiplying the estimate (5.21) by $e^{-\varepsilon_1|x_0-y|}$, $\varepsilon_1<\varepsilon$, integrating over the $x_0\in\Omega$ and arguing in the following as at the end of the proof of Theorem 3.3 we get the inequality (5.17). Proposition 5.1 is proved.

Proposition 5.2. *Under the assumptions of the previous theorem the following estimate is valid*

$$\|u_1(T)-u_2(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta,2}+\|u_1-u_2\|^2_{C^{1-\delta/2}([0,T],L^2(\Omega\cap B^1_{x_0}))}\le \tag{5.22}$$
$$\le Ce^{C_1T}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{2-\delta,2}\,dx$$

The proof of Proposition 5.2 is analogous to the prove of Proposition 5.1 only instead of the estimate (2.8) we use the estimate (2.15) and the interpolation inequality

$$\|v,\Omega\cap B^1_x\|^2_{2-\delta_1,2}\le C_\mu\|v,\Omega\cap B^1_x\|^2_{0,2}+\mu\|v,\Omega\cap B^1_x\|^2_{2-\delta,2}$$

on the left hand side of the inequality (5.19).

We will need the smoothing property below for the equation (3.1) in the following form.

Proposition 5.3. *Under the assumptions of the previous theorem the following estimate is valid:*

$$\|u_1(T)-u_2(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta,2}+\|u_1-u_2\|^2_{C^{1-\delta/2}([0,T],L^2(\Omega\cap B^1_{x_0}))}\le \tag{5.23}$$
$$\le\frac{1}{T^2}Ce^{C_1T}\int_{x\in\Omega}e^{-\varepsilon|x-x_0|}\|u_1(0)-u_2(0),\Omega\cap B^1_x\|^2_{0,2}\,dx$$

for $T>0$.

Proof. Let us denote $w(t)=tv(t)=t(u_1(t)-u_2(t))$ then

$$\partial_tw-\Delta_xw-\widehat{L}_1(t)w-\widehat{L}_2(t)\nabla_xw=v(t);\quad w\big|_{t=0}=0 \tag{5.24}$$

The equation (5.24) has the form of (5.3) with the right–hand side $v\in C([0,T],L^2)$ hence arguing as in the prove of Proposition 5.2 we obtain the estimate

$$\|w(T),\Omega\cap B^1_{x_0}\|^2_{2-\delta,2}+\|w\|_{C^{1-\delta/2}([T,T+1],L^2(\Omega\cap B^1_{x_0}))}\le \tag{5.25}$$
$$\le C\int_0^Te^{\lambda_0(t-T)}|t-T|^{-1+\delta/2}(|v(t)|^2,e^{-\varepsilon|x-x_0|})$$

Estimating the last term in (5.25) by the inequality (5.16) we obtain the assertion of the Proposition.

Remark 5.1. *Note that we cannot guarantee the uniqueness in the class of solutions growing when* $|x| \to \infty$. *Indeed, let us consider the linear parabolic equation in* $\mathbb{R}^3$ *in the simplest form*

$$\partial_t v = \Delta_x v, \quad v\big|_{t=0} = 0 \tag{5.26}$$

It is known (see for instance [23]*), that the problem* (5.26) *has a nontrivial solution* $v(t,x)$ *with the rate of growth when* $|x| \to \infty$ *not exceeding* $e^{|x|^3}$. *Making the change of variables* $\theta = e^{-|x|^4} v$ *we obtain the equation*

$$\partial_t \theta = \Delta_x \theta + K_1(x)\theta + K_2(x)\nabla_x \theta, \quad \theta(0) = 0 \tag{5.27}$$

where the coefficients K_1 *and* K_2 *have a polynomial rate of growth as* $|x| \to \infty$.

This example shows that the uniqueness can be lost (even in the class of bounded solutions) when the coefficients growth polynomially at $|x| \to \infty$. *It remains to note that the equation* (5.27) *is very similar to the class of equations of variation which we obtain from our nonlinear equation when solutions* u *with polynomial growth as* $|x| \to \infty$ *are admitted.*

We mention now that if the nonlinear function does not depend on $\nabla_x u$ and satisfies the following condition

$$\begin{cases} 1.\ f(u, \nabla_x u) \equiv f(u) \in C^1(\mathbb{R}^k, \mathbb{R}^k) \\ 2.\ f'(u) \geq -C\ \forall u \in \mathbb{R}^k \end{cases} \tag{5.28}$$

we have uniqueness without the requirement $\|g, \Omega\|_{0,2,b} < \infty$.

Proposition 5.4. *Let the conditions* (5.28) *be valid. Then the problem* (3.1) *has a unique solution in class* (4.8) *and the estimate* (5.16) *is valid for the solutions* u_i *from the class* (4.8). *Moreover, the constants* C, C_1 *in it are independent of* u_i.

The proof of this Proposition is analogous to the prove of Theorem 5.1 but simpler because in our case $\widehat{L}_2 \equiv 0$ and the term with $\widehat{L}_1$ possesses the following estimate

$$\left\langle \widehat{L}_1 v, v \right\rangle_{\{\varepsilon\}} \geq -C \|v, \Omega\|^2_{\{\varepsilon\},0,2}$$

Hence the inequality (5.15) follows immediately from this inequality and inequality (5.6) and C does not depend on u_i. Proposition 5.4 is proved. $\square$

We conclude this Section by considering the problem of differentiability with respect to the initial data for the solutions of (3.1) under the assumptions of Theorem 5.1.

Let us consider first the (formal) equation of variation for the problem.

$$\begin{cases} \partial_t w - \Delta_x w + \lambda_0 w + f'_u(u_1, \nabla_x u_1) w + f'_{\nabla_x u}(u_1, \nabla_x u_1) \nabla_x w = 0 \\ w\big|_{t=0} = w_0\ ;\ w\big|_{\partial\Omega} = 0 \end{cases} \tag{5.29}$$

Here $u_1(t)$ is a solution of the problem (3.1).

Lemma 5.1. *Let $u_1(0) \in W_b^{2-\delta,2}$ and $w_0 \in \cap_{\varepsilon>0} L^2_{\{\varepsilon\}}(\Omega)$. Then there exists the unique solution of the problem* (5.29) *which satisfies* (2.2) *for any $\varepsilon_1 > 0$ and the following estimate is valid:*

$$\|w(T), \Omega \cap B^1_{x_0}\|^2_{0,2} + \int_T^{T+1} \|w(T), \Omega \cap B^1_{x_0}\|^2_{1,2} \leq \leq Ce^{C_1 T}(e^{-\varepsilon|x-x_0|}, |w_0|^2) \tag{5.30}$$

where the constants C and C_1 depend only on $\|u_1(0), \Omega\|_{2-\delta,2,b}$.

Indeed the equation (5.29) has the form of (5.3), so the estimate (5.30) can be proved in the same way as the estimate (5.16). The existence of the solution can be deduced from the a priori estimates (5.30) in a standard way.

Theorem 5.2. *Let the nonlinear term $f(u, \nabla_x u)$ satisfy the following additional conditions:*

$$|f'_u(p_1, q_1) - f'_u(p_2, q_2)| \leq \leq Q(|p_1| + |p_2|)\,(1 + |q_1|^r + |q_2|^r)\,\left(|p_1 - p_2|^\beta + |q_1 - q_2|^\beta\right) \tag{5.31}$$

$$|f'_{\nabla_x u}(p_1, q_1) - f'_{\nabla_x u}(p_2, q_2)| \leq \leq Q(|p_1| + |p_2|)\,\left(1 + |q_1|^{r-1} + |q_2|^{r-1}\right)\,\left(|p_1 - p_2|^\beta + |q_1 - q_2|^\beta\right) \tag{5.32}$$

for any $p_1, p_2 \in \mathbb{R}^k$, $q_1, q_2 \in \mathbb{R}^{3k}$, some fixed $0 < \beta \leq 1$ and for a certain monotonic function Q. Let us suppose also that $u_2(t)$ and $u_1(t)$ are two solutions of the problem (3.1) *which satisfy the conditions of Theorem 5.1 and let $w(t)$ be the solution of the problem* (5.29) *with the initial condition $w_0 = u_2(0) - u_1(0)$. Then the following estimate is valid uniformly with respect to $x_0 \in \mathbb{R}$:*

$$\|u_1(T) - u_2(T) - w(T), \Omega \cap B^1_{x_0}\|^2_{0,2} \leq \leq Ce^{C_1 T}\left(|u_1(0) - u_2(0)|^2, e^{-\varepsilon|x-x_0|}\right)^{1+\beta} \tag{5.33}$$

for sufficiently small $\varepsilon > 0$ and C_1, C_2 depending only on $\|u_i(0), \Omega\|_{2-\delta,2,b}$.

Proof. Let us denote by $L_1(t) = f_u(u_1(t), \nabla_x u_1(t))$, $L_2(t) = f_{\nabla_x u}(u_1(t), \nabla_x u_1(t))$ and $\theta(t) = u_2(t) - u_1(t) - w(t)$. Then the function θ satisfies the following equation

$$\partial_t \theta - \Delta_x \theta + \lambda_0 \theta + L_1(t)\theta + L_2(t)\nabla_x \theta = = (L_1(t) - \widehat{L}_1(t))v + (L_2(t) - \widehat{L}_2(t))\nabla_x v \ ; \ \theta(0) = 0 \tag{5.34}$$

where $\widehat{L}_1$ and $\widehat{L}_2$ are defined by (5.4).

It can be easily obtained using the conditions (5.31) and (5.32) and inequalities (5.2) that

$$\begin{cases} |L_1(t) - \widehat{L}_1(t)| \leq C(1 + |\nabla_x u_1(t)|^r + |\nabla_x u_2(t)|^r)\,\left(|v|^\beta + |\nabla_x v|^\beta\right) \\ |L_2(t) - \widehat{L}_2(t)| \leq C(1 + |\nabla_x u_1(t)|^{r-1} + |\nabla_x u_2(t)|^{r-1})\,\left(|v|^\beta + |\nabla_x v|^\beta\right) \end{cases} \tag{5.35}$$

where the constant C depends only on $\|u, \Omega\|_{2-\delta,2,b}$.

