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Existence and uniqueness of weak solutions of an initial boundary value problem arising in laser dynamics

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Abstract

In this paper a mathematical model for the dynamical behavior of multisection DFB (distributed feedback) semiconductor lasers in the case of arbitrarily space depending carrier densities is investigated. We introduce a suitable weak formulation of the initial boundary value problem and prove existence, uniqueness and some regularity properties of the solution. The assumptions on the data are quite general, in particular, the physically relevant case of piecewise smooth, but discontinuous coefficients is included.

1 Introduction

This paper is concerned with the following system of first order differential equations

$$\partial_t n(t,z) = I(t,z) - \sigma(z)n(t,z) - G(z, n(t,z), |w(t,z)|^2), \tag{1.1}$$

$$\partial_t w(t,z) = \Big(- \partial_z w_1(t,z), \partial_z w_2(t,z)\Big) + S(z, n(t,z), |w(t,z)|^2) w(t,z), \tag{1.2}$$

supplemented by the boundary conditions

$$w_1(t,0) = r_0 w_2(t,0) \text{ and } w_2(t,1) = r_1 w_1(t,1) + a(t) \tag{1.3}$$

and the initial conditions

$$n(0,z) = n_0(z) \text{ and } w(0,z) = w_0(z). \tag{1.4}$$

The unknown real valued function n and $\mathbb{C}^2$–valued function $w = (w_1, w_2)$ depend on time $t \geq 0$ and space variable $z \in (0,1)$. From the mathematical point of view, the system (1.1), (1.2) consits of an ordinary differential equation for n (which depends parametrically on the space variable z) coupled with a hyperbolic system of two first order partial differential equations for the vector field w. System (1.1)–(1.4) is a (suitably normalized) mathematical model for the dynamical behavior of multisection DFB (distributed feedback) semiconductor lasers (cf, e.g., [7, 10, 12, 13]). The real valued function n is the carrier density of the device, whereas the complex valued functions w_1 and w_2 denote the complex amplitudes of the forward and backward traveling light waves (after averaging over the transverse plane and separating terms varying rapidly in space and time), and z is the space variable in the longitudinal direction.

The real valued functions I and σ describe the injection current and the inverse of the life time of the carriers, respectively, and G is the gain function, which is assumed to be nonnegative if n is large and nonpositive if n is small. Further, $|w|^2 \stackrel{\text{def}}{=} |w_1|^2 + |w_2|^2$ is the power of the optical field $w = (w_1, w_2)$.

The function S takes values in the space of complex 2×2–matrices, and it describes the propagation, coupling and spatial hole burning properties of the laser. Finally, the complex numbers r_0 and r_1 are the amplitude facet reflectivities, and the complex valued function a describes an external optical signal injected to the right facet of the laser.

Multsection lasers are distinguished by the property that they consist of several sections with considerably different electrical and optical properties. Hence, the coefficient functions for such lasers are discontinuous with respect to the space variable. Up to now only for multisection lasers with homogeneous sections and, hence, for models with piecewise constant coefficients, results are known concerning well-posedness of the corresponding initial boundary value problems. Moreover, in that cases simplified models are used, which describe the dynamics of the averaged (over the homogeneous sections of the laser) carrier densities (see, e.g., [4, 6, 11]). Note that in these papers the functions I and a are supposed to be differentiable with respect to time.

In this paper we consider arbitrarily space depending coefficient functions. Thus, the so-called chirping of the DFB grating in the sections is included, for example. Moreover, we consider models which describe the space dependence of the carrier densities within the sections, including the so-called hole burning effect. We introduce a suitable weak formulation of the initial boundary value problem (1.1)–(1.4) and show that it is well posed. The assumptions concerning the functions I, σ, G, S and a are quite general. In particular, the physically relevant case of piecewise smooth, but discontinuous dependence on t and z is included. Note that, even if the injected current I and the injected light signal a are smooth with respect to time, in most of the applications they are close to discontinuous one's (on and off switching of the signals), and, hence, a theory of existence, uniqueness and continuous dependence on the data for such discontinuous data is needed.

This paper is organized as follows. In Section 2 we introduce the assumptions concerning the data in (1.1)–(1.4), the appropriate notion of weak solution to (1.1)--(1.4) and the main result concerning existence and uniqueness of weak solutions. Moreover, a regularity theorem describes the regularity properties of the semiflow corresponding to (1.1)–(1.4) in the autonomous case ($a = 0$ and I independent of time). This regularity theorem will be proved using results in [2] and [3, 5].

Section 3 is concerned with weak solutions to abstract linear inhomogeneous evolution equations with nonsmooth data.

For the proof of existence of weak solutions to (1.1)–(1.4) in Section 4 an initial boundary value problem with suitably truncated functions will be introduced, which can be solved by the contraction mapping principle using the results of section 3. A priori estimates for the carrier density will be be proved for the solution of this truncated problem. With these estimates it can be shown that the solution of the truncated problem actually solves (1.1)–(1.4) provided that the truncation parameters are chosen suitablely.

2 Notation, Assumptions and Results

In what follows we denote by $\langle\cdot,\cdot\rangle$ the Hermitean scalar product in $\mathbb{C}^2$, i.e.

$$\langle u,v\rangle = u_1\overline{v_1} + u_2\overline{v_2} \text{ for all } u=(u_1,u_2), v=(v_1,v_2)\in\mathbb{C}^2,$$

and $|\cdot|$ denotes the corresponding norm in $\mathbb{C}^2$ as well as the Hermitean norm in the space $\mathbb{M}(2\times 2,\mathbb{C})$ of all complex 2×2–matrices. Further, $T>0$ is arbitrarily fixed.

