# Standard Chase on Black Swans and Canards

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**Abstract.**

The paper is devoted to the study of the relationship between integral manifolds of ordinary differential equations and duck–trajectories. We derive sufficient conditions for the existence of continuous slow integral surfaces that are devided into stable and unstable parts and propose a method of construction of surfaces consisting of duck–trajectories.

# 1 Introduction

Modelling critical phenomena in the case of an autocatalytic burning reaction led to an unexpected and striking fact: duck-trajectory appeared as a model of a critical regime. The term "canard " or " French duck " is comparatively recent in scientific literature. It has been introduced by French mathematicians investigating van der Pol's equation [3, 5]. A trajectory of a singularly perturbed system of differential equations is called a duck-trajectory, if it follows at first a stable integral manifold, and then an unstable one. In both cases the passed distances are not infinitesimally small. Bibliographies on this theme can be found in [1, 6, 13, 17]. In most papers devoted to duck-trajectories the non-standard analysis is the main tool of investigations [2, 3, 17]. Therefore, the opinion is widely used that the duck-trajectories are exotic objects and are of interest for the theory of differential equations only. Applying duck-trajectories for modelling critical regimes promoted us to solve some interesting and important problems of combustion theory [7, 8, 9, 14, 15]. Moreover, the analysis of these problems involves the necessity of proofs of new theorems on the duck-trajectories [7, 8, 15]. It should be noted that very interesting applications of ducks in models of economics and chemical kinetics were obtained in [4, 10].

# 2 Statement of the problem

We consider the following singularly perturbed system

$$\dot{x} = f(x, y, z, \varepsilon), \tag{2.1}$$

$$\dot{y} = g(x, y, z, \varepsilon), \tag{2.2}$$

$$\epsilon \dot{z} = p(x, y, z, \alpha, \varepsilon) \tag{2.3}$$

where $\varepsilon$ is a small positive parameter, $\alpha$ is a scalar parameter, $x$ and $z$ are scalar variables, $y$ is a vector of dimension $n$.

Recall that the slow surface $S$ of system (2.1)–(2.3) is the surface described by the equation

$$p(x, y, z, \alpha, 0) = 0. \tag{2.4}$$

Let $z = \phi(x, y)$ be an isolated solution of equation (2.4). We call the subset $S^s_\phi$ ($S^u_\phi$) of $S$ defined by

$$\frac{\partial p}{\partial z}(x, y, \phi(x, y), \alpha, 0) < 0 \ (> 0)$$

the stable (unstable) subset of $S$.

The subset of $S$ defined by

$$\frac{\partial p}{\partial z}(x, y, \phi(x, y), \alpha, 0) = 0$$

is called the separating surface. Its dimension is equal to $\dim y$.

In a small neighborhood of $S^s_\phi$ ($S^u_\phi$) there exists a stable (unstable) slow integral manifold. The slow integral manifold is defined as a smooth invariant surface of slow motions.

The availability of the additional scalar parameter $\alpha$ provides the possibility of gluing stable and unstable integral manifolds in a point of the separating surface. Such a point is passed by a duck-trajectory.

It should be noted that just in the first papers devoted to canards in the case $\dim y = 0$ the existence of unique duck-trajectory corresponding to unique value of parameter $\alpha = \alpha^*$ was stated (more presisely, the "duck" value of parameter $\alpha^*$ exists on an interval of order $O(e^{-1/\varepsilon})$). But in the case $\dim y = 1$ another picture is beginning to emerge. It was shown that a one-parameter family of ducks exists [15]. If we take the parameter $\alpha$ as a function of $y$ we can glue the stable and unstable integral manifolds at all points of the separating curve at the same time. This approach obviously is associated with Krasnosel'skii's method of functionalization of a parameter [11].

Consider two simple examples.

E x a m p l e 1 ( $\dim y = 0$).

As the simplest system with a duck–trajectory we propose the following system

$$\dot{x} = 1, \ \ \varepsilon \dot{z} = 2xz + \alpha.$$

It is clear that for $\alpha = 0$, the trajectory $z = 0$ is a duck–trajectory.

E x a m p l e 2 ( $\dim y = 1$).

