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Preduals of Campanato spaces and Sobolev-Campanato spaces: A general construction

Dedicated to Professor Dr. Herbert Gajewski on the occasion of his sixtieth birthday

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Abstract

In this paper we describe two limiting processes for families of Banach spaces closely related to the standard definition of projective and inductive limits. These processes lead again to Banach spaces. Information about linear operators and duality between basic families of spaces is carried over to the corresponding limit spaces.

The abstract results are shown to be applicable to Campanato spaces and Sobolev-Campanato spaces. In particular, we obtain the existence and a characterization of predual spaces. Some imbedding relations are investigated in more detail.

Introduction

When the treatment of second order elliptic boundary value problems in Sobolev spaces started, the differential equations were usually written (using the summation convention) as

$$\forall v \in C_0^1(\Omega): \quad \int_\Omega a_{ij} D_j u \, D_i v + \ldots = \int_\Omega (gv + f_i D_i v), \tag{1}$$

and requirements with respect to the right hand side of the form

$$g \in L^{q/2}(\Omega), \; f_i \in L^q(\Omega), \; i = 1, \ldots, N,$$

were made (see [LU]). Later it became clear that essential is not the representation of the right hand side of the equation by means of $g, f_1, \ldots, f_N$, but the fact that the right hand side is in $W^{-1,q}(\Omega)$ for some q. Moreover, if

$$\left.\begin{array}{l} a_{ij} \in L^\infty(\Omega), \; \|a_{ij}\|_{L^\infty(\Omega)} \leq \frac{1}{\varepsilon}, \; i, j = 1, \ldots, N; \\ a_{ij}(x)\xi_i\xi_j \geq \varepsilon|\xi|^2 \text{ for some } \varepsilon > 0, \text{ all } \xi \in \mathbb{R}^N \text{ and almost all } x \in \Omega, \end{array}\right\} \tag{2}$$

then there exists a $q_0 > 2$ depending on Ω, N and ε only such that for all $q \in [2, q_0]$ the following holds: Each solution $u \in W_0^{1,2}(\Omega)$ to (1) belongs to $W^{1,q}(\Omega)$ if and only if the right hand side belongs to $W^{-1,q}(\Omega)$. If $N > 2$ this regularity result does not imply Hölder continuity of solutions to (1).

The treatment of boundary value problems in Morrey-Campanato spaces started also with the formulation (1) but with different requirements with respect to $g, f_1, \dots, f_N$, which read for example as follows

$$g \in \mathcal{L}^{2,(\lambda-2)_+}(\Omega),\ f_i \in \mathcal{L}^{2,\lambda}(\Omega),\ \ i = 1, \dots, N;$$

here λ is a parameter and $\mathcal{L}^{2,\lambda}(\Omega)$ a corresponding Campanato space (see [Tr]). The question what, in the context of Campanato spaces, could be an appropriate substitute for the Sobolev spaces $W^{-1,q}(\Omega)$ was ignored (or considered unimportant) for a long time. Some years ago the second author (see [R]) introduced $W^{-1,2,\lambda}(\Omega)$ as the image of $W_0^{1,2,\lambda}(\Omega)$ under the duality map of $W_0^{1,2}(\Omega)$; here $W_0^{1,2,\lambda}(\Omega)$ consists of those elements of the Sobolev space $W_0^{1,2}(\Omega)$ the first derivatives of which are in the Campanato space $\mathcal{L}^{2,\lambda}(\Omega)$. It was shown that, if (2) is satisfied, there exists a $\lambda_0 > N-2$, depending on Ω, N and ε only such that for all $\lambda \in [0, \lambda_0]$ a solution $u \in W_0^{1,2}(\Omega)$ to (1) belongs to $W^{1,2,\lambda}(\Omega)$ if and only if the right hand side belongs to $W^{-1,2,\lambda}(\Omega)$. Later these results were generalized to a broader class of problems by Griepentrog and Recke in [GrR]. Note that these regularity results imply Hölder continuity (up to the boundary) of the solutions to (1) for all space dimensions N, because for $\lambda > N-2$ the space $W^{1,2,\lambda}(\Omega)$ is continuously imbedded into a Hölder space.

The definition of $W^{-1,2,\lambda}(\Omega)$ mentioned above has drawbacks: On the one hand it is difficult to decide whether a given right hand side is in the space $W^{-1,2,\lambda}(\Omega)$ or not. On the other hand one can doubt whether a definition is appropriate, which makes the solvability of equations with the duality map for right hand sides from $W^{-1,2,\lambda}(\Omega)$ a trivial consequence of the definition of $W^{-1,2,\lambda}(\Omega)$. In a forthcoming paper Griepentrog [Gr] will present another definition of $W^{-1,2,\lambda}(\Omega)$ which seems to be more natural and simpler to handle. His definition follows closely the original definition of Morrey spaces. Simultaneously the first author developped the idea to define spaces $W^{-k,p,\lambda}(\Omega)$ as dual spaces of suitably chosen other spaces. This idea came up because for Sobolev spaces one has

$$W^{-k,p}(\Omega) := (W_0^{k,p'}(\Omega))^*,\ p \in]1, \infty[. \tag{3}$$

The definition $W^{-k,p}(\Omega) := (W_0^{k,p'}(\Omega))^*$ is usually motivated by the fact that for $p \in]1, \infty[$ the Lebesgue space $L^p(\Omega)$, $1 < p < \infty$, is the dual of $L^{p'}(\Omega)$, i.e. of a space from the scale of Lebesgue spaces itself. It is this relation that allows to interpret the scale $W^{-k,p}(\Omega)$, $k \in \mathbb{N}$, as a continuation of the scale $W^{k,p}(\Omega)$, $k \in \mathbb{Z}_+$. Generally it is not true that Campanato spaces are duals of other Campanato spaces. However, it is well known (see [Le]) that for each of the Hölder spaces $C^{0,\alpha}(\overline{\Omega})$ (which are part of the scale of Campanato spaces) there exists a predual Banach space, i.e., a Banach space the dual of which is $C^{0,\alpha}(\overline{\Omega})$.

In the present paper we are going to show that for all Campanato spaces there exist predual Banach spaces. We want to convince the reader that the scale of these preduals can be interpreted in a natural way as a continuation of the scale of

Campanato spaces. More precisely, using the notation $L^{p,m,\mu}(\Omega)$ instead of Campanato's notation $\mathcal{L}_k^{(p,\lambda)}(\Omega)$ (where $m = k+1$, $\mu = \lambda/p$, cf. [Ca]), we introduce spaces $L^{p,m,-\mu}(\Omega)$ such that

$$L^{p,m,\mu}(\Omega) = (L^{p',m,-\mu}(\Omega))^*.$$

Moreover, we are going to show that for Sobolev-Campanato spaces the situation is analogous: We present spaces $W^{-k,p,m,\mu}(\Omega)$ and $W_0^{k,p,m,-\mu}(\Omega)$ such that

$$W^{-k,p,m,\mu}(\Omega) = (W_0^{k,p',m,-\mu}(\Omega))^*. \tag{4}$$

Hence, the relation (3) has a counterpart in the theory of Sobolev-Campanato spaces. The definition of $W^{-1,2,m,\mu}(\Omega)$ is closely connected to a new criterion for the right hand side of (1) which is necessary and sufficient for a solution to belong to $W^{1,2,\lambda}(\Omega)$.

It turned out that the construction of predual spaces for Sobolev-Campanato spaces is based only on a few properties of these spaces, namely:

1. The restriction of an element of a Sobolev space to a (small) subset U of the original domain of definition belongs to the corresponding Sobolev space over U.

2. Elements of Sobolev-Campanato spaces can be characterized by a "nice" dependence of (semi)norms of those restrictions on the subset U.

The essential point is that different norms can be considered simultaneously. The observation that for many results the concrete nature of the Sobolev spaces is unimportant has had great influence on the structure of our paper. We proceed as follows.

In the first section we introduce *projective and inductive systems of Banach spaces.* We show that such systems can be viewed as an "abstract" setting which allows to create new Banach spaces like, for example, Campanato spaces. In particular, we deal with duality: We make precise in which sense spaces created by means of projective systems of Banach spaces are dual to spaces created by means of inductive systems of Banach spaces.

The second section is devoted to linear operators. We show how continuity and compactness properties of mappings between the newly created spaces can be reduced to properties of mappings between the spaces of the systems we start from. Our procedure is similar to that of interpolation theory.

In Section 3 and 4 we consider – as applications of the preceding results – the classical Campanato spaces and Sobolev-Campanato spaces on open subsets of $\mathbb{R}^N$. We show that different characterizations lead to the same spaces and to equal or equivalent norms. Moreover we deal with some imbedding theorems.

We are well aware that there is a lot of further subjects, which should and could be treated: trace theorems, multiplier theorems and the behaviour of Sobolev-Campanato spaces with respect to transformation of coordinates, solvability of boundary value problems with right hand sides in Sobolev-Campanato spaces $W^{-k,p,m,\mu}(\Omega)$, to mention only a few. We omitted these points in order keep a reasonable length of this paper.

1. Projective and inductive systems

Throughout this paper we denote by $\mathcal{B}$ the class of all real Banach spaces. Sometimes $\mathcal{B}$ will be regarded as a category with the continuous linear mappings as morphisms. As usual, if E and F are Banach spaces, then $\mathcal{L}(E;F)$ denotes the space of all linear continuous mappings from E into F and E^* the dual of E.

For the time being let $\mathcal{F}$ be any set, and let $\mathcal{B}^{\mathcal{F}}$ be the set of all mappings from $\mathcal{F}$ into $\mathcal{B}$.

Definition 1.1. For $X \in \mathcal{B}^{\mathcal{F}}$ and $p \in [1,\infty]$ we introduce

$$l^p(X) := \{g = (g_U)_{U\in\mathcal{F}} \in \prod_{U\in\mathcal{F}} X(U);\ \|g\|_{l^p(X)} < \infty\},$$

where

$$\|g\|_{l^p(X)} := \Big(\sum_{U\in\mathcal{F}} \|g_U\|^p_{X(U)}\Big)^{\frac{1}{p}} \text{ if } p<\infty,\ \|g\|_{l^\infty(X)} := \sup_{U\in\mathcal{F}} \|g_U\|_{X(U)}.$$

Here $\sum_{U\in\mathcal{F}}$ is to be interpreted as the integral on $\mathcal{F}$ with respect to the counting measure, i.e., as the limit of the net of sums over finite subsets of $\mathcal{F}$.

It is easy to check – and well known – that $(l^p(X), \|\cdot\|_{l^p(X)})$ is a Banach space. The following result is also known; it can easily be deduced from the corresponding result on the standard l^p-spaces.

Theorem 1.2. *For $X \in \mathcal{B}^{\mathcal{F}}$ let $X^* \in \mathcal{B}^{\mathcal{F}}$ be defined by $X^*(U) := (X(U))^*$, $U \in \mathcal{F}$. Moreover, let $p \in [1,\infty]$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$\langle \Phi(f), g\rangle := \sum_{U\in\mathcal{F}} \langle f_U, g_U\rangle \quad \textit{for} \quad f \in l^{p'}(X^*),\ g \in l^p(X),$$

defines an isometric linear mapping Φ from $l^{p'}(X^*)$ into $(l^p(X))^*$. If $p<\infty$, then Φ is surjective.

In the sequel we shall identify f and $\Phi(f)$. In particular, $l^\infty(X^*)$ will be considered as the dual of $l^1(X)$. The notation X^* introduced in Theorem 1.2 will be used throughout the paper without further explanation.

>From now on we assume that $\mathcal{F}$ is a family of subsets of some fixed set Ω. This family will be regarded as a category: The objects of $\mathcal{F}$ are its elements, the set of morphisms from $V \in \mathcal{F}$ into $U \in \mathcal{F}$ consists of the identical imbedding $i_V^U : V \longrightarrow U$, if $V \subset U$, and is empty, if $V \not\subset U$.

We remark that the family $\mathcal{F}$ could be replaced by any ordered set. In the applications in this paper $\mathcal{F}$ will be a family of open subsets of $\mathbb{R}^N$ serving as domains of definition of functions. This is the reason why we use the letters U, V for the elements of $\mathcal{F}$. There exist, however, quite different applications. For example, the construction of new Banach spaces carried out by Gröger [G] can be interpreted as a special case of the constructions described below. In that special case the rôle of $\mathcal{F}$ had been played by the set $\{p \in \mathbb{R};\ p > 2\}$ with its usual ordering.

Definition 1.3. We call P a *projective system of Banach spaces on* $\mathcal{F}$ if it is a contravariant functor from $\mathcal{F}$ into $\mathcal{B}$. We denote by $\mathcal{P}(\mathcal{F})$ the class of all such functors.

