SHAPE OPTIMIZATION IN FREE BOUNDARY SYSTEMS

P. Neittaanmäki[1] and D. Tiba[2]*

[1]University of Jyväskylä
Department of Mathematical Information Technology
P.O. Box 35, FIN–40351 Jyväskylä, Finland
E-mail: pn@mit.jyu.fi

[2]Weierstrass Institute
Mohrenstr. 39, D–10117 Berlin, Germany
E-mail: tiba@wias-berlin.de

Abstract. We analyze existence results in constrained optimal design problems governed by variational inequalities of obstacle type. The main applications that we discuss concern the optimal packaging problem and the electrochemical machining process.

Our assumptions, in order to obtain the existence of at least one optimal domain, are just boundedness and uniform continuity (the uniform segment property) for the boundaries of the unknown regions where the free boundary problems are defined. No restrictions on the dimension are imposed.

1. Introduction

We consider the domains $C \subset \Omega \subset D$ in the Euclidean space $\mathbf{R}^d$ and some open subset $D_\Omega \subset D$ (not necessarily connected) such that $\Omega \cap D_\Omega \neq \emptyset$ has a finite number of connected components.

We define the Hilbert space:

$$V(\Omega) = \mathrm{cl}_{H^1(\Omega\setminus\overline{C})}\{v = w|_{\Omega\setminus\overline{C}} \mid w \in C_0^\infty(\mathbf{R}^d),\ w|_{Q_w\setminus K_w} = 0 \text{ for some } Q_w \text{ open and } K_w \text{ compact such that } \overline{D}_\Omega \hookrightarrow Q_w \text{ and } K_w \hookrightarrow \Omega\setminus\overline{C}\}.$$

Let $V(\Omega)_+$ denote its positive cone and

$$\Lambda(\Omega) = \{v \in V(\Omega)_+ \mid v = 1 \text{ on } \partial C\}.$$

*Permanent address: Institute of Mathematics, Romanian Academy, P.O. Box 1–764, RO–70700 Bucharest, Romania, e-mail: Dan.Tiba@imar.ro

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

We assume that $C \hookrightarrow D$ are Lipschitz domains and the above condition is imposed in the sense of traces.

Notice that, if $w_1, w_2 \in C_0^\infty(\mathbf{R}^d)$ satisfy the conditions in the definition of $V(\Omega)$ with some subsets Q_{w_1}, Q_{w_2} and K_{w_1}, K_{w_2}, then $w_1 + w_2$ will satisfy these conditions with $Q_{w_1+w_2} = Q_{w_1} \cap Q_{w_2}$ and $K_{w_1+w_2} = K_{w_1} \cup K_{w_2}$. This ensures the linear structure for $V(\Omega)$ and the closure operation gives that it is a closed subspace of $H^1(\Omega \setminus \overline{C})$, inheriting the same norm and inner product. A related definition of $V(\Omega)$, corresponding as well to mixed boundary conditions in boundary value problems, was introduced by Gröger and Rehberg [4], under different geometric assumptions. If two different open subsets D_Ω, $\widehat{D}_\Omega$ are associated to Ω, it follows, in general, that the spaces $V(\Omega)$, $\widehat{V}(\Omega)$, constructed from them, are different. However, if $D_\Omega \cap \partial\Omega = \widehat{D}_\Omega \cap \partial\Omega$, we may obtain two different representations for the same space $V(\Omega)$.

In $\Omega \setminus \overline{C}$, we define the following variational inequality of obstacle type, for $y \in \Lambda(\Omega)$:

$$\int_{\Omega\setminus C} \{\nabla y(\nabla y - \nabla v) + y(y - v)\}\, dx \le \int_{\Omega\setminus C} f(y - v)\, dx, \quad \forall v \in \Lambda(\Omega). \tag{1.1}$$

If Ω is a Lipschitz domain, the definition of $\Lambda(\Omega)$ and (1.1) correspond to the following boundary conditions of mixed type:

$$\begin{aligned} y &= 1 \quad \text{on } \partial C, \\ y &= 0 \quad \text{on } \partial\Omega \cap D_\Omega, \\ \frac{\partial y}{\partial n} &= 0 \quad \text{on } \partial\Omega \setminus D_\Omega. \end{aligned} \tag{1.2}$$

However, much weaker assumptions will be imposed in the sequel on $\partial\Omega$ (see Section 2). The existence of a unique solution $y \in \Lambda(\Omega)$ to (1.1) is standard.

A physical example described by (1.1), (1.2) is the electrochemical machining process, with a constant tension applied on ∂C, with the part $\partial\Omega \setminus D_\Omega$ of the boundary being insulated and with the metal piece being formed around $\partial\Omega \cap D_\Omega$. The variational inequality (1.1) provides the shape of the metal piece via the coincidence set $\{x \in \Omega \setminus \overline{C} \mid y(x) = 0\} = I_y$. Complete descriptions of the model may be found in Elliott and Ockendon [3] and in Barbu [2, p. 165].

