Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

To the uniqueness problem for nonlinear elliptic equations

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submitted: 29th October 1999

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Preprint No. 527
Berlin 1999

1991 Mathematics Subject Classification. 35B45, 35J25, 35J60.

Key words and phrases. Nonlinear elliptic equations, bounded solutions, uniqueness, nonstandard asymptions.

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Abstract

We prove existence and uniqueness of solutions $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ to equations of the form

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\rho(u) b_i(x, \frac{\partial u}{\partial x})] + a(x, u, \frac{\partial u}{\partial x}) = 0, \quad x \in \Omega.$$

Our nonstandard assumptions on the coefficients are such that $\log \rho(u)$ is concave and $\frac{a(x,u,\xi)}{\rho(u)}$ is increasing in u. Such assumptions are natural in view of drift diffusion processes for example in semiconductors and chemotaxis.

1 Introduction

We study existence and uniqueness of weak solutions of the problem

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\rho(u) b_i(x, \frac{\partial u}{\partial x})] + a(x, u, \frac{\partial u}{\partial x}) = 0, \quad x \in \Omega \subset I\!R^n, \tag{1}$$

$$u(x) = f(x), \quad x \in \partial\Omega. \tag{2}$$

This problem is the stationary variant of nonlinear problems that have been studied extensively by many authors (see for example papers of H. W. Alt and S. Luckhaus [1], F. Benilan and P. Wittbold [2], F. Otto [9], H. Gajewski and K. Gröger [4]). Many applied problems, especially drift-diffusion processes in porous media and semiconductors are modelled by such type of equations. We consider the problem (1), (2) under standard conditions for the functions $b(x,\xi)$, $a(x,u,\xi)$ to be formulated in Section 2. Our main specific assumptions are the following:

ρ) $\rho \in (I\!R^1 \to I\!R^1)$ *with* $\rho(u) > 0,\ u \in I\!R^1,$ *is continuous and has a piecewise continuous derivative* ρ' *such that* $\dfrac{\rho'(u)}{\rho(u)}$ *is nonincreasing on* $I\!R^1$;

a) $\dfrac{a(x,u,\xi)}{\rho(u)}$ *is nondecreasing with respect to* $u \in I\!R^1,$ *for arbitrary* $x \in \Omega,\ \xi \in I\!R^n.$

A special uniqueness result for problem (1), (2) was obtained in [3] by showing that (1) defines a so-called E-monotone operator, provided that (i) $\log \rho(u)$ is concave and (ii) $a = a(x,u)$ is nonnegative and ρ is nonincreasing or $a(x,u)$ is nonpositive and ρ is nondecreasing. Moreover, in [3] it was pointed out that such conditions resp. E-monotonicity imply uniqueness for drift-diffusion-reaction equations describing charge transport in semiconductors [4] or chemotaxis [5].

We consider the problem (1), (2) with a boundary function f satisfying

$$f \in W^{1,2}(\Omega) \cap L^\infty(\Omega). \tag{3}$$

Definition 1 *A function* $u \in W^{1,2}(\Omega)$ *is called solution of (1), (2) if*

$$\int_\Omega \rho(u) |\frac{\partial u}{\partial x}|^2 \, dx < \infty, \quad u - f \in W_0^{1,2}(\Omega) \tag{4}$$

and equation (1) is satisfied in the sense of distributions.

This definition will be justified in Section 2.

There we also prove a priori estimates of solutions u to (1), (2) in the $W^{1,2}(\Omega)$-norm. An $L^\infty(\Omega)$ estimate for u is given in Section 3. Using both these estimate we establish in Section 4 the solvability of the problem (1), (2). Our main result, uniqueness of solutions, is proved in Section 5.

The key role in our paper play special test functions ((18), (31), (60)) which us allow to analyze the behavior of solutions u on subsets of Ω, where $\rho(u)$ could tend to zero. For regular coefficients and smooth solutions uniqueness for problems like (1), (2) can be proved using results of monographs of O. A. Ladyzhenskaja, N. N. Uraltseva [7] or D. Gilbarg, N. S. Trudinger [6].

We are planing in forthcoming papers to apply our approach to problem (1), (2) with unbounded f, to corresponding parabolic problems and systems of equations describing electro-reaction-diffusion processes.