Thus, $P \in \mathcal{P}(\mathcal{F})$ means that P assigns to each $U \in \mathcal{F}$ a Banach space $P(U)$ and to each pair $U, V \in \mathcal{F}$ satisfying $V \subset U$ an operator $P_V^U := P(i_V^U) \in \mathcal{L}(P(U); P(V))$ such that P_U^U is the identity map of $P(U)$ and

$$P_W^V P_V^U = P_W^U \quad \text{if} \quad W \subset V \subset U.$$

Definition 1.4. Let $P \in \mathcal{P}(\mathcal{F})$. We define

$$\overleftarrow{l}(P) := \{f \in l^\infty(P);\ f_V = P_V^U f_U, \text{ if } V \subset U \text{ and } U, V \in \mathcal{F}\}.$$

Using the notation $l^\infty(P)$ we do not distinguish between the functor $P \in \mathcal{P}(\mathcal{F})$ and the underlying mapping $U \longmapsto P(U)$, $U \in \mathcal{F}$. Clearly, $(\overleftarrow{l}(P), \|\cdot\|_{l^\infty(P)})$ is a Banach space. It is not the standard projective limit of P (in the sense of locally convex spaces) but a good substitute for this limit if one wants to remain within the framework of Banach spaces. To simplify the notation we write $\|\cdot\|_P$ for the norm on $\overleftarrow{l}(P)$.

Definition 1.5. We call S an *inductive system of Banach spaces on* $\mathcal{F}$, if it is a covariant functor from $\mathcal{F}$ into $\mathcal{B}$. We denote by $\mathcal{S}(\mathcal{F})$ the class of all such functors.

Thus, $S \in \mathcal{S}(\mathcal{F})$ means that S assigns to each $U \in \mathcal{F}$ a Banach space $S(U)$ and to each pair $U, V \in \mathcal{F}$ satisfying $V \subset U$ an operator $S_V^U := S(i_V^U) \in \mathcal{L}(S(V); S(U))$ such that S_V^V is the identity map of $S(V)$ and

$$S_V^U S_W^V = S_W^U \quad \text{if} \quad W \subset V \subset U.$$

In the following speaking about projective or inductive systems we always have in mind projective or inductive systems of Banach spaces on $\mathcal{F}$.

Definition 1.6. Let $S \in \mathcal{S}(\mathcal{F})$. We define

$$\vec{l}(S) := l^1(S)/N(S),$$

where $N(S)$ is the closed linear subspace of $l^1(S)$ generated by those elements $g = (g_U)_{U \in \mathcal{F}}$ which, for some V, $W \in \mathcal{F}$ such that $W \subset V$, satisfy

$$g_V = -S_W^V g_W, \quad g_U = 0, \text{ if } U \neq V, W. \tag{1.1}$$

The space $\vec{l}(S)$ can be regarded as a good substitute for the inductive limit of S in the framework of Banach spaces. It is a Banach space with the usual factor space norm

$$\|g + N(S)\|_S := \inf_{h \in N(S)} \|g + h\|_{l^1(S)}.$$

Here $g + N(S)$ denotes the class of $g \in l^1(S)$ in the factor space $l^1(S)/N(S)$. An analogous notation will be used in the sequel also for other factor spaces without further explanation.

The following statement is an immediate consequence of the definitions of projective and inductive systems of Banach spaces.

If P is a projective system, then P^*, defined by

$$P^*(U) := (P(U))^*, \quad P^{*U}_V := (P^U_V)^*,$$

is an inductive system. If S is an inductive system, then S^*, defined by

$$S^*(U) := (S(U))^*, \quad S^{*U}_V := (S^U_V)^*,$$

is a projective system. We call P^* the dual of P and S^* the dual of S.

Theorem 1.7. *Let S be an inductive system on $\mathcal{F}$ and S^* the dual projective system. Then there exists a canonical linear isometric mapping from $(\vec{l}(S))^*$ onto $\overleftarrow{l}(S^*)$.*

Proof. By a standard result of linear functional analysis the dual to the factor space $l^1(S)/N(S)$ is canonically isometric to the subspace $N^0(S)$ of those elements of $(l^1(S))^* = l^\infty(S^*)$ which vanish on $N(S)$. Thus, it suffices to prove $\overleftarrow{l}(S^*) = N^0(S)$. Let $f \in l^\infty(S^*)$ and let $g \in l^1(S)$ satisfy (1.1). Then

$$\langle f, g\rangle = \langle f_W, g_W\rangle + \left\langle f_V, -S^V_W g_W\right\rangle = \left\langle f_W - (S^V_W)^* f_V, g_W\right\rangle.$$

Hence $f \in N^0(S)$ if and only if

$$f_W = (S^V_W)^* f_V \quad \text{for all} \quad V, W \in \mathcal{F} \text{ such that } W \subset V,$$

i.e., if $f \in \overleftarrow{l}(S^*)$. This is the desired result. $\square$

Remark 1.8. In the sequel we shall identify $\overleftarrow{l}(S^*)$ and $(\vec{l}(S))^*$ identifying $f \in N^0(S)$ with the functional assigning the value $\langle f, g\rangle$ to the equivalence class of $g \in l^1(S)$ in the factor space $l^1(S)/N(S)$.

Next we create in a rather simple manner new spaces by means of weight functions. We denote by $\mathcal{A}_\mathcal{F}$ the set of all positive valued functions on $\mathcal{F}$, regarded as a group with respect to the pointwise multiplication. For a, $b \in \mathcal{A}_\mathcal{F}$ we write $a \leq b$ if $a(U) \leq b(U)$ for every $U \in \mathcal{F}$. Moreover, for $\theta \in \mathbb{R}$ and $a \in \mathcal{A}_\mathcal{F}$ we define $a^\theta(U) := (a(U))^\theta$, $U \in \mathcal{F}$. The elements of $\mathcal{A}_\mathcal{F}$ will play the rôle of weight functions.

If E is any Banach space and $\alpha \in]0,\infty[$, we denote by E_α the space E equipped with the norm $\alpha\|\cdot\|_E$. Let $a \in \mathcal{A}_\mathcal{F}$. For $X \in \mathcal{B}^\mathcal{F}$ let $X_a \in \mathcal{B}^\mathcal{F}$ be defined by $X_a(U) := X(U)_{a(U)}$. For $P \in \mathcal{P}(\mathcal{F})$ we define $P_a \in \mathcal{P}(\mathcal{F})$ setting

$$P_a(U) := P(U)_{a(U)}, \quad (P_a)_V^U := P_V^U \quad \text{if} \quad U, V \in \mathcal{F},\, V \subset U.$$

Analogously we define $S_a \in \mathcal{S}(\mathcal{F})$ for $S \in \mathcal{S}(\mathcal{F})$. Obviously,

$$(P_a)^* = (P^*)_{a^{-1}} \quad \text{and} \quad (S_a)^* = (S^*)_{a^{-1}}.$$

As a consequence of Theorem 1.7 and Remark 1.8 we have the following

Corollary 1.9. *Suppose that S is an inductive system on $\mathcal{F}$ and $a \in \mathcal{A}_\mathcal{F}$. Then* $(\vec{l}(S_a))^* = \overleftarrow{l}((S^*)_{a^{-1}})$.

To conclude this section we want to compare spaces generated by means of different families $\mathcal{F}$ and $\mathcal{G}$. Let $\mathcal{G}$ be a subfamily of $\mathcal{F}$. Clearly, each $P \in \mathcal{P}(\mathcal{F})$ can be restricted to the category $\mathcal{G}$, and this restriction, denoted by $P|_\mathcal{G}$, is in $\mathcal{P}(\mathcal{G})$. We shall formulate simple sufficient conditions guaranteeing that spaces generated by means of P are canonically isomorphic to spaces generated by means of $P|_\mathcal{G}$. Analogously, we shall deal with restrictions $S|_\mathcal{G}$ of inductive system $S \in \mathcal{S}(\mathcal{F})$. As usual, we call $\mathcal{G}$ a *directed* subfamily of $\mathcal{F}$, if for arbitrary $V, W \in \mathcal{G}$ there exists $U \in \mathcal{G}$ such that $V \subset U$ and $W \subset U$.

Lemma 1.10. *Let $\mathcal{G}$ be a directed subfamily of $\mathcal{F}$. Suppose that for $P \in \mathcal{P}(\mathcal{F})$ and $c > 0$ the following holds: For every $V \in \mathcal{F}$ there exists $U \in \mathcal{G}$ such that $V \subset U$ and $\|P_V^U\|_{\mathcal{L}(P(U);P(V))} \leq c$. Then $\overleftarrow{l}(P)$ and $\overleftarrow{l}(P|_\mathcal{G})$ are canonically isomorphic as topological linear spaces.*

Proof. For $f = (f_U)_{U\in\mathcal{F}} \in \overleftarrow{l}(P)$ we define $f|_\mathcal{G} := (f_U)_{U\in\mathcal{G}}$. We want to show that the mapping $f \longmapsto f|_\mathcal{G}$ is a topological linear isomorphism from $\overleftarrow{l}(P)$ onto $\overleftarrow{l}(P|_\mathcal{G})$. Obviously, $\|f|_\mathcal{G}\|_{l^\infty(P|_\mathcal{G})} \leq \|f\|_{l^\infty(P)}$ and $f|_\mathcal{G} \in \overleftarrow{l}(P|_\mathcal{G})$. The linearity of the mapping $f \longmapsto f|_\mathcal{G}$ is also obvious. Moreover,

$$\begin{aligned}
\|f\|_P &= \sup_{V\in\mathcal{F}} \|f_V\|_{P(V)} = \sup_{V\in\mathcal{F}} \inf_{U\in\mathcal{G},\, V\subset U} \|P_V^U f_U\|_{P(V)} \\
&\leq \sup_{V\in\mathcal{F}} \inf_{U\in\mathcal{G},\, V\subset U} \|P_V^U\|_{\mathcal{L}(P(U);P(V))} \|f_U\|_{P(U)} \\
&\leq c \sup_{U\in\mathcal{G}} \|f_U\|_{P(U)} = c\, \|f|_\mathcal{G}\|_{P|_\mathcal{G}}.
\end{aligned}$$

Consequently, the mapping $f \longmapsto f|_\mathcal{G}$ is a homeomorphism onto its image. Now, let $f' \in \overleftarrow{l}(P|_\mathcal{G})$ be given. For every $V \in \mathcal{F}$ we choose $U \in \mathcal{G}$ such that $V \subset U$ and define $f_V := P_V^U f'_U$. Because $\mathcal{G}$ is a directed family it is easy to check that this definition is independent of the choice of $U \in \mathcal{G}$. The preceding estimate shows that $f := (f_U)_{U\in\mathcal{F}}$ is in $\overleftarrow{l}(P)$. Since $f|_\mathcal{G} = f'$, the mapping $f \longmapsto f|_\mathcal{G}$ is surjective. $\square$

Corollary 1.11. *Let $\mathcal{G}$ be a directed subfamily of $\mathcal{F}$. Suppose that for $P \in \mathcal{P}(\mathcal{F})$, $a \in \mathcal{A}_{\mathcal{F}}$ and $c > 0$ the following holds: For every $V \in \mathcal{F}$ there exists $U \in \mathcal{G}$ such that $V \subset U$ and $a(V)\|P_V^U\|_{\mathcal{L}(P(U);P(V))} \leq c\, a(U)$. Then $\overleftarrow{l}(P_a)$ and $\overleftarrow{l}(P_a|_{\mathcal{G}})$ are canonically isomorphic as topological linear spaces.*

Proof. In view of the elementary relation $\|P_V^U\|_{\mathcal{L}(P_a(U);P_a(V))} = \frac{a(V)}{a(U)}\|P_V^U\|_{\mathcal{L}(P(U);P(V))}$ the assertion is an immediate consequence of Lemma 1.10. $\square$

Next we state counterparts of Lemma 1.10 and Corollary 1.11 for inductive systems.