We will work with the usual notation concerning Lebesgue and Sobolev spaces and their norms. If U a Banach space, then $BV((0,T),U)$ denotes the space of all $\Phi\in L^\infty((0,T),U)$ such that there exists a constant c_Φ with

$$\Big\| \int_0^T \varphi'(t)\Phi(t)dt\Big\|_U \le c_\Phi\|\varphi\|_{L^\infty(0,T)} \text{ for all } \varphi\in C_0^\infty(0,T). \tag{2.1}$$

This is the space of all functions $\Phi:\ (0,T)\to U$ of bounded variation, which includes the piecewise smooth functions. We endow it with the norm

$$\|\Phi\|_{BV((0,T),U)} \stackrel{\text{def}}{=} \|\Phi\|_{L^\infty((0,T),U)} + \tilde{c}_\Phi,$$

where $\tilde{c}_\Phi$ is the smallest constant in (2.1).

Let us formulate our assumptions concerning the data in (1.1)–(1.4).

We suppose

$$I\in L^\infty\left((0,T)\times(0,1)\right), \tag{2.2}$$

$$\sigma\in L^\infty(0,1) \text{ with } \operatorname{ess\,inf}\sigma>0, \tag{2.3}$$

$$a\in BV((0,T),\mathbb{C}), \tag{2.4}$$

$$r_0,r_1\in\mathbb{C} \text{ with } |r_0r_1|<1. \tag{2.5}$$

$$n_0\in L^\infty(0,1) \text{ with } \operatorname{ess\,inf} n_0>0. \tag{2.6}$$

$$w_0\in L^\infty((0,1),\mathbb{C}^2). \tag{2.7}$$

The functions $G:(0,1)\times(0,\infty)\times[0,\infty)\to\mathbb{R}$ and $S:(0,1)\times(0,\infty)\times[0,\infty)\to\mathbb{M}(2\times 2,\mathbb{C})$ are supposed to satisfy the following assumptions:

$$\left.\begin{array}{rcl} G(\cdot,n,r) & \in & L^\infty(0,1)\\ S(\cdot,n,r) & \in & L^\infty((0,1),\mathbb{M}(2\times 2,\mathbb{C}))\end{array}\right\} \text{ for all } n\in(0,\infty) \text{ and } r\in[0,\infty) \tag{2.8}$$

and

$$\left.\begin{array}{rcl} G(z,\cdot,\cdot) & \in & C^1((0,\infty)\times[0,\infty))\\ S(z,\cdot,\cdot) & \in & C^1((0,\infty)\times[0,\infty),\mathbb{M}(2\times 2,\mathbb{C}))\end{array}\right\} \text{ for almost all } z\in(0,1). \tag{2.9}$$

Moreover, we suppose that for arbitrary positive δ and M there exists some $L_{\delta,M} > 0$ such that for almost all $z \in (0,1)$, all $n \in [\delta, M]$ and all $r \in [0, M]$ we have

$$
\begin{aligned}
&|G(z,n,r)| + |\partial_n G(z,n,r)| + |\partial_r G(z,n,r)| \\
&\quad + |S(z,n,r)| + |\partial_n S(z,n,r)| + |\partial_r S(z,n,r)| \le L_{\delta,M}.
\end{aligned} \tag{2.10}
$$

Finally, it is assumed that there exist positve numbers $\underline{n} \le \overline{n}$ such that for almost all $z \in (0,1)$ and all $r \in [0,\infty)$ we have

$$
G(z,n,r) \ge 0 \text{ if } n > \overline{n} \text{ and } G(z,n,r) \le 0 \text{ if } n \le \underline{n}. \tag{2.11}
$$

Now the notion of weak solutions to (1.1)–(1.4) is given.

Definition 1 *A pair of functions* $(n,w) \in L^\infty\left((0,T)\times(0,1), \mathbb{R}\times\mathbb{C}^2\right)$ *is called a weak solution to* (1.1)–(1.4)*, if* ess inf $n > 0$ *and if*

$$
\begin{aligned}
n(t,z) &= n_0(z) \\
&\quad + \int_0^t \Big(I(s,z) - \sigma(z) n(s,z) - G(z, n(s,z), |w(t,z)|^2) \Big) ds
\end{aligned} \tag{2.12}
$$

for almost all $z \in (0,1)$ *and*

$$
\begin{aligned}
\int_0^1 \langle \varphi(z), w(t,z) - w_0(z) \rangle dz &= \int_0^t \Big(\int_0^1 (\partial_z \varphi_1(z) \overline{w_1(s,z)} - \partial_z \varphi_2(z) \overline{w_2(s,z)} \\
&\quad + \langle \varphi(z), S(z, n(s,z), |w(s,z)|^2) w(s,z) \rangle) dz + \varphi_2(1) \overline{a(s)} \Big) ds
\end{aligned} \tag{2.13}
$$

for all $t \in (0,T)$ *and* $\varphi = (\varphi_1, \varphi_2) \in W^{1,2}((0,1); \mathbb{C}^2)$ *with* $\varphi_2(0) = \overline{r_0}\varphi_1(0)$ *and* $\varphi_1(1) = \overline{r_1}\varphi_2(1)$.

The following lemma explains in which sense a weak solution to (1.1)–(1.4) satisfies the system of differential equations (1.1)–(1.2), the boundary conditions (1.3) and the initial conditions (1.4). In its formulation we identify, as usual, the functions $n : (0,T)\times(0,1) \to \mathbb{R}$ and $w : (0,T)\times(0,1) \to \mathbb{C}^2$ and the corresponding function space valued maps $t \in (0,T) \mapsto n(t,\cdot)$ and $t \in (0,T) \mapsto w(t,\cdot)$.

