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Weak solutions to joined nonlinear systems of PDES

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Abstract

We establish an existence and uniqueness result for a system which consists of a finite number of coupled nonlinear systems. In each system we have two highly nonlinearly coupled equations. Such problems arise if one couples thin rods of shape memory alloys, and each of the rods is described by Falk's Landau-Ginzburg model. The two equations in each system stand for the momentum and energy balance, respectively.

1 Introduction and statement of the result

In [8, 13] the authors study a system of nonlinear partial differential equations given by

$$\rho\, u_{tt} - \sigma(\theta, u_x)_x - \nu\, u_{xxt} + R\, u_{xxxx} = f, \tag{1.1a}$$

$$c_0\theta_t - \kappa\theta_{xx} - \theta\sigma_\theta u_{xt} - \nu u_{xt}^2 = g, \tag{1.1b}$$

on $Q_T = (0,T) \times [0,1]$. This system is the Landau-Ginzburg model developed by Falk (see, e.g., [5, 6]) describing first-order martensitic phase transitions in thin rods of shape memory alloys (SMA). The first equation represents the balance of momentum, the second one the balance of energy. The boundary and initial conditions for u and θ are given by

$$u(x,t) = u_{xx}(x,t) = 0 \quad \text{on} \quad (0,T) \times \{0,1\}, \tag{1.2a}$$

$$\theta_x(x,t) = 0 \quad \text{on} \quad (0,T) \times \{0,1\}, \tag{1.2b}$$

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad \theta(x,0) = \theta_0 \quad \text{on} \quad [0,1]. \tag{1.2c}$$

In this system u represents the displacement, either longitudinal or transversal, and θ the absolute temperature. Also, ρ denotes the constant mass density, σ the stress, ν the coefficient of viscosity, R the rigidity of the material, sometimes refered to as the Ginzburg coefficient, f the distributed external load, c_0 the specific heat, κ the heat conductivity, and g the distributed heat sources and sinks. The strain or deformation $\varepsilon = u_x$ is used as the order parameter. In a Landau-Ginzburg model the stress is given by

$$\sigma(\theta, \varepsilon) = \frac{\partial}{\partial\varepsilon} G(\theta, \varepsilon, \varepsilon_x), \tag{1.3}$$

where the free energy density G is given by

$$G(\theta, \varepsilon, \varepsilon_x) = \frac{1}{2}\gamma\; (\theta - \theta_1)\,\varepsilon^2 - \frac{1}{4}\beta\,\varepsilon^4 + \frac{1}{6}\,\alpha\,\varepsilon^6 + \frac{1}{2}\,R\,\varepsilon_x^2. \tag{1.4}$$

Here α, β, γ, θ_1, R, and ν are material constants. Existence and uniqueness results to this problem are found in [8, 13]. Other authors have investigated the situation when $\nu = 0$ (see [1, 2, 3, 12]). Furthermore, control problems and numerical simulations have been worked out for related problems (see, e.g., [2, 3, 4, 9, 11, 13]). For a detailed discussion of the physical phenomena and Falk's model, we refer the reader to [2] and references cited therein.

In this paper we use the above system of PDEs to model a situation where a finite number of thin rods of different SMAs are joined together. We assume that there is no heat flux through the joints, i.e. the joints are insulating. This is a simple example of a so-called adaptive structure. There is a rapidly growing industrial interest in developing such structures.

If one studies such a situation there are in principal two distinct approaches: One can either look at the momentum balances and energy balances of each rod separately and formulate a set of joint conditions to augment the equations. This approach has been recently used to study joined linear elastic rods (see, e.g., [10]). In the nonlinear case, the joint conditions for the balances of momentum cause significant difficulties. The second method is to look at a single momentum balance and a single energy balance for the entire structure. These equations have nonsmooth coefficients, and it becomes difficult to guarantee the boundary conditions at the joint, specifically the condition that there is no heat flux through the joint.

To avoid the short-comings of these two methods we take an approach which utilizes ideas from both methods. We look at a single balance of momentum coupled with separate balances of energy for each of the rods. This is why we call this a hybrid model.

For simplicity, we only consider two coupled rods. The mathematical formulation and analysis can easily be extended to a finite number of coupled rods.

To formulate the hybrid model we need to introduce some notation. We define the following sets:

$$\Omega = (-1,1), \qquad \Omega^l = (-1,0), \qquad \Omega^r = (0,1).$$

Furthermore, let

$$\Omega_T = \Omega \times [0,T],$$

and Ω_T^l and Ω_T^r be defined in an analogous way. For functions

$$f^l : \Omega_T^l \to \mathbb{R} \qquad \text{and} \qquad f^r : \Omega_T^r \to \mathbb{R},$$

we define

$$\hat{f} : \Omega_T \to \mathbb{R} \qquad \text{by} \quad \hat{f}(x,t) = \begin{cases} f_l(x,t) & \text{for} \quad (x,t) \in \Omega_T^l, \\ f_r(x,t) & \text{for} \quad (x,t) \in \Omega_T^r. \end{cases} \tag{1.5}$$

For the remainder of the paper we will assume that $\rho = 1$ and $c_0 = 1$. We refer to the remarks at the end of this paper about these limitations.

Using these notations we write the balance of momentum for two thin SMA rods. The first rod extends over Ω^l, the second over Ω^r. The rods are joined at $x = 0$. Let $u(x,t)$ denote the displacement of the material in the rods at $(x,t) \in \Omega_T$, then u satisfies the following fourth order partial differential equation:

$$u_{tt} - \hat{\sigma}(\hat{\theta}, u_x)_x - \nu\, u_{xxt} + R\, u_{xxxx} = \hat{f} \tag{1.6}$$

for $(x,t) \in \Omega_T$. The functions $\hat{\sigma}$, $\hat{\theta}$, and $\hat{f}$ are defined as in (1.5) from functions which are defined separately on Ω^l_T and Ω^r_T. In particular, we have

$$\sigma^r(\theta^r, u_x) = \gamma_r\,(\theta^r - \theta^r_1)\, u_x - \beta_r u_x^3 + \alpha_r u_x^5 \tag{1.7}$$

and the analogous equation for the stress on the left.