Lemma 1.12. *Let $\mathcal{G}$ be a directed subfamily of $\mathcal{F}$. Suppose that for $S \in \mathcal{S}(\mathcal{F})$ and $c > 0$ the following holds: For every $V \in \mathcal{F}$ there exists $U \in \mathcal{G}$ such that $V \subset U$ and $\|S_V^U\|_{\mathcal{L}(S(V);S(U))} \leq c$. Then $\vec{l}(S)$ and $\vec{l}(S|_{\mathcal{G}})$ are canonically isomorphic as topological linear spaces.*

Proof. 1. For $g' \in l^1(S|_{\mathcal{G}})$ let $g \in l^1(S)$ be defined by

$$g_U := \begin{cases} g'_U & \text{if} \quad U \in \mathcal{G}, \\ 0 & \text{if} \quad U \notin \mathcal{G}. \end{cases}$$

Clearly, $g' \longmapsto g$ is a continuous linear mapping I from $l^1(S|_{\mathcal{G}})$ into $l^1(S)$. Moreover, it is evident that I maps the elements generating $N(S|_{\mathcal{G}})$ (cf. (1.1)) into $N(S)$. Hence

$$g' + N(S|_{\mathcal{G}}) \longmapsto g + N(S) \tag{1.2}$$

is a correctly defined mapping J from $\vec{l}(S|_{\mathcal{G}})$ into $\vec{l}(S)$. We have

$$\|g + N(S)\|_S = \inf_{h \in N(S)} \|g + h\|_{l^1(S)} \leq \inf_{h' \in N(S|_{\mathcal{G}})} \|g' + h'\|_{l^1(S|_{\mathcal{G}})} = \|g' + N(S|_{\mathcal{G}})\|_{S|_{\mathcal{G}}}.$$

2. In view of Lemma 1.10 and Theorem 1.7 we find (using the same notation as in the proof of Lemma 1.10)

$$\begin{aligned} \|g' + N(S|_{\mathcal{G}})\|_{S|_{\mathcal{G}}} &= \sup\{\langle f', g'\rangle\, ;\; f' \in \overleftarrow{l}((S|_{\mathcal{G}})^*),\ \|f'\|_{(S|_{\mathcal{G}})^*} \leq 1\} \\ &= \sup\{\langle f|_{\mathcal{G}}, g'\rangle\, ;\; f \in \overleftarrow{l}(S^*),\ \|f|_{\mathcal{G}}\|_{S^*|_{\mathcal{G}}} \leq 1\} \\ &\leq \sup\{\langle f, g\rangle\, ;\; \|f\|_{S^*} \leq c\} = c\,\|g + N(S)\|_S. \end{aligned}$$

Hence, the mapping J is a topological linear isomorphism onto its image. In particular, its image is closed in $\vec{l}(S)$.

3. Assume that for some $f \in \overleftarrow{l}(S^*)$ we have $\langle f, g\rangle = 0$ for every $g = Ig'$, $g' \in l^1(S|_{\mathcal{G}})$. Then $\langle f|_{\mathcal{G}}, g'\rangle = 0$ for every $g' \in l^1(S|_{\mathcal{G}})$. Theorem 1.7 shows that this is possible only if $f|_{\mathcal{G}} = 0$. In view of Lemma 1.10 this means that $f = 0$. By the Hahn-Banach theorem this result implies that the image of J is dense in $\vec{l}(S)$.

4. Combining the results of the preceding steps of the proof we find that J is a topological linear isomorphism from $\vec{l}(S|_{\mathcal{G}})$ onto $\vec{l}(S)$. $\square$

Corollary 1.13. *Let $\mathcal{G}$ be a directed subfamily of $\mathcal{F}$. Suppose that for $S \in \mathcal{S}(\mathcal{F})$, $a \in \mathcal{A}_{\mathcal{F}}$ and $c > 0$ the following holds: For every $V \in \mathcal{F}$ there exists $U \in \mathcal{G}$ such that $V \subset U$ and $a(U)\|S_V^U\|_{\mathcal{L}(S(V);S(U))} \leq c\,a(V)$. Then $\vec{l}(S_a)$ and $\vec{l}(S_a|_{\mathcal{G}})$ are canonically isomorphic as topological linear spaces.*

Recall that the elements of $\mathcal{F}$ are subsets of a fixed set Ω. For the final part of this section we shall assume that $\Omega \in \mathcal{F}$. In that case Ω is the unique maximal element of $\mathcal{F}$ with respect to inclusion.

Lemma 1.14. *Let $P \in \mathcal{P}(\mathcal{F})$ be such that $\|P_V^\Omega\|_{\mathcal{L}(P(\Omega);P(V))} \leq 1$ for every $V \in \mathcal{F}$ and suppose that $\Omega \in \mathcal{F}$. Then the mapping*

$$f \longmapsto f_\Omega, \quad \text{where } f = (f_V)_{V\in\mathcal{F}} \in \overleftarrow{l}(P),$$

is isometric from $\overleftarrow{l}(P)$ onto $P(\Omega)$.

Proof. The hypotheses of Lemma 1.10 are satisfied with $\mathcal{G} = \{\Omega\}$ and $c = 1$. Obviously, $\overleftarrow{l}(P|_{\mathcal{G}}) = P(\Omega)$. Moreover, for $c = 1$ the proof of Lemma 1.10 shows that the mapping from $\overleftarrow{l}(P)$ onto $\overleftarrow{l}(P|_{\mathcal{G}})$ is isometric. □

Remark 1.15. In the following, whenever the hypotheses of Lemma 1.14 are satisfied, we shall (tacitly) identify $\overleftarrow{l}(P)$ and $P(\Omega)$ identifying f and f_Ω. As a consequence, subspaces of $\overleftarrow{l}(P)$ will be treated as subspaces of $P(\Omega)$.

Lemma 1.16. *Let $S \in \mathcal{S}(\mathcal{F})$ be such that $\|S_V^\Omega\|_{\mathcal{L}(S(V);S(\Omega))} \leq 1$ for every $V \in \mathcal{F}$ and suppose that $\Omega \in \mathcal{F}$. Then the mapping*

$$g' \longmapsto g + N(S),\ g' \in S(\Omega),\ \text{where } g = (g_V)_{V\in\mathcal{F}},\ g_\Omega = g',\ g_V = 0,\ \text{if } V \neq \Omega, \quad (1.3)$$

is isometric from $S(\Omega)$ onto $\vec{l}(S)$.

Proof. One may apply Lemma 1.12 with $\mathcal{G} = \{\Omega\}$ and $c = 1$. □

Remark 1.17. In the following, whenever the hypotheses of Lemma 1.16 are satisfied, we shall (tacitly) identify $S(\Omega)$ and $\vec{l}(S)$ by means of the mapping (1.3). This corresponds to our treatment of projective systems (cf. Remark 1.15).

2. Linear operators

In the first part of this section we shall deal with linear mappings between spaces of the kind $l^p(X)$ introduced in Definition 1.1. Here $\mathcal{F}$ might be any set.

For X, $Y \in \mathcal{B}^{\mathcal{F}}$ we denote by $\mathcal{L}(X;Y)$ the linear space of all mappings A defined on $\mathcal{F}$ assigning to $U \in \mathcal{F}$ an operator $A_U \in \mathcal{L}(X(U);Y(U))$. Each $A \in \mathcal{L}(X;Y)$ can be regarded as a continuous linear mapping from $\prod_{U\in\mathcal{F}} X(U)$ into $\prod_{U\in\mathcal{F}} Y(U)$: The image Af of $f \in \prod_{U\in\mathcal{F}} X(U)$ is defined by $(Af)_U := A_U f_U$, $U \in \mathcal{F}$. For $X, Y \in \mathcal{B}^{\mathcal{F}}$ and $A \in \mathcal{L}(X;Y)$ we define $\Lambda_A \in \mathcal{A}_{\mathcal{F}}$ by

$$\Lambda_A(U) := \|A_U\|_{\mathcal{L}(X(U);Y(U))}. \tag{2.1}$$

Lemma 2.1. *Let $a, b \in \mathcal{A}_{\mathcal{F}}$, $X, Y \in \mathcal{B}^{\mathcal{F}}$ and $A \in \mathcal{L}(X;Y)$. Moreover, let $p \in [1, \infty]$. Then the following holds:*

i) *If*

$$c := \sup_{U\in\mathcal{F}} \Lambda_A(U) \frac{b(U)}{a(U)} < \infty,$$

then A maps $l^p(X_a)$ continuously into $l^p(Y_b)$, where the norm of A, considered as an element of $\mathcal{L}(l^p(X_a); l^p(Y_b))$, is bounded by c.

ii) *If, for some functions $b_0, b_1 \in \mathcal{A}_{\mathcal{F}}$, the operator A is compact as a mapping from $l^p(X_a)$ into $l^p(Y_{b_0})$ and continuous as a mapping from $l^p(X_a)$ into $l^p(Y_{b_1})$, then A is compact also as a mapping from $l^p(X_a)$ into $l^p(Y_b)$ provided that $b \le b_0^{1-\theta} b_1^{\theta}$, $\theta \in [0, 1[$.*

Proof. 1. The assertion i) is an elementary consequence of the definition of the norms involved.

2. Let $M \subset l^p(X_a)$ be bounded. We have to show that $A[M]$ is precompact in $l^p(Y_b)$. By Young's Inequality $b(U) \le \varepsilon b_1(U) + c_\varepsilon b_0(U)$ for arbitrarily chosen $\varepsilon > 0$ and an appropriate constant c_ε. For every $f \in l^p(X_a)$, $p < \infty$, we have

$$\begin{aligned} \|Af\|_{l^p(Y_b)} &= \Big(\sum_{U\in\mathcal{F}} (b(U) \|A_U f_U\|_{Y(U)})^p \Big)^{\frac{1}{p}} \\ &\le \Big(\sum_{U\in\mathcal{F}} ((\varepsilon b_1(U) + c_\varepsilon b_0(U)) \|A_U f_U\|_{Y(U)})^p \Big)^{\frac{1}{p}} \\ &\le \varepsilon \|Af\|_{l^p(Y_{b_1})} + c_\varepsilon \|Af\|_{l^p(Y_{b_0})}. \end{aligned}$$

An obvious modification of the argument shows that the estimate is true also for $p = \infty$. We choose now $f_1, \ldots, f_n$ in M such that $\inf_{1\le i\le n} \|Af - Af_i\|_{l^p(Y_{b_0})} \le \frac{\varepsilon}{c_\varepsilon}$ for every $f \in M$. This is possible because of the compactness hypothesis. The preceding estimate implies that

$$\inf_{1\le i\le n} \|Af - Af_i\|_{l^p(Y_b)} \le \varepsilon \Big(2 \sup_{g\in A[M]} \|g\|_{l^p(Y_{b_1})} + 1 \Big)$$

provided that $f \in M$. Hence, $A[M]$ has a finite ε-net in $l^p(Y_b)$ for every $\varepsilon > 0$. $\square$

Remark 2.2. Note that, if A_U is isometric from $X(U)$ into $Y(U)$ for every $U \in \mathcal{F}$, then A is isometric also as a mapping from $l^p(X_a)$ into $l^p(Y_a)$.

For arbitrary $P, Q \in \mathcal{P}(\mathcal{F})$ we denote by $\mathcal{L}(P;Q)$ the set of all natural transformations from the functor $P : \mathcal{F} \longrightarrow \mathcal{B}$ to the functor $Q : \mathcal{F} \longrightarrow \mathcal{B}$. Thus, $K \in \mathcal{L}(P;Q)$ means that K associates to each $U \in \mathcal{F}$ an operator $K_U \in \mathcal{L}(P(U); Q(U))$ such that

$$K_V P_V^U = Q_V^U K_U, \tag{2.2}$$

provided that $U, V \in \mathcal{F}$ and $V \subset U$. The class $\mathcal{P}(\mathcal{F})$ can be considered as a category with the sets $\mathcal{L}(P;Q)$, $P, Q \in \mathcal{P}(\mathcal{F})$, as the corresponding sets of morphisms.

We emphazise that, working with $\mathcal{L}(P;Q)$, we have to distinguish between the functor $P \in \mathcal{P}(\mathcal{F})$ and the underlying mapping $U \longmapsto P(U)$, $U \in \mathcal{F}$. If, for a moment, we denote this mapping by P^o, then $\mathcal{L}(P^o;Q^o)$ is strictly larger than $\mathcal{L}(P;Q)$ because its definition does not include the relation(2.2). It is clear, however, that for $K \in \mathcal{L}(P;Q)$ the function Λ_K is defined (cf. (2.1)) and that an analogue of Lemma 2.1 holds for $K \in \mathcal{L}(P;Q)$.

Theorem 2.3. *Let $a, b \in \mathcal{A}_{\mathcal{F}}$, $P, Q \in \mathcal{P}(\mathcal{F})$ and $K \in \mathcal{L}(P;Q)$. Then the following holds:*

i) *If*

$$c := \sup_{U \in \mathcal{F}} \Lambda_K(U) \frac{b(U)}{a(U)} < \infty,$$

then K maps $\overleftarrow{l}(P_a)$ continuously into $\overleftarrow{l}(Q_b)$, where the norm of K, considered as an element of $\mathcal{L}(\overleftarrow{l}(P_a); \overleftarrow{l}(Q_b))$, is bounded by c.

ii) *If, for some functions $b_0, b_1 \in \mathcal{A}_{\mathcal{F}}$, the operator K is compact as a mapping from $\overleftarrow{l}(P_a)$ into $\overleftarrow{l}(Q_{b_0})$ and continuous as a mapping from $\overleftarrow{l}(P_a)$ into $\overleftarrow{l}(Q_{b_1})$, then K is compact also as a mapping from $\overleftarrow{l}(P_a)$ into $\overleftarrow{l}(Q_b)$ provided that $b \le b_0^{1-\theta} b_1^{\theta}$, $\theta \in [0, 1[$.*

The proof of this theorem is essentially the same as that of Lemma 2.1. The relation $K[\overleftarrow{l}(P_a)] \subset \overleftarrow{l}(Q_b)$ is a consequence of (2.2). We omit the details.