2 A priori estimate in $W^{1,2}(\Omega)$

Let Ω be a bounded open set in $I\!R^n$. Let the coefficients from (1) in addition to the specific assumptions of Section 1 satisfy:

i) $a(x,u,\xi),\ b_i(x,\xi),\ i=1,...,n,$ *are measurable with respect to* x *for every* $u \in I\!R^1,\ \xi \in I\!R^n$ *and continuous with respect to* $u \in I\!R^1,\ \xi \in I\!R^n$ *for almost every* $x \in \Omega$;

ii) *there exist positive constants* $\nu_1,\ \nu_2$ *and functions* $b_0 \in L^2(\Omega),\ a_0 \in L^p(\Omega),\ p > \frac{n}{2}$, *such that for arbitrary* $x \in \Omega,\ u \in I\!R^1,\ \xi \in I\!R^n$

$ii)_1$ $\sum_{i=1}^n b_i(x,\xi)\xi_i \geq \nu_1|\xi|^2,$

$ii)_2$ $|b_i(x,\xi)| \leq \nu_2|\xi| + b_0(x),$

$ii)_3$ $|a(x,u,\xi)| \leq \nu_2(a_0(x) + |u|^{q_1} + |\xi|^{q_1})(\rho(u)+1), \quad 0 \leq q_1 < 1.$

We note some simple consequences from condition $\rho)$: Let

$$\alpha_\pm = \lim_{u \to \pm\infty} \rho(u). \tag{5}$$

Then, for nonconstant ρ at least one of the numbers $\alpha_-,\ \alpha_+$ is zero. If $\alpha_- = 0$, then

$$\rho(u) \leq R_1 \exp(\lambda_1 u) \quad for\ u \leq 0 \tag{6}$$

holds with positive numbers $R_1,\ \lambda_1$. Analogously, if $\alpha_+ = 0$, then

$$\rho(u) \leq R_2 \exp(-\lambda_2 u) \quad for\ u \geq 0 \tag{7}$$

holds with positive numbers $R_2,\ \lambda_2$. Finally

$$\rho(u) \leq R_3 \exp(\lambda_3 |u|)$$

holds with positive numbers $R_3,\ \lambda_3$ for all $u \in I\!R^1$.

From (6), (7) we get

$$\left|\int_0^{\pm\infty} \rho(s)\ ds\right| \leq R_4, \quad if\ \alpha_\pm = 0. \tag{8}$$

Remark also that we can choose a positive number N such that

$$\pm\rho'(u) < 0 \;\; for \;\; \pm u > N, \; if \; \alpha_\pm = 0. \tag{9}$$

Besides of (1) we shall consider the regularized equation

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\sigma \frac{\partial u}{\partial x_i} + \rho(u) b_i(x, \frac{\partial u}{\partial x})] + a(x, u, \frac{\partial u}{\partial x}) = 0, \quad \sigma \in [0,1], \tag{10}$$

for proving our existence theorem in Section 4.

Accordingly Definition 1, $u \in W^{1,2}(\Omega)$ is solution of (10), (2), if condition (4) is satisfied and

$$\int_\Omega \Big\{ \sum_{i=1}^{n} \Big[\sigma \frac{\partial u}{\partial x_i} + \rho(u) b_i(x, \frac{\partial u}{\partial x})\Big] \frac{\partial \varphi}{\partial x_i} + a(x, u, \frac{\partial u}{\partial x})\varphi \Big\} \; dx = 0 \;\; \forall \varphi \in C_0^\infty(\Omega). \tag{11}$$

In order to justify this definition we have to show that this integral identity is well defined. From condition ρ) we infer the inequality

$$\rho^{1/2}(u) = \rho^{1/2}(0) + 2\int_0^u \frac{\rho'(s)}{\rho(s)} \rho^{1/2}(s) \; ds \rho^{1/2}(0) + 2|\rho'(0)\rho(0) \int_0^u \frac{\rho'(s)}{\rho(s)} \rho^{1/2}(s) \; ds|. \tag{12}$$

For a function u satisfying (4) we obtain by Sobolew's embedding theorem and (12)

$$\rho^{1/2}(u) \in L^{\frac{2n}{n-2}}(\Omega). \tag{13}$$

Now (13), (4) and condition ii) show that the integral in (11) is well defined for $\varphi \in C_0^\infty(\Omega)$.

Since $C_0^\infty(\Omega)$ lies densely in $W_0^{1,2}(\Omega, \rho)$, (11) holds actually for all $\varphi \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \rho(u)(|\varphi|^2 + |\frac{\partial \varphi}{\partial x}|^2) \; dx < \infty. \tag{14}$$

Denote

$$F_0 = ess \; sup\{|f(x)|, \; x \in \Omega\}, \quad F_1 = \|f\|_{1,2}, \tag{15}$$

where $\|\cdot\|_{1,2}$ is the norm in $W^{1,2}(\Omega)$.

In what follows we will understand as known parameters all numbers from conditions ii), F_0, F_1, norms of a_0, b_0 in resp. spaces, measure of Ω, R_4 and values of the function ρ on intervals depending only on n, ν_1, ν_2, F_0, N.

Theorem 1 *Let the conditions i), ii), ρ), a), (3) be satisfied and let ρ be an unbounded function on $\mathbb{R}^1$ or $\alpha_- = \alpha_+ = 0$. Then there exists a constant M_1 depending only on known parameters and independent of σ such that each solution of (10), (2) satisfies*

$$\|u\|_{1,2} \le M_1. \tag{16}$$

PROOF:

Denote $\Omega_\pm = \{x \in \Omega : \;\; \pm[u(x) - f(x)] > 0\}$. We shall estimate the norm of $|\frac{\partial u}{\partial x}|$ in $L^2(\Omega_+)$. An estimate of this function in $L^2(\Omega_-)$ can be proved analogously.