Observe that we assume that the two materials only differ in the expressions for σ and have the same viscosity ν and rigidity R. Equation (1.6) is formally identical with the momentum equation for a single thin rod of an SMA (1.1a), the only difference being that the coefficients are discontinuous at the origin and the temperature $\hat{\theta}$ may actually be discontinuous as well.

We complement the momentum balance with the following boundary and initial conditions:

$$\begin{aligned} u(-1,t) = u(1,t) &= 0, && (1.8a)\\ u_{xx}(-1,t) = u_{xx}(1,t) &= 0, && (1.8b)\\ u(x,0) &= \hat{u}_0(x), && (1.8c)\\ u_t(x,0) &= \hat{u}_1(x). && (1.8d) \end{aligned}$$

In this model, the usually imposed joint conditions for two joined rods, i.e. that u and u_{xx} should be identical at $x = 0$, are automatically satisfied.

For the energy balances, let θ^l and θ^r — defined on Ω^l_T and Ω^r_T, respectively, — be the temperatures of the individual rods. These functions satisfy the following parabolic partial differential equations:

$$\begin{aligned} \theta^r_t - \kappa_r\, \theta^r_{xx} - \gamma_r\, \theta^r u_x u_{xt} - \nu\, u^2_{xt} &= g^r \quad \text{on} \quad \Omega^r_T, && (1.9a)\\ \theta^l_t - \kappa_l\, \theta^l_{xx} - \gamma_l\, \theta^l u_x u_{xt} - \nu\, u^2_{xt} &= g^l \quad \text{on} \quad \Omega^l_T. && (1.9b) \end{aligned}$$

The temperatures will satisfy the following boundary and initial conditions:

$$\begin{aligned} \theta^l_x(-1,t) = \theta^l_x(0,t) &= 0, && (1.10a)\\ \theta^r_x(0,t) = \theta^r_x(1,t) &= 0, && (1.10b)\\ \theta^l(x,0) &= \theta^l_0(x), && (1.10c)\\ \theta^r(x,0) &= \theta^r_0(x). && (1.10d) \end{aligned}$$

These conditions reflect the fact that the joint is thermally insulated.

We continue by stating the weak version of the above problem. To do this, let $\langle\cdot,\cdot\rangle$ denote the usual inner product on $L^2(\Omega)$, $L^2(\Omega^l)$, and $L^2(\Omega^r)$. We say that the triple $(u, \theta^r, \theta^l) \in H^{3,2}(\Omega_T) \times H^{2,1}(\Omega_T^r) \times H^{2,1}(\Omega_T^l)$ is a weak solution to (1.6)–(1.10) if it satisfies the initial conditions and

$$\int_0^T \Big(\langle u_{tt}, \phi \rangle + \langle \hat{\sigma}(\hat{\theta}, u_x), \phi_x \rangle + \nu \langle u_{xt}, \phi_x \rangle - R \langle u_{xxx}, \phi_x \rangle \Big) \, \mathrm{d}s = \int_0^T \langle \hat{f}, \phi \rangle \, \mathrm{d}s \,, \tag{1.11a}$$

$$\int_0^T \big(\langle \theta_t^r, \psi^r \rangle + \kappa_r \langle \theta_x^r, \psi_x^r \rangle - \nu \langle u_{xt}^2, \psi^r \rangle - \gamma_r \langle u_x u_{xt} \theta^r, \psi^r \rangle \big) \, \mathrm{d}s = \int_0^T \langle g^r, \psi^r \rangle \, \mathrm{d}s \,, \tag{1.11b}$$

$$\int_0^T \big(\langle \theta_t^l, \psi^l \rangle + \kappa_l \langle \theta_x^l, \psi_x^l \rangle - \nu \langle u_{xt}^2, \psi^l \rangle - \gamma_l \langle u_x u_{xt} \theta^l, \psi^l \rangle \big) \, \mathrm{d}s = \int_0^T \langle g^l, \psi^l \rangle \, \mathrm{d}s \,, \tag{1.11c}$$

$$u(-1,t) = u(1,t) = 0, \quad \forall t \in [0,T], \tag{1.11d}$$

$$u_{xx}(-1,t) = u_{xx}(1,t) = 0, \quad \text{a.e. in} \quad (0,T),$$

$$\theta_x^l(-1,t) = \theta_x^l(0,t) = \theta_x^r(0,t) = \theta_x^r(1,t) = 0, \quad \text{a.e. in} \quad (0,T), \tag{1.11e}$$

for all triples $(\phi, \psi^r, \psi^l) \in L^2(0,T; H_0^1(\Omega)) \times L^2(0,T; H_0^1(\Omega^r)) \times L^2(0,T; H_0^1(\Omega^l))$. To complete the problem we state the following additional conditions:

$$(H1) \qquad \hat{f} \in L^4(\Omega_T); \qquad g^r \in L^2(\Omega_T^r); \qquad g^l \in L^2(\Omega_T^l). \tag{1.12a}$$

$$(H2) \qquad \hat{u}_0 \in \big\{ u \in H^3(\Omega) : u(-1) = u(1) = u''(-1) = u''(1) = 0 \big\} \, ; \tag{1.12b}$$

$$\hat{u}_1(x) \in H_0^1(\Omega); \qquad \theta_0^r \in H_0^1(\Omega^r); \qquad \theta_0^l \in H_0^1(\Omega^l); \tag{1.12c}$$

$$\theta_0^r(x) > 0 \quad \text{in} \quad \Omega^r; \qquad \theta_0^l(x) > 0 \quad \text{in} \quad \Omega^l. \tag{1.12d}$$

We can now state the main result of this article.

Theorem 1.1 *Assume that (H1) and (H2) hold. Then the system (1.6)–(1.10) admits a unique weak solution in the sense of (1.11).*

Section 2 contains the proof of the existence of solutions. Uniqueness is shown in Section 3. Finally, Section 4 contains some concluding remarks about the non-viscous problem, additional regularity of the solutions and more general boundary conditions.

2 Existence of weak solutions

The proof of existence of solutions follows a standard line of argumentation. First one shows existence of solutions to a mollified version of the problem. Second one shows that these solutions converge to a solution of the problem with discontinuous coefficients. We divide the proof into several steps.