Remark 2.4. Whenever this seems desirable in order to avoid misunderstandings we shall write $K_{a,b}$ for K considered as a mapping from $\overleftarrow{l}(P_a)$ into $\overleftarrow{l}(Q_b)$.

Corollary 2.5. *Suppose that $\Omega \in \mathcal{F}$, and let the function $a \in \mathcal{A}_{\mathcal{F}}$ and the system $P \in \mathcal{P}(\mathcal{F})$ be such that $a \geq \mathrm{const} > 0$ and $\|P_V^\Omega\|_{\mathcal{L}(P(\Omega);P(V))} \leq 1$ for every $V \in \mathcal{F}$. Then $\overleftarrow{l}(P_a)$ is continuously imbedded into $P(\Omega)$ and an element $f \in P(\Omega)$ is in $\overleftarrow{l}(P_a)$ if and only if*

$$\sup_{V \in \mathcal{F}} a(V) \|P_V^\Omega f\|_{P(V)} < \infty. \tag{2.3}$$

If (2.3) *holds, then*

$$\|f\|_{P_a} = \sup_{V \in \mathcal{F}} a(V) \|P_V^\Omega f\|_{P(V)}.$$

Proof. The continuous imbedding $\overleftarrow{l}(P_a) \hookrightarrow \overleftarrow{l}(P) = P(\Omega)$ follows from Theorem 2.3, Lemma 1.14 and Remark 1.15. In view of the identification of f and f_Ω we find

$$\begin{aligned} \|f\|_{l^\infty(P_a)} &= \sup_{V \in \mathcal{F}} a(V) \|f_V\|_{P(V)} \\ &= \sup_{V \in \mathcal{F}} a(V) \|P_V^\Omega f_\Omega\|_{P(V)} = \sup_{V \in \mathcal{F}} a(V) \|P_V^\Omega f\|_{P(V)}. \end{aligned}$$

By definition of $\overleftarrow{l}(P_a)$ an element $f \in P(\Omega)$ is in $\overleftarrow{l}(P_a)$ if and only if $\|f\|_{l^\infty(P_a)}$ is finite, and in that case $\|f\|_{P_a} = \|f\|_{l^\infty(P_a)}$. $\square$

For arbitrary $S, T \in \mathcal{S}(\mathcal{F})$ we denote by $\mathcal{L}(S;T)$ the set of all natural transformations from the functor $S : \mathcal{F} \longrightarrow \mathcal{B}$ to the functor $T : \mathcal{F} \longrightarrow \mathcal{B}$. Thus, $L \in \mathcal{L}(S;T)$ means that L associates to each $U \in \mathcal{F}$ an operator $L_U \in \mathcal{L}(S(U);T(U))$ such that

$$L_U S_V^U = T_V^U L_V, \tag{2.4}$$

provided that $U, V \in \mathcal{F}$ and $V \subset U$. The class $\mathcal{S}(\mathcal{F})$ can be regarded as a category with the sets $\mathcal{L}(S;T)$, $S, T \in \mathcal{S}(\mathcal{F})$, as the corresponding sets of morphisms.

Theorem 2.6. *Let $a, b \in \mathcal{A}_\mathcal{F}$, $S, T \in \mathcal{S}(\mathcal{F})$ and $L \in \mathcal{L}(S;T)$. Then the following holds:*
i) *If*

$$c := \sup_{U \in \mathcal{F}} \Lambda_L(U) \frac{b(U)}{a(U)} < \infty,$$

then one can define $L_{a,b} \in \mathcal{L}(\vec{l}(S_a); \vec{l}(T_b))$ setting

$$L_{a,b}(g + N(S_a)) := Lg + N(T_b) \quad \textit{for} \quad g \in l^1(S_a).$$

It holds $\|L_{a,b}\|_{\mathcal{L}(\vec{l}(S_a);\vec{l}(T_b))} \le c$.
ii) *Let, for some $b_0, b_1 \in \mathcal{A}_\mathcal{F}$, the relations $\Lambda_L b_i \le \mathrm{const}\, a$, $i = 0, 1$, be satisfied. If the operator L_{a,b_0} is compact, then $L_{a,b}$ is compact provided that $b \le b_0^{1-\theta} b_1^\theta$, $\theta \in [0, 1[$.*

Proof. 1. Since $\Lambda_L\, b \le c\, a$, Lemma 2.1 proves that L maps $l^1(S_a)$ continuously into $l^1(T_b)$ and that the corresponding norm is bounded by c. In view of (2.4) the operator L maps $N(S_a)$ into $N(T_b)$. This implies that $L_{a,b}$ is correctly defined. Obviously, the norm of $L_{a,b}$ does not exceed the norm of L as a mapping from $l^1(S_a)$ into $l^1(T_b)$.
2. The compactness result can be proved as the corresponding part of Lemma 2.1.
□

Next we state as corollaries two simple consequences of the definition of the operators $L_{a,b}$. We omit the elementary proofs.

Corollary 2.7. *Let $L \in \mathcal{L}(S;T)$, where S and T are inductive systems on $\mathcal{F}$, and let $L^* \in \mathcal{L}(T^*; S^*)$ be defined by $(L^*)_U := (L_U)^*$, $U \in \mathcal{F}$. Then*

$$(L_{a,b})^* = (L^*)_{b^{-1},a^{-1}} \quad \textit{for all} \quad a, b \in \mathcal{A}_\mathcal{F}.$$

Corollary 2.8. *Suppose that S, S', S'' are inductive systems on $\mathcal{F}$. Moreover, let $L \in \mathcal{L}(S;S')$, $L' \in \mathcal{L}(S';S'')$ and $a, a', a'' \in \mathcal{A}_\mathcal{F}$ be given such that $\Lambda_L\, a' \le \mathrm{const}\, a$, $\Lambda_{L'}\, a'' \le \mathrm{const}\, a'$. Then*

$$(L'L)_{a,a''} = L'_{a',a''} L_{a,a'}.$$

Lemma 2.9. *Let $a, b \in \mathcal{A}_{\mathcal{F}}$, $b \leq \operatorname{const} a$, and let I^S denote the identity morphism of S, i.e., let I^S_U be the identity map of $S(U)$. Then $I^S_{a,b}$ maps $\vec{l}(S_a)$ (continuously) onto a dense subset of $\vec{l}(S_b))$. The adjoint $(I^S_{a,b})^*$ is the imbedding of $\overleftarrow{l}((S^*)_{b^{-1}})$ into $\overleftarrow{l}((S^*)_{a^{-1}})$.*

Proof. 1. The density statement follows from the obvious fact that

$$l_c(S) := \Big\{ g \in \prod_{U\in\mathcal{F}} S(U);\ g_U \neq 0 \text{ only for finitely many } U \in \mathcal{F} \Big\}$$

is contained and dense in $l^1(S_a)$ and $l^1(S_b)$.
2. Clearly, $(I^S)^* = I^{S^*}$, where I^{S^*} denotes the identity morphism of S^*. Therefore the second assertion follows from Corollary 2.7. □

Remark 2.10. The operator $I^S_{a,b}$ is not necessarily injective. Hence, generally we have not $\vec{l}(S_a) \hookrightarrow \vec{l}(S_b)$.

Remark 2.11. Later we shall use the following special case of Corollary 2.8: If $L \in \mathcal{L}(S;T)$, $S, T \in \mathcal{S}(\mathcal{F})$, and $b \leq \operatorname{const} a,\ a, b \in \mathcal{A}_{\mathcal{F}}$, then

$$L_{a,b} = I^T_{a,b} L_{a,a} = L_{b,b} I^S_{a,b},$$

because $L = I^T L = L I^S$.

In the following we denote by e the unit element of $\mathcal{A}_{\mathcal{F}}$, i.e., the weight function with the value 1 for every $U \in \mathcal{F}$.

Corollary 2.12. *Suppose that $\Omega \in \mathcal{F}$, and let the function $b \in \mathcal{A}_{\mathcal{F}}$ and the system $S \in \mathcal{S}(\mathcal{F})$ be such that $b \leq \operatorname{const} < \infty$ and $\|S^\Omega_V\|_{\mathcal{L}(S(V);S(\Omega))} \leq 1$ for every $V \in \mathcal{F}$. Then the operator $I^S_{e,b} \in \mathcal{L}(S(\Omega); \vec{l}(S_b))$ satisfies, for $g' \in S(\Omega)$,*

$$\|I^S_{e,b}\, g'\|_{S_b} = \inf \Big\{ \sum_{V\in\mathcal{G}} b(V) \|g_V\|_{S(V)};\ \sum_{V\in\mathcal{G}} S^\Omega_V g_V = g',\ \mathcal{G} \subset \mathcal{F} \text{ finite} \Big\}. \tag{2.5}$$

Proof. By definition of $N(S_b)$ (cf. Definition 1.6) the linear space

$$N_c(S) := \Big\{ h \in \prod_{V\in\mathcal{F}} S(V);\ \sum_{V\in\mathcal{F}} S^\Omega_V h_V = 0,\ h_V \neq 0 \text{ only for finitely many } V \in \mathcal{F} \Big\}$$

is dense in $N(S_b)$. Hence, if $g' \in S(\Omega)$ and

$$g = (g_V)_{V\in\mathcal{F}},\ g_\Omega = g',\ g_V = 0, \text{ if } V \neq \Omega,$$

then (cf. Remark 1.17 and (1.3))

$$\begin{aligned} \|I^S_{e,b}\, g'\|_{S_b} &= \|g + N(S_b)\|_{S_b} = \inf \{ \|g + h\|_{l^1(S_b)};\ h \in N_c(S) \} \\ &= \inf \Big\{ \sum_{V\in\mathcal{G}} b(V) \|g_V\|_{S(V)};\ \sum_{V\in\mathcal{G}} S^\Omega_V g_V = g',\ \mathcal{G} \subset \mathcal{F} \text{ finite} \Big\}. \ \square \end{aligned}$$

Remark 2.13. Whenever $I^S_{e,b}$ is injective we may identify $g' \in S(\Omega)$ and $I^S_{e,b}\, g'$. In this way $S(\Omega)$ becomes a (dense) subset of $\vec{l}(S_b)$ and it holds

$$\|g'\|_{S_b} = \inf\Big\{ \sum_{V\in\mathcal{G}} b(V)\|g_V\|_{S(V)};\ \sum_{V\in\mathcal{G}} S^\Omega_V g_V = g',\ \mathcal{G}\subset\mathcal{F}\text{ finite}\Big\} \text{ for } g'\in S(\Omega).$$

3. Campanato spaces

Throughout this section we assume that Ω is a fixed open and bounded subset of $\mathbb{R}^N$ and that $\mathcal{F}$ is the family of all nonempty open subsets of Ω. The diameter of a set $U \in \mathcal{F}$ (with respect to the usual Euclidean metric of $\mathbb{R}^N$) will be denoted by d_U. We shall use the weight functions $a_\mu \in \mathcal{A}_\mathcal{F}$ defined by

$$a_\mu(U) := \left(\frac{d_\Omega}{d_U}\right)^\mu \quad\text{for}\quad U\in\mathcal{F},\ \mu\in\mathbb{R}. \tag{3.1}$$

We define $\mathbb{P}_m$, $m \in \mathbb{N}$, as the space of polynomials of degree less than m with respect to the coordinates of the argument $x \in \mathbb{R}^N$. For $m = 0$ we define $\mathbb{P}_m := \{0\}$.

In the following measurability, integrability and integrals will always be understood with respect to the N-dimensional Lebesgue measure. If E is a measurable subset of $\mathbb{R}^N$, then $|E|$ denotes its measure. The letter p will always denote a number from $[1,\infty]$. For given p the dual exponent p' is defined by $\frac{1}{p}+\frac{1}{p'}=1$. The spaces $L^p(U)$, $U \in \mathcal{F}$, will be equipped with their standard norms, denoted by $\|\cdot\|_{p,U}$ or simply $\|\cdot\|_p$.

For $m \in \mathbb{Z}_+$ we define a projective system $P^{p,m}$ and an inductive systems $S^{p,m}$ setting for $U, V \in \mathcal{F}$ such that $V \subset U$:

$$\left.\begin{array}{l} P^{p,m}(U) := L^p(U)/\mathbb{P}_m,\ (P^{p,m})^U_V(u+\mathbb{P}_m) := u|_V + \mathbb{P}_m,\ \text{if } U\neq\Omega,\\ P^{p,m}(\Omega) := L^p(\Omega),\ (P^{p,m})^\Omega_V u := u|_V + \mathbb{P}_m,\ \text{if } V\neq\Omega. \end{array}\right\} \tag{3.2}$$

and

$$\left.\begin{array}{l} S^{p,m}(U) := \{u\in L^p(U);\ \int_U uw = 0 \text{ for all } w\in\mathbb{P}_m\},\ \text{if } U\neq\Omega,\\ S^{p,m}(\Omega) := L^p(\Omega),\quad (S^{p,m})^U_V v := v^U. \end{array}\right\} \tag{3.3}$$

Here and in the sequel v^U denotes the extension of v from its original domain of definition (which will always be a subset of U) to U by 0. In (3.2), (3.3) the space $\mathbb{P}_m$ is to be regarded as a subspace of $L^p(U)$, $L^p(V)$ and $L^{p'}(U)$, respectively. Similarly, $\mathbb{P}_m$ will be used below as a subspace of different function spaces (with different domains of definition). This should not lead to misunderstandings.