The aim of the present work is to study shape optimization problems associated to (1.1). It is obvious that the form of the obtained metal piece depends on Ω and it is also of interest to find a domain Ω with minimal area that satisfies the physical requirements. Therefore, one mathematical formulation of interest is:

$$\text{Min} \int_{\Omega\setminus C} dx \tag{1.3}$$

subject to (1.1) and to the constraints

$$y \le 0 \quad \text{a.e. in } E \setminus C. \tag{1.4}$$

The minimization parameter in (1.3) is $\Omega \in \mathcal{O}$, a class of domains to be precised in the next section. The set $E \subset D$ from (1.4) is fixed and represents the desired shape of the metal piece and $\Omega \supset E \cup C$ for any $\Omega \in \mathcal{O}$. Taking into account the definition of $\Lambda(\Omega)$, it yields that

$$y = 0 \quad \text{a.e. in } E \setminus C,$$

i.e. (1.4) says that $E \setminus C \subset I_y$.

Finally, the physical interpretation of the optimal design problem (1.3), (1.4), (1.1) is to find the domain $\Omega \in \mathcal{O}$ of minimal measure such that the metal piece, obtained via the electrochemical machining process in $\Omega \setminus C$, includes the set $E \setminus C$.

In Section 2, we introduce the family $\mathcal{O}$ of nonsmooth domains and we study its convergence properties. Section 3 is devoted to the continuity of the solution of the free boundary problem (1.1), with respect to domain perturbations. As a corollary, we obtain the main result of this paper establishing the existence of at least one optimal domain for the problem (1.3), (1.4), (1.1). Other possible applications are mentioned as well. Recently, Tiba [10], Liu, Neittaanmäki and Tiba [7] proved similar results in the case of linear elliptic equations with Dirichlet or Neumann boundary conditions. A control problem for free boundary problems in a fixed domain and with mixed boundary conditions was studied by Barbu [2, p. 238].

2. Domain convergence

We define now the class $\mathcal{O}$ of admissible domains in $\mathbf{R}^d$, for the problem (1.3). We say that an open set $\Omega \subset \mathbf{R}^d$ is of class C if there is a family $\mathcal{F}_\Omega$ of continuous functions $g\colon \overline{S(0,k_\Omega)} \to \mathbf{R}$, with $S(0,k_\Omega) \subset \mathbf{R}^{d-1}$ being the open ball of center 0 and radius $k_\Omega > 0$, such that

$$\partial\Omega = \bigcup_{g \in \mathcal{F}_\Omega} \{R_g(s,0) + o_g + g(s)y_g \mid s \in S(0,k_\Omega)\}. \tag{2.1}$$

In relation (2.1), $o_g \in D$ is a translation vector and $R_g\colon \mathbf{R}^d \to \mathbf{R}^d$ is a rotation operator, associated with the local chart $g \in \mathcal{F}_\Omega$. The "vertical" vector y_g of the local system of axes, centered in o_g, is given by $y_g = R_g(0,0,...,0,1)$ and has length 1. This is a slight modification of the standard definition from Maz'ya [8].

We shall also assume that there is $r_\Omega \in]0,k_\Omega[$ such that

$$\partial\Omega = \bigcup_{g \in \mathcal{F}_\Omega} \{R_g(s,0) + o_g + g(s)y_g \mid s \in \overline{S(0,r_\Omega)}\}. \tag{2.2}$$

To simplify the notation, in the sequel, we shall write directly the local coordinates $(s,g(s))$ corresponding to the local chart g and the other elements R_g, o_g, y_g are no more mentioned. For any Ω of class C, constants $k_\Omega > r_\Omega > 0$ can be found. The segment property, Adams [1], is also true:

$$(s,g(s)-t) \in \Omega, \qquad t \in]0,a_\Omega[,\ s \in S(0,k_\Omega), \tag{2.3}$$

$$(s,g(s)+t) \in D \setminus \overline{\Omega}, \quad t \in]0,a_\Omega[,\ s \in S(0,k_\Omega). \tag{2.4}$$

Here $a_\Omega > 0$ is a constant depending on Ω.

For any Ω, we associate $D_\Omega \subset D$, open subset, and we assume that each of its (finitely many) connected components, denoted by D_Ω^i, $i = \overline{1, i_\Omega}$, are of class C as described in (2.1)–(2.4) with families of local charts denoted by $\mathcal{F}_\Omega^i$, $i = \overline{1, i_\Omega}$.