Step 1: A mollified version of the problem. We start by defining for $n \in \mathbb{N}$ the function

$$M_n(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 - \cos(xn\pi) & \text{for} \quad 0 \le x \le \frac{1}{n}, \\ 1 & \text{for} \quad x > \frac{1}{n}. \end{cases} \tag{2.1}$$

Using this we can mollify the coefficient $\hat{\gamma}$ as follows. Let

$$\gamma_n(x) = \gamma_l \, M_n(-x) + \gamma_r \, M_n(x). \tag{2.2}$$

Then this coefficient is a continuously differentiable function which converges pointwise on $\Omega \backslash \{0\}$ to the coefficient of the original problem. Furthermore, we have that $\gamma_n(0) = \gamma_n'(0) = 0$, which implies that

$$\gamma_n(\hat{\theta} - \hat{\theta}_1)$$

is also a continuously differentiable function at the origin if $\hat{\theta}$ is differentiable everywhere but the origin. For the remaining coefficients and the external load $\hat{f}$ we use the following mollifier:

$$Q_n(x) = \begin{cases} 0 & \text{for} \quad x < -\frac{1}{n}, \\ \frac{1}{2}\left(1 + \sin\left(\frac{xn\pi}{2}\right)\right) & \text{for} \quad -\frac{1}{n} \le x \le \frac{1}{n}, \\ 1 & \text{for} \quad x > \frac{1}{n}. \end{cases} \tag{2.3}$$

We define

$$\beta_n(x) = \beta_l(1 - Q_n(x)) + \beta_r Q_n(x) \quad \text{and} \quad \alpha_n(x) = \alpha_l(1 - Q_n(x)) + \alpha_r Q_n(x). \tag{2.4}$$

These coefficients will be bounded away from zero. Together with $\gamma_n(x)$ we get that

$$\sigma_n(\hat{\theta}, u_x) = \gamma_n(\hat{\theta} - \hat{\theta}_1) u_x - \beta_n u_x^3 + \alpha_n u_x^5 \tag{2.5}$$

is continuously differentiable at the origin. For the external load let

$$f_n(x,t) = (f_l(x,t) + f_l(-x,t))(1 - Q_n(x)) + (f_r(x,t) + f_r(-x,t))Q_n(x), \tag{2.6}$$

where $f_l(-x,t)$ and $f_r(-x,t)$ serve to extend f_l and f_r to the entire domain Ω_T. One can now formally write down the mollified system:

$$(u_n)_{tt} - \sigma_n(\hat{\theta}_n, (u_n)_x)_x - \nu (u_n)_{xxt} + R(u_n)_{xxxx} = f_n, \tag{2.7a}$$

$$(\theta_n^r)_t - \kappa_r(\theta_n^r)_{xx} - \gamma_n \theta_n^r (u_n)_x (u_n)_{xt} - \nu (u_n)_{xt}^2 = g^r \quad \text{in} \quad \Omega_T^r, \tag{2.7b}$$

$$(\theta_n^l)_t - \kappa_l(\theta_n^l)_{xx} - \gamma_n \theta_n^l (u_n)_x (u_n)_{xt} - \nu (u_n)_{xt}^2 = g^l \quad \text{in} \quad \Omega_T^l. \tag{2.7c}$$

Here $\hat{\theta}_n$ is constructed from θ_n^r and θ_n^l as in (1.5). The system is augmented by the appropriate initial and boundary conditions:

$$u_n(-1,t) = u_n(1,t) = 0, \tag{2.8a}$$

$$(u_n)_{xx}(-1,t) = (u_n)_{xx}(1,t) = 0, \tag{2.8b}$$

$$u_n(x,0) = \hat{u}_0(x), \qquad (u_n)_t(x,0) = \hat{u}_1(x), \tag{2.8c}$$

$$(\theta_n^l)_x(-1,t) = (\theta_n^l)_x(0,t) = (\theta_n^r)_x(0,t) = (\theta_n^r)_x(1,t) = 0, \tag{2.8d}$$

$$\theta_n^l(x,0) = \theta_0^l(x), \qquad \theta_n^r(x,0) = \theta_0^r(x). \tag{2.8e}$$

Step 2: Global existence of smooth solutions to the mollified problem. For every fixed value n we can now apply the results of [8, 13] to the mollified problem (2.7)–(2.8). The fact that α_n, β_n, and γ_n are differentiable functions of x does not impede this process nor does the fact that we have two energy balances. We define

$$\mathcal{B} := H^{4,2}(\Omega_T) \times H^{2,1}(\Omega_T^r) \times H^{2,1}(\Omega_T^l) \tag{2.9}$$

and refer the reader to [8, 13] for the proof of the following lemma.

Lemma 2.1 (K.-H. Hoffmann, A. Zochowski, 1993) *For any fixed n and any finite T there exists a unique solution $(u_n, \theta_n^r, \theta_n^l) \in \mathcal{B}$ to the problem (2.7)–(2.8). Moreover, one has*

$$\int_{\Omega_r} \theta_n^r(x,t)\,\mathrm{d}x \geq 0 \quad \textit{and} \quad \int_{\Omega_l} \theta_n^l(x,t)\,\mathrm{d}x \geq 0 \tag{2.10}$$

for all $t \in [0,T]$.

For the remainder of the proof of Theorem 1.1 we assume that Lemma 2.1 holds. Furthermore, all constants denoted by C and C_i, $i \in \mathbb{N}$, are positive constants which may depend on T but not on n, and $\|.\|$ will denote the L^2-norm in Ω, Ω^r, or Ω^l. One needs to remark that $(u_n, \theta_n^r, \theta_n^l)$ is bounded in $\mathcal{B}$ for every n, but these bounds may depend on n and thus may not stay bounded as $n \to \infty$. However, we can redo some of the a priori estimates of [8, 13] to show that they are independent of n.

It is important to mention that the only way in which we can explicitly introduce n into the a priori bounds is to differentiate one of the mollified quantities α_n, β_n, γ_n, or f_n with respect to x.