Let us multiply the equation (5.34) by $e^{-\varepsilon|x-x_0|}\theta$ and integrate over $x \in \Omega$. Then we get

$$
\begin{aligned}
(5.36)\quad & \partial_t(|\theta(t)|^2, e^{-\varepsilon|x-x_0|}) + \left(|\nabla_x\theta(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \lambda_0\left(|\theta|^2, e^{-\varepsilon|x-x_0|}\right) \le \\
& \le \left(|L_1(t)|\cdot|\theta(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \left(|L_2(t)|\cdot|\nabla_x\theta(t)|\cdot|\theta(t)|, e^{-\varepsilon|x-x_0|}\right) + \\
& \left(|L_1(t)-\widehat{L}_1(t)||v(t)||\theta(t)|, e^{-\varepsilon|x-x_0|}\right) + \left(|L_2(t)-\widehat{L}_2(t)||\nabla_x v(t)||\theta(t)|, e^{-\varepsilon|x-x_0|}\right)
\end{aligned}
$$

Arguing as in the proof of Theorem 5.1 we obtain the following estimates for the first two terms of the right–hand side of (5.36)

$$
\begin{aligned}
(5.37)\quad & \left(|L_1(t)|\cdot|\theta(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \left(|L_2(t)||\nabla_x\theta(t)||\theta(t)|, e^{-\varepsilon|x-x_0|}\right) \\
& \le C_\mu\left(|\theta(t)|^2, e^{-\varepsilon|x-x_0|}\right) + \mu\left(|\nabla_x\theta(t)|^2, e^{-\varepsilon|x-x_0|}\right)
\end{aligned}
$$

for an arbitrary positive μ. Let us estimate the other two terms. Due to the inequality (5.35) we get

$$
\begin{aligned}
(5.38)\quad & \left(|L_2(t)-\widehat{L}_2(t)||\nabla_x v(t)||\theta(t)|, e^{-\varepsilon|x-x_0|}\right) \le \\
& \le C\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}(|\nabla_x u|^{r-1}+1)(|v|^\beta + |\nabla_x v|^\beta)|\nabla_x v||\theta|\,dx
\end{aligned}
$$

Let us estimate the most complicated term in the right–hand side of (5.38) using Holder inequality with the exponents 6, $2/(1+\beta)$ and l $(1/l = 1/3 - \beta/2)$, the estimate (5.9), the fact that $l(r-1) \le 6(r-1) < 3r$ if $\beta \le 1$ and Sobolev embedding $W^{1,2} \subset L^6$. Indeed,

$$
\begin{aligned}
(5.39)\quad & \int_0^T \left(|\nabla_x u(t)|^{r-1}|\nabla_x v(t)|^{1+\beta}, |\theta(t)|e^{-\varepsilon|x-x_0|}\right) dt \le \\
& \le C\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\int_0^T \| \,|\nabla_x v(t)|^{1+\beta}|\nabla_x u(t)|^{r-1}\,|\theta(t)|, V_x\|_{0,1}\,dt\,dx \le \\
& \le C_1 \sup_{t\in[0,T]}\{\|u,\Omega\|^{r-1}_{b,1,l(r-1)}\}\times \\
& \int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\left(\int_0^T \|\theta(t), V_x\|^2_{1,2}\,dt\right)^{1/2}\left(\int_0^T \|v(t), V_x\|^2_{1,2}\,dt\right)^{(1+\beta)/2} dx \le \\
& \le C_\mu\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\left(\int_0^T \|v(t), V_x\|^2_{1,2}\,dt\right)^{1+\beta} dx + \\
& \qquad + \mu\int_0^T\int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\|\theta, V_x\|^2_{1,2}\,dx\,dt
\end{aligned}
$$

Using the estimate (5.16) and the estimate (1.3) we get

$$
\begin{aligned}
& \int_{x\in\Omega} e^{-\varepsilon|x-x_0|}\left(\int_0^T \|v(t), V_x\|^2_{1,2}\,dt\right)^{(1+\beta)} dx \le \\
& \qquad \le Ce^{C_1 T}\left(|u_1(0)-u_2(0)|^2, e^{-\varepsilon_1|x-x_0|}\right)^{1+\beta}
\end{aligned}
$$

where $\varepsilon_1 = \frac{\varepsilon}{1+\beta}$. Thus,

$$\int_0^T \left(|\nabla_x u(t)|^{r-1} |\nabla_x v(t)|^{1+\beta}, |\theta(t)| e^{-\varepsilon|x-x_0|} \right) dt \leq$$
$$\leq C_\mu e^{C_1 T} \left(|u_1(0) - u_2(0)|^2, e^{-\varepsilon_1|x-x_0|} \right)^{1+\beta} + \mu \int_0^T \left(|\nabla_x \theta(t)|^2, e^{-\varepsilon|x-x_0|} \right) dt$$

Estimating the rest of the terms in the expression (5.38) in the same way we will get

$$\int_0^T \left(|L_2(t) - \widehat{L}_2(t)||\nabla_x v(t)||\theta(t)|, e^{-\varepsilon|x-x_0|} \right) dt \leq \tag{5.40}$$
$$\leq C_\mu e^{C_1 T} \left(|u_1(0) - u_2(0)|^2, e^{-\varepsilon_1|x-x_0|} \right)^{1+\beta} + \mu \int_0^T \left(|\nabla_x \theta(t)|^2, e^{-\varepsilon|x-x_0|} \right) dt$$

The third term in the right–hand side of (5.36) can be estimated analogously

$$\int_0^T \left(|L_1(t) - \widehat{L}_1(t)||v(t)||\theta(t)|, e^{-\varepsilon|x-x_0|} \right) dt \leq \tag{5.41}$$
$$\leq C_\mu e^{C_1 T} \left(|u_1(0) - u_2(0)|^2, e^{-\varepsilon_1|x-x_0|} \right)^{1+\beta} + \mu \int_0^T \left(|\nabla_x \theta(t)|^2, e^{-\varepsilon|x-x_0|} \right) dt$$

Integrating the inequality (5.36) over $t \in [0,T]$ and replacing the right–hand side of it by the estimates (5.37),(5.40) and (5.41) for sufficiently small $\mu > 0$ we obtain the estimate (5.33). Theorem 5.2 is proved.

Proposition 5.5. *Under the assumptions of the previous theorem the following estimate is valid*

$$\|u_1(T) - u_2(T) - w(T)\|_{2-\delta,2} \leq \tag{5.42}$$
$$\leq C e^{C_1 T} \left(\int_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u_1(0) - u_2(0), B_x^1\|_{2-\delta,2}^2 \, dx \right)^{1+\beta}$$

Where the constants C and C_1 depends only on $\|u_i(0), \Omega\|_{2-\delta,2,b}$.

The proof of this Proposition is analogous to the proof of Propositions 5.1 and 5.2 only instead of the estimate (5.16) we should use the estimate (5.33).

Part 2. The attractors.

This part of the paper is devoted to a study of the long-time behaviour of solutions of (0.1) in weighted Sobolev spaces.

The attractors $\mathcal{A}_{\langle\alpha\rangle}$ of the problem (0.1) in the scale of weighted Sobolev spaces $W^{s,p}_{\langle\alpha\rangle}$ are constructed in Section 6.

In Section 7 we present a new construction of the infinite dimensional unstable manifold which does not require hyperbolicity of the corresponding equilibrium point.

This construction allows us to construct in Section 8 a large number of equations of the form (0.1) which possess the infinite dimensional attractors.

The finite dimensionality of the attractors in the case where $\alpha \geq 0$ will be proved in Section 9.

§6 Attractors of the nonlinear equation in weighted Sobolev spaces.

In this Section we obtain the existence of the attractor for the equation (3.1) in weighted Sobolev spaces. Following the tradition we restrict ourselves by considering only the case of power weighted Sobolev spaces $W^{s,p}_{\langle\alpha\rangle}$ and consequently we suppose that

$$g \in L^2_{\langle\alpha\rangle}(\Omega) \quad \text{for a some } \alpha \in \mathbb{R} \tag{6.1}$$

Definition 6.1. *Let us define the phase space for the problem* (3.1) *by the following expression*

$$\Phi_{\langle\alpha\rangle}(\Omega) = W^{2-\delta,2}_{\langle\alpha\rangle}(\Omega) \text{ if } \alpha \geq 0 \tag{6.2}$$

and

$$\Phi_{\langle\alpha\rangle}(\Omega) = C_{\langle\alpha/2\rangle}(\Omega) \cap L^{(2,\infty)}_{\{\alpha\}}(\Omega) \cap W^{2-\delta,2}_{\langle K\alpha\rangle}(\Omega) \text{ if } \alpha < 0 \tag{6.3}$$

Here the constant $\delta > 0$ and $K \geq 1$ were defined in Theorem 3.3.

Note that all results of this Section can be straightforwardly extended to the case $g \in L^2_\phi(\Omega)$ for any weight function ϕ with the rate of growth μ if μ is small enough. The phase space Φ_ϕ in this case is given by

$$\Phi_\phi(\Omega) = C_{\phi/2}(\Omega) \cap L^{(2,\infty)}_\phi(\Omega) \cap W^{2-\delta,2}_{\widehat{\phi}}(\Omega)$$

with $\widehat{\phi}(x) = \min\{\phi(x), \phi(x)^K\}$. But for simplicity we consider below only the case where $\phi(x) \equiv \phi_{\langle\alpha\rangle}(x) = (1+|x|^2)^{\alpha/2}$.

Theorem 6.1. *Let the conditions of Theorem 3.3 hold,* (6.1) *be valid and $u_0 \in \Phi_{\langle\alpha\rangle}(\Omega)$ for some $\alpha \in \mathbb{R}$. Then any solution u of the problem* (3.1) *satisfied* (4.8) *belongs to the space $C_b(\mathbb{R}_+, \Phi_{\langle\alpha\rangle}(\Omega))$ and the following estimate is valid for some $\gamma > 0$*

$$\begin{aligned} \|u(T), \Omega\|^2_\Phi \leq C(\|u(0), \Omega\|^2_\Phi + \|u(0), \Omega\|^{2K}_\Phi)e^{-\gamma T} + \\ + C(\|g, \Omega\|^2_{\langle\alpha\rangle,0,2} + \|g, \Omega\|^{2K}_{\langle\alpha\rangle,0,2}) \end{aligned} \tag{6.4}$$

Moreover the following estimate holds uniformly with respect to $R \to \infty$ for some $\beta, \gamma > 0$

$$\begin{aligned} \|u(T), \Omega \cap \{|x| > R\}\|^2_\Phi \leq C(\|u(0), \Omega\|^2_\Phi + \|u(0), \Omega\|^{2K}_\Phi)e^{-\gamma T - \beta R} + \\ + C(\|u(0), \Omega \cap \{|x| > R/2\}\|^2_\Phi + \|u(0), \Omega \cap \{|x| > R/2\}\|^{2K}_\Phi)e^{-\gamma T} + \\ + C_1(\|g, \Omega\|^2_{\langle\alpha\rangle,0,2} + \|g, \Omega\|^{2K}_{\langle\alpha\rangle,0,2})e^{-\beta R} + \\ + C_2(\|g, \Omega \cap \{|x| > R/2\}\|^2_{\langle\alpha\rangle,0,2} + \|g, \Omega \cap \{|x| > R/2\}\|^{2K}_{\langle\alpha\rangle,0,2}) \end{aligned} \tag{6.5}$$

Proof. Let us suppose first that $\alpha < 0$. Then multiplying the estimate (3.4) by $\phi_{\langle\alpha\rangle}(x_0)$ and taking $\sup_\Omega$ of both sides of the obtained inequality we will have

$$\begin{aligned} \sup_{x_0 \in \Omega}\{\phi_{\langle\alpha\rangle}(x_0)|u(T, x_0)|^2\} \leq \\ \leq C \sup_{x_0 \in \Omega} \left\{ \phi_{\langle\alpha\rangle}(x_0) \sup_{x \in \Omega}\{e^{-\varepsilon|x-x_0|}|u(0,x)|^2\} \right\} e^{-\gamma T} + \\ + C \sup_{x_0 \in \Omega}\{\phi_{\langle\alpha\rangle}(x_0)(|g|^2, e^{-\varepsilon|x-x_0|})\} \end{aligned}$$

Applying the estimates (1.6) and (1.3) with $q = \infty$ to the obtained inequality we will have

$$\|u(T)\|^2_{C_{\langle \alpha/2\rangle}(\Omega)} \le C\|u(0)\|^2_{C_{\langle \alpha/2\rangle}(\Omega)} e^{-\gamma T} + C\|g\|^2_{L^2_{\langle\alpha\rangle}(\Omega)}$$

which coincides with the part of (6.4) for the $C_{\langle \alpha/2\rangle}$–norm.