Lemma 1 *Let* (n,w) *be a weak solution to* (1.1)–(1.4)*. Then the following holds:*
(i) $n \in W^{1,\infty}\left((0,T), L^\infty(0,1)\right)$, (1.1) *is satisfied for all* $t \in (0,T)$ *and almost all* $z \in (0,1)$*, and* $n(0,z) = n_0(z)$ *for almost all* $z \in (0,1)$.
(ii) *System* (1.2) *is satisfied in the sense of distributions, i.e.*

$$
\begin{aligned}
\int_0^T \int_0^1 \Big(&\langle \partial_t \varphi(t,z), w(t,z) \rangle + \partial_z \varphi_1(t,z) w_1(t,z) - \partial_z \varphi_2(t,z) w_2(t,z) \\
&+ \langle \varphi(t,z), S(z, n(t,z), |w(t,z)|^2) w(t,z) \rangle \Big) dz dt = 0
\end{aligned}
$$

for all $\varphi \in C_0^\infty((0,T)\times(0,1),\mathbb{C}^2)$.
(iii) *For all* $t \in (0,T)$ *we have*

$$u(t) \stackrel{\text{def}}{=} \int_0^t w(s)ds \in W^{1,2}((0,1);\mathbb{C}^2)$$

and

$$u_1(t,0) = r_0 u_2(t,0) \text{ and } u_2(t,1) = r_1 u_1(t,1) + \int_0^t a(s)ds. \tag{2.14}$$

(iv) *The function* w *is weakly continuous as a map from* $(0,T)$ *into* $L^2((0,1),\mathbb{C}^2)$, *and we have* $w(0,z) = w_0(z)$ *for almost all* $z \in (0,1)$.

Now we formulate our main result:

Theorem 1 *There exists a unique weak solution* (n,w) *to* (1.1)–(1.4). *Moreover, the estimates*

$$\begin{aligned} e^{-\sigma(z)t} \operatorname{ess\,inf} n_0 \le n(t,z) \le \\ e^{-\sigma(z)t}\, n_0(z) + \max\{\overline{n}, \|\sigma^{-1}I\|_{L^\infty((0,T)\times(0,1))}\} \end{aligned} \tag{2.15}$$

hold for all $t \in (0,T)$ *and almost all* $z \in (0,1)$.

Of course, if the external signal a in (1.3) vanishes, the injection current I is independent of time and the initial function w_0 satisfies the corresponding homogeneous boundary conditions, then the weak solution to (1.1)–(1.4) has more regularity. This is described in the next theorem:

Theorem 2 *Suppose* $a = 0$, $w_0 = (w_{01}, w_{02}) \in W^{1,2}((0,1),\mathbb{C}^2)$, $w_{01}(0) = r_0 w_{02}(0)$ *and* $w_{02}(0) = r_1 w_{01}(1)$. *Then the weak solution* (n,w) *to* (1.1)–(1.4) *satisfies*

$$w \in C^1\left([0,T], L^2((0,1),\mathbb{C}^2)\right) \cap C\left([0,T], W^{1,2}((0,1),\mathbb{C}^2)\right),$$

and $w_1(t,0) = r_0 w_2(t,0)$, $w_2(t,1) = r_1 w_1(t,1)$ *for all* $t \in (0,T)$. *If, moreover,* I *is independent of time, then*

$$n \in C^1\left([0,T], L^\infty(0,1)\right).$$

3 Linear inhomogeneous evolution equations with discontinuous data

In this section a general concept of weak solutions to abstract linear inhomogeneous evolution equations is given, which is suitable for linear inhomogeneous first order initial boundary value problems, where the boundary data may be discontinuous in time . These solutions are "very weak", because they do not satisfy the variation of constants formula, in general.

Througout this section let $\mathcal{X}$ be an arbitrary Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{X}}$, and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{X} \to \mathcal{X}$ is the generator of a strongly continuous semigroup $\exp(t\mathcal{B})$ $(t \geq 0)$ in $\mathcal{X}$. By $\mathcal{B}^*$ we denote the dual operator to $\mathcal{B}$, and $D(\mathcal{B}^*)$ is the domain of definition of $\mathcal{B}^*$, i.e. $v \in D(\mathcal{B}^*)$ iff there exists a constant $c_{\mathcal{B}} > 0$ such that

$$|\langle \mathcal{B}u, v\rangle_{\mathcal{X}}| \leq c_{\mathcal{B}} \|u\|_{\mathcal{X}} \text{ for all } u \in D(\mathcal{B}).$$

The space $\mathcal{Y} \stackrel{\text{def}}{=} D(\mathcal{B}^*)$ is endowed with the norm

$$\|u\|_{\mathcal{Y}}^2 \stackrel{\text{def}}{=} \|u\|_{\mathcal{X}}^2 + \|\mathcal{B}^* u\|_{\mathcal{X}}^2 \text{ for all } u \in \mathcal{Y}.$$

We denote by $[\cdot,\cdot]_{\mathcal{Y}}$ the dual pairing between $\mathcal{Y}$ and $\mathcal{Y}^*$.

In this section we consider the abstract linear inhomogeneous initial value problem

$$\dot{w} = \mathcal{B}w + f + \Phi, \quad w(0) = w_0. \tag{3.1}$$

(Note that $\mathcal{X}$ can be imbedded into $\mathcal{Y}^* = (D(\mathcal{B}^*))^*$.)

Definition 2 *Let $w_0 \in \mathcal{X}$, $f \in L^1((0,T),\mathcal{X})$ and $\Phi \in BV((0,T),\mathcal{Y}^*)$. Then $w \in L^\infty((0,T),\mathcal{X})$ is called a weak solution to* (3.1) *iff for all $t \in (0,T)$ and $\varphi \in \mathcal{Y}$ one has*

$$\langle \varphi, w(t) - w_0\rangle_{\mathcal{X}} = \int_0^t \Big(\langle \mathcal{B}^*\varphi, w(s)\rangle_{\mathcal{X}} + \langle \varphi, f(s)\rangle_{\mathcal{X}} + [\Phi(s), \varphi(s)]_{\mathcal{Y}} \Big) ds.$$