Consider the system

$$\dot{x} = 1, \ \ \dot{y} = 0, \ \ \varepsilon \dot{z} = 2xz + \alpha - y.$$

If $\alpha$ is a parameter then the different duck–trajectories are determined by

$$\dot{x} = 1, \ \ y = y_0, \ \ z = 0,$$

that is they pass through the unique point of glueing $x = 0$, $y = y_0$, $z = 0$ on the separating curve $x = 0$ of the slow surface $2xz + y_0 - y = 0$ for $\alpha = y_0$.

If $\alpha$ is a function of the variable $y$ then for $\alpha = y$ the integral manifold $z = 0$ is stable for $x < 0$ and unstable for $x > 0$.

# 3 A model of combustion

Duck–trajectories are of greatest interest in models of combustion of a rarefied gas mixture in an inert porous or in a dust-laden medium.

Let us consider the case of uniform temperature distribution and phase-to-phase heat exchange. The chemical conversion kinetics are represented by one-stage and irreversible reaction. The dimensionless model in this case has the form

$$\epsilon\dot{\Theta} = \Psi(\eta)exp(\Theta/\left(1+\beta\Theta\right)) - \alpha(\Theta-\Theta_c) - \delta\Theta,$$

$$\gamma_c\dot{\Theta}_c = \alpha(\Theta-\Theta_c),$$

$$\dot{\eta} = \Psi(\eta)exp(\Theta/\left(1+\beta\Theta\right)),$$

$$\eta(0) = \eta_0/\left(1+\eta_0\right), \;\; \Theta(0) = \Theta_c(0) = 0.$$

Here, $\Theta$ and $\Theta_c$ are the dimensionless temperatures of the reactant phase and of the inert one; $\eta$ is the depth of conversion; $\eta_0$ is the criterion of autocatalyticity; $\beta$ and $\epsilon$ are small parameters. The term $-\delta\Theta$ characterizes the external heat dissipation. The parameter $\gamma_c$ reflects the physical features of the inert phase. The parameter $\alpha$ characterizes the physical properties of the system and determines the dynamics of the process. Depending on $\alpha$ the reaction either changes to a slow regime with decay of reaction, or into a regime of selfacceleration with progressive temperature growth. The last phenomenon is called as "thermal explosion". The transition region from the slow regime to explosive one exists due to the continuous dependence of system (2.1)—(2.3) on the parameter $\alpha$. The case $\alpha = \alpha^*$ represents the optimal technological regime: to increase the temperature as high as possible but without explosion. We have to note that this regime is critical, and it corresponds to a chemical reaction separating the domain of selfaccelerating reactions ($\alpha < \alpha^*$) and the domain of slow reactions ($\alpha > \alpha^*$).

The following cases are considered:

$$\Psi(\eta) = \begin{cases} 1-\eta, & \text{– first-order reaction } (\eta_0 = 0) \\ \eta(1-\eta), & \text{– autocatalytic reaction.} \end{cases}$$

In the absence of external heat dissipation ($\delta = 0$) the system of differential equations possesses a first integral and therefore we obtain $\dim x = 1$ in (2.1)—(2.3). In this case, the asymptotic expansion of the duck–trajectory passing through the selfintersection point of the slow curve has been derived in [14].

In the case $\delta \neq 0$, the existence of a duck–trajectory corresponding to $\alpha = \alpha^*$, has been established in [7, 15]. This duck–trajectory describes the critical regime, and $\alpha^*$ is the critical parameter value.

The asymptotic expansions of duck-trajectories modelling critical regimes of chemical systems depending on parameter has been derived in [7, 15].

# 4 Black swans

In this paper we use the standard approach to study slow integral surfaces of variable stability (or black swans). These surfaces are considered as natural generalizations of the notion of a canard.

We suggest to use the term "black swan" by two reasons. The first one is that a swan is a bird of the family of ducks. The second one is connected with the usual meaning of "black swan" in the sense of a rare phenomenon. It should be noted that the French term "canard" is used in the sense of false rumour in English.

In order to glue the stable and unstable parts of a duck–trajectory an additional parameter is used. To glue integral manifolds whose dimension is greater than one we need an additional function. The argument of this function is a vector variable parametrizing the separating surface.