The last descriptive assumption is a compatibility condition on the intersections between Ω and D_Ω^i, $i = \overline{1, i_\Omega}$ (assumed to be nonvoid). If $x \in \partial\Omega \cap \partial D_\Omega^{i_0}$, there is V_x, a neighbourhood of x, such that $\partial(\Omega \cup D_\Omega^{i_0}) \cap V_x$ and $\partial(\Omega \cup C_{\overline{D}_\Omega^{i_0}}) \cap V_x$ can be represented (in the same local system of axes) by continuous functions which "extend" the representation of $\partial D_\Omega^{i_0}$ around x, from $\partial D_\Omega^{i_0} \setminus \Omega$ to the above mentioned sets. That is:

$$\partial(\Omega \cup D_\Omega^{i_0}) \cap V_x = \{(s, \tilde{g}_x(s)) \mid s \in S(0, k_\Omega^{i_0})\}, \tag{2.5}$$

$$\partial(\Omega \cup C_{\overline{D}_\Omega^{i_0}}) \cap V_x = \{(s, \hat{g}_x(s)) \mid s \in S(0, k_\Omega^{i_0})\}, \tag{2.6}$$

and $\hat{g}_x(s) \le \tilde{g}_x(s)$ for all $s \in S(0, k_\Omega^{i_0})$. They are equal on $\partial D_\Omega^{i_0} \cap [\partial(\Omega \cup D_\Omega^{i_0}) \cap V_x] = \partial D_\Omega^{i_0} \cap [\partial(\Omega \cup C_{\overline{D}_\Omega^{i_0}}) \cap V_x]$ where they coincide with some local chart $g_x \in \mathcal{F}_\Omega^{i_0}$ of $\partial D_\Omega^{i_0}$.

Moreover,

$$\begin{aligned} z \in \partial\Omega \setminus \bigcup_x \Big[&\{(s, \tilde{g}_x(s)) \mid s \in \overline{S(0, r_\Omega^{i_0})}\} \cup \\ &\cup \{(s, \hat{g}_x(s)) \mid s \in \overline{S(0, r_\Omega^{i_0})}\}\Big] \Rightarrow \operatorname{dist}(z, D_\Omega^{i_0}) \ge c_\Omega^{i_0} > 0. \end{aligned} \tag{2.7}$$

Here the union is taken with respect to all $x \in \partial\Omega \cap \partial D_\Omega$. The inside and outside segment property is also valid in (2.5), (2.6) with segments of length a_Ω. The conditions (2.5)–(2.7) show, in particular, that $\Omega \cup D_\Omega$ and $\Omega \cup C_{\overline{D}_\Omega}$ are of class C.

The family $\mathcal{O}$ that we consider consists of all domains Ω contained in D such that the corresponding open sets D_Ω are also contained in the bounded domain D and such that all the constants appearing in (2.1)–(2.7) satisfy $a_\Omega \ge a > 0$, $c_\Omega^i > c > 0$, $k_\Omega \ge k > 0$, $k_\Omega^i \ge k > 0$, $r_\Omega \le r$, $r_\Omega^i \le r$, $0 < r < k$. The associated continuous mappings from $\mathcal{F} = \bigcup_{\Omega \in \mathcal{O}} \left[\mathcal{F}_\Omega \cup \bigcup_{i=1}^{i_\Omega} \mathcal{F}_\Omega^i\right]$ that give the local representations in (2.1)–(2.7) are all supposed to be bounded and equiuniformly continuous.

Remark. The hypotheses (2.5), (2.6) should be understood in the context that the open sets $D_\Omega = \bigcup_{i=1}^{i_\Omega} D_\Omega^i$ are to be chosen just in order to specify the part of $\partial\Omega$ where the generalized Dirichlet condition is valid. This gives a big flexibility in checking (2.5), (2.6). Simple examples in $\mathbf{R}^2$ show that, if the intersection of $\partial\Omega$ with small balls centered in $x \in \partial\Omega \cap \partial D_\Omega$ is convex, then (2.5), (2.6) are fulfilled.

We shall prove that the class $\mathcal{O}$ of domains, with uniform constants and equi uniformly continuous local charts is compact with respect to the Hausdorff–Pompeiu distance, applied to the complementary sets:

$$\rho(\Omega_1, \Omega_2) = \operatorname{dist}(\overline{D} \setminus \Omega_1, \overline{D} \setminus \Omega_2), \quad \forall \Omega_1, \Omega_2 \in \mathcal{O} \tag{2.8}$$

(see Kuratowski [5], Pironneau [9]). We denote by Hlim or $\xrightarrow{H}$, the limit in the sense of (2.8). It is known that, if $\{\Omega_n\}$ is bounded, on a subsequence we may assume that $\Omega_{n_l} \xrightarrow{H} \Omega_0$ (some open bounded set) for $l \to \infty$. If $\text{dist}(\overline{\Omega}_0, \overline{\Omega}_n) + \text{dist}(\overline{\overline{\Omega}}_0, \overline{\overline{\Omega}}_n) \to 0$, then $\text{dist}(\overline{\Omega}_0 \cup \overline{\overline{\Omega}}_0, \overline{\Omega}_n \cup \overline{\overline{\Omega}}_n) \to 0$ for $n \to \infty$.