Lemma 2 *Let $w_0 \in \mathcal{X}$, $f \in L^1((0,T),\mathcal{X})$ and $\Phi \in BV((0,T),\mathcal{Y}^*)$, and let w be a weak solution to* (3.1). *Then w is weakly continuous as a map from $[0,T]$ into $\mathcal{X}$, and $w(0) = w_0$.*

Proof Take $\varphi \in \mathcal{X}$ arbitrary. Since $\mathcal{B}$ is densely defined and closed on a Hilbert space, $\mathcal{Y} = D(\mathcal{B}^*)$ is dense in $\mathcal{X}$. Hence there exists a sequence $\varphi_n \in \mathcal{Y}$ with

$$\|\varphi_n - \varphi\|_{\mathcal{X}} \stackrel{n\to\infty}{\longrightarrow} 0. \tag{3.2}$$

For all n we have

$$u_n \stackrel{\text{def}}{=} \langle \varphi_n, w(\cdot)\rangle_{\mathcal{X}} \in C([0,T],\mathbb{R}) \text{ and } u_n(0) = \langle \varphi_n, w_0\rangle_{\mathcal{X}}. \tag{3.3}$$

By (3.2) it follows that

$$u_n(t) \stackrel{n\to\infty}{\longrightarrow} u(t) \stackrel{\text{def}}{=} \langle \varphi, w(t)\rangle_{\mathcal{X}}$$

uniformly with respect ot t. Hence, we get from (3.3) that $\langle \varphi, w(\cdot)\rangle_{\mathcal{X}} \in C([0,T],\mathbb{R})$ and $\langle \varphi, w(0)\rangle_{\mathcal{X}} = \langle \varphi, w_0\rangle_{\mathcal{X}}$.

Theorem 3 *Let $w_0 \in \mathcal{X}$, $f \in L^1((0,T),\mathcal{X})$ and $\Phi \in BV((0,T),\mathcal{Y}^*)$. Then there exists a unique weak solution w to (3.1). Moreover,*

$$\|w(t)\|_{\mathcal{X}} \leq c_T\Big(\|w_0\|_{\mathcal{X}} + \|f\|_{L^1((0,T),\mathcal{X})} + \|\Phi\|_{BV((0,T),\mathcal{Y}^*)}\Big),$$

for all $t \in [0,T]$, where the constant c_T does not depend on u_0, f and Φ.

Proof First uniqueness is shown. For this purpose it suffices to consider the homogeneous case, i.e. $w_0 = 0$, $f = 0$ and $\Phi = 0$. Suppose $\mathbf{w} \in L^\infty((0,T),\mathbf{X})$ solves the corresponding homogeneous problem. Then $u(t) \overset{\text{def}}{=} \int_0^t w(s)ds$ obeys

$$\frac{d}{dt}\langle\varphi, u(t)\rangle_{\mathcal{X}} = \langle\varphi, w(t)\rangle_{\mathcal{X}} = \langle\mathcal{B}^*\varphi, u(t)\rangle_{\mathcal{X}},$$

i.e. $u \in C([0,T],\mathcal{X})$ is a weak solution of $\partial_t u(t) = \mathcal{B}u(t)$ in the sense of [2], and hence $u(t) = \exp(t\mathcal{B})(w(0)) = 0$, which completes the proof of uniqueness.

Now we prove existence. The idea is approximate Φ by functions which are smooth with respect to time, apply the variation of constants formula and pass to the limit. Let $\omega_n \in C_0^\infty(-1/n, 0)$, $n \in \mathbb{N}$ be a mollifier with the property $\int_{\mathbb{R}} \omega_n(t)dt = 1$ and define $\Phi_n \in C^\infty([0,T],\mathcal{Y}^*)$ by

$$\Phi_n(t) \overset{\text{def}}{=} \int_0^T \omega_n(t-s)\Phi(s)ds$$

By Riesz' lemma applied to $\mathcal{Y}$ there exists a unique $G_n \in C^\infty([0,T],\mathcal{Y})$ with

$$[\Phi_n(t),\varphi]_{\mathcal{Y}} = \langle\varphi, G_n(t)\rangle_{\mathcal{X}} + \langle\mathcal{B}^*\varphi, \mathcal{B}^*G_n(t)\rangle_{\mathcal{X}} \text{ for all } \varphi \in \mathcal{Y} \text{ and } t \in (0,T). \tag{3.4}$$

Now, let

$$\begin{aligned} w_n(t) &\overset{\text{def}}{=} \exp(t\mathcal{B})(w_0 + \mathcal{B}^*G_n(0)) \\ &\quad + \int_0^t \exp((t-s)\mathcal{B})\left[f(s) + G_n(s) + \mathcal{B}^*\partial_t G_n(s)\right] ds - \mathcal{B}^*G_n(t) \end{aligned} \tag{3.5}$$

Because of (3.4) and (3.5) one has for all $\varphi \in \mathcal{Y}$

$$\begin{aligned} &\frac{d}{dt}\langle\varphi, w_n(t)\rangle_{\mathcal{X}} \\ &\quad = \langle\mathcal{B}^*\varphi, \exp(t\mathcal{B})(w_0 + \mathcal{B}^*G_n(0))\rangle_{\mathcal{X}} + \langle\varphi, f(t) + G_n(t)\rangle_{\mathcal{X}} \\ &\quad + \int_0^t \langle\mathcal{B}^*\varphi, \exp((t-s)\mathcal{B})\left(f(s) + G_n(s) + \mathcal{B}^*\partial_t G_n(s)\right)\rangle_{\mathcal{X}} ds \\ &\quad = \langle\mathcal{B}^*\varphi, w_n(t) + \mathcal{B}^*G_n(t)\rangle_{\mathcal{X}} + \langle\varphi, f(t) + G_n(t)\rangle_{\mathcal{X}} \\ &\quad = \langle\mathcal{B}^*\varphi, w_n(t)\rangle_{\mathcal{X}} + \langle\varphi, f(t)\rangle_{\mathcal{X}} + [\Phi_n(t),\varphi]_{\mathcal{Y}}. \end{aligned}$$