We will use following notations

$$v_k(x) := [v(x)]_k = \min\{v(x), k\}, \; k \in \mathbb{R}^1, \;\; [v(x)]_+ = \max\{v(x), 0\}, \tag{17}$$

for an arbitrary function v defined on Ω.

Let us consider firstly the case $\alpha_+ = 0$. Inserting the test function

$$\varphi = \frac{1}{\rho(u_k)} \Big[\int_m^{u_k} \rho(s)\ ds \Big]_+, \quad k > m = \max\{F_0, N\}, \tag{18}$$

into (11) we get

$$\int_{\{m<u<k\}} \sum_{i=1}^n \Big\{ \sigma \frac{\partial u}{\partial x_i} + \rho(u) b_i(x, \frac{\partial u}{\partial x}) \Big\} \frac{\partial u}{\partial x_i} \Big\{ 1 - \frac{\rho'(u)}{\rho(u)} \varphi \Big\} dx + \int_{\{u>m\}} a(x, u, \frac{\partial u}{\partial x}) \varphi\ dx = 0, \tag{19}$$

where $\{m < u < k\} = \{x \in \Omega :\ m < u(x) < k\}$ and the set $\{u > m\}$ is analogously defined. Now condition ρ) for $u > m$ implies

$$-\frac{\rho'(u)}{\rho(u)} \int_m^u \rho(s)\ ds \geq -\int_m^u \rho'(s)\ ds = \rho(m) - \rho(u). \tag{20}$$

Further condition a) and (8), (9) imply

$$\begin{aligned} \frac{a(x,u,\frac{\partial u}{\partial x})}{\rho(u_k)} \int_m^{u_k} \rho(s)\ ds &= \frac{a(x,u,\frac{\partial u}{\partial x})}{\rho(u)} \frac{\rho(u)}{\rho(u_k)} \int_m^{u_k} \rho(s)\ ds \\ &\geq \frac{a(x,0,\frac{\partial u}{\partial x})}{\rho(0)} \frac{\rho(u)}{\rho(u_k)} \int_m^{u_k} \rho(s)\ ds \geq -R_4 \frac{|a(x,0,\frac{\partial u}{\partial x})|}{\rho(0)}. \end{aligned} \tag{21}$$

Using (20) and (21), we get from (19)

$$\rho(m) \int_{\{m<u<k\}} \sum_{i=1}^n b_i(x, \frac{\partial u}{\partial x}) \frac{\partial u}{\partial x_i}\ dx \leq c_1 \int_{\{u>m\}} |a(x, 0, \frac{\partial u}{\partial x})|\ dx, \tag{22}$$

and passing to the limit $k \to \infty$ and applying the monotone convergence theorem, we obtain

$$\int_{\{u>m\}} \sum_{i=1}^n b_i(x, \frac{\partial u}{\partial x}) \frac{\partial u}{\partial x_i}\ dx \leq c_2 \int_{\{u>m\}} |a(x, 0, \frac{\partial u}{\partial x})|\ dx. \tag{23}$$

Here and in what follows $c_l,\ l = 1, 2, ...,$ denote positive constants depending only on known parameters. Estimating the left hand side of (23) by $ii)_1$ and the right hand side of (23) by $ii)_3$ and Young's inequality we obtain

$$\int_{\{u>m\}} |\frac{\partial u}{\partial x}|^2\ dx \leq c_3. \tag{24}$$

In order to estimate the integral of $|\frac{\partial u}{\partial x}|^2$ over the set $\{f < u < m\}$, we insert the test function

$$\varphi(x) = \min\{[u(x) - f(x)]_+, m + F_0\} \tag{25}$$

into (11). We obtain by standard calculations

$$\int_{\{0<u-f<m+F_0\}} |\frac{\partial u}{\partial x}|^2\ dx \leq c_4 \int_{\{u>f\}} \Big[1 + a_0(x) + |\frac{\partial u}{\partial x}| \Big]\ dx. \tag{26}$$

Using (24), (26), we get the estimate

$$\int_{\Omega_+} |\frac{\partial u}{\partial x}|^2\ dx \leq c_3 + c_4 \int_{\{u>f\}} \Big[1 + a_0(x) + |\frac{\partial u}{\partial x}| \Big]\ dx,$$

and hence the estimate

$$\int_{\Omega_+} |\frac{\partial u}{\partial x}|^2\ dx \leq c_5, \tag{27}$$

which completed the proof for the case $\alpha_+ = 0$.