Analogously, multiplying the estimate (3.13) by $\phi_{\langle\alpha\rangle}(x_0)$ integrating over Ω and using the estimate (1.3) we obtain the part of (6.4) for the $L^{(2,\infty)}_{\langle\alpha\rangle}$–norm.

Multiplying the estimate (3.14) by $\phi_{\langle\alpha\rangle}(x_0)^K$, integrating over Ω and using the evident inequality $\phi_{\langle\alpha\rangle} \ge \phi^K_{\langle\alpha\rangle}$ (since $\alpha < 0$) we will have

$$\begin{aligned}
&\int_{x_0\in\Omega} \phi_{\langle\alpha\rangle}(x_0)^K \|u(T), \Omega\cap B^1_{x_0}\|^2_{2-\delta,2}\, dx_0 \le \\
&\quad \le Ce^{-\gamma T}\left(\int_{x_0\in\Omega} \phi_{\langle K\alpha\rangle}(x_0) \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u_0, \Omega\cap B^1_x\|^2_{2-\delta,2}\, dx\, dx_0 + \right. \\
&\qquad + \int_{x_0\in\Omega} \phi_{\langle\alpha\rangle}(x_0) \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u_0, \Omega\cap B^1_x\|^2_{0,\infty}\, dx\, dx_0 + \\
&\quad \left. + \int_{x_0\in\Omega} \phi_{\langle\alpha\rangle}(x_0)^K \left\{ \int_{x\in\Omega} e^{-\varepsilon|x-x_0|} \|u_0, \Omega\cap B^1_x\|^2_{0,\infty}\, dx \right\}^K dx_0 \right) e^{-\gamma T} + \\
&\qquad + C\int_{x_0\in\Omega} \phi_{\langle\alpha\rangle}(x_0) \int_{x\in\Omega} |g(x)|^2 e^{-\varepsilon|x-x_0|} dx\, dx_0 + \\
&\qquad\quad + C\int_{x_0\in\Omega} \phi_{\langle\alpha\rangle}(x_0)^K \left(\int_{x\in\Omega} |g(x)|^2 e^{-\varepsilon|x-x_0|} dx \right)^K dx_0
\end{aligned}$$

Applying the estimate (1.3) to this inequality we obtain

$$\begin{aligned}
&\|u(T), \Omega\|^2_{\langle K\alpha\rangle, 2-\delta,2} \le \\
&\quad \le Ce^{-\gamma T}\left(\|u(0), \Omega\|^2_{\langle K\alpha\rangle,2-\delta,2} + \|u(0),\Omega\|^2_{\langle\alpha\rangle,(2,\infty)} + \|u(0),\Omega\|^{2K}_{\langle\alpha\rangle,(2,\infty)} \right) + \\
&\qquad\qquad + C_1\left(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2} \right)
\end{aligned}$$

which coincides with the $W^{2-\delta,2}_{\langle K\alpha\rangle}$-part of the estimate (6.4). Thus, the estimate (6.4) is proved when $\alpha < 0$.

Let us suppose now that $\alpha \ge 0$. In this case $\phi_{\langle\alpha\rangle} \le \phi^K_{\langle\alpha\rangle} = \phi_{\langle K\alpha\rangle}$ hence we could multiply the inequality (3.14) by $\phi_{\langle\alpha\rangle}(x_0)$ and integrate over Ω. Then after using the estimates (1.3) and Sobolev embedding theorem $W^{2-\delta,2} \subset C$ we obtain the inequality (6.4) for $\alpha \ge 0$.

The estimate (6.5) could be proved analogously, only instead of integrating over Ω we should integrate over $\Omega\cap\{|x| > R\}$ and use the estimates (1.4) and (1.7) instead of (1.3) and (1.6).

Thus we proved that $u \in L^\infty(\mathbb{R}_+, \Phi_{\langle\alpha\rangle}(\Omega))$.

The continuity of u follows immediately from the continuity of $u\big|_{\Omega\cap B^R_0}$ for every $R > 0$, the estimate (6.5) and the result of Theorem 1.4. Theorem 6.1 is proved. $\square$

Proposition 6.1. *Let the conditions of the previous Theorem be valid. Then* $u \in C^{1-\delta/2}(\mathbb{R}_+, L^2_{\langle P\alpha\rangle}(\Omega))$ *where* $P = 1$ *for* $\alpha \geq 0$ *and* $P = K$ *for* $\alpha < 0$ *and the following estimate is valid:*

$$\|u\|^2_{C^{1-\delta/2}([T,T+1],L^2_{\langle P\alpha\rangle}(\Omega))} \leq C(\|u(0),\Omega\|^2_\Phi + \|u(0),\Omega\|^{2K}_\Phi)e^{-\gamma T} + \\ + C(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2}) \tag{6.6}$$

Moreover the following estimate is valid uniformly as $R \to \infty$ *for some* $\beta, \gamma > 0$

$$\|u\|^2_{C^{1-\delta/2}([T,T+1],L^2_{\langle P\alpha\rangle}(\Omega\cap\{|x|>R\}))} \leq C(\|u(0),\Omega\|^2_\Phi + \\ + \|u(0),\Omega\|^{2K}_\Phi)e^{-\gamma T-\beta R} + \\ + C(\|u(0),\Omega\cap\{|x|>R/2\}\|^2_\Phi + \|u(0),\Omega\cap\{|x|>R/2\}\|^{2K}_\Phi)e^{-\gamma T} + \\ + C_1(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2})e^{-\beta R} + \\ + C_2(\|g,\Omega\cap\{|x|>R/2\}\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\cap\{|x|>R/2\}\|^{2K}_{\langle\alpha\rangle,0,2}). \tag{6.7}$$

The proof of this Proposition is analogous to the proof of Theorem 6.1, only instead of the estimate (3.14) we should use its analogue for the $C^{1-\delta/2}([T,T+1],L^2)$–norm, obtained in Corollary 3.3.

Now we are in position to study the long-time behaviour for the solutions of the problem (3.1). As we mentioned in the previous Section we do not have uniqueness when $\alpha < 0$, in general. Therefore we will use the concept of a trajectory attractor developed in [9], [10], [34].

Definition 6.2. *Let us define the space* $\Theta^+_{\langle\alpha\rangle}$ *by the following expression*

$$\Theta^+_{\langle\alpha\rangle} = \Theta^+_{\langle\alpha\rangle}(\mathbb{R}_+,\Omega) = C_{\rm loc}(\mathbb{R}_+,\Phi_{\langle\alpha\rangle}(\Omega)) \cap C^{1-\delta/2}_{\rm loc}(\mathbb{R}_+, L^2_{\langle P\alpha\rangle}(\Omega))) \tag{6.8}$$

where the constant P *is the same as in previous Proposition.*

Evidently the space (6.8) *is a metrizable* F*–space and the system of seminorms in it is given by the following expression*

$$\|u,[T,T+1],\Omega\|_{\Theta^+} \equiv \|u\big|_{[T,T+1]}\|_{\Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega)},\ T \in \mathbb{R}_+ \tag{6.9}$$

Here

$$\Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega) = C([T,T+1],\Phi_{\langle\alpha\rangle}) \cap C^{1-\delta/2}([T,T+1],L^2_{\langle P\alpha\rangle}(\Omega))) \tag{6.10}$$

The following assertion is an immediate corollary of Theorem 6.1 and Proposition 6.1

Corollary 6.1. *Let* u *be a solution and let the assumptions of Theorem 6.1 be valid. Then* $u \in \Theta^+_{\langle\alpha\rangle}$ *and*

$$\|u,[T,T+1],\Omega\|^2_{\Theta^+} \leq C(\|u(0),\Omega\|^2_\Phi + \|u(0),\Omega\|^{2K}_\Phi)e^{-\gamma T} + \\ + C(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2}). \tag{6.11}$$

Moreover the following estimate is valid uniformly with respect to $R \to \infty$ *for some* $\beta, \gamma > 0$

$$
\begin{aligned}
(6.12)\quad \|u, [T, T+1], \Omega \cap \{|x| > R\}\|^2_{\Theta^+} \leq \\
\leq C(\|u(0), \Omega\|^2_\Phi + \|u(0), \Omega\|^{2K}_\Phi) e^{-\gamma T - \beta R} + \\
+ C(\|u(0), \Omega \cap \{|x| > R/2\}\|^2_\Phi + \|u(0), \Omega \cap \{|x| > R/2\}\|^{2K}_\Phi) e^{-\gamma T} + \\
+ C_1(\|g, \Omega\|^2_{\langle\alpha\rangle,0,2} + \|g, \Omega\|^{2K}_{\langle\alpha\rangle,0,2}) e^{-\beta R} + \\
+ C_2(\|g, \Omega \cap \{|x| > R/2\}\|^2_{\langle\alpha\rangle,0,2} + \|g, \Omega \cap \{|x| > R/2\}\|^{2K}_{\langle\alpha\rangle,0,2})
\end{aligned}
$$

Indeed the estimates (6.11) and (6.12) follow immediately from the estimates (6.4)–(6.7)

Let us denote by $K^+_{\langle\alpha\rangle}$ the set of all solutions of the problem (3.1) with an arbitrary initial condition $u_0 \in \Phi_{\langle\alpha\rangle}(\Omega)$. We endow the set $K^+_{\langle\alpha\rangle}$ by the topology induced by the embedding

$$
(6.13)\qquad K^+_{\langle\alpha\rangle} \subset \Theta^+_{\langle\alpha\rangle}
$$

Since the equation (3.1) is translation invariant with respect to t then the semigroup $\{T_s, s \geq 0\}$ of positive shifts along the t axis acts in the space $K^+_{\langle\alpha\rangle}$, i.e.

$$
(6.14)\qquad T_s K^+_{\langle\alpha\rangle} \subset K^+_{\langle\alpha\rangle}\ ; \quad s \geq 0\ ,\ (T_s u)(t) \equiv u(t+s)
$$

Definition 6.3. *The attractor* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ *of the semigroup* T_s *acting in the metric space* $K^+_{\langle\alpha\rangle}$ *is called the trajectory attractor of the equation* (3.1)*, i.e. the set* $\mathcal{A}^{tr}_{\langle\alpha\rangle} \subset K^+_{\langle\alpha\rangle}$ *is the trajectory attractor for the problem* (3.1) *if*

1. *The set* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ *is compact in* $K^+_{\langle\alpha\rangle}$.
2. *The set* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ *is strictly invariant under the* T_s*–action, i.e.* $T_s \mathcal{A}^{tr}_{\langle\alpha\rangle} = \mathcal{A}^{tr}_{\langle\alpha\rangle}$ *for every* $s \geq 0$.
3. *The set* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ *is an attracting set for the semigroup* T_s*, i.e. for every bounded subset of solutions* $B \subset K^+_{\langle\alpha\rangle}$ *and for every neighborhood* $\mathcal{O}(\mathcal{A}^{tr}_{\langle\alpha\rangle})$ *of the set* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ *there exists the number* $S = S(B, \mathcal{O})$ *such that*

$$
(6.15)\qquad T_s B \subset \mathcal{O}(\mathcal{A}^{tr}_{\langle\alpha\rangle}) \text{ for every } s \geq S
$$

Theorem 6.2. *Let the conditions of Theorem 6.1 be valid. Then the equation* (3.1) *possesses the trajectory attractor in the space* $\Theta^+_{\langle\alpha\rangle}$ *which could be represented in the following form*

$$
(6.16)\qquad \mathcal{A}^{tr}_{\langle\alpha\rangle} = \Pi_+ K_{\langle\alpha\rangle}
$$

where $K_{\langle\alpha\rangle}$ *means the set of all bounded solutions* $u \in C_b(\mathbb{R}, \Phi_{\langle\alpha\rangle}(\Omega))$ *and* Π_+ *– the restriction operator to the semiaxis* $\mathbb{R}_+$.