Since $w_n(0) = w_0$, this yields

$$\langle\varphi, w_n(t) - w_0\rangle_{\mathcal{X}} = \int_0^t \left(\langle\mathcal{B}^*\varphi, w_n(s)\rangle_{\mathcal{X}} + \langle\varphi, f(s)\rangle_{\mathcal{X}} + [\Phi_n(s),\varphi]_{\mathcal{Y}}\right) ds. \tag{3.6}$$

Next, an L^1-bound on $\partial_t \Phi$ is given. Suppose $\varphi \in C_0^\infty((0,1),\mathbb{C}^2)$. Then

$$\Big\| \int_0^T \varphi(t)\partial_t\Phi_n(t)dt \Big\|_{\mathcal{Y}^*} = \Big\| \int_0^T \partial_t(\varphi * \omega_n)(t)\Phi(t)dt \Big\|_{\mathcal{Y}^*}$$
$$\leq \|\Phi\|_{BV((0,T),\mathcal{Y})}\|\varphi * \omega_n\|_{L^\infty((0,T),\mathbb{C}^2)} \leq \|\Phi\|_{BV((0,T),\mathcal{Y})}\|\varphi\|_{L^\infty((0,T),\mathbb{C}^2)}.$$

Hence

$$\|\partial_t \Phi_n\|_{L^1((0,T),\mathcal{Y}^*)} \leq \|\Phi\|_{BV(0,T,\mathcal{Y}^*)}. \tag{3.7}$$

Now it is shown that w_n is uniformly bounded in $L^\infty((0,T),\mathcal{X})$. From (3.7) follows

$$\|G_n\|_{W^{1,1}((0,T),\mathcal{Y})} \leq \|\partial_t\Phi_n\|_{L^1((0,T),\mathcal{Y}^*)} + \|\Phi_n\|_{L^1((0,T),\mathcal{Y}^*)} \leq C_{1,T}\|\Phi\|_{BV(0,T,\mathcal{Y}^*)}$$

With (3.5) this implies

$$\|w_n\|_{L^\infty((0,T),\mathcal{X})} \leq \|w_0\|_{\mathcal{X}} + \|G_n\|_{W^{1,1}((0,T),D(\mathcal{B}^*))} + \|f\|_{L^1((0,T),\mathcal{X})}$$
$$\leq C_{2,T}\left(\|\Phi\|_{BV((0,T),\mathcal{Y}^*)} + \|w_0\|_{\mathcal{X}} + \|f\|_{L^1((0,T),\mathcal{X})}\right).$$

By this estimate there exists some $w \in L^\infty((0,T),\mathcal{X})$ and a subsequence still labeled by $w_n, n \in \mathbb{N}$, such that

$$w_n \overset{n\to\infty}{\longrightarrow} w \text{ in } L^\infty((0,T),\mathcal{X}) \text{ weak-}*. \tag{3.8}$$

Since $\Phi_n \overset{n\to\infty}{\longrightarrow} \Phi$ in $L^\infty((0,T),\mathcal{Y}^*)$-weak$*$, it follows easily from (3.6) and (3.8) that

$$\langle \varphi, w(t) - w_0\rangle_{\mathcal{X}} = \int_0^t \left(\langle \mathcal{B}^*\varphi, w(s)\rangle_X + \langle \varphi, f(s)\rangle_X + [\Phi(s),\varphi]_{\mathcal{Y}}\right) ds, \tag{3.9}$$

Hence w is a weak solution to (3.1).

4 Proof of Existence and Regularity

In this section we prove the Lemma 1 and the Theorems 1 and 2.

In order to use the results of Section 3 let us introduce the Hilbert space

$$X \overset{\text{def}}{=} L^2\left((0,1),\mathbb{C}^2\right)$$

with its usual scalar product $\langle\cdot,\cdot\rangle_X$. Further, we define the unbounded linear operator B on X by

$$Bw \overset{\text{def}}{=} (-w_1', w_2') \text{ (differentiation with respect to } z \in (0,1)) \tag{4.1}$$

with domain of definition

$$D(B) \overset{\text{def}}{=} \{(w_1,w_2) \in W^{1,2}\left((0,1),\mathbb{C}^2\right) : \ w_1(0) = r_0 w_2(0), \ w_2(1) = r_1 w_1(0)\}.$$

By means of assumption (2.5) it is easy to show (cf., e.g., [9, 1, 8] that B is the generator of a strongly continuous contraction semigroup $\exp(tB)$ $(t \geq 0)$ in X. Moreover, $\exp(tB)$ maps $L^\infty((0,1),\mathbb{C}^2)$ into $L^\infty((0,1),\mathbb{C}^2)$, and there exists an $\alpha > 0$ such that

$$\|\exp(tB)w\|_{L^\infty((0,1),\mathbb{C}^2)} \leq \exp(-\alpha t)\|w\|_{L^\infty((0,1),\mathbb{C}^2)}. \tag{4.2}$$

The adjoint operator B^* is defined on the domain

$$Y \stackrel{\text{def}}{=} \{(w_1,w_2) \in W^{1,2}\left((0,1),\mathbb{C}^2\right):\ w_2(0) = \overline{r_0}w_1(0),\ w_1(1) = \overline{r_1}w_2(0)\}. \tag{4.3}$$

The space Y is a Hilbert space with respect to the scalar product

$$\langle u,v\rangle_X + \langle B^*u, B^*v\rangle_{\mathcal{X}} \text{ for } u,v \in Y.$$

In order to take into account the inhomogeneous boundary condition (1.3), we define the functional $\Phi \in BV((0,T)),Y^*)$ by

$$[\Phi(t),\varphi]_Y \stackrel{\text{def}}{=} \varphi_2(1)\overline{a(t)} \text{ for all } \varphi \in Y, \tag{4.4}$$

where $[\cdot,\cdot]_Y$ denotes the dual pairing between Y^* and Y.