It remains to prove (24) for the case $\alpha_+ = \infty$. For this purpose we insert the test function

$$\varphi(x) = [u(x) - \mu]_+, \quad \mu > F_0 + 1,$$

into (11) and obtain after simple calculations

$$\int_{\{u>\mu\}} \rho(u)|\frac{\partial u}{\partial x}|^2 \, dx \le c_6 \int_{\{u>\mu\}} (\rho(u)+1)\Big[a_0 + |u|_1^q + |\frac{\partial u}{\partial x}|^{q_1}\Big](u-\mu) \, dx. \tag{28}$$

Now we want to estimate the terms containing $\rho(u)$ on the right hand side of (28). By the embedding theorem and condition ρ) we get

$$\begin{aligned} \int_{\{u>\mu\}} \rho(u)u^{1+q_1} \, dx &\le \mu^{q_1-1} \int_{\{u>\mu\}} \rho(u)u^2 \, dx \le 2\mu^{q_1-1} \int_{\{u>\mu\}} \Big\{|\rho^{\frac{1}{2}}(u)u - \rho^{\frac{1}{2}}(\mu)\mu|^2 + \rho(\mu)\mu^2\Big\} \, dx \\ &\le c_7\Big\{\rho(\mu)\mu^2 + \mu^{q_1-1} \int_{\{u>\mu\}} |\nabla(\rho(u)^{\frac{1}{2}}u)|^2 \, dx\Big\} \\ &\le c_7\Big\{\rho(\mu)\mu^2 + \mu^{q_1-1}(\frac{\rho'(0)}{2\rho(0)}+1)^2 \int_{\{u>\mu\}} \rho(u)|\frac{\partial u}{\partial x}|^2 \, dx\Big\}. \end{aligned}$$

Hence the desired estimate of $|\frac{\partial u}{\partial x}|$ on the set $\{u > \mu\}$ follows, provided

$$\mu \ge K \tag{29}$$

where K is a constant depending only on known parameters. Now we fix μ satisfying (24) and prove the corresponding estimate on the set $\{f < u < \mu\}$ analogously to the case $\alpha_+ = 0$. $\square$

Remark 1 *From the proof of Theorem 1 it follows that its assertion is true for bounded function ρ satisfying* $\sup\{\rho(u): \; u \in I\!R^1\} > K$ *where K is the number from (29).*

3 A priori estimate in $L^\infty(\Omega)$

In this Section we will prove a $L^\infty(\Omega)$ a priori estimate:

Theorem 2 *Let the conditions i), ii), ρ), a), (3) be satisfied. Then there exists a constant M_0 depending only on known parameters and $\|u\|_{1,2}$ being independent of σ, such that each solution of (10), (2) satisfies*

$$ess \; sup\{|u(x)|: \; x \in \Omega\} \le M_0. \tag{30}$$

PROOF:
We keep the notations of Section 2 and estimate the maximum of $|u(x)|$ on the set Ω_+. Again the proof for Ω_- runs analogously.

Starting with the case $\alpha_+ = 0$, we insert the test function

$$\varphi(x) = \frac{[u_k(x)-m]^r}{\rho(u_k(x))}\Big[\int_m^{u_k(x)} \rho(s) \, ds\Big]_+, \quad k > m, \quad r \ge 0, \tag{31}$$

into (11) and obtain

$$\begin{aligned} \int_{\{m<u<k\}} \sum_{i=1}^n \Big[\sigma|\frac{\partial u}{\partial x_i}|^2 &+ \rho(u)b_i(x, \frac{\partial u}{\partial x})\frac{\partial u}{\partial x_i}\Big] \\ &\times \Big[r\frac{[u(x)-m]^{r-1}}{\rho(u(x))}\int_m^{u(x)} \rho(s) \, ds + [u(x)-m]^r[1 - \frac{\rho'(u(x))}{\rho^2(u(x))}\int_m^{u(x)} \rho(s) \, ds]\Big] \, dx \\ &+ \int_{\{u>m\}} a(x,u,\frac{\partial u}{\partial x})\frac{[u_k(x)-m]^r}{\rho(u_k(x))}\int_m^{u_k(x)} \rho(s) \, ds \, dx = 0. \end{aligned} \tag{32}$$

Now (20) implies

$$r\frac{[u(x)-m]^{r-1}}{\rho(u(x))}\int_m^{u(x)}\rho(s)\;ds+[u(x)-m]^r[1-\frac{\rho'(u(x))}{\rho^2(u(x))}\int_m^{u(x)}\rho(s)\;ds]]$$
$$\geq\;\frac{\rho(m)}{\rho(u(x))}[u(x)-m]^r. \tag{33}$$

On the other hand (21) and $ii)_3$ give

$$\int_{\{u>m\}}a(x,u,\frac{\partial u}{\partial x})\frac{[u_k(x)-m]^r}{\rho(u_k(x))}\int_m^{u_k(x)}\rho(s)\;ds\;dx$$
$$\geq\;-c_8\int_{\{u>m\}}[a_0+|\frac{\partial u}{\partial x}|][u_k-m]^r\;dx. \tag{34}$$