The notation introduced here will be used throughout this section without further explanation.

Lemma 3.1. *If $1 \le p < \infty$, then $(S^{p,m})^* = P^{p',m}$. The space $P^{1,m}(U)$ is isometrically imbedded into $(S^{\infty,m}(U))^*$ for every $U \in \mathcal{F}$.*

Proof. We have $(L^p(U))^* = L^{p'}(U)$ for $p \in [1, \infty[$, and $L^1(U)$ is isometrically imbedded into $(L^\infty(U))^*$. Moreover, $\mathbb{P}_m$ is the annihilator of

$$\left\{u \in L^p(U);\ \int_U uw = 0 \text{ for all } w \in \mathbb{P}_m\right\}$$

in $L^{p'}(U)$. These facts prove the lemma. $\square$

Definition 3.2. For $\mu \in \mathbb{R}_+$ we define the Banach spaces

$$L^{p,m,\mu}(\Omega) := \overleftarrow{l}\,(P^{p,m}_{a_\mu}) \quad \text{and} \quad L^{p,m,-\mu}(\Omega) := \vec{l}(S^{p,m}_{a_{-\mu}}),$$

(cf. (3.1) for the definition of the weight functions a_μ and $a_{-\mu}$).

Theorem 3.3. *For $\mu \in \mathbb{R}_+$ it holds $(L^{p,m,-\mu}(\Omega))^* = L^{p',m,\mu}(\Omega)$, $1 \le p < \infty$, and $L^{1,m,\mu}(\Omega)$ is isometrically imbedded into $(L^{\infty,m,-\mu}(\Omega))^*$.*

Proof. The theorem is a consequence of Lemma 3.1 and Corollary 1.9 (cf. also Remark 2.2). $\square$

Remark 3.4. Since $V \subset \Omega$, $\|(P^{p,m})^\Omega_V\|_{\mathcal{L}(P(\Omega);P(V))} \le 1$ for all $V \in \mathcal{F}$ and $a_\mu \ge$ const > 0 for $\mu \in \mathbb{R}_+$, the hypotheses of Corollary 2.5 are satisfied. Consequently, $L^{p,m,\mu}(\Omega)$ is to be regarded as a subspace of $L^p(\Omega)$, and we have

$$\|u\|_{L^{p,m,\mu}(\Omega)} = \max\left\{\|u\|_{p,\Omega}, \sup_{V \in \mathcal{F}} \inf_{w \in \mathbb{P}_m} \left(\frac{d_\Omega}{d_V}\right)^\mu \|u|_V - w\|_{p,V}\right\}. \tag{3.4}$$

An element of $L^p(\Omega)$ is in $L^{p,m,\mu}(\Omega)$ if and only if the right hand side of (3.4) is finite. In the following we shall write simply u instead of $u|_V$; it should be clear that for $\|u - w\|_{p,V}$ the function u is to be restricted to V.

Remark 3.5. From Lemma 1.10 it follows that $\|u\|_{L^{p,m,\mu}(\Omega)}$, $\mu \in \mathbb{R}^+$, is equivalent to the norm

$$|u|_{p,m,\mu,\Omega} := \max\left\{\|u\|_{p,\Omega}, \sup_{r>0,\, x \in \Omega} \inf_{w \in \mathbb{P}_m} r^{-\mu}\|u - w\|_{p,B_r(x)\cap\Omega}\right\}. \tag{3.5}$$

(As usual, $B_r(x)$ denotes the open ball of radius r centered at x.) Indeed, one can apply Lemma 1.10 to $\mathcal{G} := \{\Omega \cap B_r(x);\ r > 0,\ x \in \Omega\}$ because $\mathcal{G} \subset \mathcal{F}$ and

$$V \in \mathcal{F} \Longrightarrow V \subset W := \Omega \cap B_{d_V}(x),\ x \in V.$$

Obviously, we could define another equivalent norm replacing $B_r(x)$ in (3.5) by the cube of side length r centered at x with edges parallel to the coordinate axes in $\mathbb{R}^N$.

Remark 3.6. If Ω is a bounded domain in $\mathbb{R}^N$, then $L^{p,m,\mu}(\Omega)$, $1 \le p < \infty$, $m \in \mathbb{Z}_+$, $\mu \in \mathbb{R}_+$, is the well known scale of Campanato spaces. We changed, however, the notation of these spaces and replaced the original norms by equivalent norms (cf. [Ca]). Our notation also differs from that adopted by Triebel [T]. As mentioned already in the introduction we replace Campanato's notation $\mathcal{L}_k^{(p,\lambda)}(\Omega)$ by $L^{p,m,\mu}(\Omega)$, where $m = k+1$, $\mu = \lambda/p$. Our notation allows to express the duality result of Theorem 3.3 in a very simple way. This result would look more complicated with Campanato's or Triebel's notation. The change of norms compared to those in [Ca] allows a simpler description of the predual spaces $L^{p',m,-\mu}(\Omega)$. The original Campanato norm differs only slightly from the norm $|\cdot|_{p,m,\mu,\Omega}$ introduced in the preceding remark.

Remark 3.7. Our notation suggests that all the spaces defined above should be considered as parts of one scale of spaces. This point of view will be justified by some of the results below. Since both $L^{p,m,0}(\Omega)$ and $L^{p,m,-0}(\Omega)$ coincide with $L^p(\Omega)$ (including the norm), our notation does not cause problems for $\mu = 0$.

Remark 3.8. Campanato [Ca] proved that $L^{p,m,\mu}(\Omega) = \mathbb{P}_m$ if Ω is a bounded domain and $\mu > m + \frac{N}{p}$. On the other hand, it is easy to prove by means of Taylor's Formula that $C_c^m(\Omega)$ (the space of functions on Ω with compact support having continuous derivatives up to the order m) is contained in $L^{p,m,\mu}(\Omega)$ provided that $\mu \le m + \frac{N}{p}$. Because of this fact and the duality theorem above we shall be interested in the spaces $L^{p,m,\mu}(\Omega)$ mainly for $\mu \in [-m - \frac{N}{p'}, m + \frac{N}{p}]$.

Remark 3.9. In the sequel the number m in the notation for spaces and norms will be omitted if it is 0. We write, for example, shortly $L^{p,\mu}(\Omega)$ instead of $L^{p,0,\mu}(\Omega)$. For $u \in L^p(\Omega)$ and $\mu \in \mathbb{R}_+$ we have (cf. Remark 3.4)

$$\|u\|_{L^{p,\mu}(\Omega)} = \sup_{V \in \mathcal{F}} \left(\frac{d_\Omega}{d_V}\right)^\mu \|u\|_{p,V}.$$

The spaces $L^{p,\mu}(\Omega)$, $\mu \in \mathbb{R}_+$, were introduced by Morrey; they are now called *Morrey spaces* (see [KJF]).

Lemma 3.10. *Let $\mu \in [0, m + \frac{N}{p'}]$. Then the operator $I_{e,a_{-\mu}}^{S^{p,m}}$ treated in Lemma 2.9 is injective. The space $L^p(\Omega)$ can be regarded as a dense subset of $L^{p,m,-\mu}(\Omega)$, and it holds*

$$\|u\|_{L^{p,m,-\mu}(\Omega)} = \inf\Big\{ \sum_{V \in \mathcal{G}} \left(\frac{d_V}{d_\Omega}\right)^\mu \|v_V\|_{p,V};\ u = \sum_{V \in \mathcal{G}} v_V^\Omega,\ v_V \in S^{p,m}(V),\ \mathcal{G} \subset \mathcal{F} \text{ finite}\Big\}$$

for every $u \in L^p(\Omega)$ (see (3.3) for the definition of the spaces $S^{p,m}(V)$ used here).

Proof. Let $I_{e,a_{-\mu}}^{S^{p,m}} u = 0$ for some $u \in L^p(\Omega)$. For every $f \in C_c^m(\Omega) \subset L^{p',m,\mu}(\Omega)$ we have (cf. Lemma 2.9)

$$\int_\Omega f u = \left\langle (I_{e,a_{-\mu}}^{S^{p,m}})^* f, u \right\rangle = \left\langle f, I_{e,a_{-\mu}}^{S^{p,m}} u \right\rangle = 0.$$

Because of the arbitrariness of f this implies that $u = 0$, i.e., $I_{e,a_{-\mu}}^{S^{p,m}}$ is injective. For the proof of the remaining assertions we refer to Remark 2.13. $\square$

Remark 3.11. In the sequel, whenever $\mu \in [0, m + \frac{N}{p'}]$, we shall consider $L^p(\Omega)$ as a subspace of $L^{p,m,-\mu}(\Omega)$.

Theorem 3.12. *Let $1 \le q \le p \le \infty$, $\lambda := \frac{N}{q} - \frac{N}{p}$ and $-m - \frac{N}{q'} \le \nu \le \lambda + \mu \le m + \frac{N}{q}$. Moreover, let $c := \omega_N^{\lambda/N}$, where ω_N denotes the measure of the unit ball in $\mathbb{R}^N$. Then the following holds:*
i) *If $\mu \ge 0$, then $L^{p,m,\mu}(\Omega) \hookrightarrow L^{q,m,\nu}(\Omega)$ and the norm of the corresponding imbedding operator does not exceed cd_Ω^λ.*
ii) *If $\mu < 0$ then $L^p(\Omega)$ is dense in $L^{p,m,\mu}(\Omega)$ and $L^p(\Omega) \hookrightarrow L^{q,m,\nu}(\Omega)$. The imbedding of $L^p(\Omega)$ into $L^{q,m,\nu}(\Omega)$ can uniquely be extended to a continuous (linear) mapping from $L^{p,m,\mu}(\Omega)$ into $L^{q,m,\nu}(\Omega)$ the norm of which does not exceed cd_Ω^λ.*

Proof. We choose $a := a_\mu$ and $b := a_\nu$. For $u \in L^q(U)$, $U \in \mathcal{F}$, we have

$$\|u\|_{q,U} \le |U|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{p,U} \le cd_U^\lambda \|u\|_{p,U}. \tag{3.6}$$

We distinguish four cases:

1. Case $\mu \ge 0$, $\nu \ge 0$: Let K_U be the natural imbedding of $P^{p,m}(U)$ into $P^{q,m}(U)$, $U \in \mathcal{F}$. Then $K \in \mathcal{L}(P^{p,m}; P^{q,m})$ and $\Lambda_K(U) \le cd_U^\lambda$. This follows easily from (3.6). Theorem 2.3 yields $L^{p,m,\mu}(\Omega) \subset L^{q,m,\nu}(\Omega)$ and, for $u \in L^{p,m,\mu}(\Omega)$,

$$\|u\|_{L^{q,m,\nu}(\Omega)} = \|u\|_{P_b^{q,m}} \le cd_\Omega^\lambda \|u\|_{P_a^{p,m}} = cd_\Omega^\lambda \|u\|_{L^{p,m,\mu}(\Omega)}.$$

2. Case $\mu \le 0$, $\nu \le 0$: In this case let L_U be the natural imbedding of $S^{p,m}(U)$ into $S^{q,m}(U)$. Then $\Lambda_L(U) \le cd_U^\lambda$. The hypotheses with respect to μ and ν imply that $L^p(\Omega)$ is contained and dense in $L^{p,m,\mu}(\Omega)$ and $L^{q,m,\nu}(\Omega)$ (cf. Lemma 3.10). By Theorem 2.6 we have, for $u \in L^p(\Omega)$,

$$\|u\|_{L^{q,m,\nu}(\Omega)} = \|u\|_{S_b^{q,m}} \le cd_\Omega^\lambda \|u\|_{S_a^{p,m}} = cd_\Omega^\lambda \|u\|_{L^{p,m,\mu}(\Omega)}.$$

3. Case $\mu \ge 0$, $\nu \le 0$: Using step 2 of this proof with $(0,0)$ instead of (λ, μ) and step 1 with 0 instead of ν we find that $L^{q,m,\nu}(\Omega) \subset L^q(\Omega) \subset L^{p,m,\mu}(\Omega)$ and, for $u \in L^{p,m,\mu}(\Omega)$,

$$\|u\|_{L^{q,m,\nu}(\Omega)} \le \|u\|_{L^{q,m,0}(\Omega)} \le cd_\Omega^\lambda \|u\|_{L^{p,m,\mu}(\Omega)}.$$