Proposition 2.1. *If $\Omega_n \xrightarrow{H} \Omega_0$, it yields that $D \setminus \overline{\Omega}_n \xrightarrow{H} D \setminus \overline{\Omega}_0$.*

Proof. We have that $\text{dist}(\overline{D} \setminus \Omega_n, \overline{D} \setminus \Omega_0) \to 0$ and we assume that $\text{dist}(\overline{\Omega}_n, P) \to 0$ for some closed set $P \hookrightarrow D$ (which exists by the compactness property of the Hausdorff–Pompeiu distance). We also have $\overline{D} = (\overline{D} \setminus \Omega_n) \cup \overline{\Omega}_n = \lim[(\overline{D} \setminus \Omega_n) \cup \overline{\Omega}_n] = (\overline{D} \setminus \Omega_0) \cup P$. This shows that $\Omega_0 \subset P$. We show that $P = \overline{\Omega}_0$, by contradiction. Assume that there is $\hat{x} \in P \setminus \overline{\Omega}_0$. Then $\text{dist}(\hat{x}, \overline{\Omega}_0) > 0$ and there is $\lambda > 0$ such that the ball $B(\hat{x}, \lambda) \subset D \setminus \overline{\Omega}_0$. By the assumption of uniform continuity of the local charts for $\Omega \in \mathcal{O}$ and the first convergence, it yields that $B(\hat{x}, \lambda/2) \subset D \setminus \overline{\Omega}_n$ for $n \geq \hat{n}$.

But $\hat{x} \in P$ gives that there is $x_n \in \overline{\Omega}_n$ such that $x_n \to \hat{x}$, by the definition of P. This contradicts the above statement that $B(\hat{x}, \lambda/2) \subset D \setminus \overline{\Omega}_n$ for $n \geq \hat{n}$ and the proof is finished.

Remark. The statement of Proposition 2.1 is not true if $\partial\Omega_n$ are not uniformly continuous as simple examples with oscillating boundaries show.

Theorem 2.2. *If $\Omega_n \in \mathcal{O}$ and $\Omega_n \xrightarrow{H} \Omega_0$, then $\Omega_0 \in \mathcal{O}$ and the characteristic functions satisfy $\chi_{\Omega_n} \to \chi_{\Omega_0}$ a.e. in D.*

Proof (sketch). It is known that Ω_0 is of class C and fulfils conditions (2.1)–(2.4) with the same constants k, r, a as Ω_n (see Liu, Neittaanmäki and Tiba [7], Tiba [10]). The same follows for the associated domains D_{Ω_n} and we may assume that $D_{\Omega_n} \xrightarrow{H} D_{\Omega_0}$ (at least on a subsequence) with D_{Ω_0} of class C (D_{Ω_0} is a new notation). The convergence of the characteristic functions is again obtained as in the above mentioned works.

Here, we just indicate the proof of the stability of the compatibility properties (2.5), (2.6) with respect to the passage to the limit.

Let us take an arbitrary $x \in \partial\Omega_0 \cap \partial D^{i_0}_{\Omega_0}$, $1 \leq i_0 \leq i_{\Omega_0}$, with i_{Ω_0} being the number of the connected components of D_{Ω_0}, which is finite since the interior segment property gives a measure bounded from below for each $D^i_{\Omega_0}$. By the definition of the convergence, there is $x_n \in \partial\Omega_n$ and $x_n \to x$.

Assuming that x_n cannot be represented in the restricted local charts around $\partial\Omega_n \cap \partial D_{\Omega_n}$, we get that $\text{dist}(x_n, D_{\Omega_n}) \geq c > 0$, for all n. This is absurd, therefore there is n_x such that, for $n \geq n_x$, we have

$$x_n = (s_n, g_n(s_n)), \quad s_n \in \overline{S(0,r)} \tag{2.9}$$

and $g_n \colon S(0,k) \to \mathbf{R}$ is some appropriate compatibility local chart around $\partial\Omega_n \cap \partial D_{\Omega_n}$, as defined in (2.5) or in (2.6). We consider both types of local charts from (2.5) and (2.6) and we denote them by $\tilde{g}_n$, respectively $\hat{g}_n$, defined on $S(0,k)$ and such that (2.9) is valid for $g_n = \tilde{g}_n$ or for $g_n = \hat{g}_n$. By the uniform continuity and boundedness assumptions, we may suppose that $\tilde{g}_n \to \tilde{g}$, $\hat{g}_n \to \hat{g}$ and $\hat{g}(s) \le \tilde{g}(s)$ in $S(0,k)$.