Step 3: Energy estimate. First we get an energy estimate for the mollified problem. We multiply (2.7a) by $(u_n)_t$ and integrate over Ω. Furthermore, we integrate (2.7b) over Ω^r and (2.7c) over Ω^l, add the three integrals and obtain, using Gronwall's inequality,

$$\int_\Omega \left(\frac{1}{2}(u_n)_t^2(s) + \frac{1}{2}(u_n)_{xx}^2(s) + C_1 (u_n)_x^2(s) \right) \mathrm{d}x + \int_{\Omega^l} \theta_n^l(s)\,\mathrm{d}x + \int_{\Omega^r} \theta_n^r(s)\,\mathrm{d}x \;\leq\; C_E \tag{2.11}$$

for all $s \in [0,T]$. In this process we never differentiate the mollified constants. The constant C_E only depends on the initial amount of energy in the system and the energy added to the system and is independent of n. The positive constant C_1 comes from a lower bound on the polynomial G of (1.4). This bound can be chosen independent of n even if the coefficients of G depend on n, since the coefficients stay bounded for all n.

Step 4: A priori estimates for $(u_n)_{xt}$, θ_n^r and θ_n^l. This is the crucial step of the proof. Its results are summarized in the next lemma.

Lemma 2.2 *The following bounds depend only on C_E and T and not on n.*

$$\int_{\Omega^r} (\theta_n^r(s))^4 \,\mathrm{d}x\,\mathrm{d}s \le C, \qquad \int_{\Omega^l} (\theta_n^l(s))^4 \,\mathrm{d}x\,\mathrm{d}s \le C, \tag{2.12a}$$

$$\int_{\Omega_t} (u_n)_{xt}^4 \,\mathrm{d}x\,\mathrm{d}s \le C, \tag{2.12b}$$

$$\left\|\theta_n^r\right\|_{H^{2,1}(\Omega_t^r)} \le C, \qquad \left\|\theta_n^l\right\|_{H^{2,1}(\Omega_t^l)} \le C, \tag{2.12c}$$

$$\left\|\theta_n^r\right\|_{L^\infty(\Omega_t^r)} \le C, \qquad \left\|\theta_n^l\right\|_{L^\infty(\Omega_t^l)} \le C. \tag{2.12d}$$

Proof of the Lemma: The proof of this lemma consists of carefully redoing Steps 3 and 4 of the proof of Theorem 1 in [8] in the current setting. We introduce the notation $\Omega_t = \Omega \times [0,t]$ for $t \in (0,T]$ and analogously Ω_t^r and Ω_t^l.

As in Step 1 of the proof of Theorem 1 in [8] one can rewrite (2.7a) as a system of two parabolic equations as follows:

$$\begin{aligned} &(w_n)_t - b(w_n)_{xx} = f_n + \sigma_n(\hat{\theta}_n, (u_n)_x)_x, \\ &w_n(x,0) = w_0(x) = u_1(x) - a(u_0)_{xx}, \\ &w_n(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &(u_n)_t - a(u_n)_{xx} = w_n, \\ &u_n(x,0) = u_0(x), \\ &u_n(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T]. \end{aligned} \tag{2.14}$$

The numbers a and b are chosen two be two positive real numbers with

$$a + b = \nu \quad \text{and} \quad a \cdot b = R. \tag{2.15}$$

Following a trick of [7], w_n can be written as a sum $w_n = w_n^1 + (w_n^2)_x$, where

$$\begin{aligned} &(w_n^1)_t - b(w_n^1)_{xx} = f_n, \\ &w_n^1(x,0) = w_0, \\ &(w_n^1)_x(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \end{aligned}$$

and

$$
\begin{aligned}
&(w_n^2)_t - b(w_n^2)_{xx} = \sigma_n \\
&w_n^2(x,0) = 0, \\
&(w_n^2)_x(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T].
\end{aligned}
$$

Note that the right-hand-side of the equation for w_n^2 does not contain the derivative of σ_n, but σ_n itself. Applying standard parabolic regularity estimates one gets:

$$
\begin{aligned}
\left\|w_n^1\right\|_{W_4^{2,1}(\Omega_T)} &\le C_1 + C_2 \left\|f_n\right\|_{L^4(\Omega_T)}, \\
\left\|w_n^2\right\|_{W_4^{2,1}(\Omega_T)} &\le C_3 \left\|\sigma_n\right\|_{L^4(\Omega_T)}.
\end{aligned}
$$

The constants in these inequalities are independent of the temperature and n. This implies that

$$
\|(w_n)_x\|_{L^4(\Omega_T)} \le C_4 + C_5 \|\sigma_n\|_{L^4(\Omega_T)}. \tag{2.16}
$$

Since $u_n \in H^{4,2}(\Omega_T)$ for all n, it follows that $(u_n)_x \in W_4^{2,1}(\Omega_T)$ (although the bounds may depend on n). Therefore, $v_n = (u_n)_x$ may satisfy the equation

$$
\begin{aligned}
&(v_n)_t - a(v_n)_{xx} = (w_n)_x, \\
&v_n(x,0) = (u_0)_x, \\
&(v_n)_x(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T]
\end{aligned}
$$

in $W_4^{2,1}(\Omega_T)$. This gives a bound for $(u_n)_{xt}$ that does not depend on $(\theta_n^r)_x$ or $(\theta_n^l)_x$:

$$
\|(u_n)_x\|_{W_4^{2,1}(\Omega_T)} \le C_5 \left(1 + \|(w_n)_x\|_{L^4(\Omega_T)}\right) \le C_6 + C_7 \|\sigma_n\|_{L^4(\Omega_T)}. \tag{2.17}
$$