Taking (33) and (34) into account, we get from (32)

$$\int_{\{m<u<k\}}[u(x)-m]^r|\frac{\partial u}{\partial x}|^2\;dx\leq c_9\int_{\{u>m\}}[u(x)-m]^r[a_0(x)+|\frac{\partial u}{\partial x}|]\;dx. \tag{35}$$

Let us denote

$$I(r)=\int_{\{u>m\}}[u(x)-m]^r|\frac{\partial u}{\partial x}|^2\;dx,\;\;J(r)=\int_{\{u>m\}}[u(x)-m]^r[a_0(x)+|\frac{\partial u}{\partial x}|]\;dx \tag{36}$$

and suppose that $J(r)<\infty$ for some r, then, by taking the limit $k\to\infty$ we see that $I(r)<\infty$. Now, by Theorem 1 we have $J(r_0)<\infty$ for $r_0=1$ and hence $I(r_0)<\infty$. This implies $[u-m]_+^{\frac{r_0}{2}+1}\in W^{1,2}(\Omega)$ and $[u-m]_+^{\frac{r_0}{2}+1}\in L^{\frac{2n}{n-2}}(\Omega)$. Thus we have $J(r_1)<\infty$ for $r_1=\gamma r_0$ with $\gamma=\min\{\frac{n-1}{n-2},\frac{n}{p'(n-2)}\}>1$. Iterating this we see that $I(r),\;J(r)$ are finite for each positive number r and

$$\int_{\{u>m\}}[u(x)-m]^r|\frac{\partial u}{\partial x}|^2\;dx\leq c_9\int_{\{u>m\}}[u(x)-m]^r[a_0(x)+|\frac{\partial u}{\partial x}|]\;dx \tag{37}$$

and hence

$$\int_{\{u>m\}}[u(x)-m]^r|\frac{\partial u}{\partial x}|^2\;dx\leq c_{10}\int_{\{u>m\}}[u(x)-m]^r[1+a_0(x)]\;dx. \tag{38}$$

From this we obtain the desired estimate for the maximum of $[u-m]_+$ by Moser's (comp. [8]) iteration technique in the case. $\alpha_+=0$.

In the case $\alpha_+>0$ we insert the test function

$$\varphi(x)=[u(x)_k-m]^{r+1},\;\;r>0, \tag{39}$$

into (35) and use Moser's iteration to prove the result. □

4 Existence

In order to prove existence of a solution to (1), (2) we must replace condition $ii)_1$, by a monotonicity condition. In view of the next section we assume a stronger condition as needed here:

$ii)^*$ *condition $ii)$ holds with*

$$ii)_1^*\;\;\sum_{i=1}^n[b_i(x,\xi)-b_i(x,\eta)](\xi_i-\eta_i)\geq\nu_1|\xi-\eta|^2,\;\;\forall\,x\in\Omega,\;\xi,\;\eta\in I\!R^n \tag{40}$$

instead of $ii)_1$.

Theorem 3 *Let the conditions i), ii)*, ρ), a), (3) be satisfied and let ρ be unbounded or* $\alpha_- = \alpha_+ = 0$. *Then the boundary value problem (1), (2) has at least one solution* $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

PROOF:
In the case that ρ is an unbounded function we shall modify the functions ρ and a in the following way: Denote

$$R_0 = \max\{\rho(u) : \ |u| \le M_0\}, \tag{41}$$

where M_0 is the constant from Theorem 2. We can choose a number m^* depending only on known parameters such that

$$\rho(u) \ge K + R_0, \ \ \pm \rho'(u) > 0 \ for \ \pm u > m^* \ if \ \alpha_\pm \ne 0, \tag{42}$$

where K is the constant from (29). Then we define functions

$$\rho^*(u) = \rho(\min\{u, m^*\}), \tag{43}$$

$$a^*(x,u,\xi) = a(x, \min\{u,m^*\}, \xi) \tag{44}$$

which satisfy the conditions ρ), a), i), ii), with the same parameters as ρ and a.

Now we consider for $t, \ \sigma \in [0,1]$ the following parametric family of boundary value problems

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}[\sigma \frac{\partial u}{\partial x_i} + \rho^*(u) b_i(x, \frac{\partial u}{\partial x})] + t a^*(x,u,\frac{\partial u}{\partial x}) = 0, \quad x \in \Omega, \tag{45}$$

$$u(x) = t f(x), \quad x \in \partial\Omega. \tag{46}$$

By Theorems 1 and 2 we get a priori estimates for solutions to (45), (46)

$$\|u\|_{1,2} \le M_1, \quad ess\ sup\{|u(x)| : \ x \in \Omega\} \le M_0 \tag{47}$$

with constants $M_0, \ M_1$ independent of $t, \ \sigma$.

From (41)-(44), (47) we see that a solution to (45), (46) with $t=1, \ \sigma = 0$ is a solution to (1), (2). We shall prove firstly existence of a solution to (45), (46) for $t = 1, \ \sigma > 0$ and after that take the limit $\sigma \to 0$, to prove Theorem 3.