By the assumptions (2.1)–(2.5), $\Omega_n \cup D_{\Omega_n}$ are domains of class C, with uniform continuous local charts and uniform constants. By Proposition 2.1, we have that $\Omega_0 \cup D_{\Omega_0}$ is of class C and satisfies (2.1)–(2.5) with the same constants k, r, a. The same argument may be used for (2.6). The stability of (2.7) follows by a direct passage to the limit and this ends the proof.

Remark. The estimate which ensures the stability of the segment property may be found in Tiba [10], Liu, Neittaanmäki and Tiba [7].

Proposition 2.3. *Let $\Omega_n \overset{H}{\to} \Omega_0$ and $D_{\Omega_n} \overset{H}{\to} D_{\Omega_0}$ and let $K \subset \Omega_0 \cup D_{\Omega_0}$ be any compact subset. There is $\hat{n} = n(k)$ such that $K \subset \Omega_n \cup D_{\Omega_n}$ for $n \ge \hat{n}$.*

Proof. We have that $\Omega_n \cup D_{\Omega_n} \overset{H}{\to} \Omega_0 \cup D_{\Omega_0}$ according to the previous results. Then, the conclusion is a special case of the Γ-property for the metric (2.8), Pironneau [9], Liu, Neittaanmäki and Tiba [7].

3. Optimization

We study first the continuous dependence on the domain of the generalized solution to (1.1). We shall need the following characterization of the space $V(\Omega)$:

Theorem 3.1. *Let $v \in V(\Omega)$ and $\tilde{v}$ be its extension by 0 to $(\Omega \setminus \overline{C}) \cup D_\Omega$. Then $\tilde{v} \in H^1((\Omega \setminus \overline{C}) \cup D_\Omega)$. Conversely, if Ω satisfies (2.1)–(2.7) and $w \in H^1((\Omega \setminus \overline{C}) \cup D_\Omega)$ is such that $w = 0$ a.e. in $D_\Omega \setminus \Omega$, then $w|_{\Omega \setminus \overline{C}} \in V(\Omega)$.*

Proof. $v \in V(\Omega) \Rightarrow v = \lim v_n|_{\Omega\setminus\overline{C}}$ in $H^1(\Omega \setminus \overline{C})$ and $v_n \in C_0^\infty(\mathbf{R}^d)$, $v_n = 0$ in $Q_n \setminus K_n$, $Q_n \hookleftarrow D_\Omega$, $K_n \hookrightarrow \Omega$. Then, $v_n = 0$ in $D_\Omega \setminus K_n$ and in $D_\Omega \setminus \Omega$. Consequently, $v_n|_{D_\Omega \cup (\Omega\setminus\overline{C})}$ is a Cauchy sequence in $H^1((\Omega \setminus \overline{C}) \cup D_\Omega)$. We denote its limit by $\tilde{v} \in H^1((\Omega \setminus \overline{C}) \cup D_\Omega)$ and we have $v = \tilde{v}|_{\Omega\setminus\overline{C}}$, $\tilde{v}|_{D_\Omega\setminus\Omega} = 0$.

Conversely, we consider an open covering of $\partial\Omega$ given by a finite number of neighbourhoods V_j, $j = \overline{1,l}$, which contains a covering of $\partial\Omega \cap \partial D_\Omega$ as indicated in (2.5), (2.6) via neighbourhoods V_x.

We take one more open set $V_0 \hookrightarrow \Omega$ such that

$$\Omega \subset \bigcup_{j=0}^{l} V_j. \tag{3.1}$$

To this covering, we associate a partition of unity $\psi_j \in C_0^\infty(\mathbf{R}^d)$, $0 \le \psi_j \le 1$, $\operatorname{supp} \psi_j \subset V_j$ and

$$\sum_{j=0}^{l} \psi_j(x) = 1, \quad x \in \Omega. \tag{3.2}$$

We denote by $w_j(x) = \psi_j(x) w(x)$, $x \in \Omega$, with $w_0 \in V(\Omega)$, clearly. We show that this is valid for all w_j, $j = \overline{1,l}$.