Proof. The proof of this Theorem is based on the theorem from [2], [19], [32] which gives sufficient conditions for the existence of an attractor for abstract semigroups. To apply this theorem to our semigroup T_s we should check the following conditions:

1. The semigroup $T_s : K^+_{\langle\alpha\rangle} \to K^+_{\langle\alpha\rangle}$ is continuous for every fixed $s \geq 0$.
2. The semigroup $T_s : K^+_{\langle\alpha\rangle} \to K^+_{\langle\alpha\rangle}$ possesses a compact attracting set P in the space $\Theta^+_{\langle\alpha\rangle}$.

The continuity of T_s is evident. Thus it remains to verify only the existence of the compact attracting set. To this end we need the following Lemma.

Lemma 6.1. *Let the above assumptions be valid. Then*

$$u \in C([T,T+1], W^{2-\delta/2}_{\langle P\alpha\rangle}(\Omega)) \cap C^{1-\delta/4}([T,T+1], L^2_{\langle P\alpha\rangle}(\Omega)) \equiv W_T(\Omega)$$

for any $T \geq 1$. Here P is the same as in Proposition 6.1. Moreover, the following estimate is valid

$$\|u\|^2_{W_T(\Omega)} \leq C_3(\|u(0),\Omega\|^2_\Phi + \|u(0),\Omega\|^{2K}_\Phi)e^{-\gamma T} + \\ + C_4(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2}) \tag{6.17}$$

The proof of this Lemma is the same as the proof of Theorem 6.1 only instead of the estimate (3.14) one should use the estimate (3.31).

The end of the proof of Theorem 6.2. Let us define the set

$$P_1 = \big\{u \in \Theta^+_{\langle\alpha\rangle} \,:\, \|u,[T,T+1],\Omega\cap\{|x|>R\}\|^2_{\Theta^+} \leq \\ \leq 2M^2C_1(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2})e^{-\beta R} + \\ + 2M^2C_2(\|g,\Omega\cap\{|x|>R/2\}\|^2_{\langle\alpha\rangle,0,2} + \\ + \|g,\Omega\cap\{|x|>R/2\}\|^{2K}_{\langle\alpha\rangle,0,2}),\ \forall T\in\mathbb{R}_+;\ R\in\mathbb{R}_+\big\} \tag{6.18}$$

Here the constants C_1, C_2 and β are the same as in the estimate (6.12) and $M>0$ is a sufficiently large positive number which will be defined below. Let us introduce also the set

$$P_2 = \big\{u \in \Theta^+_{\langle\alpha\rangle} \,:\, \|u\|^2_{W_T(\Omega)} \leq 2M^2C_4(\|g,\Omega\|^2_{\langle\alpha\rangle,0,2} \\ + \|g,\Omega\|^{2K}_{\langle\alpha\rangle,0,2})\ ,T\in\mathbb{R}_+\big\} \tag{6.19}$$

where the constant C_4 is the same as in (6.17).

We claim that the set $P = P_1 \cap P_2 \subset \Theta^+_{\langle\alpha\rangle}$ with a sufficiently large M is the compact attracting set for the semigroup T_s on $K^+_{\langle\alpha\rangle}$. Indeed, since

$$\Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega\cap B^R_0) \subset\subset W_T(\Omega\cap B^R_0)$$

for every $T\geq 0$ and $R>0$ then due to (6.19) the restriction $P\big|_{x\in\Omega\cap B^R_0}$ of the set P to any ball B^R_0 is compact in $\Theta^+_{\langle\alpha\rangle}\big|_{x\in\Omega\cap B^R_0}$. Moreover, since $g\in L^2_{\langle\alpha\rangle}(\Omega)$ then (6.18) implies that

$$\|P,[T,T+1],\Omega\cap\{|x|>R\}\|_{\Theta^+} \to 0 \text{ when } R\to\infty. \tag{6.20}$$

Thus, the set P is compact in $\Theta^+_{\langle\alpha\rangle}(\Omega)$. So, it remains to check that P is the attracting set for T_s on $K^+_{\langle\alpha\rangle}$. To this end we introduce the family of cut-off functions $\psi_R(x)\in C^\infty_0(\mathbb{R}^3)$, $0\leq\psi\leq 1$, $R>1$ such that

$$\psi_R(x)=1 \text{ if } |x|\leq R-1,\ \psi_R(x)=0 \text{ if } |x|>R \text{ and } \|\psi_R\|_{C^2}\leq C \tag{6.21}$$

with C independent of R and the corresponding family of cut-off operators $\Pi_R v \equiv \psi_R v$. Then since C in (6.21) is independent of R then there exists a constant M such that the following estimates are valid uniformly with respect to $R>1$ and $T\geq 0$

$$\|\Pi_R\|_{W_T(\Omega)\to W_T(\Omega)} \leq M\,,\ \|\Pi_R\|_{\Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega)\to\Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega)} \leq M \tag{6.22}$$

Let us consider now the arbitrary bounded subset $B \subset K^+_{\langle\alpha\rangle}$. It means particulary that the set $B_0 = \{u(0) : u \in B\}$ is bounded in the phase space $\Phi_{\langle\alpha\rangle}$. Moreover, it follows from the estimates (6.12), (6.17) and (6.22) and from our choice of the set P that for every $R > 1$ there exists $S = S(R, B)$ such that

$$\Pi_R(T_s B) \subset P \text{ for } s \geq S$$

and consequently for $s \geq S$ and any $T \in \mathbb{R}_+$

$$\begin{aligned}(6.23)\quad \operatorname{dist}\{T_s B, P\}_{\Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega)} &\leq \|(1-\Pi_R)(T_s B), [T, T+1], \Omega\|_{\Theta^+} \leq \\ &\leq M\|T_s B, [T, T+1], \Omega \cap \{|x| > R-1\}\|_{\Theta^+} = \\ &= M\|B, [T+s, T+s+1], \Omega \cap \{|x| > R-1\}\|_{\Theta^+}\end{aligned}$$

Here we denote by $\operatorname{dist}\{X, Y\}_H$ the Hausdorff distance between the sets $X, Y \subset H$, i.e.

$$(6.24)\qquad \operatorname{dist}\{X, Y\}_H = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_H$$

Note, that the estimate (6.12) together with the boundedness of B_0 imply that the right-hand side of (6.23) could be taken arbitrary small by choosing R and s large enough. Thus, P is a compact attracting set for the semigroup $\{T_s, s \geq 0\}$ on $K^+_{\langle\alpha\rangle}$. This proves Theorem 6.2. $\square$

One might naturally ask the relationship between the trajectory attractor and a global attractor in the case of uniqueness.

Consider now the case $\alpha \geq 0$ and suppose that the conditions of Theorem 5.1 are valid. Then the problem (3.1) has the unique solution $u \in C(\mathbb{R}_+, \Phi_{\langle\alpha\rangle})$. Hence one could define the semigroup $S_t : \Phi_{\langle\alpha\rangle} \to \Phi_{\langle\alpha\rangle}$ by the formula

$$(6.25)\qquad S_t u(0) = u(t)\ ,\ u \text{ is the solution of the problem (3.1)}$$

Corollary 6.2. *Let $\alpha \geq 0$ and the conditions of Theorem 5.1 be valid. Then the semigroup* (6.25) *possesses a (global) attractor $\mathcal{A}_{\langle\alpha\rangle} = \mathcal{A}^{gl}_{\langle\alpha\rangle}$ in the space $\Phi_{\langle\alpha\rangle}$. Moreover*

$$(6.26)\qquad \mathcal{A}^{gl}_{\langle\alpha\rangle} = \mathcal{A}^{tr}_{\langle\alpha\rangle}\big|_{t=0}$$

Indeed, multiplying the estimate (5.22) by $\phi_{\langle\alpha\rangle}(x_0)$ and integrating over $x_0 \in \Omega$ we obtain after simple calculations that

$$\|u_1 - u_2, [T, T+1], \Omega\|^2_{\Theta^+} \leq Ce^{KT}\|u_1(0) - u_2(0)\|^2_{\Phi_{\langle\alpha\rangle}}$$

for every two solutions u_1 and u_2 and for certain positive constants C and K (which depend in general on the $\Phi_{\langle\alpha\rangle}$ norms of the initial values $u_1(0)$ and $u_2(0)$). This estimate implies that the trace operator $\Pi_0 : \Theta^+_{\langle\alpha\rangle} \to \Phi_{\langle\alpha\rangle}$ defined by the formula $\Pi_0 u = u(0)$ realizes a Lipshitz continuous homeomorphism between the spaces $K^+_{\langle\alpha\rangle}$ and $\Phi_{\langle\alpha\rangle}$. Moreover under the conditions of Theorem 5.2 this homeomorphism will be in fact a C^1-diffeomorphism (it can be derived analogously, using the estimate (5.42)). Thus, the semigroups S_h and T_h, defined by (6.25) and (6.14) correspondingly, are conjugate by homeomorphism and consequently the assertion of Corollary 6.2 follows immediately from Theorem 6.2.

Remark 6.1. *In the case when we do not have the uniqueness theorem for the equation* (3.1). *The set* $\mathcal{A}^{gl}_{\langle\alpha\rangle}$, *defined by* (6.26), *could be interpreted as the attractor of a multivalued semigroup, defined by* (6.25) *(see* [3], [5]*).*

Let us consider now the dependency on α for the attractors $\mathcal{A}^{tr}_{\langle\alpha\rangle}$.

Theorem 6.3. *Let the assumptions of Theorem 3.3 be valid and let*

$$g \in L^2_{\langle\alpha\rangle}(\Omega). \tag{6.27}$$

Then any solution $u \in \cap_{\varepsilon>0} C_b(\mathbb{R}, W^{2-\delta,2}_{\{\varepsilon\}}(\Omega))$ *belongs to the space*

$$u \in C_b(\mathbb{R}, \Phi_{\langle\alpha\rangle}(\Omega)) \tag{6.28}$$

Moreover the following estimates are valid uniformly with respect to $t \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}$*:*

$$\begin{cases} \|u(t), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} \le C\left((|g|^2, e^{-\varepsilon|x-x_0|}) + (|g|^2, e^{-\varepsilon|x-x_0|})^K\right) \\ |u(t,x_0)|^2 \le C(|g|^2, e^{-2\varepsilon|x-x_0|}) \end{cases} \tag{6.29}$$

Here K *is the same as in Theorem 3.3.*

Proof. The estimates (6.29) follow from (3.14) and (3.4) by passing to the limit $T \to \infty$. The assertion (6.28) follows from the estimates (6.29) as in the proof of Theorem 6.1.