For fixed number $\sigma \in (0,1]$ we consider the parametric family of operators $A_t \in (W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*), \ t \in [0,1]$, defined by

$$< A_t, v, \varphi >= \int_\Omega \Big\{ \sum_{i=1}^n \Big[\sigma \frac{\partial(tf+v)}{\partial x_i} + \rho^*(tf+v) b_i(x, \frac{\partial(tf+v)}{\partial x})\Big] \frac{\partial \varphi}{\partial x_i} + t a^*(x,u,\frac{\partial(tf+v)}{\partial x})\varphi \Big\} \, dx = 0. \tag{48}$$

Easily to check that the operator A_t satisfies the following condition:

$$\begin{aligned} &for\ arbitrary\ sequences\ v_j \in W_0^{1,2}(\Omega)\ and\ t_j \in [0,1]\ such\ that \\ &v_j \rightharpoonup v_0 \in W_0^{1,2}(\Omega)\ (weakly),\ t_j \to t_0\ and \lim_{j\to\infty} < A_{t_j} v_j, v_j - v_0 > \le 0, \\ &it\ follows\ that\ v_j \to v_0\ (strongly). \end{aligned} \tag{49}$$

That means that the operator $A_t, \ t$ fixed satisfies the condition (S_+).

For proving existence we will apply the degree theory for (S_+) operators (I. V. Skrypnik [10]). For this purpose we consider operators A_t on the ball $B = \{v \in W_0^{1,2}(\Omega) : \ \|v\|_{1,2} \le R\}$, where $R = F_1 + M_1 + 1$ and $F_1, \ M_1$ are the constants from (15), (47). By (47)

$$A_t v \ne 0 \ \ for \ \ v \in \partial B, \ \ t \in [0,1] \tag{50}$$

and consequently the family $\{A_t\}$ realizes homotopy of the operators A_0 and A_1.

Since $< A_0 v, v >> 0$ for $v \neq 0$ it follows from [10], Theorem 4.4, chapter 2, that $Deg(A_0, \bar{B}, 0) = 1$. This implies $Deg(A_1, \bar{B}, 0) = 1$ and by the principle of non-zero degree ([10], Corollary 4.1, chapter 2) the existence of a solution v to the equation $A_1 v = 0$ on B. This means that the problem (10), (2) has the solution $u = f + v$.

Consider now the sequence $\sigma_j = \frac{1}{j}$ and let $u_j \in W^{1,2}(\Omega)$ be a solution of (10), (2) for $\sigma = \sigma_j$. Then by (47)

$$\|u_j\|_{1,2} \leq M_1, \quad ess\ sup\{|u_j(x)| : \ x \in \Omega\} \leq M_0 \tag{51}$$

and we can assume that $u_j \rightharpoonup u_0 \in W^{1,2}(\Omega)$. From the condition ii)* we have

$$\|u_j - u_0\|_{1,2} \leq c_{11} \int_\Omega \sum_{i=1}^n \rho^*(u_j)\Big[b_i(x, \frac{\partial u_j}{\partial x}) - b_i(x, \frac{\partial u_0}{\partial x})\Big] \frac{\partial (u_j - u_0)}{\partial x_i}\ dx. \tag{52}$$

Using the integral identity (11) we can see that the right hand side of (52) tends to zero for $j \to \infty$. Hence $u_j \to u_0$ in $W^{1,2}(\Omega)$. Now we can pass to the limit $\sigma = \sigma_j \to 0$ in (11) in order to verify that u_0 is solution to (1), (2). $\square$

5 Uniqueness

In this section the main result of our paper is established. We need now the following local Lipschitz continuity condition:

iii) *there exist a positive nondecreasing function* $\mu \in (\mathbb{R}^1 \to \mathbb{R}^1)$ *and functions* $a_1 \in L^p(\Omega),\ a_2 \in L^{2p}(\Omega),\ p > \frac{n}{2}$, *such that*

$$|a(x,u,\xi) - a(x,v,\xi)| \leq [\mu(N) + a_1(x) + |\xi|^{\frac{2}{p}}]|u - v|, \tag{53}$$

$$|a(x,u,\xi) - a(x,u,\eta)| \leq [\mu(N) + a_2(x)]|\xi - \eta| \tag{54}$$

hold for arbitrary $N > 0$ *and* $x \in \Omega,\ |u|,\ |v| \leq N,\ \xi,\ \eta \in \mathbb{R}^n$.