There are three situations to be discussed. If $\text{dist}(V_j, D_\Omega) > 0$, we extend w_j by 0 to $\mathbf{R}^d$ (denoted by $\tilde{w}_j$) and $\tilde{w}_j \in H^1(\mathbf{R}^d \setminus (V_j \cap \partial\Omega))$. We perform a translation in the outside direction to $\partial\Omega$ corresponding to V_j (which exists since $\partial\Omega$ is of class C) and we denote by $\tilde{w}_j^t \in H^1(\mathbf{R}^d \setminus (V_j \cap \partial\Omega)_t)$ the shifted mapping (with $(V_j \cap \partial\Omega)_t$ being a translation of $V_j \cap \partial\Omega$ by the same vector and $t > 0$ being the length of the translation vector). For small t, it follows that $\text{dist}((V_j \cap \partial\Omega)_t, \overline{D}_\Omega) > 0$ and $\text{dist}((V_j \cap \partial\Omega)_t, \overline{\Omega}) > 0$. Then, $\tilde{w}_j^t|_\Omega \in H^1(\Omega)$ and $\lim_{t\to 0} \tilde{w}_j^t|_\Omega = \tilde{w}_j|_\Omega = w_j$ in $H^1(\Omega \setminus \overline{C})$, by the continuity in the mean of the integral.

The above mentioned positivity of distances allow to take a smoothing $\tilde{w}_j^{t,\varepsilon}$ of $\tilde{w}_j^t$ with a Friedrichs mollifier of order $\varepsilon > 0$. Then $\tilde{w}_j^{t,\varepsilon} \in C_0^\infty(\mathbf{R}^d)$ and the above convergences are preserved since $\tilde{w}_j^{t,\varepsilon} \to \tilde{w}_j^t$ strongly in $H^1(\Omega \setminus \overline{C})$ for $\varepsilon \to 0$. Moreover, $\tilde{w}_j^{t,\varepsilon} \in V(\Omega)$ for ε small, by its construction. This gives that $w_j \in V(\Omega)$ in this case.

If V_j is of type V_x, $x \in \partial\Omega \cap \partial D_\Omega$, we make a translation in the "vertical" direction of the local chart provided by hypotheses (2.5), (2.6), after the extension by 0 to $\tilde{w}_j$ (as before). The obtained shifted mapping, again denoted by $\tilde{w}_j^t \in H^1(\mathbf{R}^d \setminus (\partial\Omega \cap V_j)_t)$ and its support is at positive distance from $\overline{D}_\Omega \setminus \Omega$. Then, $\tilde{w}_j^t|_\Omega \in H^1(\Omega)$ and the regularization $\tilde{w}_j^{t,\varepsilon}$ will preserve these properties, for ε small. The convergence discussion is as in the previous case and we again obtain $w_j \in V(\Omega)$. In the last case, we may assume that $V_j \subset D_\Omega \cap \Omega$. The $\tilde{v}_j$, the extension by 0 of w_j to $\mathbf{R}^d$ satisfies $\tilde{w}_j \in H^1(\mathbf{R}^d)$. We perform a translation in the direction interior to $\Omega \cup C_{D_\Omega}$ given by (2.6). Then a smoothing can be performed without perturbing the support properties. The obtained mapping, again denoted by $\tilde{w}_j^{t,\varepsilon} \in C_0^\infty(\mathbf{R}^d) \cap V(\Omega)$ and is convergent to w_j in $H^1(\Omega \setminus \overline{C})$. Therefore, in all possible cases, $w_j \in V(\Omega)$, which yields that $w \in V(\Omega)$ as well, by (3.2). This ends the proof.

Remark. Theorem 3.1 is a direct extension of the classical result on the density of $C_0^\infty(\mathbf{R}^d)$ in $H^1(\Omega)$ if Ω is of class C, Adams [1].

Theorem 3.2. *Let $\Omega_n \in \mathcal{O}$ and $\Omega_n \to \Omega_0$ in the sense of (2.8). Denote by $y_n \in \Lambda(\Omega_n)$ and by $y_0 \in \Lambda(\Omega_0)$, the solutions of (1.1) associated to Ω_n, respectively Ω_0. Then, for any $K \subset \Omega_0 \setminus \overline{C}$ compact, there is n_k such that $K \subset \Omega_n \setminus \overline{C}$ for $n \geq n_k$ and $y_0|_K = \lim y_n|_K$ in $H^1(K)$ weak.*

Proof. The first statement follows by the Γ-property as in Proposition 2.3. For $n \geq n_k$, we can rewrite (1.1) as follows

$$(3.3)\quad \int_{K\setminus C} [\nabla y_n(\nabla y_n - \nabla v) + y_n(y_n - v)] - \int_{K\setminus C} (y_n - v) f \leq$$
$$\leq - \int_{(\Omega_n\setminus C)\setminus K} [\nabla y_n(\nabla y_n - \nabla v) + y_n(y_n - v)] + \int_{(\Omega_n\setminus C)\setminus K} (y_n - v) f$$

for any $v \in \Lambda(\Omega_n)$. In the sequel, we shall work with test functions from $C^1(\overline{D}) \cap \Lambda(\Omega_0)$, obtained by the definition of $V(\Omega_0)$. They have the restrictions to $\Omega_n \setminus \overline{C}$

in $\Lambda(\Omega_n)$ since, in particular, they are 0 on $D_{\Omega_n} \setminus \Omega_n$ for n big enough (by the convergence of Ω_n).