Finally observe that

$$
\begin{aligned}
\|\sigma_n\|^4_{L^4(\Omega_T)} &= \int_{\Omega_t} \left(\gamma_n \left(\hat{\theta}_n - \hat{\theta}_1\right)(u_n)_x + \beta_n (u_n)_x^3 - \alpha_n (u_n)_x^5\right)^4 \,\mathrm{d}x\,\mathrm{d}s \\
&\le C_8 + C_9 \left(\int_{\Omega_t^r} (\theta_n^r)^4 \,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t^l} (\theta_n^l)^4 \,\mathrm{d}x\,\mathrm{d}s\right),
\end{aligned}
$$

where the constants are independent of n. This implies

$$
\int_{\Omega_t} (u_n)_{xt}^4 \,\mathrm{d}x\,\mathrm{d}s \le C_{10} + C_{11} \left(\int_{\Omega_t^r} (\theta_n^r)^4 \,\mathrm{d}x\,\mathrm{d}s + \int_{\Omega_t^l} (\theta_n^l)^4 \,\mathrm{d}x\,\mathrm{d}s\right). \tag{2.18}
$$

From Lemma A2 of the appendix of [8],

$$
\int_{\Omega_t^r} (\theta_n^r)^4 \,\mathrm{d}x\,\mathrm{d}s \le C_{12} + C_{13} \int_{\Omega_t^r} (\theta_n^r)_x^2 \,\mathrm{d}x\,\mathrm{d}s \quad \text{and} \tag{2.19a}
$$

$$
\int_{\Omega_t^l} (\theta_n^l)^4 \,\mathrm{d}x\,\mathrm{d}s \le C_{12} + C_{13} \int_{\Omega_t^l} (\theta_n^l)_x^2 \,\mathrm{d}x\,\mathrm{d}s, \tag{2.19b}
$$

where the constants only depend on C_E, which in turn does not depend on n. To continue we multiply (2.7b) by θ_n^r, integrate over Ω_t^r, and obtain

$$
\begin{aligned}
&\frac{1}{2} \int_{\Omega_t^r} \frac{d}{dt} (\theta_n^r)^2 \,\mathrm{d}x\,\mathrm{d}s + \kappa_r \int_{\Omega_t^r} (\theta_n^r)_x^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad = \int_{\Omega_t^r} \left(\nu \theta_n^r (u_n)_{xt}^2 + \gamma_n (\theta_n^r)^2 (u_n)_x (u_n)_{xt} + g_r \theta_n^r\right) \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{2.20}
$$

One can estimate the terms on the right as follows:

$$\begin{aligned}
&\int_{\Omega_t^r} (\theta_n^r)^2 \, |(u_n)_x (u_n)_{xt}| \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad \le C_{14} \int_{\Omega_t^r} (\theta_n^r)^2 \, |(u_n)_{xt}| \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad \le C_{15} \left(\int_{\Omega_t^r} (\theta_n^r)^4 \, \mathrm{d}x \, \mathrm{d}s \int_{\Omega_t^r} (u_n)_{xt}^2 \, \mathrm{d}x \, \mathrm{d}s \right)^{\frac{1}{2}} \\
&\qquad \le C_{16} \, t^{\frac{1}{4}} \left(\int_{\Omega_t^r} (\theta_n^r)^4 \, \mathrm{d}x \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\int_{\Omega_t^r} (u_n)_{xt}^4 \, \mathrm{d}x \, \mathrm{d}s \right)^{\frac{1}{4}} \\
&\qquad \le C_{17} + C_{18} \, t^{\frac{1}{4}} \int_{\Omega_t^r} (\theta_n^r)^4 \, \mathrm{d}x \, \mathrm{d}s \, ,
\end{aligned}$$

where we used (2.18). The constants depend only on C_E. In a similar manner we can estimate the other terms on the right-hand-side of (2.20) to get

$$\begin{aligned}
\frac{1}{2} \int_{\Omega^r} (\theta_n^r)^2 \, \mathrm{d}x \, + \kappa_r \int_{\Omega_t^r} (\theta_n^r)_x^2 \, \mathrm{d}x \, \mathrm{d}s \quad &\le \quad C_{19} + C_{20} \, t^{\frac{1}{4}} \int_{\Omega_t^r} (\theta_n^r)^4 \, \mathrm{d}x \, \mathrm{d}s \\
&\le \quad C_{21} + C_{22} \, t^{\frac{1}{4}} \int_{\Omega_t^r} (\theta_n^r)_x^2 \, \mathrm{d}x \, \mathrm{d}s \, ,
\end{aligned}$$

where the constants are independent of n. Analogous esitmates for (θ_n^l) give

$$\begin{aligned}
\frac{1}{2} \int_{\Omega^l} (\theta_n^l)^2 \, \mathrm{d}x \, + \kappa_l \int_{\Omega_t^l} (\theta_n^l)_x^2 \, \mathrm{d}x \, \mathrm{d}s \quad &\le \quad C_{19} + C_{20} \, t^{\frac{1}{4}} \int_{\Omega_t^l} (\theta_n^l)^4 \, \mathrm{d}x \, \mathrm{d}s \\
&\le \quad C_{21} + C_{22} \, t^{\frac{1}{4}} \int_{\Omega_t^l} (\theta_n^l)_x^2 \, \mathrm{d}x \, \mathrm{d}s \, . \qquad (2.21)
\end{aligned}$$

We can choose t^* such that $C_{22}(t^*)^{1/4} = \frac{1}{2} \min\{\kappa_r, \kappa_l\}$. This yields, for all $t \le t^*$,

$$\int_{\Omega^r} (\theta_n^r(s))^2 \, \mathrm{d}x \, \le C_{23}, \quad \int_{\Omega^l} (\theta_n^l(s))^2 \, \mathrm{d}x \, \le C_{23}, \quad \forall s \in [0,t], \quad (2.22a)$$

$$\int_{\Omega_t^r} (\theta_n^r)_x^2 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{24}, \quad \int_{\Omega_t^l} (\theta_n^l)_x^2 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{24}, \qquad (2.22b)$$

where C_{23} and C_{24} only depend on C_E and not on n. This implies

$$\int_{\Omega^r} (\theta_n^r(s))^4 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{25}, \quad \int_{\Omega^l} (\theta_n^l(s))^2 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{25}, \qquad (2.23a)$$

$$\int_{\Omega_t} (u_n)_{xt}^4 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{25}. \qquad (2.23b)$$