Using this notation we get the following: If (n,w) is a weak solution to (1.1)–(1.4) (in the sense of Definition 1), then

$$\langle\varphi, w(t)-w_0\rangle_X = \int_0^t \Big(\langle B^*\varphi, w(t)\rangle_X + \langle\varphi, S(n(s),|w(s)|^2)w(s)\rangle_X + [\Phi(s),\varphi]_Y\Big)ds$$

for all $\varphi \in Y$ and all $t \in [0,T]$. Here we use the same symbol S for the Nemycki operator as for the function (introduced in (2.8) and (2.9)) generating this Nemycki operator.

Proof of Lemma 1 Let (n,w) be a weak solution to (1.1)–(1.4). Because of (2.2), (2.3), (2.8) and (2.10), the integrand in (2.12) belongs to $L^\infty((0,T)\times(0,1))$. Therefore $n \in W^{1,\infty}((0,T),L^\infty(0,1))$, in particular $n \in C([0,T],L^\infty(0,1))$. Using (2.12) again, we get $n(0,z) = n_0(z)$ for almost all $z \in (0,1)$. Thus, assertion (i) is proved.

Assertion (ii) follows easily from (2.13): Insert for the test function φ a test function $\partial_t\psi$ with $\psi \in C_0^\infty((0,T)\times(0,1),\mathbb{C}^2)$ and integrate over $t \in (0,T)$.

Now, let us prove assertion (iv). Denote

$$f(t) = (f_1(t),f_2(t)) \stackrel{\text{def}}{=} S(n(t),|w(t)|^2)w(t) \text{ for } t \in (0,T).$$

Then we have $f \in L^\infty((0,T),X)$, and w is a weak solution (in the sense of Definition 2) to $\dot{w} = Bw+f+\Phi$, $w(0) = w_0$. Hence, Lemma 2 yields that w is weakly continuous as a map from $[0,T]$ into X, and $w(0) = w_0$.

Finally, in order to prove (iii), denote $u(t) \overset{\text{def}}{=} \int_0^t w(s)ds$ for $t \geq 0$. Then (2.13) yields

$$\int_0^1 \left(\varphi_1' \overline{u_1(t,z)} - \varphi_2' \overline{u_2(t,z)}\right) dz$$
$$= \left\langle \varphi, w(s) - w_0 - \int_0^t f(s)ds \right\rangle_X \text{ for all } t > 0 \text{ and } \varphi \in C_0^\infty((0,T),X).$$

Hence, for all $t > 0$ we get $u(t) \in W^{1,2}((0,1),\mathbb{C}^2)$ and

$$\begin{aligned} -\partial_z u_1(t) &= w_1(t) - w_{01} - \textstyle\int_0^t f_1(s)ds, \\ \partial_z u_2(t) &= w_2(t) - w_{02} - \textstyle\int_0^t f_2(s)ds. \end{aligned}$$

Using this, it follows from (2.13) that

$$\int_0^1 \left(\varphi_1' \overline{u_1(t,z)} - \varphi_2' \overline{u_2(t,z)}\right) dz$$
$$= \int_0^1 \left(- \varphi_1 \overline{\partial_z u_1(t,z)} + \varphi_2 \overline{\partial_z u_2(t,z)}\right) dz + \varphi_2(1)\overline{a(t)} \text{ for all } t > 0 \text{ and } \varphi \in Y.$$

Because of (4.3) this yields (2.14) for all $t \in (0,T)$.

Lemma 3 *Let $w^0 \in L^\infty([0,T],X)$ be the weak solution of*

$$\partial_t w = Bw + \Phi, \quad w(0) = w_0$$

in the sense of Definition 2. Then $w^0 \in L^\infty\left((0,T), L^\infty((0,1),\mathbb{C}^2)\right)$.

Proof Let Φ_n, $\mathbf{G}_n$ and $w_n$be defined as in the proof of Theorem 3. By the definition (3.4) of G_n one has

$$\langle \varphi, G_n(t)\rangle_X + \langle B^*\varphi, B^*G_n(t)\rangle_X = 0 \text{ for all } \varphi \in C_0^\infty((0,1),\mathbb{C}^2) \text{ and } t \in (0,T).$$

Therefor from (4.1) it follows that $B^*G_n(t) \in W^{1,2}\left((0,1),\mathbb{C}^2\right)$ and $(B^*G_n(t))' = \operatorname{diag}(1,-1)G_n(t)$ and hence,

$$\|B^*G_n(t)\|^2_{W^{1,2}((0,1),\mathbb{C}^2)} = \|G_n(t)\|^2_{L^2((0,1),\mathbb{C}^2)} + \|B^*G_n(t)\|^2_{L^2((0,1),\mathbb{C}^2)}.$$

By the continuous embedding $W^{1,2}((0,1),\mathbb{C}^2) \hookrightarrow L^\infty((0,1),\mathbb{C}^2$ it follows

$$\|G_n(t)\|_{L^\infty((0,1),\mathbb{C}^2)} + \|B^*G_n(t)\|_{L^\infty((0,1),\mathbb{C}^2)}$$
$$\leq c_1 \left(\|G_n(t)\|_{L^2((0,1),\mathbb{C}^2)} + \|B^*G_n(t)\|_{L^2((0,1),\mathbb{C}^2)}\right) \leq c_2 \|\Phi_n(t)\|_{Y^*}$$

and analogously

$$\|\partial_t G_n(t)\|_{L^\infty((0,1),\mathbb{C}^2)} + \|B^*\partial_t G_n(t)\|_{L^\infty((0,1),\mathbb{C}^2)} \leq c_2 \|\partial_t \Phi_n(t)\|_{Y^*}.$$

Now, it follows from (3.5), (3.7), and property (4.2) of B that

$$\|w_n\|_{L^\infty((0,T),L^\infty((0,1),\mathbb{C}^2))} \leq c_3.$$

By (3.8) this completes the proof.