Corollary 6.3. *Let the assumptions of Theorem 3.3 be valid and let*

$$g \in L^2_b(\Omega). \tag{6.30}$$

Then any solution $u \in \cap_{\varepsilon>0} C_b(\mathbb{R}, W^{2-\delta,2}_{\{\varepsilon\}}(\Omega))$ *belongs to the space*

$$u \in C_b(\mathbb{R}, W^{2-\delta,2}_b(\Omega)). \tag{6.31}$$

Moreover the following estimate is valid uniformly with respect to $t \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}$*:*

$$\|u(t), \Omega \cap B^1_{x_0}\|^2_{2-\delta,2} \le C\left(\|g, \Omega\|^2_{0,2,b} + \|g, \Omega\|^{2K}_{0,2,b}\right). \tag{6.29}$$

Here K *is the same as in Theorem 3.3.*

Corollary 6.3 is evident and we leave its proof to the pedant reader.

Corollary 6.4. *Let the conditions of Theorem 6.3 be valid and let* $g \in L^2_{\langle\alpha_1\rangle}(\Omega) \cap L^2_{\langle\alpha_2\rangle}(\Omega)$. *Then*

$$\mathcal{A}^{tr}_{\langle\alpha_1\rangle} = \mathcal{A}^{tr}_{\langle\alpha_2\rangle} \text{ and } \mathcal{A}^{gl}_{\langle\alpha_1\rangle} = \mathcal{A}^{gl}_{\langle\alpha_2\rangle} \tag{6.32}$$

Indeed the equalities (6.32) follow immediately from the representation (6.16) and from the assertion of the previous Theorem.

Remark 6.2. *The equalities* (6.32) *mean that the attractors* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ *are independent of* α. *Thus we sometimes omit below the index* $\langle\alpha\rangle$ *and write* $\mathcal{A}^{tr}$ *instead of* $\mathcal{A}^{tr}_{\langle\alpha\rangle}$.

§7 The unstable sets.

In this Section we study the unstable sets of the equilibrium points for the equation (3.1). For simplicity we restrict ourselves to a scalar case ($k = 1$) and assume that the nonlinear term f is independent of $\nabla_x u$,

$$f(u, \nabla_x u) \equiv f(u) \in C^3(\mathbb{R}). \tag{7.1}$$

Let z_0 be an equilibrium point for the problem (3.1), i.e.

$$\begin{cases} -\Delta_x z_0 + f(z_0) + \lambda_0 z_0 = g \\ z_0\big|_{\partial\Omega} = 0 \end{cases} \tag{7.2}$$

Moreover the right–hand side g is assumed to belong to the space $L^2_b(\Omega)$ and consequently $z_0 \in W^{2,2}_b(\Omega)$.

Let us consider a function $w = u - z_0$. Then

$$\begin{cases} \partial_t w - \Delta_x w + f'(z_0)w + \lambda_0 w = -[f(w + z_0) - f(z_0) - f'(z_0)w] \equiv F(w) \\ w\big|_{\partial\Omega} = 0. \end{cases} \tag{7.3}$$

Notice, that by the definition of F,

$$F(0) = 0 \text{ and } F'(0) = 0 \text{ consequently } F(w) = \Phi(x, w)w^2 \tag{7.4}$$

where $\Phi \in C(\overline{\Omega}, C^1(\mathbb{R}))$.

Definition 7.1. *The unstable set of the equilibrium point z_0 is defined to be the following set:*

$$\begin{aligned} \mathcal{M}^+(z_0) = \{w_0 \in W^{1,2}(\Omega) \cap C(\overline{\Omega}) : \exists w \in C(\mathbb{R}_-, W^{1,2}(\Omega) \cap C(\overline{\Omega})), \\ w(0) = w_0, \quad w\text{- satisfies (7.3) and } \lim_{t\to-\infty} w(t) = 0\} \end{aligned} \tag{7.5}$$

The main task of this Section is to prove that under natural assumptions the unstable set $\mathcal{M}^+(z_0)$ contains C^1-manifolds of an arbitrary large dimension. We shall use this result in the next Section to construction examples of equations (3.1) with infinite dimensional attractors.

We are going to apply the implicit function theorem to the equation (7.3). To do it we study first the linear nonhomogeneous problem of view (7.3)

$$\partial_t w + Lw = h(t) \tag{7.6}$$

Here $L = -\Delta_x + f'(z_0) + \lambda_0$. For every $\gamma \in \mathbb{R}$ we introduce the spaces

$$\begin{cases} H = W^{1,2}(\Omega) \cap C(\overline{\Omega}) \\ C_\gamma(H) = \{u \in C(\mathbb{R}_-, H) : \ \|u\|_\gamma = \sup_{t\in\mathbb{R}_-} e^{\gamma t}\|u(t)\|_H < \infty\} \\ C_\gamma(L^2) = \{u \in C(\mathbb{R}_-, L^2(\Omega)) : \ \|u\|_\gamma = \sup_{t\in\mathbb{R}_-} e^{\gamma t}\|u(t)\|_{L^2} < \infty\} \end{cases} \tag{7.7}$$

Lemma 7.1. *Suppose that $\gamma \in \mathbb{R}$ does not belong to the spectrum of L in $L^2(\Omega)$. Let also P be the spectral projector–valued measure, which corresponds to the selfadjoint operator L in $L^2(\Omega)$ (see [14]). Then for any $h \in C_\gamma(L^2)$ the problem (7.6) with the initial condition $P((-\infty,\gamma])w = 0$ has a unique solution $w \in C_\gamma(H)$, i.e. this problem defines the linear continuous mapping*

$$\mathcal{R}_\gamma : C_\gamma(L^2) \to C_\gamma(H)\,,\ \mathcal{R}_\gamma(h) = w \tag{7.8}$$

Proof. The assertion of this Lemma can be derived by standard arguments. For the convenience of the reader we give below a sketch of this proof.

According to the spectral theorem for selfadjoint operators (see [14]), there exists a measured space $(\mathbb{M},\nu)$, a measured function $l(m)$ and a unitary transformation $U : L^2(\Omega) \to L^2(\mathbb{M},\nu)$ such that $ULU^{-1} = l(m)\cdot$, i.e. the operator L is equivalent to the multiplication operator $l(m)$ in the space $L^2(\mathbb{M},\nu)$. Applying the operator U to the equation (7.6) we obtain an equivalent equation in the space $\mathbb{M}$.

$$\partial_t \widehat{w}(t) + l(m)\widehat{w}(t) = \widehat{h}(t) \tag{7.9}$$

where $\widehat{w} = Uw$, $\widehat{h} = Uh$.

Without loss of generality we can assume that $\gamma = 0 \notin \sigma(L)$. Let $\mathbb{M}_+ = \{m \in \mathbb{M}, l(m) > 0\}$ and $\mathbb{M}_- = \{m \in \mathbb{M}, l(m) < 0\}$. Since $0 \notin \sigma(L)$ then there exists $\mu > 0$ such that $l(m) \geq \mu > 0$ for almost all $m \in \mathbb{M}_+$ and $l(m) \leq -\mu < 0$ for almost all $m \in \mathbb{M}_-$ and the condition $P((-\infty,\gamma))w(0) = 0$ is equivalent to $\widehat{w}(0) = 0$ for $m \in \mathbb{M}_-$.

Moreover, it is easy to verify that $W_0^{1,2}(\Omega) = L^2(\mathbb{M}, |l|\nu)$ and

$$\|w\|^2_{1,2} = \int_{\mathbb{M}} |l||\widehat{w}|^2 \nu(dm). \tag{7.10}$$

Let us consider the equation (7.9) separately for $m \in \mathbb{M}_+$ and for $m \in \mathbb{M}_-$. First let $m \in \mathbb{M}_+$. Then all solutions of the problem (7.9) are given by the following expression

$$\widehat{w}(t) = \widehat{C}e^{-l(m)t} + \int_{-\infty}^{t} e^{-l(m)(t-s)}\widehat{h}(s)\,ds\,,\ \widehat{C} \in \mathbb{R} \tag{7.11}$$

Let us estimate the integral $I_1(t) = \int_{-\infty}^{t} e^{-l(m)(t-s)}\widehat{h}(s)\,ds$ on the right–hand side of (7.11) by the Holder inequality

$$|I_1(t)|^2 \leq \int_{-\infty}^{t} |\widehat{h}(s)|^2 e^{-l(m)(t-s)}ds \int_{-\infty}^{t} e^{-l(m)(t-s)}ds \leq \tag{7.12}$$

$$\leq l(m)^{-1} \int_{-\infty}^{t} |\widehat{h}(s)|^2 e^{-\mu(t-s)}\,ds.$$

Consequently,

$$\int_{\mathbb{M}_+} l(m)|I_1(t)|^2\nu(dm) \leq \int_{-\infty}^{t} \|h(s)\|^2_{0,2} e^{-\mu(t-s)}\,ds \leq C\|h\|_{C_0(L^2)}. \tag{7.13}$$

Thus, the unique solution of (7.9) for $m \in \mathbb{M}_+$ which is bounded with respect to $t \to -\infty$ is given by (7.11) with $\widehat{C} = 0$.

Let us consider now the equation (7.9) for $m \in \mathbb{M}_-$. Then the unique solution with zero initial conditions is given by

$$\widehat{w}(t) = \int_t^0 e^{-l(m)(t-s)}\widehat{h}(s)\,ds. \tag{7.14}$$

Arguing as in the derivation of the estimate (7.13) we obtain that

$$-\int_{\mathbb{M}_-} l(m)|\widehat{w}(t)|^2\nu(dm) \le C\|h\|_{C_0(L^2)}. \tag{7.15}$$

The estimates (7.13), (7.15) and (7.10) imply that our problem has a unique solution $w \in C_0(W_0^{1,2})$ and the following estimate is valid

$$\|w\|_{C_0(W_0^{1,2})} \le C\|h\|_{C_0(L^2)}. \tag{7.16}$$

To prove the lemma it is sufficient to deduce from (7.16) the estimate for the C-norm of w. Let us consider $w_1(t) = (t-T-1)w(t)$, $T \le 0$. Then

$$\partial_t w_1 - \Delta_x w_1 + \lambda_0 w_1 = -(t-T-1)f'(z_0)w + w(t) + h(t)\ ,\ w_1(-T-1) = 0$$

Applying the estimate (2.25) for this equation at the point $t = T$ and using the fact that $z_0 \in C_b(\Omega)$ we obtain that

$$\sup_{x\in\Omega}|w(T,x)|^2 \le \sup_{t\in[-T-1,-T]}\{\|w(t)\|_{0,2}^2 + \|h(t)\|_{0,2}^2\} \tag{7.17}$$

The assertion of the Lemma is an immediate corollary of the estimates (7.16) and (7.17). This proves Lemma 7.1.