Multiplying the energy balances by $(\theta_n^r)_t$ and $(\theta_n^l)_t$, respectively, one gets, after integration and using standard estimating techniques on the right, the following:

$$\int_{\Omega_t^r} (\theta_n^r)_t^2 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{26}, \quad \int_{\Omega_t^l} (\theta_n^l)_t^2 \, \mathrm{d}x \, \mathrm{d}s \, \le C_{26}, \qquad (2.24a)$$

$$\int_{\Omega^r} (\theta_n^r(s))_x^2 \, \mathrm{d}x \, \le C_{27}, \quad \int_{\Omega^l} (\theta_n^l(s))_x^2 \, \mathrm{d}x \, \le C_{27}, \quad \forall s \in [0,t], \quad (2.24b)$$

and therefore

$$\left\|\theta_n^r\right\|_{H^{2,1}(\Omega_t^r)} \le C_{28}, \qquad \left\|\theta_n^l\right\|_{H^{2,1}(\Omega_t^l)} \le C_{28}, \tag{2.25a}$$

$$\left\|\theta_n^r\right\|_{L^\infty(\Omega_t^r)} \le C_{29}, \qquad \left\|\theta_n^l\right\|_{L^\infty(\Omega_t^l)} \le C_{29}, \tag{2.25b}$$

where the constants only depend on C_E and not on n. Since t^* depends only on C_E and the constants κ_r and κ_l, we can cover $[0,T]$ by a finite number of intervals of length t^*. Therefore, the previous estimates can be extended to $[0,T]$.

□

Step 5: Higher a priori estimates. The authors of [8] obtain higher a priori estimates. However, they use the L^2-norm of $(\sigma_n(\hat{\theta}_n,(u_n)_x))_x$, which depends on n, to obtain these estimates. We avoid this problem by using a priori estimates similar to the ones used in [3, 12].

Lemma 2.3 *The following estimates are independent of n:*

$$\sup_{0\le s\le T}\left(\|(u_n)_{xt}(s)\|^2+\|(u_n)_{xxx}(s)\|^2\right)+\int_{\Omega_T}(u_n)_{xxt}^2\,\mathrm{d}x\,\mathrm{d}s \quad\le\quad C, \tag{2.26a}$$

$$\|(u_n)_t\|_{L^\infty(\Omega_T)}+\|(u_n)_{xx}\|_{L^\infty(\Omega_T)} \quad\le\quad C, \tag{2.26b}$$

$$\int_{\Omega_T}(u_n)_{tt}^2\,\mathrm{d}x\,\mathrm{d}s \quad\le\quad C. \tag{2.26c}$$

Proof of the Lemma: The second assertion follows immediately from the first assertion. To prove the first assertion we follow [12] and multiply the balance of momentum (2.7a) by $-(u_n)_{xxt}$. After integrating over Ω_t and integrating by parts we have:

$$\begin{aligned}
&\frac{1}{2}\left(\|(u_n)_{xt}(t)\|^2+R\,\|(u_n)_{xxx}(t)\|^2\right)+\nu\int_{\Omega_t}(u_n)_{xxt}^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\frac{1}{2}\left(\|(u_n)_{xt}(0)\|^2+R\,\|(u_n)_{xxx}(0)\|^2\right)-\int_{\Omega_t}f_n(u_n)_{xxt}\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\int_{\Omega_t}\sigma_n(\hat{\theta}_n,(u_n)_x)(u_n)_{xxxt}\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

The only critical part is the last term on the right. For this we integrate by parts in t as follows:

$$\begin{aligned}
&\int_{\Omega_t}\sigma_n(\hat{\theta}_n,(u_n)_x)(u_n)_{xxxt}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_{\Omega}\left(\sigma_n(\hat{\theta}_n(t),(u_n)_x(t))(u_n)_{xxx}(t)-\sigma_n(\hat{\theta}_n,(u_n)_x(0))(u_n)_{xxx}(0)\right)\,\mathrm{d}s\\
&\qquad-\int_{\Omega_t}\left(\sigma_n(\hat{\theta}_n,(u_n)_x)\right)_t(u_n)_{xxx}\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

The terms at t only contain $\hat{\theta}_n$ and $(u_n)_x$ which are already bounded in $L^\infty(\Omega)$. The term $(u_n)_{xxx}(t)$ can be brought to the left-hand-side using Young's inequality. For the last term observe that

$$\|(\sigma_n)_t\|^2_{L^2(\Omega_t)} \le C_1 \|(\theta^r_n)_t\|^2_{L^2(\Omega^r_t)} + C_2 \left\|(\theta^l_n)_t\right\|^2_{L^2(\Omega^l_t)} + C_3 \|(u_n)_{xt}\|^2_{L^2(\Omega_t)} \tag{2.27}$$

with constants that are independent of n. The terms on the right are bounded independently of n by Lemma 2.2. This means we can use Hölder's and Gronwall's inequalities to treat this last term, and the first assertion of the lemma follows.

To prove the last assertion of the lemma we multiply the equation by $(u_n)_{tt}$ and integrate over Ω_t to get

$$\int_{\Omega_t} \left((u_n)^2_{tt} - (\sigma_n)_x(u_n)_{tt} - \nu\,(u_n)_{xxt}(u_n)_{tt} + R\,(u_n)_{xxxx}(u_n)_{tt}\right)\,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_{\Omega_t} f_n\,(u_n)_{tt}\,\mathrm{d}x\,\mathrm{d}s\,.$$

The term involving σ_n is again treated by integrating by parts in x and then integrating by parts in t and using very similar estimates as above. The difficult term in this estimate is the last one on the left. Observe that

$$\begin{aligned}\int_{\Omega_t}(u_n)_{xxxx}(u_n)_{tt}\,\mathrm{d}x\,\mathrm{d}s &= -\int_{\Omega_t}(u_n)_{xxx}(u_n)_{xtt}\,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_{\Omega}(u_n)_{xxx}(t)(u_n)_{xt}(t)\,\mathrm{d}x + \int_{\Omega}(u_n)_{xxx}(0)(u_n)_{xt}(0)\,\mathrm{d}x \\ &\quad + \int_{\Omega_t}(u_n)_{xxxt}(u_n)_{xt}\,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_{\Omega}(u_n)_{xxx}(t)(u_n)_{xt}(t)\,\mathrm{d}x + \int_{\Omega}(u_n)_{xxx}(0)(u_n)_{xt}(0)\,\mathrm{d}x \\ &\quad - \int_{\Omega_t}(u_n)_{xxt}(u_n)_{xxt}\,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned} \tag{2.28}$$

The terms in the last line are all bounded independently of n by the previous estimates. The third assertion of the lemma follows now using Hölder's inequality and previous estimates.