Now the existence of solutions to a suitablely truncated problem will be proved, and it will be shown that its solution is actually a weak solution to (1.1)–(1.4) using suitable a prori estimates. Let δ and M be positive constants. Then the truncated equations read as

$$\begin{aligned} \partial_t n(t,z) &= I(t,z) - \sigma(z)n(t,z) - G_{\delta,M}\left(z, n(t,z), |H_M(w(t,z))|^2\right), \\ \partial_t w(t,z) &= Bw(t,z) + S_{\delta,M}\left(z, n(t,z), |H_M(w(t,z))|^2\right) H_M(w(t,z)). \end{aligned} \tag{4.5}$$

Here

$$G_{\delta,M}(z,y,r) \stackrel{\text{def}}{=} \begin{cases} G(z,y,r) & \text{if } y \in [\delta, M], \\ G(z,\delta,r) & \text{if } y \in (-\infty, \delta], \\ G(z,M,r) & \text{if } y \in [M, \infty). \end{cases} \tag{4.6}$$

The definition of $S_{\delta,M}$ is analogous. The function $H_M : \mathbb{C}^2 \to \mathbb{C}^2$ is globally Lipschitz continuous and bounded with the property that

$$|H_M(u)| \leq \min\{|u|, M\} \text{ for all } u \in \mathbb{C}^2 \text{ and } H_M(u) = u \text{ if } |u| \leq M. \tag{4.7}$$

The notion of weak solutions to (4.5), (1.3), (1.4) is analogous to Definition 1.

Lemma 4 *There exists a unique weak solution to* (4.5), (1.3), (1.4).

Proof: Let $w^0 \in C_w([0,T], X)$ be the solution of $\partial_t w = Bw + \Phi, \quad w(0) = w(0)$ as in lemma 3 in the sense of Definition 2. First $n \in L^\infty((0,T), L^\infty(0,1))$ and $w \in L^\infty((0,T), L^\infty((0,1),\mathbb{C}^2))$ solve (4.5), (1.3), (1.4) if and only if n and $u \stackrel{\text{def}}{=} w - w^0$ satisfy

$$n(t) = n_0 + \int_0^t \left[I(s) - \sigma n(s) - G_{\delta,M}\left(z, n(s), |H_M(u(s) + w^0(s))|^2\right)\right] ds$$

and

$$\langle \varphi, u(t)\rangle_X = \int_0^t \Big(\langle B^*\varphi, u(t)\rangle_X + \langle \varphi, S(n(s), |H_M(u(s)+w^0(s))|^2) w(s)\rangle_X\Big) ds \tag{4.8}$$

for all $\varphi \in Y$. By the result in [2] it follows that (4.8) is fulfilled if and only the variation of constants formula

$$\begin{aligned} u(t) = \int_0^t &\exp((t-s)B) \\ &\left[S_{\delta,M}\left(z, n(s), |H_M(u(s)+w^0(s))|^2\right) H_M(u(s)+w^0(s))\right] ds \end{aligned}$$

holds.

This means that $(n,u) \in \mathcal{S} \stackrel{\text{def}}{=} L^\infty((0,T), L^\infty((0,1), \mathbb{R}\times\mathbb{C}^2)) = L^\infty((0,T)\times(0,1), \mathbb{R}\times\mathbb{C}^2)$ has to be a fixed-point of the operator $A : \mathcal{S} \to \mathcal{S}$ defined by

$$A(n,u) \stackrel{\text{def}}{=} (\tilde{n}, \tilde{u})$$

with

$$\tilde{n}(t) \stackrel{\text{def}}{=} n_0 + \int_0^t \left[I(s) - \sigma n(s) - G_{\delta,M}\left(z, n(s), |H_M(u(s)+w^0(s))|^2\right)\right] ds$$

and

$$\tilde{u}(t) \stackrel{\text{def}}{=} \int_0^t \exp((t-s)B) \\ \left[S_{\delta,M}\left(z, n(s), |H_M(u(s)+w^0(s))|^2\right) H_M(u(s)+w^0(s))\right] ds.$$

Due to the truncation the nonlinear functions occurring in A are globally Lipschitz continuous with respect to u. Therefore it follows easily from (4.2) that A is a contraction in $\mathcal{S}$ with respect to the norm

$$|(n,u)|_L \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \left(\exp(-tL)\|(n(t),u(t))\|_{L^\infty((0,1),\mathbb{R}\times\mathbb{C}^2)}\right) \text{ for } L \in (0,\infty)$$

provided that $L > 0$ is chosen large enough. Hence A has a unique fixed point $(n,u) \in \mathcal{S}$. Finally, $(n, u + w_0)$ solves the truncated problem. This completes the proof.

The aim of the following considerations is to show that the weak solution (n,w) of (4.5), (1.3), (1.4) is actually a solution of (1.1)-(1.4) provided that δ is sufficently small and M sufficiently large. This completes the proof of Theorem 1.

Theorem 4 *For all $\delta > 0$ and $M > 0$ the weak solution (n,w) of* (4.5), (1.3), (1.4) *satisfies the estimates*

$$n(t,z) \le \ max\, \{\overline{n}, \|\sigma^{-1} I\|_{L^\infty((0,T)\times(0,1))}\} + \exp(-t\sigma(z)) n_0(z), \tag{4.9}$$

$$n(t,z) \ge \operatorname{ess\,inf} n_0 \exp(-\sigma(z)t) \tag{4.10}$$

and

$$|w(t,x)| \le M_0(T, n_0, w_0)$$

for all $t \in (0,T)$ and almost all $z \in (0,1)$. Here the constant $M_0(T,n_0,w_0)$ is independent of δ and M.