4. Case $\mu \le 0$, $\nu \ge 0$: In this case we can refer neither to Theorem 2.3 nor to Theorem 2.6. (It is this case which indicates that it is natural to consider the spaces defined by means of $P^{p,m}$ and of $S^{p,m}$ as one scale.) Note that $-\mu \le -\nu + \lambda \le \frac{N}{p'}$. Therefore $L^p(\Omega)$ is contained and dense not only in $L^{p,m,\mu}(\Omega)$ but also in $L^{p,\mu}(\Omega)$ (cf. again Lemma 3.10). For $u \in L^p(\Omega)$ it holds $\|u\|_{L^{q,m,\nu}(\Omega)} \le \|u\|_{L^{q,\nu}(\Omega)}$ and $\|u\|_{L^{p,\mu}(\Omega)} \le \|u\|_{L^{p,m,\mu}(\Omega)}$. Therefore it suffices to prove the estimate

$$\|u\|_{L^{q,\nu}(\Omega)} \le cd_\Omega^\lambda \|u\|_{L^{p,\mu}(\Omega)} \quad \text{for} \quad u \in L^p(\Omega). \tag{3.7}$$

Let

$$u = \sum_{V\in\mathcal{G}} v_V^\Omega,\ v_V \in L^p(V),\ \mathcal{G} \text{ finite subset of } \mathcal{F}.$$

Then we obtain for $W \in \mathcal{F}$:

$$\begin{aligned} b(W)\|u\|_{q,W} &\leq \left(\frac{d_\Omega}{d_W}\right)^\nu \sum_{V\in\mathcal{G}} \|v_V^\Omega\|_{q,V\cap W} \\ &\leq \left(\frac{d_\Omega}{d_W}\right)^\nu \sum_{V\in\mathcal{G}} c d_{V\cap W}^\lambda \|v_V^\Omega\|_{p,V\cap W} \\ &\leq c d_\Omega^\lambda \sum_{V\in\mathcal{G}} \left(\frac{d_\Omega}{d_V}\right)^\mu \|v_V^\Omega\|_{p,\Omega} \\ &= c d_\Omega^\lambda \sum_{V\in\mathcal{G}} a(V)\|v_V^\Omega\|_{p,\Omega}. \end{aligned}$$

From this relation the desired estimate (3.7) follows (cf. Lemma 3.10). $\square$

Remark 3.13. For $\nu \geq 0$ part i) of the theorem had been proved already by Campanato [Ca]. Note that the extended operator in part ii) of the theorem is not necessarily injective.

We want to conclude this section with a result essentially due to Campanato [Ca]. To state this result we need the following definition.

Definition 3.14. A bounded set Ω in $\mathbb{R}^N$ is said to be *of type* A, $A > 0$, if for every $x \in \Omega$ and every $r \in]0, d_\Omega]$ we have $|\Omega \cap B_r(x)| \geq A r^N$.

Theorem 3.15. *Let Ω be a bounded domain of type $A > 0$. Then $L^{p,m,\mu}(\Omega)$ and $L^{p,n,\mu}(\Omega)$ coincide as linear topological spaces, if $n < m$ and $-n - \frac{N}{p'} < \mu < n + \frac{N}{p}$.*

Proof. 1. Let $\mu \geq 0$. For $1 \leq p < \infty$ the assertion has been proved by Campanato [Ca]. An inspection of his proof shows that it remains valid also for $p = \infty$.
2. Let $\mu < 0$. By the first step of this proof the spaces $L^{p',m,-\mu}(\Omega)$ and $L^{p',n,-\mu}(\Omega)$ coincide as topological linear spaces. Hence Theorem 3.3 allows to regard $L^{p,m,\mu}(\Omega)$ and $L^{p,n,\mu}(\Omega)$ as closed subspaces of the dual to $L^{p',m,-\mu}(\Omega)$. Because we know from Lemma 3.10 that $L^p(\Omega)$ is dense in $L^{p,m,\mu}(\Omega)$ as well as in $L^{p,n,\mu}$ these spaces must be equal as topological linear spaces. $\square$

4. Sobolev-Campanato spaces

As in the preceding section we assume that an open bounded set $\Omega \subset \mathbb{R}^N$ is fixed and that $\mathcal{F}$ is the family of all nonempty open subsets of Ω. Throughout this section k and m denote numbers from $\mathbb{Z}_+$, and p denotes a number in $]1, \infty[$. We supplement the definitions of the preceding section setting $\mathbb{P}_n := \{0\}$ and $L^{p,n,\mu}(U) := L^{p,\mu}(U)$ if n is a negative integer. This will help to simplify the presentation.

We want to define spaces of functions with derivatives in Campanato spaces. We use the possibility to introduce such spaces in the same way as we introduced the Campanato spaces in the preceding section, namely by means of appropriate projective and inductive systems. We shall use the same weight functions a_μ as for Campanato spaces (cf. (3.1)).

The spaces $W^{k,p}(U)$, $U \in \mathcal{F}$, are the usual Sobolev spaces equipped with their standard norms, denoted by $\|\cdot\|_{k,p,U}$ or shortly $\|\cdot\|_{k,p}$. We define $W_0^{k,p}(U)$ as the closure of the set $\{u \in W^{k,p}(U);\ \operatorname{supp} u \subset U\}$ in $W^{k,p}(U)$ and $W^{-k,p'}(U)$ as the dual of $W_0^{k,p}(U)$. For $k = 0$ this means that we identify $(L^p(U))^*$ and $L^{p'}(U)$.

We define projective and inductive systems $P^{k,p,m}$ and $S^{k,p,m}$, respectively, setting for $U, V \in \mathcal{F}$ such that $V \subset U$:

$$\left.\begin{array}{l} P^{k,p,m}(\Omega) := W^{k,p}(\Omega),\ P^{k,p,m}(U) := W^{k,p}(U)/\mathbb{P}_{m+k},\ \text{ if } U \neq \Omega, \\ (P^{k,p,m})_V^\Omega u := u|_V + \mathbb{P}_{m+k}, \quad \text{if } V \neq \Omega, \\ (P^{k,p,m})_V^U (u + \mathbb{P}_{m+k}) := u|_V + \mathbb{P}_{m+k},\ \text{ if } U \neq \Omega. \end{array}\right\} \tag{4.1}$$

and

$$\left.\begin{array}{l} S^{k,p,m}(U) := \{u \in W_0^{k,p}(U);\ \int_U uw = 0 \text{ for all } w \in \mathbb{P}_{m-k}\},\ \text{ if } U \neq \Omega, \\ S^{k,p,m}(\Omega) := W_0^{k,p}(\Omega), \quad (S^{k,p,m})_V^U v := v^U. \end{array}\right\} \tag{4.2}$$

Definition 4.1. For $\mu \in \mathbb{R}_+$ we introduce

$$W^{k,p,m,\mu}(\Omega) := \overleftarrow{l}(P_{a_\mu}^{k,p,m}), \quad W_0^{k,p,m,-\mu}(\Omega) := \vec{l}(S_{a_{-\mu}}^{k,p,m}).$$

Remark 4.2. According to Remark 1.15 the space $W^{k,p,m,\mu}(\Omega)$ will be regarded as a subspace of $W^{k,p}(\Omega)$. The norm in $W^{k,p,m,\mu}(\Omega)$ is

$$\|u\|_{W^{k,p,m,\mu}(\Omega)} = \max\left\{\|u\|_{k,p,\Omega},\ \sup_{U\in\mathcal{F}}\left\{\left(\frac{d_\Omega}{d_U}\right)^\mu \inf_{w\in\mathbb{P}_{m+k}} \|u-w\|_{k,p,U}\right\}\right\}, \tag{4.3}$$

and $W^{k,p,m,\mu}(\Omega)$ consists of all elements of $W^{k,p}(\Omega)$ for which the right hand side of (4.3) is finite (cf. Corollary 2.5).

Remark 4.3. If $\mu \in \mathbb{R}_+$, $|\alpha| \le k$, $l \le k - |\alpha|$, $n + l \ge m + k - |\alpha|$, $l, n \in \mathbb{Z}_+$, then

$$D^\alpha \in \mathcal{L}(W^{k,p}(U); W^{l,p}(U)),\ \|D^\alpha\|_{\mathcal{L}(W^{k,p}(U);W^{l,p}(U))} \le 1 \text{ and } D^\alpha[\mathbb{P}_{m+k}] \subset \mathbb{P}_{n+l}.$$

Consequently, D^α is a linear bounded operator from $P^{k,p,m}(U)$ into $P^{l,p,n}(U)$ with norm not larger than 1 for all $U \in \mathcal{F}$, and hence (cf. Theorem 2.3)

$$D^\alpha \in \mathcal{L}(W^{k,p,m,\mu}(\Omega); W^{l,p,n,\nu}(\Omega)),\ \|D^\alpha\|_{\mathcal{L}(W^{k,p,m,\mu}(\Omega);W^{l,p,n,\nu}(\Omega))} \le 1$$

provided that $0 \le \nu \le \mu$. Moreover, $D^\alpha \in \mathcal{L}(W^{k,p,m,\mu}(\Omega); W^{l,p,n,\nu}(\Omega))$ is compact, whenever $0 \le \nu < \mu$ and $D^\alpha \in \mathcal{L}(W^{k,p}(\Omega); W^{l,p}(\Omega))$ is compact, which is the case if

$l < k - |\alpha|$ and Ω is not too bad.

Remark 4.4. The preceding remark and Remark 3.8 show that $W^{k,p,m,\mu}(\Omega)$ consists of elements which are locally polynomials, if $\mu > m + \frac{N}{p}$. Therefore we are interested in the spaces $W^{k,p,m,\mu}(\Omega)$ mainly if $0 \le \mu \le m + \frac{N}{p}$.

Remark 4.5. If $\mu \in \mathbb{R}_+$, $|\alpha| \le k$, $l \le k - |\alpha|$, $n - l \le m - k + |\alpha|$, $l, n \in \mathbb{Z}_+$, then

$$D^\alpha \in \mathcal{L}(W_0^{k,p}(U); W_0^{l,p}(U)), \ \|D^\alpha\|_{\mathcal{L}(W_0^{k,p}(U); W_0^{l,p}(U))} \le 1$$

and

$$\int_U w\, D^\alpha u = (-1)^{|\alpha|} \int_U u\, D^\alpha w = 0 \ \text{ if } \ u \in S^{k,p,m}(U), \ w \in \mathbb{P}_{n-l}.$$

Hence D^α induces an operator in $\mathcal{L}(W_0^{k,p,m,-\mu}(\Omega); W_0^{l,p,n,-\nu}(\Omega))$ if $0 \le \nu \le \mu$, which is compact if $0 \le \mu < \nu$ and $l < k - |\alpha|$ (cf. Theorem 2.6).

Remark 4.6. Let $I_{e,a_{-\mu}}^{S^{k,p,m}} u = 0$ for some $u \in W_0^{k,p}(\Omega)$. Then (cf. Remark 2.11)

$$I_{e,a_{-\mu}}^{S^{p,m}} D^\alpha u = D^\alpha_{a_{-\mu}, a_{-\mu}} I_{e,a_{-\mu}}^{S^{k,p,m}} u = 0,$$

where $D^\alpha_{a_{-\mu}, a_{-\mu}}$ denotes the operator induced by D^α, considered as an element of $\mathcal{L}(S^{k,p,m}; S^{p,m})$. We have seen in Lemma 3.10 that the operator $I_{e,a_{-\mu}}^{S^{p,m}}$ is injective provided that $\mu \in [0, m + \frac{N}{p'}]$. In that case $D^\alpha u = 0$, $|\alpha| = k$. Since $u \in W_0^{k,p}(\Omega)$ this implies that $u = 0$. Hence under the hypothesis $\mu \in [0, m + \frac{N}{p'}]$ the operator $I_{e,a_{-\mu}}^{S^{k,p,m}}$ is injective, $W_0^{k,p}(\Omega)$ is a dense subset of $W_0^{k,p,m,-\mu}(\Omega)$ and, for $u \in W_0^{k,p}(\Omega)$, we have (cf. Remark 2.13)

$$\|u\|_{W^{k,p,m,-\mu}(\Omega)} = \inf\Big\{ \sum_{V \in \mathcal{G}} \Big(\frac{d_V}{d_\Omega}\Big)^\mu \|v_V\|_{k,p,V};\ u = \sum_{V \in \mathcal{G}} v_V^\Omega,\ v_V \in S^{k,p,m}(V),\ \mathcal{G} \subset \mathcal{F} \text{ finite}\Big\}.$$

For the next results we introduce some more notation.

Definition 4.7. For any bounded open set $V \subset \mathbb{R}^N$ we introduce

$$\varrho(V) := \sup\Big\{ \frac{|E|}{|B|};\ E \subset V \subset B, \ \text{ where } E,\ B \text{ are balls}\Big\}.$$

A bounded open set $U \subset \mathbb{R}^N$ is said to be of *class* $\delta > 0$ if $\varrho(B_r(\xi) \cap U) \ge \delta$ provided that $\xi \in U$ and $r > 0$. We denote by $\mathcal{F}_\delta$ the family of all $U \in \mathcal{F}$ of class $\delta > 0$.