□

Step 6: Passage to the limit. By the previous four steps we established that the sequence

$$(u_n, \theta^r_n, \theta^l_n) \in H^{3,2}(\Omega_T) \times H^{2,1}(\Omega^r_T) \times H^{2,1}(\Omega^l_T) \tag{2.29}$$

is bounded in this Hilbert space independent of n. By Alaoglu's theorem, the sequence has a weakly convergent subsequence. We will use the same notation for this subsequence. Furthermore, $(u_n)_t$ and $(u_n)_x$ are both bounded in $H^{2,1}(\Omega_T)$. Hence, after passing to subsequences if necessary, these sequences converge weakly in these spaces. $H^{2,1}(\Omega_T)$ is compactly imbedded into $L^\infty(\Omega_T)$ and the analogous results hold for $H^{2,1}(\Omega^r_T)$ and $H^{2,1}(\Omega^l_T)$. This implies that $(u_n)_t$ and $(u_n)_x$

both converge strongly in $L^\infty(\Omega_T)$. For the same reason, θ_n^r and θ_n^l also converge strongly in $L^\infty(\Omega_T^r)$ and $L^\infty(\Omega_T^l)$, respectively. Finally, $(\theta_n^r)_x$ converges strongly in $L^2(0,T;L^\infty(\overline{\Omega^r}))$. This implies that $\int_0^T(\theta_n^r)_x^2(0,s)\,\mathrm{d}s$ converges to zero and the same holds for the remaining boundary conditions.

We have now that the triple $(u_n,\theta_n^r,\theta_n^l)$ satisfies the initial conditions and

$$\begin{aligned}
&\int_0^T \Big(\langle (u_n)_{tt},\phi\rangle + \langle \hat\sigma_n(\hat\theta_n,(u_n)_x),\phi_x\rangle + \nu\langle (u_n)_{xt},\phi_x\rangle - R\langle (u_n)_{xxx},\phi_x\rangle\Big)\,\mathrm{d}s \\
&\qquad\qquad = \int_0^T \langle f_n,\phi\rangle\,\mathrm{d}s\,,\\
&\int_0^T \big(\langle(\theta_n^r)_t,\psi^r\rangle + \kappa_r\langle(\theta_n^r)_x,\psi_x^r\rangle - \nu\langle(u_n)_{xt}^2,\psi^r\rangle - \gamma_r\langle(u_n)_x(u_n)_{xt}\theta_n^r,\psi^r\rangle\big)\,\mathrm{d}s\\
&\qquad\qquad = \int_0^T \langle g^r,\psi^r\rangle,\\
&\int_0^T \big(\langle(\theta_n^l)_t,\psi^l\rangle + \kappa_l\langle(\theta_n^l)_x,\psi_x^l\rangle - \nu\langle(u_n)_{xt}^2,\psi^l\rangle - \gamma_l\langle(u_n)_x(u_n)_{xt}\theta_n^l,\psi^l\rangle\big)\,\mathrm{d}s\\
&\qquad\qquad = \int_0^T \langle g^l,\psi^l\rangle\,\mathrm{d}s\,,\\
&u_n(-1,t) = u_n(1,t) = 0, \qquad \forall t\in[0,T],\\
&(u_n)_{xx}(-1,t) = (u_n)_{xx}(1,t) = 0, \qquad \text{a.e. in} \quad (0,T)\\
&(\theta_n^l)_x(-1,t) = (\theta_n^l)_x(0,t), \qquad \text{a.e. in} \quad (0,T),\\
&(\theta_n^r)_x(0,t) = (\theta_n^r)_x(1,t) = 0, \qquad \text{a.e. in} \quad (0,T),
\end{aligned}$$

for all functions $(\phi,\psi^r,\psi^l)\in L^2(0,T;H_0^1(\Omega))\times L^2(0,T;H_0^1(\Omega^r))\times L^2(0,T;H_0^1(\Omega^l))$. The strong convergences above guarantee that these equations converge to (1.11). The limit (u,θ^r,θ^l) is the desired solution.

3 Uniqueness of solutions

The uniqueness is an immediate consequence of the following stability result which is similar to a result in [1].

Lemma 3.1 *Let $(u_1,\theta_1^r,\theta_1^l)$ and $(u_2,\theta_2^r,\theta_2^l)$ be two solutions to (1.11a)–(1.11c). Then*

$$\begin{aligned}
&\|(u_1)_t(t)-(u_2)_t(t)\|^2 + \|(u_1)_{xx}(t)-(u_2)_{xx}(t)\|^2\\
&\quad + \|\theta_1^r(t)-\theta_2^r(t)\|^2 + \|\theta_1^r(t)-\theta_2^r(t)\|^2\\
&\qquad \le \Big(\|(u_1)_t(0)-(u_2)_t(0)\|^2 + \|(u_1)_{xx}(0)-(u_2)_{xx}(0)\|^2\\
&\qquad\qquad + \|\theta_1^r(0)-\theta_2^r(0)\|^2 + \|\theta_1^r(0)-\theta_2^r(0)\|^2\Big)e^{Ct}
\end{aligned} \tag{3.1}$$

holds for all $t\in[0,T]$.