Theorem 4 *Let the conditions i), ii)*, iii), ρ), a), (3) be satisfied and let* $b_i(x, 0) = 0,\ i = 1, ..., n$. *Then the boundary value problem (1), (2) has a unique solution* $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ *satisfying (4).*

PROOF:
The Theorems 1-3 guaranty existence of a bounded solution to (1), (2). Now we will assume the existence of two solutions $u_1,\ u_2 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and show that necessarily $u_1 = u_2$. By Theorems 1, 2 we have

$$\|u_j\|_{1,2} \leq M_1, \quad ess\ sup\{|u_j(x)| : \ x \in \Omega\} \leq M_0,\ j = 1, 2. \tag{55}$$

Denote $v = u_2 - u_1$ and suppose contradictorily

$$M = ess\ sup\{|v(x)| : \ x \in \Omega\} > 0. \tag{56}$$

It is sufficient to prove that the positive part $[v]_+$ of v vanishes. The functions $u_j,\ j = 1, 2$, satisfy the following integral identities

$$\int_\Omega \Big\{ \sum_{i=1}^n \Big[\rho(u_j) b_i(x, \frac{\partial u_j}{\partial x})\Big] \frac{\partial \varphi}{\partial x_i} + a(x, u_j, \frac{\partial u_j}{\partial x})\varphi \Big\}\ dx = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega), \quad j = 1, 2. \tag{57}$$

We insert the test function

$$\varphi = [v - m]_+, \quad m \in [0, M] \tag{58}$$

into (57), $j = 2$, and obtain

$$\int_{\{v>m\}} \Big\{\rho(u_1+v)\sum_{i=1}^n b_i(x,\frac{\partial(u_1+v)}{\partial x})\frac{\partial v}{\partial x_i} + a(x,u_1+v,\frac{\partial(u_1+v)}{\partial x})(v-m)\Big\}\ dx = 0. \tag{59}$$

Additionally we insert the test function

$$\varphi = \frac{1}{\rho(u_1)}\Big[\int_{u_1+m}^{u_1+v}\rho(s)\ ds\Big]_+ \tag{60}$$

into (57), $j = 1$. Thus we get

$$\int_{\{v>m\}} \Big\{\sum_{i=1}^n b_i(x,\frac{\partial u_1}{\partial x})\Big[\frac{\partial(u_1+v)}{\partial x_i}\rho(u_1+v) - \frac{\partial u_1}{\partial x_i}\rho(u_1+m)\Big] \\ -\sum_{i=1}^n b_i(x,\frac{\partial u_1}{\partial x})\frac{\partial u_1}{\partial x_i}\rho'(u_1)\varphi + a(x,u_1,\frac{\partial u_1}{\partial x})\varphi\Big\}\ dx = 0. \tag{61}$$

Taking the difference of (59) and (61), we obtain

$$\int_{\{v>m\}} \Big\{\rho(u_1+v)\sum_{i=1}^n \Big[b_i(x,\frac{\partial(u_1+v)}{\partial x}) - b_i(x,\frac{\partial u_1}{\partial x})\Big]\frac{\partial v}{\partial x_i} \\ -\sum_{i=1}^n b_i(x,\frac{\partial u_1}{\partial x})\frac{\partial u_1}{\partial x_i}[\rho(u_1+v)-\rho(u_1+m)] + \sum_{i=1}^n b_i(x,\frac{\partial u_1}{\partial x})\frac{\partial u_1}{\partial x_i}\rho'(u_1)\varphi \\ +a(x,u_1+v,\frac{\partial(u_1+v)}{\partial x})(v-m) - a(x,u_1,\frac{\partial u_1}{\partial x})\varphi\Big\}\ dx = 0. \tag{62}$$

Now condition ρ) implies

$$\rho'(u_1)\varphi \geq \int_{u_1+m}^{u_1+v}\rho'(s)\ ds = \rho(u_1+v) - \rho(u_1+m). \tag{63}$$

Further condition a) yields

$$-a(x,u_1,\frac{\partial u_1}{\partial x})\varphi \geq -\int_{u_1+m}^{u_1+v} a(x,s,\frac{\partial u_1}{\partial x})\ ds. \tag{64}$$

Moreover, (40) along with $b_i(x,0) = 0,\ i = 1,...,n$, give

$$\sum_{i=1}^n b_i(x,\frac{\partial u_1}{\partial x})\frac{\partial(u_1)}{\partial x_i} \geq 0. \tag{65}$$

Thus, using (63)-(65) we obtain from (62)

$$\int_{\{v>m\}} \Big\{\rho(u_1+v)\sum_{i=1}^n \Big[b_i(x,\frac{\partial(u_1+v)}{\partial x}) - b_i(x,\frac{\partial u_1}{\partial x})\Big]\frac{\partial v}{\partial x_i} \\ +a(x,u_1+v,\frac{\partial(u_1+v)}{\partial x})(v-m) - \int_{u_1+m}^{u_1+v} a(x,s,\frac{\partial u_1}{\partial x})\ ds\Big\}\ dx \leq 0. \tag{66}$$