Remark 4.8. The family of sets which are of class δ for some $\delta > 0$ is rather large: It is invariant with respect to bi-Lipschitz transformations and contains the class of domains with Lipschitz boundary (we refer to [GR] for a detailed discussion of various types of domains). On the other hand, each element of this family is of type A for some $A > 0$.

Remark 4.9. If $\Omega \in \mathcal{F}_\delta$, then

$$\overleftarrow{l}(P_{a_\mu}^{k,p,m}) = \overleftarrow{l}(P_{a_\mu}^{k,p,m}|_{\mathcal{F}_\delta}) \ \text{ and } \ \vec{l}(S_{a_{-\mu}}^{k,p,m}) = \vec{l}(S_{a_{-\mu}}^{k,p,m}|_{\mathcal{F}_\delta}),$$

where the spaces are to be understood as linear topological spaces, i.e., restricting ourselves in the definition of the norms of $W^{k,p,m,\mu}(\Omega)$ and $W_0^{k,p,m,-\mu}(\Omega)$ to sets $U \in \mathcal{F}_\delta$ we arrive at norms which are equivalent to those defined by means of all $U \in \mathcal{F}$. Indeed, by definition of $\mathcal{F}_\delta$ for each $U \in \mathcal{F}$ we have $U \subset B_{d_U}(\xi) \cap \Omega \in \mathcal{F}_\delta$ for every $\xi \in U$. Therefore the claim follows from Lemma 1.10 and Lemma 1.12.

Theorem 4.10. *Let Ω be of class δ for some $\delta > 0$, and let $\mu \in \mathbb{R}_+$. Then*

$$W^{k,p,m,\mu}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^{p,m+k-|\alpha|,\mu}(\Omega),\ |\alpha| \le k\}$$

and the norm $\|\cdot\|_{W^{k,p,m,\mu}(\Omega)}$ is equivalent to $\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^{p,m+k-|\alpha|,\mu}(\Omega)}$.

We postpone the proof of this theorem and proceed with some auxiliary results.

Lemma 4.11. *Let E and B be two open balls in $\mathbb{R}^N$ such that $E \subset B$. Then there exists a finite number c depending on N, p, m and $\frac{|E|}{|B|}$ only such that*

$$\forall w \in \mathbb{P}_m: \quad \|w\|_{p,B} \le c\|w\|_{p,E}.$$

Proof. It suffices to prove the lemma under the additional assumption that B is the unit ball. If the assertion were wrong, then there would exist functions $w_n \in \mathbb{P}_m$ and balls $B_r(\xi_n) \subset B$ of the same radius $r > 0$ such that

$$\|w_n\|_{p,B} = 1, \quad \int_{B_r(\xi_n)} |w_n|^p \longrightarrow 0 \text{ as } n \to \infty.$$

Without loss of generality we may assume that $w_n \longrightarrow w$ in $C(\overline{B})$ (since $\mathbb{P}_m$ is finite dimensional) and that $\xi_n \longrightarrow \xi$. Then

$$\|w\|_{p,B} = 1 \quad \text{and} \quad \int_{B_r(\xi)} |w|^p = \lim_{n\to\infty} \int_{B_r(\xi_n)} |w_n|^p = 0.$$

Because w is a polynomial this is impossible. The contradiction completes the proof. $\square$

Lemma 4.12. *Let $U \in \mathcal{F}$. Then, for every $u \in W^{k,p}(U)$ there exists $w \in \mathbb{P}_{m+k}$ such that, for $|\alpha| \le k$,*

$$\|D^\alpha(u-w)\|_{p,U} \le c \inf_{w_\alpha \in \mathbb{P}_{m+k-|\alpha|}} \|D^\alpha u - w_\alpha\|_{p,U},$$

where c is independent of u and depends on U via $\varrho(U)$ only (cf. Definition 4.7).

Proof. 1. First let $E = \{x \in \mathbb{R}^N;\ |x| < 1\}$. For a given $u \in W^{k,p}(E)$ we choose w as the unique element of $\mathbb{P}_{m+k}$ such that

$$\forall \widetilde{w} \in \mathbb{P}_{m+k}: \quad \int_E w\,\widetilde{w} = \int_E u\,\widetilde{w}.$$

We define

$$g_\alpha(x) := (1-|x|^2)^{|\alpha|} \quad \text{for} \quad x \in E.$$

Using partial integration and the definition of w, we find, for any $w_\alpha \in \mathbb{P}_{m+k-|\alpha|}$,

$$\int_E D^\alpha(u-w)(D^\alpha w - w_\alpha)g_\alpha = (-1)^{|\alpha|}\int_E (u-w)D^\alpha((D^\alpha w - w_\alpha)g_\alpha) = 0.$$

Hence

$$\int_E |D^\alpha w - w_\alpha|^2 g_\alpha = \int_E (D^\alpha u - w_\alpha)(D^\alpha w - w_\alpha)g_\alpha \le \|D^\alpha u - w_\alpha\|_{p,E}\|D^\alpha w - w_\alpha\|_{p',E}.$$

Since $\widetilde{w} \mapsto \int_E \widetilde{w}^2 g_\alpha$ is a norm on $\mathbb{P}_{m+k-|\alpha|}$ and all norms on $\mathbb{P}_{m+k-|\alpha|}$ are equivalent, this gives

$$\|D^\alpha w - w_\alpha\|_{p,E} \le c_0\|D^\alpha u - w_\alpha\|_{p,E},$$

where c_0 is independent of u. Consequently,

$$\|D^\alpha(u-w)\|_{p,E} \le \|D^\alpha u - w_\alpha\|_{p,E} + \|w_\alpha - D^\alpha w\|_{p,E} \le (1+c_0)\|D^\alpha u - w_\alpha\|_{p,E}.$$

Because this is true for every $w_\alpha \in \mathbb{P}_{m+k-|\alpha|}$, we have (with $c_1 := 1 + c_0$)

$$\|D^\alpha(u-w)\|_{p,E} \le c_1 \inf_{w_\alpha \in \mathbb{P}_{m+k-|\alpha|}} \|D^\alpha u - w_\alpha\|_{p,E}.$$

This is the assertion for the special case $U = E$. A simple scaling argument shows that the assertion holds for every ball U.

2. For arbitrary $U \in \mathcal{F}$ we choose balls E and B such that $E \subset U \subset B$ and $\frac{|E|}{|B|} \le 2\varrho(U)$. Let $u \in W^{k,p}(U)$ be given. Using the first step of the proof we choose $w \in \mathbb{P}_{m+k}$ such that for $|\alpha| \le k$

$$\|D^\alpha(u-w)\|_{p,E} \le c_1 \inf_{w_\alpha \in \mathbb{P}_{m+k-|\alpha|}} \|D^\alpha u - w_\alpha\|_{p,E}.$$

Then, for arbitrary $w_\alpha \in \mathbb{P}_{m-|\alpha|}$, we find

$$\begin{aligned}
\|D^\alpha(u-w)\|_{p,U} &\le \|D^\alpha u - w_\alpha\|_{p,U} + \|w_\alpha - D^\alpha w\|_{p,B} \\
&\le \|D^\alpha u - w_\alpha\|_{p,U} + c\|w_\alpha - D^\alpha w\|_{p,E} \\
&\le (1+c)\|D^\alpha u - w_\alpha\|_{p,U} + c\|D^\alpha u - D^\alpha w\|_{p,E} \\
&\le (1+c+cc_1)\|D^\alpha u - w_\alpha\|_{p,U},
\end{aligned}$$

where c depends on U via $\varrho(U)$ only (cf. Lemma 4.11). $\square$

Proof of Theorem 4.10. Let $\Omega \in \mathcal{F}_\delta$. According to Remark 4.9 we can define an equivalent norm in $W^{k,p,m,\mu}(\Omega)$ as follows:

$$\|u\|^{\mathcal{F}_\delta}_{W^{k,p,m,\mu}(\Omega)} := \max\left\{\|u\|_{k,p,\Omega}, \sup_{U \in \mathcal{F}_\delta}\left(\frac{d_\Omega}{d_U}\right)^\mu \inf_{w \in \mathbb{P}_{m+k}} \|u-w\|_{k,p,U}\right\}.$$

By the preceding lemma

$$\begin{aligned}
\|u\|^{\mathcal{F}_\delta}_{W^{k,p,m,\mu}(\Omega)} &\le \max\left\{\|u\|_{k,p,\Omega}, c\sup_{U\in\mathcal{F}_\delta}\sum_{|\alpha|\le k}\left(\frac{d_\Omega}{d_U}\right)^\mu \inf_{w_\alpha\in\mathbb{P}_{m+k-|\alpha|}}\|D^\alpha u - w_\alpha\|_{p,U}\right\}\\
&\le c\sum_{|\alpha|\le k}\max\left\{\|D^\alpha u\|_{p,\Omega}, \sup_{U\in\mathcal{F}}\left(\frac{d_\Omega}{d_U}\right)^\mu \inf_{w_\alpha\in\mathbb{P}_{m+k-|\alpha|}}\|D^\alpha u - w_\alpha\|_{p,U}\right\}\\
&= c\sum_{|\alpha|\le k}\|D^\alpha u\|_{L^{p,m+k-|\alpha|,\mu}(\Omega)}.
\end{aligned}$$

Since the converse inequality had been obtained already with Remark 4.3 the proof is complete. $\square$

Next we define projective systems $P^{-k,p,m}$ and inductive systems $S^{-k,p,m}$ setting for $U, V\in\mathcal{F}$ such that $V\subset U$,

$$\left.\begin{aligned}
&P^{-k,p,m}(\Omega) := W^{-k,p}(\Omega),\ P^{-k,p,m}(U) := W^{-k,p}(U)/\mathbb{P}_{m-k},\ \text{ if } U\ne\Omega,\\
&(P^{-k,p,m})^\Omega_V f := f|_V + \mathbb{P}_{m-k},\ \text{ if } V\ne\Omega,\\
&(P^{-k,p,m})^U_V(f+\mathbb{P}_{m-k}) := f|_V + \mathbb{P}_{m-k},\ \text{ if } U\ne\Omega.
\end{aligned}\right\}\tag{4.4}$$

and

$$\left.\begin{aligned}
&S^{-k,p,m}(U) := \{g\in (W^{k,p'}(U))^* :\ \langle g, w\rangle = 0 \text{ for all } w\in\mathbb{P}_{m+k}\},\ \text{ if } U\ne\Omega,\\
&S^{-k,p,m}(\Omega) := (W^{k,p'}(\Omega))^*,\quad (S^{-k,p,m})^U_V g := g^U.
\end{aligned}\right\}\tag{4.5}$$

Here $f|_V$ for $f\in W^{-k,p}(U)$ is defined by $\langle f|_V, v\rangle := \left\langle f, v^U\right\rangle$ for $v\in W^{k,p'}_0(V)$, and the extension g^U of $g\in (W^{k,p'}(V))^*$ is defined by $\left\langle g^U, u\right\rangle := \langle g, u|_V\rangle$ for $u\in W^{k,p'}(U)$. Obviously,

$$P^{-k,p,m} = (S^{k,p',m})^* \quad\text{and}\quad S^{-k,p,m} = (P^{k,p',m})^*,$$

where $S^{k,p',m}$ and $P^{k,p',m}$ are given by (4.2), (4.1).

Definition 4.13. For $\mu\in\mathbb{R}_+$ we introduce

$$W^{-k,p,m,\mu}(\Omega) := \overleftarrow{l}\,(P^{-k,p,m}_{a_\mu}),\quad W^{-k,p,m,-\mu}_*(\Omega) := \vec{l}(S^{-k,p,m}_{a_{-\mu}}).$$

Remark 4.14. According to Remark 1.15 the space $W^{-k,p,m,\mu}(\Omega)$ will be regarded as a subspace of $W^{-k,p}(\Omega)$. The norm in $W^{-k,p,m,\mu}(\Omega)$ is

$$\|f\|_{W^{-k,p,m,\mu}(\Omega)} = \max\left\{\|f\|_{-k,p,\Omega},\ \sup_{U\in\mathcal{F}}\left\{\left(\frac{d_\Omega}{d_U}\right)^\mu\inf_{w\in\mathbb{P}_{m-k}}\|f-w\|_{-k,p,U}\right\}\right\},\tag{4.6}$$

and $W^{-k,p,m,\mu}(\Omega)$ consists of all elements of $W^{-k,p}(\Omega)$ for which the right hand side of (4.6) is finite (cf. Corollary 2.5). The last statement can also be expressed as follows: A functional $f\in W^{-k,p}(\Omega)$ is in $W^{-k,p,m,\mu}(\Omega)$ if and only if there exists a constant c such that the hypotheses

$$v\in W^{k,p'}_0(\Omega),\ \operatorname{supp} v\subset U \ \text{ and }\ \int_\Omega vw = 0 \ \text{ for all } w\in\mathbb{P}_{m-k}$$

imply that

$$|\langle f, v\rangle| \leq c d_U^{\mu} \|v\|_{k,p',\Omega}.$$

By means of Corollary 1.9 one obtains immediately

Theorem 4.15. *For $\mu \in \mathbb{R}_+$ it holds*

$$W^{-k,p,m,\mu}(\Omega) = (W_0^{k,p',m,-\mu}(\Omega))^* \quad \textit{and} \quad W^{k,p,m,\mu}(\Omega) = (W_*^{-k,p',m,-\mu}(\Omega))^*.$$

Remark 4.16. Combining the first part of this theorem with Theorem 3.3 and Remark 4.5 we obtain the following result: If $f_\alpha \in L^{p,m-k+|\alpha|,\mu}(\Omega)$, $|\alpha| \leq k$, and

$$\langle f, v\rangle := \sum_{|\alpha|\leq k} \langle f_\alpha, D^\alpha v\rangle \quad \text{for} \quad v \in W_0^{k,p'}(\Omega),$$

then $f \in W^{-k,p,m,\mu}(\Omega)$.