Proof: To abbreviate the notation let $v = u_1 - u_2$, $\vartheta^r = \theta_1^r - \theta_2^r$, and $\vartheta^l = \theta_1^l - \theta_2^l$. Observe that v satisfies the equation

$$v_{tt} - \nu\, v_{xxt} + R\, v_{xxxx} = \hat{\sigma}(\hat{\theta}_1, (u_1)_x)_x - \hat{\sigma}(\hat{\theta}_2, (u_2)_x)_x .$$

Since both sides of this equation are integrable (see the remark in the next section), we do not need to consider a mollified version for the following estimates. We multiply this equation by v_t and integrate over Ω. After integration by parts and applying Young's inequality we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v_t\|^2 + \|v_{xx}\|^2\right) + \nu\,\|v_{xt}\|^2 \\ &\quad = -\int_\Omega \left(\hat{\sigma}(\hat{\theta}_1, (u_1)_x) - \hat{\sigma}(\hat{\theta}_2, (u_2)_x)\right) v_{xt}\,\mathrm{d}x \\ &\qquad\qquad \leq \delta\,\|v_{xt}\|^2 + \frac{C_1}{\delta}\int_\Omega \left(\hat{\sigma}(\hat{\theta}_1, (u_1)_x) - \hat{\sigma}(\hat{\theta}_2, (u_2)_x)\right)^2 \mathrm{d}x\,. \end{aligned}$$

For the second term observe that $\hat{\sigma}$ is a polynomial in $\hat{\theta}$ and u_x, and therefore locally Lipschitz continuous in both variables. It follows that

$$\int_\Omega \left(\hat{\sigma}(\hat{\theta}_1, (u_1)_x) - \hat{\sigma}(\hat{\theta}_2, (u_2)_x)\right)^2 \mathrm{d}x \leq C_2\,\|v_x\|^2 + C_3\,\|\vartheta^r\|^2 + C_4\,\|\vartheta^l\|^2 ,$$

where the constants only depend on the coefficients of $\hat{\sigma}$ and the norms of $(u_1, \theta_1^r, \theta_1^l)$ and $(u_2, \theta_2^r, \theta_2^l)$. To continue, observe that ϑ^r satisfies

$$\vartheta^r_t - \kappa^r \vartheta^r_{xx} = \nu (u_1)^2_{xt} - \nu (u_2)^2_{xt} + \alpha_r \theta_1^r (u_1)_x (u_1)_{xt} - \alpha_r \theta_2^r (u_2)_x (u_2)_{xt} .$$

We multiply this by ϑ^r and integrate over Ω^r. After using the Lipschitz continuity of the right-hand-side we get

$$\frac{1}{2}\frac{d}{dt}\|\vartheta^r\|^2 + \kappa^r\,\|\vartheta^r_x\|^2 \leq \delta\,\|v_{xt}\|^2 + C_5\,\|v_x\|^2 + C_6\,\|\vartheta^r\|^2 .$$

We repeat this process for ϑ^l to get the analogous estimate. Finally, since $v(-1,t) = 0 = v(1,t)$, there exists for every $t \in [0,T]$ an $x \in \Omega$ such that $v_x(x,t) = 0$, i.e. we may apply Poincaré's inequality to v_x to get

$$\|v_x\|^2 \leq C\,\|v_{xx}\|^2 ,$$

as in [3]. We use this on the right-hand-sides. Next we combine the inequalities, choose δ sufficiently small and integrate over $[0,t]$ on both sides. The result follows by applying Gronwall's inequality.

□

Now for the initial condition $(v(x,0), \vartheta^r(x,0), \vartheta^l(x,0)) = (0,0,0)$ we have that $(v(x,t), \vartheta^r(x,t), \vartheta^l(x,t)) = (0,0,0)$, which in turn implies uniqueness.

4 Concluding remarks

The non-viscous case: The case when $\nu = 0$ is considerably more difficult. To prove existence of weak solutions in this case one can either follow the approach of [3, 12], or try to get the a priori estimates in this article independent of ν. The first method runs into problems right after the energy estimate. The authors of [3, 12] need to differentiate the term

$$\theta \, \sigma_\theta \, u_{xt}$$

with respect to x to get a priori estimates for θ. Thus the first estimate for θ depends on n. However, after this first estimate, all the other estimates up to Lemma 2.3 in this paper will work without difficulty. In other words, if one can show that

$$\sup_{0 \le s \le T} ||\theta^r||^2_{L^2(\Omega^r)} \le C \tag{4.1}$$

is independent of n, Theorem 1.1 would also be valid for $\nu = 0$.

Despite the fact that the estimates in Section 2 do not explicitely depend on ν, there is an implicit dependency. We used parabolic regularity theory in Step 4 of the proof. These estimates depend on the coefficients a and b in the equations (2.13) and (2.14), and indirectly on ν. So for the second approach one needs a new set of a priori estimates.

Regularity of the weak solutions: The weak solutions of Theorem 1.1 have considerable regularity. However, the regularity is less than the corresponding solutions in previous papers ([3, 8, 12, 13]), i.e. we only have $u \in H^{3,2}(\Omega_T)$. This result can be improved slightly. Since the derivative of the mollifier $M_n(x)$ (2.1) is bounded independently of n in $L^\infty(0,T;L^1(\Omega))$, one can easily show that

$$(u_n)_{xxxx} \in L^2(0,T;L^1(\Omega)) \tag{4.2}$$

independently of n. This implies the same regularity for u_{xxxx}. We already used this fact in the proof of uniqueness. However, one cannot show $(u_n) \in H^{4,2}(\Omega_T)$ independently of n.

More general boundary conditions: The present result can be easily extended to more general boundary conditions on θ^r and θ^l. In particular we may allow boundary conditions like

$$\theta^r_x(1,t) = \delta_r \left(\theta^r(1,t) - \theta^r_\Gamma(t)\right), \tag{4.3a}$$

$$\theta^l_x(-1,t) = -\delta_l \left(\theta^l(-1,t) - \theta^l_\Gamma(t)\right), \tag{4.3b}$$

without adding any additional difficulties.

Different values for the rigidity R, the viscosity ν and the density ρ in the different rods: As we remarked in the introduction we only treated the case when the rigidity, the viscosity, and the density are the same in all rods. One can always divide by ρ to remove the explicit dependence on the density. However, this dependence will then implicitly appear in ν and R. In [8] the authors remark that their proof is also valid when R and ν are smooth functions in the spatial variable x. However, if these coefficients are discontinuous, like the coefficients in σ, our process can not be applied. The terms involving R and ν are subject to integration by parts in many of the a priori estimates. We would obtain terms that contain the derivatives of the mollifiers, which are not uniformly bounded independently of n.

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