To prove the existence of a slow integral manifold with changing stability we reduce the system (2.1)—(2.3) to form

$$\frac{dy}{dx} = Y(x,y,z,\varepsilon), \qquad y \in R^n, \qquad x \in R; \tag{4.5}$$

$$\varepsilon\frac{dz}{dx} = 2xz + a + Z(x,y,z,a,\varepsilon), \qquad |z| \le r; \qquad |a| \le a_0, \tag{4.6}$$

where $r$ and $a_0$ are positive constants. It is supposed that the functions $Y, Z$ are continuous and satisfy the following inequalities for $x \in R$, $y \in R^n$, $|z| \le r$, $|a| \le a_0$, $\varepsilon \in [0, \varepsilon_0]$:

$$\|Y(x,y,z,\varepsilon)\| \le k, \quad |Z(x,y,z,a,\varepsilon)| \le M\left(\varepsilon^2 + \varepsilon|z| + |z|^2\right), \tag{4.7}$$

$$\|Y(x,y,z,\varepsilon) - Y(x,\bar{y},\bar{z},\varepsilon)\| \le M(\|y-\bar{y}\| + |z-\bar{z}|), \tag{4.8}$$

$$\begin{aligned} &|Z(x,y,z,a,\varepsilon) - Z(x,\bar{y},\bar{z},\bar{a},\varepsilon)| \le M\left\{(\varepsilon + |\tilde{z}|)|z-\bar{z}| + \right. \\ &\left. +(\varepsilon^2 + \varepsilon|\tilde{z}| + |\tilde{z}|^2)\|y-\bar{y}\| + \varepsilon|a-\bar{a}|\right\}, \quad |\tilde{z}| = \max\{|z|, |\bar{z}|\}, \end{aligned} \tag{4.9}$$

where $\|\cdot\|$ denotes the usual norm in $R^n$ and $|\cdot|$ denotes the absolute value of a scalar, $k$ and $M$ are positive constants.

Let us consider $a$ as a function $a = a(y,\varepsilon)$. Let $F$ be the complete metric space of functions $a(y,\varepsilon)$ continuous with respect to $y$ and satisfying for $\varepsilon \in (0, \varepsilon_0]$

$$|a(y,\varepsilon)| \le \varepsilon^2 K, \quad |a(y,\varepsilon) - a(\bar{y},\varepsilon)| \le \varepsilon^2 L\|y-\bar{y}\|, \tag{4.10}$$

where $K$ and $L$ are positive constants, with the metric defined by

$$\rho(a,\bar{a}) = \sup_{y \in R^n} |a(y,\varepsilon) - \bar{a}(y,\varepsilon)|.$$

Let $H$ be the complete metric space of functions $h(x,y,\varepsilon)$ mapping $R \times R^n$ to $R$ continuous with respect to $x, y$ and satisfying for $\varepsilon \in (0, \varepsilon_0]$

$$|h(x,y,\varepsilon)| \le \varepsilon^{\frac{3}{2}} q, \tag{4.11}$$

$$|h(x,y,\varepsilon) - h(x,\bar{y},\varepsilon)| \le \varepsilon^{\frac{3}{2}} \delta \|y - \bar{y}\|, \tag{4.12}$$

where $q$ and $\delta$ are positive constants, with the metric

$$\rho(h, \bar{h}) = \sup_{x \in R, y \in R^n} |h(x,y,\varepsilon) - \bar{h}(x,y,\varepsilon)|.$$

On the space $H$ we define an operator $T$ by the formula

$$Th(x,y,\varepsilon) = \begin{cases} -\varepsilon^{-1} \int\limits_x^{\infty} e^{(x^2-s^2)/\varepsilon} [Z(\cdot) + a(\varphi(s,x),\varepsilon)] ds & , \quad x \ge 0 \\ \varepsilon^{-1} \int\limits_{-\infty}^{x} e^{(x^2-s^2)/\varepsilon} [Z(\cdot) + a(\varphi(s,x),\varepsilon)] ds & , \quad x < 0. \end{cases}$$

where $Z(\cdot) = Z(s, \varphi(s,x), h(s,\varphi(s,x),\varepsilon), a(\varphi(s,x),\varepsilon), \varepsilon)$, and $\varphi(s,x)$ is defined as follows. For any element $h \in H$, we consider the initial value problem

$$\frac{d\varphi}{ds} = Y(s, \varphi, h(s,\varphi,\varepsilon), \varepsilon), \tag{4.13}$$

$$\varphi(x) = y. \tag{4.14}$$

The solution of this problem is denoted by $\Phi(s,x,y,\varepsilon|h) = \varphi(s,x)$. In case the operator $T$ possesses a fixed point $h(x,y,\varepsilon)$ in $H$ then the surface $z = h(x,y,\varepsilon)$ is a slow integral manifold with changing stability (black swan).