Let us notice that $\{|y_n|_{H^1(\Omega_n \setminus \overline{C})}\}$ is bounded due to the coercivity of the elliptic operator, independently with respect to n.

We construct a mapping $\tilde{y} \in H^1(\Omega_0 \setminus \overline{C})$ by taking a sequence of compacts $K_m \hookrightarrow \Omega_0 \setminus \overline{C}$, $\cup_{m=1}^{\infty} K_m = \Omega_0 \setminus \overline{C}$ and such that

$$\tilde{y}|_{K_m} = \lim_{n\to\infty} y_n|_{K_m}, \quad n \geq n_m$$

in the weak topology of $H^1(K_m)$.

By the weak lower semicontinuity of the norm, we have:

$$\liminf \int_{K\setminus C} [\nabla y_n(\nabla y_n - \nabla v) + y_n(y_n - v)] \geq \geq \int_{K\setminus C} [\nabla \tilde{y}(\nabla \tilde{y} - \nabla v) + \tilde{y}(\tilde{y} - v)], \tag{3.4}$$

$$\int_{K\setminus C} (y_n - v)f \to \int_{K\setminus C} (\tilde{y} - v)f. \tag{3.5}$$

If $v \in C^1(\overline{D}) \cap \Lambda(\Omega_0)$, the right-hand side of (3.3) can be estimated as follows

$$-\int_{(\Omega_n\setminus C)\setminus K} [\nabla y_n(\nabla y_n - \nabla v) + y_n(y_n - v)] + \int_{(\Omega_n\setminus C)\setminus K} (y_n - v)f \leq \leq \int_{(\Omega_n\setminus C)\setminus K} (\nabla y_n \nabla v + y_n v) + \int_{(\Omega_n\setminus C)\setminus K} (y_n - v)f,$$

(since the quadratic terms are positive),

$$\int_{(\Omega_n\setminus C)\setminus K} (\nabla y_n \nabla v + y_n v + y_n f - vf) \leq \leq |v|_{C^1(\overline{D})} \int_{(\Omega_n\setminus C)\setminus K} [|\nabla y_n| + |y_n| + |f|] \leq \leq M|v|_{C^1(\overline{D})} \operatorname{meas}[(\Omega_n \setminus C) \setminus K]^{\frac{1}{2}} \tag{3.6}$$

(by the Hölder inequality). Notice that, by the convergence of the characteristic functions (see Theorem 2.2), we have

$$\operatorname{meas}[(\Omega_n \setminus C) \setminus K]^{\frac{1}{2}} \to \operatorname{meas}[(\Omega_0 \setminus C) \setminus K]^{\frac{1}{2}}. \tag{3.7}$$

Combining (3.4)–(3.7), we can pass to the limit in (3.3) to obtain

$$\int_{K\setminus C} [\nabla \tilde{y}(\nabla \tilde{y} - \nabla v) + \tilde{y}(\tilde{y} - v)] - \int_{K\setminus C} f(\tilde{y} - v) \leq \leq M|v|_{C^1(\overline{D})} \operatorname{meas}[(\Omega_0 \setminus C) \setminus K]^{\frac{1}{2}}. \tag{3.8}$$

In (3.8), we can take a sequence of compacts $K_m \to \Omega_0 \setminus \overline{C}$, as described above, and we get

$$\int_{\Omega_0 \setminus C} [\nabla \tilde{y}(\nabla \tilde{y} - \nabla v) + \tilde{y}(\tilde{y} - v)] \le \int_{\Omega_0 \setminus C} f(\tilde{y} - v) \tag{3.9}$$

for any $v \in C^1(\overline{D}) \cap \Lambda(\Omega_0)$.

Since C is fixed and Lipschitz, we have that $\tilde{y}|_{\partial C} = 1$, by the trace theorem. It is also clear that $\tilde{y} \in H^1(\Omega_0 \setminus C)_+$ due to the positivity of y_n.

By the definition of $V(\Omega_0)$, $C^1(\overline{D}) \cap \Lambda(\Omega_0)$ is dense in $\Lambda(\Omega_0)$ (the positive part is continuous in H^1 and smoothing preserves positivity) and (3.9) is valid for any $v \in \Lambda(\Omega_0)$ since we can pass to the limit with respect to v in the $H^1(\Omega_0 \setminus C)$ topology.