Definition 7.2. *Let z_0 be an equilibrium point of the equation (3.1), i.e. let z_0 be a solution of (7.2). Then the unstable index Ind_{z_0} of the equilibria point z_0 is defined to be the dimension of the spectral subset of L which corresponds to the negative part of the spectrum*

$$\mathrm{Ind}_{z_0} = \dim \mathrm{Im}\, P(-\infty, 0) = \dim L^2(\mathbb{M}_-, \nu) \tag{7.18}$$

(see Lemma 7.1).

Theorem 7.1. *Let (7.1) hold and z_0 be the equilibria point of (3.1) such that $\mathrm{Ind}_{z_0} = \infty$. Then the instable set $\mathcal{M}^+(z_0)$ contains C^1–submanifolds of H with an arbitrary large dimension. (Note that the hyperbolicity of z_0 is not assumed.)*

Proof. Let $\gamma_{ess} = \inf\{\lambda :\ \lambda \in \sigma_{ess}(L)\}$. Then $\mathrm{Ind}_{z_0} = \infty$ implies that $\gamma_{ess} \le 0$. First consider the case when $\gamma_{ess} = 0$. Then the negative part of the spectrum $\sigma(L)$ consists of a countable infinite number of normal eigenvalues with finite multiplicities. Moreover the set $\sigma_\varepsilon(L) = \{\lambda \in \sigma(L) : \lambda < -\varepsilon\}$ contains only a finite number of eigenvalues for any $\varepsilon > 0$. Using the standard technique (see [2]) one can prove that for any $\varepsilon > 0$, $-\varepsilon \notin \sigma(L)$ and for a sufficiently small $\beta > 0$ the set

$$\mathcal{M}^+_{\varepsilon,\beta}(z_0) = \{w_0 \in H :\ \exists w \in C(\mathbb{R}_-, W^{1,2}(\Omega)\cap C(\overline{\Omega})),\ w(0) = w_0,$$
$$w\text{– satisfies (7.3)},\ \lim_{t\to-\infty} e^{-\varepsilon t}\|w(t)\|_H < \infty \text{ and } \|w(t)\|_H < \beta\}$$

is a C^1-manifold in H with $\dim \mathcal{M}^+_{\varepsilon,\beta}(z_0) = \dim \mathrm{Im}\, P(\sigma_\varepsilon(L))$. The evident assertion $\dim \mathrm{Im}\, P(\sigma_\varepsilon(L)) \to \infty$ when $\varepsilon \to 0$ proves Theorem in this case.

Thus, the main problem is to prove the Theorem in the case when $\gamma_{ess} < 0$ (especially in the case when L has only continuous spectrum without lacunas). To this end we need one more lemma.

Lemma 7.2. *Let $\gamma \in \mathbb{R}$ and let $H^-_\gamma = P((-\infty,\gamma])L^2(\Omega)$. Then the problem*

$$\begin{cases} \partial_t w + Lw = 0 \\ P((-\infty,\gamma])w(0) = w_0 \end{cases} \tag{7.19}$$

has a unique solution $w \in C_\gamma(H)$. Thus, the problem (7.19) defines the linear continuous mapping

$$\mathcal{S}_\gamma : H^-_\gamma \to C_\gamma(H)\,,\ \mathcal{S}_\gamma w_0 = w \tag{7.20}$$

The assertion of this Lemma can be derived analogously to the proof of the previous Lemma but even more simply and hence we omit the proof here.

The end of the proof of Theorem 7.1. Recall that it remains to consider the case $\gamma_{ess} < 0$. Let us fix $\gamma_{ess} < \gamma_0 < 0$ such that $2\gamma_0 < \gamma_{ess}$ and $2\gamma_0 \notin \sigma(L)$. It is possible to do this because the part of $\sigma(L)$ which satisfies $\lambda < \gamma_{ess}$ is discrete. Moreover due to $\gamma_0 > \gamma_{ess}$, $\dim H^-_{\gamma_0} = \infty$. We are going to prove that the set $\mathcal{M}^+_{-\gamma_0,\beta}$ is an infinite dimensional manifold in H for sufficiently small $\beta > 0$.

We should find the solutions of the equation (7.3) in C_{γ_0}. To this end we rewrite this equation in the following form:

$$w = \mathcal{S}_{\gamma_0} w_0 + \mathcal{R}_{2\gamma_0} F(w) \tag{7.21}$$

where $w_0 \in H^-_{\gamma_0}$. Indeed it follows from (7.4) that $F(w) \in C_{2\gamma_0}(L^2)$ for $w \in C_{\gamma_0}(H)$. Moreover simple checking implies that every solution w of the equation (7.3) from the space $C_{\gamma_0}(H)$ satisfies (7.21) with an appropriate w_0 and the inverse assertion also holds.

Let us apply the implicit function theorem to equation (7.21). To be more rigorous, define a function

$$\Phi : C_{\gamma_0}(H) \times H^-_{\gamma_0} \to C_{\gamma_0}(H)\,,\ \Phi(w, w_0) = w - \mathcal{S}_{\gamma_0} w_0 - \mathcal{R}_{2\gamma_0} F(w)$$

It is not difficult to check using (7.4) that $\Phi \in C^1$ and that $\partial_w \Phi(0,0) = Id$. Hence due to the implicit function Theorem (see, for instance [36], [29]), there exist neighborhoods $H_\beta \subset H^-_{\gamma_0}$ and $V_\beta \subset C_{\gamma_0}(H)$ such that the unique solution to the problem (7.21) in V_β is given by the function $W : H_\beta \to V_\beta$ i.e.

$$\Phi(W(w_0), w_0) = 0 \tag{7.22}$$

Consider the function $M(w_0) = W(w_0)\big|_{t=0} : H_\beta \to H$. Then it follows from (7.22) that $M \in C^1(H_\beta, H)$ and

$$P(\lambda \le \gamma_0) D_{w_0} M(0) = Id\,,\ P(\lambda > \gamma_0) D_{w_0} M(0) = 0$$

and consequently, the set $\mathcal{M}^+_{\gamma_0,\beta}(z_0) = M(H_\beta)$ is an infinite dimensional C^1-manifold, diffeomorphic to $H^-_{\gamma_0}$. Theorem 7.1 is proved.

§8 A DIMENSION OF THE ATTRACTOR. THE CASE OF INFINITE DIMENSION.

In this Section we will study the dimension of the attractor $\mathcal{A}_{\langle\alpha\rangle}$ of the equation (3.1) in the case $\alpha < 0$. It is also assumed throughout this Section that the equation (3.1) is scalar ($k = 1$) and the nonlinear term f is independent of $\nabla_x u$ and satisfies (7.1).

For the reader's convenience we recall shortly the definition Hausdorff dimension and some simple properties of it.

Definition 8.1. *Let X be a compact set in metric space $\mathcal{H}$. Then for any $\varepsilon > 0$, $d \geq 0$ Hausdorff (d, ε)–measure is defined to be the following number:*

$$\mu_H(X, d, \varepsilon) = \inf\{\sum_{i=1}^{\infty} r_i^d : X \subset \cup_{i=1}^{\infty} B_{x_i}^{r_i},\ |r_i| < \varepsilon\} \tag{8.1}$$

$B_{x_i}^{r_i}$ means a ball of radius r_i centered in $x_i \in \mathcal{H}$ and the infinum is taken over all coverings of the set X.

The Hausdorff d-measure $\mu_H(X, d)$ of X and the Hausdorff dimension $\dim_H(X)$ is defined to be the following numbers:

$$\begin{cases} \mu_H(X, d) = \sup_{\varepsilon > 0} \mu_H(d, \varepsilon) \in [0, \infty] \\ \dim_H(X) = \inf\{d : \mu_H(X, d) = 0\} \in [0, \infty] \end{cases} \tag{8.2}$$

A detailed study of the concept of Hausdorff dimension is given for instance in (see [32] and the references therein).

Proposition 8.1. *The following properties of Hausdorff dimension can be easily deduced from Definition 8.1:*

1. Let $X_1, X_2 \subset \mathcal{H}$ and let $X_1 \subset X_2$. Then

$$\dim_H(X_1) \leq \dim_H(X_2) \tag{8.3}$$

2. Let X be a Lipshitz manifold in H with dimension N. Then

$$\dim_H(X) = N \tag{8.3}$$

3. Let $L : \mathcal{H} \to \mathcal{H}_1$ be a Lipshitz mapping ($\mathcal{H}, \mathcal{H}_1$ are metric spaces). Then

$$\dim_H(L(X)) \leq \dim_H(X) \tag{8.4}$$

Let us suppose now that the assumptions (7.1) and (3.2) are valid and suppose the right–hand side g of the equation (3.1) satisfies the following condition:

$$g \in L_b^2(\Omega) \cap L_{\langle\alpha\rangle}^2(\Omega) \tag{8.5}$$

for a certain $\alpha < 0$. Then according to Theorem 6.2 the problem (3.1) possesses a trajectory attractor $\mathcal{A}^{tr}$ in the space $\Theta_{\langle\alpha\rangle}^+(\Omega)$ defined by (6.8). Let us define also the set $\mathcal{A}^{gl}$ by formula (6.26). Then Corollary 6.3 implies that $\mathcal{A}^{gl}$ is a bounded set in $W_b^{2-\delta,2}(\Omega)$. Hence it follows from Theorem 5.1 that the solving operator S for the problem realizes a one-to-one correspondence between $\mathcal{A}^{gl}$ and $\mathcal{A}^{tr}$:

$$S : \mathcal{A}^{gl} \to \mathcal{A}^{tr} \tag{8.6}$$

(it means particularly that the problem (3.1) has a *unique* solution for any $u_0 \in \mathcal{A}^{gl}$). Moreover if we endow the attractors $\mathcal{A}^{gl}$ and $\mathcal{A}^{tr}$ with the topology induced by the embeddings $\mathcal{A}^{gl} \subset \Phi_{\langle\alpha\rangle}(\Omega)$ and $\mathcal{A}^{tr} \subset \Theta^+_{\langle\alpha\rangle}$ respectively then Proposition 5.2 implies that (8.6) is a Lipshitz continuous isomorphism (see also the proof of Corollary 6.2) and consequently, due to Proposition 8.1,

$$\dim_H \left(\mathcal{A}^{tr}\big|_{[T,T+1]\times\Omega}, \Theta^+_{\langle\alpha\rangle}([T,T+1],\Omega)\right) = \dim_H \left(\mathcal{A}^{gl}, \Phi_{\langle\alpha\rangle}\right) \tag{8.7}$$

for any $T \in \mathbb{R}_+$. Thus, instead of studying the dimension of trajectory attractor $\mathcal{A}^{tr}_{\langle\alpha\rangle}$ in the space $\Theta^+_{\langle\alpha\rangle}$, we will to study the dimension of $\mathcal{A}^{gl}$ in the space $\Phi_{\langle\alpha\rangle}(\Omega)$.

Theorem 8.1. *Let the above assumptions be valid and let* $\mathrm{Ind}_{z_0} = \infty$ *for some equilibrium point of the equation* (3.1). *Then the attractor* $\mathcal{A}^{gl}$ *of the equation* (3.1) *has infinite Hausdorff dimension in* $\Phi_{\langle\alpha\rangle}$. *Moreover for every* $N \in \mathbb{N}$ *it contains a* C^1*-manifold* $\mathcal{M}_N$ *of dimension* N.