It should be noted that we use a modification of the usual technique of the integral manifold theory [12, 16]. The following statement is true.

**T h e o r e m.** *Let the conditions (4.7)–(4.9)are satisfied. Then there are numbers* $\varepsilon_0 > 0$ *and* $K, L, q, \delta$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there exist functions* $a(y,\varepsilon) \in F$ *and* $h(x,y,\varepsilon) \in H$ *such that* $z = h(x,y,\varepsilon)$ *is a slow integral manifold.*

## 4.1 Auxiliary estimates

To prove the Theorem let us introduce the following functions:

$$\varphi_1(s,x) = \Phi(s,x,\bar{y},\varepsilon|h),$$

$$\varphi_2(s,x) = \Phi(s,x,y,\varepsilon|\bar{h}).$$

From (4.13), (4.14) it follows that

$$\varphi(s,x) = y + \int\limits_x^s Y(\eta, \varphi(\eta,x), h(\eta,\varphi(\eta,x),\varepsilon), \varepsilon) d\eta,$$

$$\varphi_1(s,x) = \bar{y} + \int\limits_x^s Y(\eta, \varphi_1(\eta,x), h(\eta, \varphi_1(\eta,x),\varepsilon),\varepsilon)d\eta,$$

$$\varphi_2(s,x) = y + \int\limits_x^s Y(\eta, \varphi_2(\eta,x), \bar{h}(\eta, \varphi_2(\eta,x),\varepsilon),\varepsilon)d\eta.$$

Using these relations, inequalities (4.8), (4.12) and the Gronwall – Bellman inequality we obtain the estimates

$$\|\varphi(s,x) - \varphi_1(s,x)\| \le \|y - \bar{y}\| e^{M(1+\varepsilon^{3/2}\delta)|s-x|}, \tag{4.15}$$

$$\|\varphi(s,x) - \varphi_2(s,x)\| \le \rho(h,\bar{h}) \frac{1}{(1+\varepsilon^{3/2}\delta)} (e^{M(1+\varepsilon^{3/2}\delta)|s-x|} - 1). \tag{4.16}$$

## 4.2 Existence of the function $a(y,\varepsilon)$

For any fixed function $h \in H$ consider the following integral–operator equation

$$\int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon}[Z(s,\varphi(s,0),h(s,\varphi(s,0),\varepsilon),a(\varphi(s,0),\varepsilon),\varepsilon) + a(\varphi(s,0),\varepsilon)]ds = 0 \tag{4.17}$$

with respect to a function $a(y,\varepsilon)$, where $\varphi(s,0) = \Phi(s,0,y,\varepsilon|h)$. This equation is obtained from the condition of continuity of $Th$ at $x = 0$.

It is convenient to rewrite (4.17) in the form

$$Aa(y,\varepsilon) = Qa(y,\varepsilon).$$

Here

$$Aa(y,\varepsilon) \equiv \frac{1}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon} a(\varphi(s,0),\varepsilon)ds,$$

$$Qa(y,\varepsilon) \equiv -\frac{1}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon} Z(s,\varphi(s,0),h(s,\varphi(s,0),\varepsilon),a(\varphi(s,0),\varepsilon),\varepsilon)ds.$$

The last expressions define on $F$ a linear operator $A : a(y,\varepsilon) \to Aa(y,\varepsilon)$ and a nonlinear operator $Q : a(y,\varepsilon) \to Qa(y,\varepsilon)$. It is convenient to represent the operator $A$ as a sum of two operators $A = I + R$, where $I$ is the identity and $R$ is defined by the formula

$$Ra(y,\varepsilon) = \frac{1}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon}[a(\varphi(s,0),\varepsilon) - a(y,\varepsilon)]ds.$$