As $y_n \in V(\Omega_n)$, they may be extended by 0 to $(\Omega_n \setminus \overline{C}) \cup D_{\Omega_n}$, which, in turn, converges to $(\Omega_0 \setminus \overline{C}) \cup D_{\Omega_0}$ in the complementary Hausdorff–Pompeiu topology. Denote by $\hat{y}_n$ these extensions and, clearly, if $K \hookrightarrow (\Omega_0 \setminus \overline{C}) \cup D_{\Omega_0}$ is compact $\hat{y}_n \to \hat{y}$ in $H^1(K)$, weakly (see Proposition 2.3). In this way, we construct a mapping $\hat{y} \in H^1((\Omega_0 \setminus \overline{C}) \cup D_{\Omega_0})$ such that $\hat{y}|_{D_{\Omega_0} \setminus \Omega_0} = 0$ and $\hat{y}|_{\Omega_0 \setminus \overline{C}} = \tilde{y}$. Theorem 3.1 gives, then, that $\tilde{y} \in V(\Omega_0)$, which ends the proof.

Remark. By Theorem 3.2, we see that variational inequalities of obstacle type (and the corresponding free boundary problems) are wellposed with respect to domain perturbations, even in the case of mixed boundary conditions!

Corollary 3.3. *The shape optimization problem (1.3), (1.4), (1.1) has at least one minimizer* $\Omega^* \in \mathcal{O}$.

Proof. If $\{\Omega_n\}$ is a minimizing sequence, we may assume that $\Omega_n \to \Omega_0 \in \mathcal{O}$ in the Hausdorff–Pompeiu complementary topology. We have the convergence of the cost values (to the minimum value) due to the convergence of the associated characteristic functions. We also have the convergence of the corresponding solutions to (1.1), (1.2), as explained to the limit in (1.4), i.e. Ω_0 is admissible.

Remark. More general cost functionals can be as well studied, Tiba [10], Liu and Rubio [6].

Finally, we indicate an example with another possible application, the optimal packaging problem, Tiba [11].

Consider a variable membrane $\Omega \subset D$ in possible contact with a rigid obstacle G described by some function $\varphi \in H^1(D)$. We introduce the (nonvoid) closed convex set

$$K(\Omega) = \{v \in H_0^1(\Omega) \mid v \ge \varphi \text{ a.e. in } \Omega\}$$

and the variational inequality describing the membrane deflection $u(\Omega)$

$$\int_\Omega \nabla u(\Omega) \cdot \nabla [u(\Omega) - v]\, dx \le \int_\Omega f(u(\Omega) - v)\, dx,$$

for any $v \in K(\Omega)$ and with $u(\Omega) \in K(\Omega)$ and $f \in L^2(D)$ being the applied load.

We denote by $z(\Omega) = \{x \in \Omega \mid u(\Omega)(x) = \varphi(x)\}$ the contact region (equivalently, the coincidence set).

The optimization problem is again the minimization of the measure of $\Omega \in \mathcal{O}$

$$\mathrm{Min} \int_{\Omega} dx$$

such that the set $z(\Omega) \supset E$, a given subset with $E \subset \Omega$ for any $\Omega \in \mathcal{O}$. Obviously, this is a variant of the problem (1.3), (1.4), (1.1) and similar results may be proved.

References

1. Adams, R., *Sobolev spaces*, Academic Press, New York, 1975.
2. Barbu, V., *Optimal control of variational inequalities*, RNM 100, Pitman, London, 1984.
3. Elliott, C. M. and Ockendon, J. R., *Weak and variational methods for moving boundary problems*, RNM 59, Pitman, London, 1982.
4. Gröger, K. and Rehberg, J., *Resolvent estimates in $W^{-1,p}$ for second order elliptic differential operators in case of mixed boundary conditions*, Math. Ann. **285** (1989), 105–113.
5. Kuratowski, K., *Introduction to set theory and topology*, Pergamon Press, Oxford, 1961.
6. Liu, W. B. and Rubio, J. E., *Local convergence and optimal shape design*, SIAM J. Control Optim. **30** (1992), 49–62.
7. Liu, W. B., Neittaanmäki, P. and Tiba, D., *Existence for shape optimization problems in arbitrary dimension*, Preprint 208, Dept. of Mathematics, Univ. of Jyväskylä (1999).
8. Maz'ya, V., *Sobolev spaces*, Springer, Berlin, 1985.
9. Pironneau, O., *Optimal shape design for elliptic systems*, Springer, Berlin, 1984.
10. Tiba, D., *A property of Sobolev spaces and existence in optimal design*, Preprint 5, Institute of Mathematics, Romanian Academy, Bucharest (1999).
11. Tiba, D., *Optimal control of nonsmooth distributed parameter systems*, LNM 1459, Springer, Berlin, 1990.