Proof Suppose $\delta > 0$, $M > 0$ and that $(n,w) \in L^\infty((0,T), L^\infty(0,1))$ solves (4.5), (1.3), (1.4). Let $m \stackrel{\text{def}}{=} \max\, \{\overline{n}, \|\sigma^{-1} I\|_{L^\infty((0,T)\times(0,1))}\}$ and $h(y) \stackrel{\text{def}}{=} [y-m]^+$ for $y \in \mathbb{R}$. It follows from the property (2.11) of G that $G_{\delta,M}(z, n(t,z)) \ge 0$ for all $(t,z) \in (0,T)\times(0,1)$ with $n(t,z) > m$. Hence (4.5) yields

$$\partial_t h(n) = h'(n)\left[I - \sigma n - G_{\delta,M}\left(z, n, |H_M(w)|^2\right)\right]$$

$$\leq -\sigma h'(n)[n - \sigma^{-1} I] \leq -\sigma h(n).$$

This implies the upper a priori bound $h(n(t,z)) \leq \exp(-t\sigma(z)) h(n_0(z))$, and hence

$$n(t,z) \leq m + h(n(t,z)) \leq m + \exp(-t\sigma(z))[n_0(z) - m]^+,$$

whence (4.9).

Next a lower bound is proved. Let $\underline{n}$ as in assumption (2.10). Define $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ by $g_\varepsilon(u) \stackrel{\text{def}}{=} \underline{n}^{-1}$ if $u \geq \underline{n}$, $g_\varepsilon(u) \stackrel{\text{def}}{=} u^{-1}$ if $\varepsilon \leq u \leq \underline{n}$, and $g_\varepsilon(u) \stackrel{\text{def}}{=} \varepsilon^{-1}$ if $u \leq \varepsilon$ for $\varepsilon > 0$. Since $g' \leq 0$, $I \geq 0$ and $G_{\delta,M}(y) \leq 0$ if $y \leq \underline{n}$, it follows from (4.5) that

$$\partial_t g_\varepsilon(n) = g'_\varepsilon(n) \left[I - \sigma n - G_{\delta,M}\left(z, n, |H_M(w)|^2\right) \right]$$

$$\leq -\sigma g'_\varepsilon(n) n \leq \sigma g_\varepsilon(n).$$

Hence,

$$g_\varepsilon(n(t,z)) \leq (\inf n_0)^{-1} \exp(t\sigma(z))$$

provided that $\varepsilon \leq \inf n_0$. Letting $\varepsilon \to 0$ we obtain estimate (4.10).

It remains to show the upper bound for the field w. By assumption (2.10), (4.9) and (4.10) one has

$$|S_{\delta,M}\left(z, n(t,z), |H_M(w(t,z))|^2\right)| \leq C_1 \text{ for all } t \in (0,T), x \in (0,1) \tag{4.11}$$

with some $C_1 \in (0,\infty)$ independent of δ, M. Recall that

$$w(t) = w^0(t) + \int_0^t \exp((t-s)B) \left[S_{\delta,M}\left(z, n(s), |H_M(w(s))|^2\right) H_M(w(s)) \right] ds \quad .$$

Now it follows from (4.2), Lemma 3 and (4.11) that

$$\|w(t)\|_{L^\infty((0,1),\mathbb{C}^2)} \leq C_2 + C_2 \int_0^t \|S_{\delta,M}\left(z, n(s), |H_M(w(s))|^2\right) H_M(w(s))\|_{L^\infty((0,1),\mathbb{C}^2)} ds$$

$$\leq C_3 \left(1 + \int_0^t \|w(s)\|_{L^\infty((0,1),\mathbb{C}^2)} ds \right).$$

This implies by Gronwall's lemma that

$$\|w(t)\|_{L^\infty((0,1),\mathbb{C}^2)} \leq C_4 \text{ for all } t \in (0,T). \tag{4.12}$$

with some C_4 independent of δ, M. Note that the constants may depend on n_0, w_0 and T. Since (4.9), (4.10) and (4.12) are independent of δ, M, this completes the proof of Theorem 4.

This section is closed by the proof of Theorem 2 concerning the regularity of the solution in the case of no input-signal.

Proof of Theorem 2 In the case $a = 0$ (and hence $\Phi = 0$) the function $w \in C([0,\infty), X)$ is a weak solution of the evolution problem

$$\partial_t w = Bw + S(z, n, |w|^2)w, \quad w(0) = w_0$$

in the sense of [2].

Since $n \in L^\infty((0,T) \times (0,1))$, $w \in L^\infty((0,T) \times (0,1))$ and $n^{-1} \in L^\infty((0,T) \times (0,1))$ one can introduce by truncation a globally bounded function $F : (0,1) \times \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2$, which is globally Lipschitz continuous with respect to (n, w) uniformly in z, such that

$$S(z, n(t,z), |w(t,z)|^2)w(t,z) = F(z, n(t,z), w(t,z)) \text{ for all } t \in (0,T), z \in (0,1).$$

Now let $f : (0,T) \times X \to X$ be defined by

$$(f(t,u))(z) \stackrel{\text{def}}{=} F(z, n(t,z), u(t,z)) \text{ for } t \in (0,T), z \in (0,1).$$

Then w satisfies the variation of constants formula, see [2]

$$w(t) = \exp(tB)w_0 + \int_0^t \exp((t-s)B)f(s, w(s))ds. \tag{4.13}$$

Recall that

$$n \in W^{1,\infty}((0,T), L^\infty(0,1)). \tag{4.14}$$

Since F is globally Lipschitz continuous with respect to (n, w), it follows easily that $f : (0,T) \times X \to X$ is Lipschitz continuous in both variables. Therefore we obtain from [5, Theorem 1.6, sect. 6] that w is a strong solution, i.e. $\partial_t w \in L^1((0,T), X)$. By (4.14) this implies $f(\cdot, w(\cdot)) \in W^{1,1}((0,T), X)$.
Finally the assertion follows from the regularity theorem [3, Proposition 4.1.6].

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