The inequalities (4.8), (4.10) imply

$$|Ra(y,\varepsilon)| \le \frac{1}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon}\varepsilon^2 L \| \int\limits_0^s Y(\eta,\varphi(\eta,0),h(\eta,\varphi(\eta,0),\varepsilon),\varepsilon)d\eta\| ds \le$$

$$\le \varepsilon^{5/2}\frac{Lk}{\sqrt{\pi}}.$$

For $\varepsilon^{5/2}\frac{Lk}{\sqrt{\pi}} < 1$ there exists the linear operator $(I+R)^{-1}$ and the following inequality is true

$$\|(I+R)^{-1}\| \le \frac{1}{1-\varepsilon^{5/2}Lk/\sqrt{\pi}}. \tag{4.18}$$

Let us introduce the operator $P$ on $F$ by the formula

$$Pa = (I+R)^{-1}Qa.$$

In the sequel we will prove that the operator $P$ maps $F$ into itself and is contracting. For $Q$ we get

$$|Qa(y,\varepsilon)| \le \frac{1}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon} M\left(\varepsilon^2+\varepsilon|z|+|z|^2\right) ds \le$$

$$\le M\left(\varepsilon^2+\varepsilon^{5/2}q+\varepsilon^3 q^2\right).$$

Using the last inequality and (4.18), we obtain

$$|Pa(y,\varepsilon)| \le \frac{M\left(\varepsilon^2+\varepsilon^{5/2}q+\varepsilon^3q^2\right)}{1-\varepsilon^{5/2}Lk/\sqrt{\pi}}.$$

Under the condition

$$\frac{\varepsilon^{5/2}kL}{\sqrt{\pi}} \le \frac{1}{2}, \tag{4.19}$$

the inequality

$$|Pa(y,\varepsilon)| \le 2M\left(1+\sqrt{\varepsilon}q+\varepsilon q^2\right)\varepsilon^2.$$

is true. Is easy to verify the estimate

$$|Qa(y,\varepsilon)-Qa(\bar{y},\varepsilon)| \le \frac{1}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon} MS\|\varphi(s,0)-\varphi_1(s,0)\|ds \le$$

$$\le \frac{MS}{\sqrt{\varepsilon\pi}} \int\limits_{-\infty}^{\infty} e^{-s^2/\varepsilon} e^{M(1+\varepsilon^{3/2}\delta)|s|}\|y-\bar{y}\|ds,$$

where $S = \left(\varepsilon^2+\varepsilon^{5/2}q+\varepsilon^3q^2+\varepsilon^{3/2}\delta(\varepsilon+\varepsilon^{3/2}q)+\varepsilon^3L\right)$.

Using (4.15) and the error integral $erf(\lambda)$ it is possible to obtain for $\sqrt{\varepsilon}M(1+\varepsilon^{3/2}\delta)<1$

$$|Qa(y,\varepsilon)-Qa(\bar{y},\varepsilon)|<3MS\|y-\bar{y}\|.$$

Taking into account(4.19) we obtain

$$|Pa(y,\varepsilon)-Pa(\bar{y},\varepsilon)|\leq 6MS\|y-\bar{y}\|.$$

Consiquently, if the inequality (4.19) and the inequalities

$$2M\left(1+\sqrt{\varepsilon}q+\varepsilon q^2\right)\leq K,$$

$$\sqrt{\varepsilon}M(1+\varepsilon^{3/2}\delta)<1,$$

$$6M\left(1+\sqrt{\varepsilon}q+\varepsilon q^2+\sqrt{\varepsilon}\delta(1+\sqrt{\varepsilon}q)+\varepsilon L\right)\leq L,$$

hold then $P:F\to F$.

Now we derive conditions assuring $P$ to be a contracting operator. At first let us estimate the difference $Qa-Q\bar{a}$.

$$|Qa(y,\varepsilon)-Q\bar{a}(y,\varepsilon)|\leq\frac{1}{\sqrt{\varepsilon\pi}}\int\limits_{-\infty}^{\infty}e^{-s^2/\varepsilon}\varepsilon M\rho(a,\bar{a})ds=\varepsilon M\rho(a,\bar{a}).$$

Taking into account (4.19) we obtain

$$|Pa(y,\varepsilon)-P\bar{a}(y,\varepsilon)|\leq 2\varepsilon M\rho(a,\bar{a}).$$

If

$$2\varepsilon M<1,$$

holds then $P$ is a contracting operator in $F$ and therefore the equation $a=Pa$, which is equivalent to (4.17), possesses a unique solution in $F$.