Proof. This Theorem is a corollary of Theorem 7.1. Indeed, according to Theorem 7.1 for every $N \in \mathbb{N}$ the unstable set $\mathcal{M}^+(z_0)$ (see Definition 7.2) contains a C^1-manifold $\mathcal{M}_N$ of dimension N. To complete the proof it remains to verify that $\mathcal{M}^+(z_0) \subset \mathcal{A}^{gl}$. Let $u_0 \in M^+(z_0)$. By definition it means that there exists a solution $u(t)$ for $t \le 0$ for the equation (3.1) such that $u(t) - z_0 \in C_b(\mathbb{R}_-, W^{1,2}(\Omega) \cap C(\Omega))$ and $u(t) \to z_0$ when $t \to -\infty$. Recall also that due to Theorem 6.3 $z_0 \in \Phi_{\langle\alpha\rangle}(\Omega) \cap C_b(\Omega)$. Since $\alpha < 0$ then $u \in C_b(\mathbb{R}_-, W^{1,2}_{\langle\alpha\rangle}(\Omega) \cap C_b(\Omega))$. Rewriting the equation (3.1) in the form of a linear equation

$$\partial_t u = \Delta_x u - \lambda_0 u - h(t) \text{ with } h(t) = f(u(t)) - g \ , \ t \le 0 \tag{8.8}$$

and applying the smoothing property for the equation (8.8) (as in the proof of Proposition 5.3 one can easily obtain that $u \in C_b(\mathbb{R}_-, \Phi_{\langle\alpha\rangle}(\Omega))$. Consequently, $u_0 \in \Phi_{\langle\alpha\rangle}(\Omega)$. Hence, according to Theorems 4.2 and 6.1 there exists a solution $u(t)$, $t \ge 0$, $u(0) = u_0$ of the problem (3.1) and $u(t) \in C_b(\mathbb{R}_+, \Phi_{\langle\alpha\rangle}(\Omega))$. Thus, we construct a solution $u \in C_b(\mathbb{R}, \Phi_{\langle\alpha\rangle}(\Omega))$ and $u(0) = u_0$. Theorem 6.2 implies now that $u_0 \in \mathcal{A}^{gl}$. Theorem 8.1 is proved.

We conclude this Section by constructing explicit examples of the equations in (3.1) which have an infinite dimensional attractor. For simplicity we assume below that $\Omega = \mathbb{R}^3$

Theorem 8.2. *Let* $\Omega = \mathbb{R}^3$, $\alpha < 0$ *and the nonlinear term* f *satisfy* (7.1). *Let us suppose also that the equation* (3.1) *is nonmonotonic, i.e. there exists* $\xi \in \mathbb{R}$ *such that*

$$f'(\xi) + \lambda_0 < 0. \tag{8.9}$$

Then there exists a right–hand side $g \in L^2_{\langle\alpha\rangle}(\Omega) \cap L^2_b(\Omega)$ *of the equation* (3.1) *such that the attractor of this equation* $\mathcal{A}^{gl}$ *has an infinite dimension in* $\Phi_{\langle\alpha\rangle}(\Omega)$.

Proof. Due to the previous Theorem it is sufficient to construct the right–hand side g in such way that the equation (3.1) possesses an equilibrium point z_0 with infinite unstable index.

We begin by constructing an equilibrium z_0 and afterwards we define the right–hand side g by the formula

$$g = -\Delta_x z_0 + \lambda_0 z_0 + f(z_0). \tag{8.10}$$

For every $R > 0$ consider a smooth function $Q_R(z)$, $z \in \mathbb{R}$ such that $Q_R(z) = 1$ for $|z| \le R$, $Q_R(z) = 0$ for $|z| > R+1$ and $0 \le Q_R(z) \le 1$ for $R < |z| < R+1$. Consider also a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}^3$ such that

$$|x_i - x_j| > 2R + 2 \text{ for } i \neq j. \tag{8.11}$$

Define the equilibrium point z_0 by the formula

$$z_0(x) = \xi \sum_{n=1}^{\infty} Q_R(|x - x_n|) \tag{8.12}$$

Then due to conditions (8.11), $z_0 \in C_b(\mathbb{R}^3)$ and $|z_0| \le \xi$.

Lemma 8.1. *The unstable index* Ind_{z_0} *of the operator*

$$L_{z_0} = -\Delta_x + f'(z_0) + \lambda_0 \tag{8.13}$$

is equal to ∞ *for sufficiently large* R *and for an arbitrary sequence* $\{x_n\}$ *which satisfies* (8.12).

Proof. We will use the min-max principle (see [32]) for the calculation of the unstable index of the self-adjoint operator L_{z_0} in the following form: suppose there exists N-dimensional subspace $V_N \subset W^{1,2}(\mathbb{R}^3)$ such that for a certain $\varepsilon > 0$

$$(L_{z_0}v, v) + \varepsilon(v, v) \le 0 \text{ for any } v \in V_N. \tag{8.14}$$

Then $\mathrm{Ind}_{z_0} \ge N$.

For $k > 0$ consider the function

$$v_k(x) = \begin{cases} \sin(x_1/k)\sin(x_2/k)\sin(x_3/k) & \text{if } x \in [0, k\pi]^3 \\ 0 & \text{if } x \notin [0, k\pi]^3 \end{cases} \tag{8.15}$$

Then it is not difficult to verify $v_k \in W^{1,2}(\mathbb{R}^3)$ and

$$(\Delta_x v_k, v_k) = -\frac{3}{k^2}(v_k, v_k). \tag{8.16}$$

Let us fix now the constant k such that $\frac{3}{k^2} + \lambda_0 + f'(\xi) < 0$ and let the constant R be chosen such that $3\pi k < R$. Fix also a sequence $\{x_n\}$ satisfying (8.11) and define a sequence of functions

$$e_n(x) = v_k(x - x_n) \tag{8.17}$$

Then $\operatorname{supp} e_i \cap \operatorname{supp} e_j = \varnothing$ for $i \neq j$ and $z_0(x) = \xi$ when $x \in \operatorname{supp} e_n$. Consequently

$$-(L_{z_0}e_n, e_n) = ((\Delta_x - a - f'(\xi))v_k, v_k) = -(\frac{3}{k^2} - \lambda_0 - f'(\xi))(v_k, v_k) > 0 \tag{8.18}$$

Taking $V_N = \operatorname{span}\{e_1, \cdots, e_N\}$ and applying (8.14) we obtain that $\mathrm{Ind}_{z_0} = \infty$. Lemma 8.1 is proved.

To complete the proof of Theorem 8.2 it remains to fix the sequence $\{x_n\}$ in such a way that the function g defined by (8.10) belongs to the space $L^2_{\langle\alpha\rangle}(\mathbb{R}^3)$.

Lemma 8.2. *Let* $x_1 = (1, 1, 1)$ *and* $x_n = Re^n x_1$ *for* $n > 1$ *(*R *was defined in Lemma 8.1). Then the function* g *defined by* (8.10) *belongs to the space* $L^2_{\langle\alpha\rangle}(\Omega)$ *for any* $\alpha < 0$.

The proof of this Lemma is based on the fact $f(0) = 0$ and can be obtained by a direct computation. Lemma 8.2 is proved. Theorem 8.2 is proved.

§9 A dimension of the attractor. The case of finite dimension.

In this Section, under some natural assumptions on the nonlinear term $f(u, \nabla_x u)$ we prove that in the case when the right–hand side g of the equation (3.1) belongs to the space $L^2_{\langle\alpha\rangle}(\Omega)$ for a some $\alpha \geq 0$ the attractor $\mathcal{A}^{gl}$ of this equation has finite Hausdorff dimension in $\Phi_{\langle\alpha\rangle}$. For simplicity we will consider below only the case when $\alpha = 0$ and $\Omega = \mathbb{R}^3$. The case $\alpha > 0$ could be treated analogously.

Let us suppose that the assumptions of Proposition 5.3 are valid. Then, integrating the estimate (5.23) over $x_0 \in \Omega = \mathbb{R}^n$ and using the estimates (1.3) and (1.10) with $\alpha = 0$ we obtain that the following estimate holds uniformly with respect to $u_1(0), u_2(0) \in \mathcal{A}^{gl}$,

$$\|u_1(t) - u_2(t), \mathbb{R}^3\|^2_{2-\delta,2} \leq C\frac{e^{\alpha t}}{t^2}\|u_1(0) - u_2(0), \mathbb{R}^3\|^2_{0,2}. \tag{9.1}$$

Here $u_i(t)$, $i = 1, 2$ are solutions of the problem (3.1). Using the invariance property of the attractor ($S_t\mathcal{A}^{gl} = \mathcal{A}^{gl}$) and the third assertion of Proposition 8.1 one can easily deduce from (9.1) that

$$\dim_H(\mathcal{A}^{gl}, \Phi_{\langle 0\rangle}(\mathbb{R}^3)) = \dim_H(\mathcal{A}^{gl}, L^2(\mathbb{R}^3)). \tag{9.2}$$

So, instead of estimating the dimension of $\mathcal{A}^{gl}$ in the space $\Phi_{\langle 0\rangle}(\mathbb{R}^3) = W^{2-\delta,2}(\mathbb{R}^3)$ we estimate below its dimension in a simpler space $L^2(\mathbb{R}^3)$. To this end we need the following definition

Definition 9.1. *A map $S : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is a subset of certain Banach space X is called uniformly quasidifferentiable on $\mathcal{A}$ if for any $x \in X$ there exists a linear operator $S'(x) : X \to X$ (quasidifferential) such that*

$$\|S(x + v) - S(x) - S'(x)v\|_X = \overline{\overline{o}}(\|v\|_X) \tag{9.3}$$

holds uniformly with respect to $x \in X$, $x + v \in X$.

The estimation of the dimension of the attractor $\mathcal{A}$ is based on the following theorem.

Theorem 9.1 [32]. *Let S_t be a semigroup in a certain Hilbert space H and let $\mathcal{A} \subset H$ be a compact strictly invariant set of this semigroup ($S_t\mathcal{A} = \mathcal{A}$). Let us suppose also that S_t is uniformly quasidifferentiable on $\mathcal{A}$ for any fixed t and the following inequality holds for some $T > 0$*

$$\omega_d(\mathcal{A}) = \sup_{x\in\mathcal{A}} \omega_d\left(S'_T(x)\right) < 1 \tag{9.4}$$

where $\omega_d(L) \equiv \|\Lambda^d L\|_{\Lambda^d H}$ is the norm of d-th exterior power of the operator L in Hilbert space $\Lambda^d H$ (see [32]). Then the Hausdorff dimension of the set $\mathcal{A}$ is finite in H. Moreover,

$$\dim_H(\mathcal{A}, H) \leq d \tag{9.4}$$

Lemma 9.1. *Let all of the assumptions of Theorem 5.2 be valid and let $g \in L^2(\Omega)$ then the semigroup $S_t : \Phi_{\langle 0\rangle} \to \Phi_{\langle 0\rangle}$ generated by the equation (3.1) is uniformly quasidifferentiable in the space $L^2(\Omega)$ on the attractor $\mathcal{A}$.*

Proof. The assertion of the lemma is a corollary of the estimate (5.33). Indeed let $u_1(0), u_2(0) \in \mathcal{A}$ and let $w(t)$ be the solution of the problem (5.29) with $w(0) = u_1(0) -$

$u_2(0)$. Then integrating this estimate over $x_0 \in \mathbb{R}^3$ we obtain after simple computations that

$$\|u_1(T) - u_2(T) - w(T), \mathbb{R}^3\|^2_{0,2} \le C_1 e^{C_2 T} \|u_1(0) - u_2(0), \mathbb{R}^3\|^{2(1+\beta)}_{0,2} \tag{9.5}$$

where the constants C_1 and C_2 depends only on norms $\|u_i(t)\|_\Phi$ which remain bounded on the attractor.