We estimate summands from (66) containing the function a by using condition iii) and (55):

$$\begin{aligned} &|a(x,u_1+v,\frac{\partial(u_1+v)}{\partial x})(v-m) - \int_{u_1+m}^{u_1+v} a(x,s,\frac{\partial u_1}{\partial x})\ ds| \\ = &\ |\int_{u_1+m}^{u_1+v}[a(x,u_1+v,\frac{\partial(u_1+v)}{\partial x}) - a(x,s,\frac{\partial u_1}{\partial x})]\ ds| \\ &\leq c_{12}\Big\{[a_2(x)+1]|\frac{\partial v}{\partial x}| + [a_1(x)+1+|\frac{\partial u_1}{\partial x}|^{\frac{2}{p}}](v-m)\Big\}(v-m), \quad x\in\{v>m\}. \end{aligned} \tag{67}$$

Thus by $ii)*_1$ we get from (66)

$$\int_{\{v>m\}} |\frac{\partial v}{\partial x}|^2 \, dx \leq c_{13} \int_{\{v>m\}} \Big\{[a_2(x)+1]|\frac{\partial v}{\partial x}| + [a_1(x)+1+|\frac{\partial u_1}{\partial x}|^{\frac{2}{p}}](v-m)\Big\}(v-m) \, dx.$$

Using Young's and Hölder's inequalities, we obtain

$$\int_{\{v>m\}} |\frac{\partial v}{\partial x}|^2 \, dx \leq c_{14} \int_{\{v>m\}} \Big\{[v-m]^{2p'} \, dx\Big\}^{\frac{1}{p'}}, \quad p' = \frac{p}{p-1}, \tag{68}$$

where c_{14} depends only on known parameters and norms of the functions a_1, a_2 in $L^p(\Omega)$, $L^{2p}(\Omega)$ respectively.

Let now $q < 2$ be defined by $\frac{nq}{n-q} = 2p'$. Using the embedding theorem and Hölder's inequality we can evaluate the right hand side of (68) in the following way

$$\begin{aligned} \Big\{\int_{\{v>m\}} [v-m]^{2p'} \, dx\Big\}^{\frac{1}{p'}} &\leq c_{15}\Big\{\int_{\{v>m\}} |\frac{\partial v}{\partial x}|^q \, dx\Big\}^{\frac{2}{q}} \\ &= c_{15}\Big\{\int_{\{m<v<M\}} |\frac{\partial v}{\partial x}|^q \, dx\Big\}^{\frac{2}{q}} \\ &\leq c_{15}\Big[meas\{m<v<M\}\Big]^{\frac{2}{q}-1} \int_{\{v>m\}} |\frac{\partial v}{\partial x}|^2 \, dx. \end{aligned} \tag{69}$$

We used here (56) and the fact that $|\frac{\partial v}{\partial x}| = 0$ almost every where on the set $\{v = M\}$. From (68), (69) we conclude

$$\int_{\{v>m\}} |\frac{\partial v}{\partial x}|^2 \, dx \leq c_{16}\Big[meas\{m<v<M\}\Big]^{\frac{2}{q}-1} \int_{\{v>m\}} |\frac{\partial v}{\partial x}|^2 \, dx. \tag{70}$$

Since the measure of the set $\{M - \frac{1}{j} < v < M\}$ tends to zero for $j \to \infty$, (70) implies that for sufficiently large j_0

$$\int_{\{v>m(j_0)\}} |\frac{\partial v}{\partial x}|^2 \, dx = 0, \quad m(j_0) = M - \frac{1}{j_0} > 0. \tag{71}$$

Using Friedrich's inequality we get from (71)

$$\int_{\{v>m(j_0)\}} |v - m(j_0)|^2 \, dx = 0$$

and consequently $v \leq m(j_0)$ almost every where on Ω. This contradicts (56). $\Box$

We conclude this section showing that our conditions for unicity are sharp in some sense. More precisely, we give a counter example concerning the function a_0 from condition $ii)_3$. To this end let us consider the special boundary value problem

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[e^{2u}\frac{\partial u}{\partial xi}] + a_\epsilon(x,u,|\frac{\partial u}{\partial x}|) = 0, \quad x \in \Omega = B_1 = \{x \in I\!R^n : \ |x| < 1\}, \tag{72}$$

$$u = 1, \quad x \in \partial B_1, \tag{73}$$

with

$$a_\epsilon(x,u,s) = ne^{2u}|x|^{\epsilon-2}\min\{s, s^\epsilon\}, \quad s \geq 0, \quad \epsilon \in (0,1).$$

It is easy to see that (72) satisfies all conditions of our paper except

$$a_0 \in L^p(\Omega), \quad p > \frac{n}{2}. \tag{74}$$

Indeed $ii)_3$ holds with $a_0 = |x|^{\frac{\epsilon-2}{1-\epsilon}}$, i.e., a_0 violates (74) for sufficient small ϵ.
On the other hand the problem (72), (73) has the two different solutions

$$u_1(x) = 1, \quad u_2(x) = \ln(e|x|).$$

Consequently, (74) cannot be weakened as

$$a_0 \in L^p(\Omega), \quad p > \nu, \ \nu < \frac{n}{2}.$$

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