Thus, we have derived conditions for the existence a unique solution of (4.17) in $F$.

Now we study the dependence of the fixed point $a$ of $P$ on $h$. Let $a(y,\varepsilon)$ ( $\bar{a}(y,\varepsilon)$) be a solution of (4.17) corresponding to the functions $h$ ( $\bar{h}$). Then we have $Aa=Qa$ or $(I+R)a=Qa$ and $\bar{A}\bar{a}=\bar{Q}\bar{a}$ or $(I+\bar{R})\bar{a}=\bar{Q}\bar{a}$, where

$$\bar{R}\bar{a}(y,\varepsilon)=\frac{1}{\sqrt{\varepsilon\pi}}\int\limits_{-\infty}^{\infty}e^{-s^2/\varepsilon}[\bar{a}(\varphi_2(s,0),\varepsilon)-\bar{a}(y,\varepsilon)]ds,\ \bar{Q}\bar{a}(y,\varepsilon)=$$

$$=-\frac{1}{\sqrt{\varepsilon\pi}}\int\limits_{-\infty}^{\infty}e^{-s^2/\varepsilon}Z(s,\varphi_2(s,0),\bar{h}(s,\varphi_2(s,0),\varepsilon),\bar{a}(\varphi_2(s,0),\varepsilon),\varepsilon)ds.$$

After some elementary transformation we obtain

$$(I+R)(a-\bar{a})=Qa-\bar{Q}\bar{a}+(\bar{R}-R)\bar{a}$$

or

$$a - \bar{a} = (I + R)^{-1}[Qa - \bar{Q}\bar{a} + (\bar{R} - R)\bar{a}].$$

The expressions in the square brackets will be estimated at first.

$$|Qa(y,\varepsilon) - \bar{Q}\bar{a}(y,\varepsilon)| \le \frac{M}{\sqrt{\varepsilon\pi}} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon}[S\|\varphi(s,0) - \varphi_2(s,0)\| +$$

$$+(\varepsilon + \varepsilon^{3/2}q)\rho(h,\bar{h}) + \varepsilon\rho(a,\bar{a})]ds].$$

Taking into account (4.16) we obtain

$$|Qa(y,\varepsilon) - \bar{Q}\bar{a}(y,\varepsilon)| \le M\left[\varepsilon\rho(a,\bar{a}) + (\varepsilon + \varepsilon^{3/2}q + 2S)\rho(h,\bar{h})\right]$$

and

$$|(\bar{R} - R)\bar{a}(y,\varepsilon)| = \frac{1}{\sqrt{\varepsilon\pi}}\left|\int_{-\infty}^{\infty} e^{-s^2/\varepsilon}[\bar{a}(\varphi_2(s,0),\varepsilon) - \bar{a}(\varphi(s,0),\varepsilon)]ds\right| \le$$

$$\le \frac{1}{\sqrt{\varepsilon\pi}} \int_{-\infty}^{\infty} e^{-s^2/\varepsilon}\varepsilon^2 L\|\varphi(s,0) - \varphi_2(s,0)\|ds \le$$

$$\le \frac{2}{\sqrt{\varepsilon\pi}} \int_0^{\infty} e^{-s^2/\varepsilon}\frac{\varepsilon^2 L}{1+\varepsilon^{3/2}\delta}(e^{M(1+\varepsilon^{3/2}\delta)s} - 1)\rho(h,\bar{h})ds \le$$

$$\le 2\varepsilon^2 L\rho(h,\bar{h}).$$

Thus, under the conditions (4.19) and $4\varepsilon M < 1$ the following estimate is true

$$\rho(a,\bar{a}) \le 4\left[M\left(\varepsilon + \varepsilon^{3/2}q + 2S\right) + 2\varepsilon^2 L\right]\rho(h,\bar{h}). \tag{4.20}$$