The estimate (9.5) implies by definition that the semigroup S_t is uniformly quasidifferentiable on the attractor in the space $L^2(\mathbb{R}^3)$ and its quasidifferential coincides with a solving operator for the problem (5.29). Lemma 9.1 is proved.

Thus, to estimate the dimension of the attractor it remains to estimate d-th exterior powers of the solving operator for the problem (5.29).

Lemma 9.2. *Let the assumptions of Lemma 9.1 be valid. Then*

$$\omega_d(S'_T(u_0)) \le e^{\int_0^T \mathrm{Tr}_d\{L(u(t))\}\,dt} \tag{9.6}$$

where $u(t)$ is a solution of (3.1) with $u(0) = u_0 \in \mathcal{A}$,

$$L(\theta) = \Delta_x - \lambda_0 - f'_u(\theta, \nabla_x\theta)) - f'_{\nabla_x u}(\theta, \nabla_x\theta)) \tag{9.7}$$

and Tr_d means a d-dimensional trace of the upper semibounded linear operator L, i.e.

$$\mathrm{Tr}_d(L) \equiv \sup\{\sum_{i=1}^d (Lv_i, v_i) \ :\ \|v_i\|_{0,2} = 1\,,\ i = 1...d;\ (v_i, v_j) = 0 \text{ for } i \ne j\} \tag{9.8}$$

The proof of this Lemma can be found for instance in [32].

Lemma 9.3. *Let the operator $L(\cdot)$ be defined by formula (9.7). Then for any $\theta = \theta(x) \in \mathcal{A}$ and for any $d \in \mathbb{N}$ the following estimate is valid:*

$$\mathrm{Tr}_d\{L(\theta)\} \le -\frac{\lambda_0 d}{2} + Q(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2} \tag{9.9}$$

where Q is a certain monotonic function independent of d

Proof. Let $\{v_i\}_{i=1}^d$ be the orthonormal system in the space $L^2(\mathbb{R}^3)$. Then, due to the Holder inequality,

$$\begin{aligned}
\sum_{i=1}^d (Lv_i, v_i) = &- \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |\nabla_x v_i(x)|^2\right) dx - \lambda_0 \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |v_i(x)|^2\right) - \\
&- \int_{\mathbb{R}^3} f'_u(\theta(x), \nabla_x\theta(x)) \left(\sum_{i=1}^d |v_i(x)|^2\right) dx - \\
&- \int_{\mathbb{R}^3} f'_{\nabla_x u}(\theta(x), \nabla_x\theta(x)) \left(\sum_{i=1}^d \nabla_x v_i(x) v_i(x)\right) dx \le \\
&- \frac{1}{2} \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |\nabla_x v_i(x)|^2\right) dx - \lambda_0 \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |v_i(x)|^2\right) dx + \\
&+ \int_{\mathbb{R}^3} \left(-f'_u(\theta(x), \nabla_x\theta(x)) + \frac{1}{2}|f'_{\nabla_x u}(\theta(x), \nabla_x\theta(x))|^2\right) \left(\sum_{i=1}^d |v_i(x)|^2\right) dx.
\end{aligned} \tag{9.10}$$

Let us estimate the last term in the right–hand side of (9.10). To this end we note that the first assumption of (3.2) implies that

$$f_u(0,0) \geq 0 \text{ and } f_{\nabla_x u}(0,0) = 0. \tag{9.11}$$

Since $f \in C^1$ then there exists $\beta > 0$ such that $\frac{1}{2}|f'_v(u,v)|^2 - f'_u(u,v) \leq \lambda_0/2$ for $|u| \leq \beta$ and $|v| \leq \beta$. Let us fix such $\beta > 0$ and introduce the set

$$\Omega_\beta = \{x \in \mathbb{R}^3 \, : |\theta(x)| \leq \beta, |\nabla_x \theta(x)| < \beta\}. \tag{9.12}$$

Then, by definition

$$I(x) \equiv \frac{1}{2}|f'_{\nabla_x \theta}(\theta(x), \nabla_x \theta(x))|^2 - f'_\theta(\theta(x), \nabla_x \theta(x)) \leq \lambda_0/2 \tag{9.13}$$

for every $x \in \Omega_\beta$. Therefore, using (9.13), Lieb-Thirring inequality (see [24])

$$\int_{\mathbb{R}^3} \left(\sum_{i=1}^d |v_i(x)|^2 \right)^{5/3} dx \leq C \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |\nabla_x v_i(x)|^2 \right) dx,$$

and Holder inequality with the exponents $5/3$ and $5/2$ we obtain that

$$\begin{aligned} \int_{\mathbb{R}^3} I(x) \left(\sum_{i=1^d} |v_i(x)|^2 \right) dx \leq \lambda_0/2 \int_{\Omega_\beta} \left(\sum_{i=1}^d |v_i(x)|^2 \right) dx + \\ + \int_{\mathbb{R}^3 \setminus \Omega_\beta} |I(x)| \left(\sum_{i=1}^d |v_i(x)|^2 \right) dx \leq \\ \leq \lambda_0/2 \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |v_i(x)|^2 \right) dx + 1/2 \int_{\mathbb{R}^3} \left(\sum_{i=1}^d |\nabla_x v_i(x)|^2 \right) dx + \\ + \int_{\mathbb{R}^3 \setminus \Omega_\beta} |I(x)|^{5/2} dx \end{aligned} \tag{9.14}$$

Thus, it remains to estimate the last integral into the right-hand side of (9.14). To this end we recall that $\theta \in \mathcal{A}$, since according to (6.29) we infer that

$$\begin{cases} \|\theta, \mathbb{R}^3\|^2_{0,2} + \|\nabla_x \theta, \mathbb{R}^3\|^2_{0,2} + \|\theta, \mathbb{R}^3\|^2_{2-\delta,2} \leq Q_1(\|g, \mathbb{R}^3\|_{0,2}) \|g, \mathbb{R}^3\|^2_{0,2} \\ \|\theta, \mathbb{R}^3\|_{0,\infty} \leq C \|g, \mathbb{R}^3\|_{0,2} \end{cases} \tag{9.15}$$

for a certain monotonic function Q_1.

The estimates (9.15) imply that

$$\text{mes}\left(\mathbb{R}^3 \setminus \Omega_\beta\right) \leq Q_2(\|g, \mathbb{R}^3\|_{0,2}) \|g, \mathbb{R}^3\|^2_{0,2} \tag{9.16}$$

Moreover, it follows from the assumptions (5.1) that

$$|I(x)|^{5/2} \leq Q(\|\theta, \mathbb{R}^3\|_{0,\infty})(1 + |\nabla_x \theta(x)|^{5r/2}) \leq Q_3(\|g, \mathbb{R}^3\|_{0,2})(1 + |\nabla_x \theta|^l) \tag{9.17}$$

with $l = \max\{2, 5r/2\}$. Estimating the last integral in the right-hand side of (9.14) using (9.16) and (9.17) we will have

$$\int_{\mathbb{R}^3\setminus\Omega_\beta} |I(x)|^{5/2} \le Q_3 \,\mathrm{mes}\{\mathbb{R}^3\setminus\Omega_\beta\} + Q_3 \int_{\mathbb{R}^3} |\nabla_x \theta(x)|^l dx \tag{9.18}$$
$$\le Q_4(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2} + Q_3\|\theta, \mathbb{R}^3\|^l_{1,l}$$
$$\le Q_4(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2} + CQ_3\|\theta, \mathbb{R}^3\|^l_{2-\delta,2} \le Q(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2}$$

Here we also use the Sobolev embedding $W^{1,l} \subset W^{2-\delta,2}$ which holds since $\delta < \frac{1}{r} - \frac{1}{2}$, and the estimate (9.15).

Combining the estimates (9.10), (9.14) and (9.18) we obtain that

$$\sum_{i=1}^{d} (Lv_i, v_i) \le -\frac{\lambda_0 d}{2} + Q_2(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2}. \tag{9.19}$$

The estimate (9.9) is an immediate corollary of (9.19) and (9.8). Lemma 9.3 is proved.

Theorem 9.2. *Let the previous assumptions be valid. Then the attractor $\mathcal{A}^{gl}$ has the finite Hausdorff dimension:*

$$\dim(\mathcal{A}^{gl}, L^2(\mathbb{R}^3)) \le Q(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2} \tag{9.20}$$

where Q is a certain monotonic function.

The assertion of this Theorem is an immediate corollary of Theorem 9.1 and Lemmata 9.1,9.2 and 9.3.

Remark 9.1. *Let us consider now a slighty modified equation of the form* (3.1)

$$\partial_t u = \nu\Delta_x u - \lambda_0 u - f(u, \nabla_x u) + g \tag{9.21}$$

where $\nu > 0$ is a small parameter. Then arguing as in the proof of Theorem 9.2 one can obtain the following estimate for the Hausdorff dimension of the attractor $\mathcal{A}_\nu$ of (9.21)*:*

$$\dim(\mathcal{A}_\nu, L^2(\mathbb{R}^3) \le \frac{Q(\|g, \mathbb{R}^3\|_{0,2})\|g, \mathbb{R}^3\|^2_{0,2}}{\nu^P} \tag{9.22}$$

where the monotonic function Q is independent of ν and the exponent $P = P(r)$ can be expressed explicitly.

Remark 9.2. *If the nonlinear term $f(u, \nabla_x u)$ is independent of $\nabla_x u$ ($f(u, \nabla_x u) \equiv f(u)$) then arguing as in the proof of Theorem 9.2 we can obtain a sharper estimate for the dimension of the attractor to the equation* (9.21) *(see* [15]*).*

$$\dim(\mathcal{A}_\nu, L^2(\mathbb{R}^3)) \le C_f \frac{\|g, \mathbb{R}^3\|^2_{0,2}}{\nu^{3/2}}. \tag{9.23}$$

Moreover, this estimate cannot be improved, i.e. there exists a sequence of right–hand sides $g_\nu \in L^2(\mathbb{R}^3)$, $\nu \to 0$ such that

$$C'_f \frac{\|g_\nu, \mathbb{R}^3\|^2_{0,2}}{\nu^{3/2}} \le \dim(\mathcal{A}_\nu, L^2(\mathbb{R}^3)) \le C_f \frac{\|g_\nu, \mathbb{R}^3\|^2_{0,2}}{\nu^{3/2}}. \tag{9.24}$$

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