Theorem 4.17. *Suppose that there exists a bijective mapping $\Phi\colon \Omega \longrightarrow \widetilde{\Omega}$, where $\widetilde{\Omega} \subset \mathbb{R}^N$ is convex and Φ, Φ^{-1} are Lipschitzian. Moreover, let $\mu \in [0, m + \frac{N}{p}]$. Then*

$$\forall u \in W^{k,p}(\Omega): \quad \|u\|_{L^{p,m+k,\mu+k}(\Omega)} \leq \|u\|_{p,\Omega} + c \sum_{|\alpha|=k} \|D^\alpha u\|_{L^{p,m,\mu}(\Omega)},$$

where c is a constant independent of u.

For the proof of this theorem we need some auxiliary results.

Lemma 4.18. *Let U be a bounded open convex subset of $\mathbb{R}^N$ and let $u \in W^{1,p}(U)$ be such that $\int_U u = 0$. Then*

$$\|u\|_{p,U} \leq 2^N d_U \sum_{i=1}^{N} \|D_i u\|_{p,U}. \tag{4.7}$$

Proof. It suffices to prove the inequality (4.7) under the additional assumption that $u \in C^1(U) \cap W^{1,p}(U)$. We obtain for $x, y \in U$

$$\begin{aligned} u(x) &= u(y) + \int_0^1 \frac{d}{dt} u(y + t(x-y))dt \\ &= u(y) + \int_0^1 \sum_{i=1}^{N} (D_i u)(y + t(x-y))(x_i - y_i)dt. \end{aligned}$$

Integration with respect to y gives

$$|U| u(x) = \int_U \int_0^1 \sum_{i=1}^{N} (D_i u)(y + t(x-y))(x_i - y_i)dt\, dy.$$

Hence

$$|U||u(x)| \leq (|U|N)^{\frac{1}{p'}} d_U \Big(\int_U \int_0^1 \sum_{i=1}^N |(D_i u)(y + t(x-y))|^p dt\, dy \Big)^{\frac{1}{p}}$$

and

$$\begin{aligned}
\|u\|_{p,U} &\leq |U|^{-\frac{1}{p}} N^{\frac{1}{p'}} d_U \Big(\int_0^1 \int_U \int_U \sum_{i=1}^N |(D_i u)(y + t(x-y))|^p dy\, dx\, dt \Big)^{\frac{1}{p}} \\
&\leq |U|^{-\frac{1}{p}} N^{\frac{1}{p'}} d_U \Big(\int_0^{\frac{1}{2}} \int_U \int_U \sum_{i=1}^N |(D_i u)(z)|^p \frac{dz}{(1-t)^N}\, dx\, dt \\
&\qquad\qquad + \int_{\frac{1}{2}}^1 \int_U \int_U \sum_{i=1}^N |(D_i u)(z)|^p \frac{dz}{t^N}\, dy\, dt \Big)^{\frac{1}{p}} \\
&\leq 2^{\frac{N}{p}} N^{\frac{1}{p'}} d_U \sum_{i=1}^N \|D_i u\|_{p,U} \leq 2^N d_U \sum_{i=1}^N \|D_i u\|_{p,U}.
\end{aligned}$$

This is the desired estimate. $\square$

Lemma 4.19. *Suppose that $\Phi : \Omega \longrightarrow \widetilde{\Omega}$ is bijective, where $\widetilde{\Omega} \subset \mathbb{R}^N$ is convex and Φ, Φ^{-1} are Lipschitzian with the Lipschitz constant L. Let*

$$\mathcal{G} := \{W \in \mathcal{F};\ \Phi(W) = B_r(\tilde{\xi}) \cap \Phi(\Omega) \text{ for some } \tilde{\xi} \in \Phi(\Omega),\ r > 0\}.$$

Then, for every $V \in \mathcal{F}$ there exists $W \in \mathcal{G}$ such that $V \subset W$ and $d_W \leq 2L^2 d_V$. Moreover, there exists a $\delta > 0$ such that $\varrho(W) \geq \delta$ for every $W \in \mathcal{G}$ (cf. Definition 4.7).

Proof. 1. We have $d_{\Phi(V)} \leq L d_V$. We fix $\tilde{\xi} \in \Phi(V)$ arbitrarily and define $W \in \mathcal{G}$ by

$$\Phi(W) := B_{L d_V}(\tilde{\xi}) \cap \Phi(\Omega).$$

Then $\Phi(V) \subset \Phi(W)$ and therefore $V \subset W$. Moreover $d_{\Phi(W)} \leq 2L d_V$. Consequently, it holds $d_W \leq 2L^2 d_V$.
2. Elementary considerations show that $\varrho(B_r(\tilde{\xi}) \cap \Phi(\Omega)) \geq (r_0/d_\Omega)^N$ if r_0 is the radius of a ball contained in $\Phi(U)$. Since the image under Φ^{-1} of a ball of radius r contains a ball of radius r/L and is contained in a ball of radius Lr the last relation implies that $\varrho(W) \geq (r_0/L^2 d_\Omega)^N$. $\square$

Lemma 4.20. *Under the hypotheses of the preceding lemma there exists a constant c_k such that*

$$\forall W \in \mathcal{G},\ \forall u \in W^{k,p}(W) : \quad \inf_{w \in I\!P_k} \|u - w\|_{p,W} \leq c_k d_W^k \sum_{\alpha = k} \|D^\alpha u\|_{p,W}.$$

Proof. We prove the assertion by induction with respect to k.

1. Let $k=1$. Let $\widetilde{W} := \Phi(W)$ and $\tilde{u}(\tilde{x}) := u(\Phi^{-1}(\tilde{x}))$ for $\tilde{x} \in \widetilde{W},\ u \in W^{1,p}(W)$. Using Lemma 4.18 we find, setting $a := \frac{1}{|\widetilde{W}|} \int\limits_{\widetilde{W}} \tilde{u}$,

$$\begin{aligned}
\|u-a\|_{p,W} &= \left(\int_W |u(x)-a|^p dx\right)^{\frac{1}{p}} = \left(\int_{\widetilde{W}} |\tilde{u}(\tilde{x})-a|^p \det{(\Phi^{-1})'(\tilde{x})} d\tilde{x}\right)^{\frac{1}{p}} \\
&\le L^{\frac{N}{p}} \|\tilde{u}-a\|_{p,\widetilde{W}} \le 2^N L^{\frac{N}{p}} d_{\widetilde{W}} \sum_{i=1}^N \|\widetilde{D}_i \tilde{u}\|_{p,\widetilde{W}} \\
&\le 2^N L^{\frac{N}{p}+1} d_W \sum_{i=1}^N \left\| \sum_{j=1}^N (D_j u) \circ \Phi^{-1} \cdot \widetilde{D}_i (\Phi^{-1})_j \right\|_{p,\widetilde{W}} \\
&\le 2^N L^{\frac{2N}{p}+2} N d_W \sum_{j=1}^N \|D_j u\|_{p,W}.
\end{aligned}$$

Hence

$$\inf_{w \in \mathbb{P}_1} \|u-w\|_{p,W} \le c_1 d_W \sum_{j=1}^N \|D_j u\|_{p,W},$$

where c_1 depends on L, N and p only.

2. We prove the assertion for k under the hypothesis that it has been proved already for $k-1$ instead of k. Let $u \in W^{k,p}(W)$. We define

$$v(x) := u(x) - \sum_{|\beta|=k-1} \frac{a_\beta x^\beta}{\beta!},$$

where the a_β, $|\beta| = k-1$, are chosen such that

$$\|D^\beta v\|_{p,W} \le c_1 d_W \sum_{i=1}^N \|D_i D^\beta v\|_{p,W}$$

(cf. step 1 of this proof). Using this result and the induction hypothesis we find

$$\begin{aligned}
\inf_{w \in \mathbb{P}_{k-1}} \|v-w\|_{p,W} &\le c_{k-1} d_W^{k-1} \sum_{|\beta|=k-1} \|D^\beta v\|_{p,W} \\
&\le c_{k-1} d_W^{k-1} c_1 d_W \sum_{i=1}^N \sum_{|\beta|=k-1} \|D_i D^\beta v\|_{p,W} \\
&\le c_{k-1} c_1 N d_W^k \sum_{|\alpha|=k} \|D^\alpha v\|_{p,W}.
\end{aligned}$$

In view of the relation between u and v this proves the assertion with $c_k := c_{k-1} c_1 N$.
□

Proof of Theorem 4.17. We introduce $\mathcal{G}$ as in Lemma 4.19. Let $u \in W^{k,p}(U)$ and $W \in \mathcal{G}$ be fixed. We choose $w \in \mathbb{P}_{m+k}$ such that for $|\alpha| = k$

$$\|D^\alpha(u-w)\|_{p,W} \le c \inf_{w_\alpha \in \mathbb{P}_m} \|D^\alpha u - w_\alpha\|_{p,W},$$

where c is independent of u and W (cf. Lemma 4.12 and Lemma 4.19). Next we use Lemma 4.20 to obtain

$$\inf_{\widetilde{w}\in\mathbb{P}_k} \|u-w-\widetilde{w}\|_{p,W} \leq c d_W^k \sum_{|\alpha|=k} \|D^\alpha(u-w)\|_{p,W} \leq c d_W^k \sum_{|\alpha|=k} \inf_{w_\alpha\in\mathbb{P}_m} \|D^\alpha u-w_\alpha\|_{p,W}.$$

In view of Lemma 1.10 and Lemma 4.19 this estimate implies that

$$\begin{aligned}\|u\|_{L^{p,m+k,\mu+k}(\Omega)} &\leq \max\Big\{\|u\|_{p,\Omega}, c\sup_{W\in\mathcal{G}} d_W^{-\mu-k} \inf_{w\in\mathbb{P}_{m+k}} \|u-w\|_{p,W}\Big\} \\ &\leq \max\Big\{\|u\|_{p,\Omega}, c\sup_{W\in\mathcal{G}} \sum_{|\alpha|=k} d_W^{-\mu} \inf_{w_\alpha\in\mathbb{P}_m} \|D^\alpha u-w_\alpha\|_{p,W}\Big\} \\ &\leq \|u\|_{p,\Omega} + c\sum_{|\alpha|=k} \|D^\alpha u\|_{p,m,\mu,\Omega}.\end{aligned}$$

This is the desired estimate. □

Remark 4.21. Let the hypotheses of Theorem 4.17 be satisfied and let $l \in \mathbb{Z}_+$ be such that $0 \leq l \leq k$. If $|\beta| = k-l$, then $D^\beta u \in W^{p,l}(\Omega)$ and $D^\alpha(D^\beta u) \in L^{p,m,\mu}(\Omega)$ for $|\alpha| = l$. Therefore, the theorem can be applied with $D^\beta u$ and l instead of u and k. One obtains

$$\begin{aligned}\|D^\beta u\|_{p,m+l,\mu+l} &\leq \|D^\beta u\|_p + c\sum_{|\alpha|=k} \|D^\alpha u\|_{p,m,\mu} \\ &\leq c\Big(\|u\|_p + \sum_{|\alpha|=k} \|D^\alpha u\|_{p,m,\mu}\Big).\end{aligned}$$

This result implies in particular that $D^\beta u \in L^{p,m+k-|\beta|,\mu}(\Omega)$ for all β, $|\beta| \leq k$, which is equivalent to $u \in W^{k,p,m,\mu}(\Omega)$ by Theorem 4.10. Thus, $u \in W^{k,p,m,\mu}(\Omega)$ if and only if $D^\alpha u \in L^{p,m,\mu}(\Omega)$ for $|\alpha| = k$.

Remark 4.22. For $u \in W_0^{k,p}(\Omega)$ the assertion of Theorem 4.17 is true for every bounded open set $\Omega \subset \mathbb{R}^N$. This follows immediately from the fact that u can be extended by 0 to the convex hull of Ω without any change of the relevant norms.

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