## 4.3 Existence of slow manifold

We derive now conditions guaranteeing that $Th(x,y)$ satisfies the inequalities (4.11), (4.12) . For $x \ge 0$ we have

$$|Th(x,y,\varepsilon)| \le \varepsilon^{-1}\int_x^{\infty} e^{(x^2-s^2)/\varepsilon}[|Z(s,\varphi(s,x),h(s,\varphi(s,x),\varepsilon),a(\varphi(s,x),\varepsilon),\varepsilon)| +$$

$$+|a(\varphi(s,x),\varepsilon)|]ds \le \varepsilon^{3/2}\frac{\sqrt{\pi}}{2}(K + M(1 + \sqrt{\varepsilon}q + \varepsilon q^2)).$$

$$|Th(x,y,\varepsilon) - Th(x,\bar{y}),\varepsilon| \le \varepsilon^{-1}\int_x^{\infty} e^{(x^2-s^2)/\varepsilon}\left(MS + \varepsilon^2 L\right)\|\varphi(s,x) - \varphi_1(s,x)\|ds \le$$

$$\leq \varepsilon^{-1}\left(MS+\varepsilon^2 L\right)\|y-\bar{y}\|\int\limits_x^\infty e^{(x^2-s^2)/\varepsilon}e^{M(1+\varepsilon^{3/2}\delta)(s-x)}ds \leq$$

$$\leq \frac{3\sqrt{\pi}}{2\sqrt{\varepsilon}}\left(MS+\varepsilon^2 L\right)\|y-\bar{y}\|.$$

Thus, under the conditions

$$\frac{\sqrt{\pi}}{2}(K+M(1+\sqrt{\varepsilon}q+\varepsilon q^2)) \leq q \leq r,$$

$$\frac{3\sqrt{\pi}}{2}\left[M\left(1+\sqrt{\varepsilon}q+\varepsilon q^2+\sqrt{\varepsilon}\delta(1+\sqrt{\varepsilon}q)\right)+L(1+\varepsilon M)\right] \leq \delta$$

the operator $T$ maps $H$ into itself.

Now we prove $T$ is a contracting operator. By (4.16) and (4.20) we have

$$|Th(x,y,\varepsilon)-T\bar{h}(x,y,\varepsilon)| \leq \varepsilon^{-1}\int\limits_x^\infty e^{(x^2-s^2)/\varepsilon}\Big([MS+\varepsilon^2 L]\|\varphi(s,x)-\varphi_2(s,x)\|+$$

$$+(1+\varepsilon M)\rho(a,\bar{a})+M(\varepsilon+\varepsilon^{3/2}q)\rho(h,\bar{h})\Big)ds \leq$$

$$\leq \varepsilon^{-1}\int\limits_x^\infty e^{(x^2-s^2)/\varepsilon}\left[A+B\left(\varepsilon^{M(1+\varepsilon^{3/2}\delta)|s-x|}-1\right)\right]ds\ \rho(h,\bar{h}) \leq$$

$$\leq \frac{\sqrt{\pi}}{2\sqrt{\varepsilon}}(A+2B)\rho(h,\bar{h}),$$

where

$$A = M(\varepsilon+\varepsilon^{3/2}q)+4(1+\varepsilon M)\left[M(\varepsilon+\varepsilon^{3/2}q+2S)+2\varepsilon^2 L\right],$$

$$B = MS+\varepsilon^2 L.$$

Thus, under the condition

$$\frac{\sqrt{\pi}}{2\sqrt{\varepsilon}}(A+2B) < 1$$

$T$ is a contracting mapping in $H$.

R e m a r k 1. Usually, the conditions (4.7)–(4.9) are fulfilled for $|x| \leq r_1$, $\|y\| \leq r_2$ only. Integral manifolds in this case are local.

R e m a r k 2. Let the functions $Y$ and $Z$ in (4.5)–(4.6) are sufficiently smooth, then for the functions $h$ and $a(y,\varepsilon)$ asymptotical expansions can be derived

$$a(y,\varepsilon) = \sum \varepsilon^i a_i(y),\, z = h(x,y,\varepsilon) = \sum \varepsilon^i h_i(x,y).$$

R e m a r k 3. Systems of the type (2.1)–(2.3) can be reduced to the form (4.5)–(4.6) in a neighborhood of the first approximation of the slow integral manifold.

# 5 Integral manifolds and duck–trajectories

Let us discuss a connection between slow integral manifolds and duck–trajectories. At first, we consider system (2.1)–(2.3) in the case $\dim y = 0$. This system can be reduced to the form (4.6), where $Z$ does not depend on $y$, and $a$ is a parameter. In this case the slow integral manifold is one–dimensional. If the variables $t$ and $x$ increase simultaneously and $x$ passes through zero then this integral manifold contains a duck–trajectory.

Statements of type "The life of canard is very short" can be found usually in papers. This means that values of parameter $a$ corresponding to duck–trajectories belong to an interval of order $O(e^{-1/\kappa\varepsilon})$ where $\kappa$ is some positive number. It is not difficult to give examples of ducks living for centenaries.

E x a m p l e 3. Consider the system

$$\dot{x} = z$$

$$\varepsilon\dot{z} = x^2 + z^2 - a^2$$

The circle $(x + \frac{\varepsilon}{2})^2 + z^2 = a^2 - \frac{\varepsilon^2}{4}$ is a duck-trajectory. The upper semicircle is unstable and the lower one is stable. This duck exists for all $a^2 > \varepsilon^2/4$.

If there exists a gluing function $a(y, \varepsilon)$ then every trajectory on the slow integral manifold is a duck–trajectory if it crosses the surface $x = 0$ from the stable part ($x < 0$) to unstable one ($x > 0$). Thus, in Example 2 for $\alpha = y$, every trajectory on the slow integral manifold $z = 0$, is a duck–trajectory. The analogous situation takes place for the model of combustion in an inert porous medium. But it follows from the physical reasons that the gluing function has to be constant: $a(y, \varepsilon) = a(y_0, \varepsilon)$. In this case, the stable and unstable parts of the integral manifold can be glued in one point $y = y_0$ only. The duck- trajectory passes just through this point. A natural generalization of this situatiom can be done. Let the gluing function $a = a(y, \varepsilon)$ be given. On the $n$–dimensional separating surface let some $n_1$–dimensional surface $y = \chi(u), u \in R^{n_1}$ of lower dimension ($n_1 < n$) be given. If the gluing function $a(y, \varepsilon)$ is restricted to $y = \chi(u)$ then the gluing of the stable and unstable parts of slow integral surfaces can be realized in points of the surface $y = \chi(u)$ only. That permits us to construct slow integral manifolds with changing stability of various forms and dimensions.

E x a m p l e 4. Consider the following system

$$\dot{x} = 1;$$

$$\dot{y} = 0, \ \ y \in R^n;$$

$$\varepsilon\dot{z} = 2xz + a(y, \varepsilon) + p(y) + xq(y) + x^2 r(y).$$

Here $p, q, r$ are scalar continuous functions of the vector variable $y$. By setting $a(y, \varepsilon) = -p(y) - \varepsilon r(y)$, we obtain $h = -q(y) - xr(y)$. Let $y = \chi(u), u \in R^{n_1}$ be any surface, then the system

$$\dot{x} = 1;$$

$$\dot{y} = 0,\ y \in R^n;$$
$$\varepsilon\dot{z} = 2xz - p(\chi(u)) - \varepsilon r(\chi(u)) + p(y) + xq(y) + x^2 r(y).$$

possesses the higher–dimensional cylindrical slow integral surface $z = -q(\chi(u)) - xr(\chi(u))$, and every element of this cylindrical surface is a duck.

In conclusion a higher–dimension generalization of the Example 3 will be given.

E x a m p l e 5. Consider the differential system

$$\dot{x} = z$$
$$\dot{y}_i = z,\ i = 1, \ldots, n$$
$$\varepsilon\dot{z} = x^2 + \sum_{i=1}^{n} y_i^2 + z^2 - a^2$$

It is a straighforward exercise now to see that the higher–dimensional sphere

$$(x + \varepsilon/2)^2 + \sum_{i=1}^{n} (y_i + \varepsilon/2)^2 + z^2 = a^2 - \frac{n+1}{4}\varepsilon^2$$

is a slow integral manifold, one part of it ($z < 0$) is stable and other one ($z > 0$) is unstable. This black swan lives for all $a^2 > \frac{n+1}{4}\varepsilon^2$.

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