# SINGULAR BEHAVIOUR OF FINITE APPROXIMATIONS TO THE ADDITION MODEL

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**Abstract**

Instantaneous gelation in the addition model with superlinear rate coefficients is investigated. The conjectured post-gelation solution is shown to arise naturally as the limit of solutions to some finite approximations as the number of equations grows to infinity. Non-existence of continuous solutions to the addition model is also established in that case.

# 1 Introduction

One approach to describe irreversible aggregation in the dynamics of cluster growth involves a coupled infinite system of ordinary differential equations first introduced by Smoluchowski [1] which reads

$$\frac{dc_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} a_{j,i-j}\ c_j\ c_{i-j} - c_i \sum_{j=1}^{\infty} a_{i,j}\ c_j, \quad i \geq 1.$$

Here $c_i$ denotes the concentration of $i$-clusters (i.e. clusters made of $i$ particles), $i \geq 1$ and the coagulation rates $a_{i,j}$ are nonnegative real numbers satisfying $a_{i,j} = a_{j,i}$ and characterising the reaction between $i$- and $j$-clusters, producing $i+j$-clusters. In the above equation, the first term of the right hand side accounts for the formation of $i$-clusters by coagulation of smaller clusters while the second term represents the loss of $i$-clusters due to coalescence with other clusters. Notice that since particles are neither destroyed nor created in the coagulation process described above the total density of clusters $\sum_{i=1}^{\infty} ic_i$ is expected to remain constant through time evolution. However it is well-known that this is not always the case and that the total density of clusters may decrease after some time

$$\sum_{i=1}^{\infty} ic_i(t) < \sum_{i=1}^{\infty} ic_i(0) \quad \text{for} \ \ t > T_{gel}, \tag{1.1}$$

a phenomenon known as gelation [2, 3]. The gelation phenomenon is said to take place instantaneously if $T_{gel} = 0$ in (1.1).

In this paper we discuss some mathematical properties of the so-called addition model which may be obtained from the Smoluchowski coagulation equation under the additional assumption that the only active reactions are those involving monoclusters. From a mathematical point of view, this assumption simply reads

$$a_{i,j} = 0 \quad \text{whenever} \quad \min\{i,j\} \geq 2.$$

Introducing

$$a_{i,1} = a_{1,i} = a_i \quad \text{if} \ \ i \geq 2 \quad \text{and} \ \ a_{1,1} = 2a_1,$$

the addition model reads [4]

$$\begin{cases} \dfrac{dc_1}{dt} = -a_1 c_1^2 - \displaystyle\sum_{i=1}^{\infty} a_i c_1 c_i, \\ \\ \dfrac{dc_i}{dt} = a_{i-1} c_1 c_{i-1} - a_i c_1 c_i, \quad i \geq 2, \end{cases} \tag{1.2}$$

$$c_i(0) = c_i^0, \quad i \geq 1. \tag{1.3}$$

Let us mention that (1.2)-(1.3) may also be seen as a particular case of the Becker-Döring cluster equations [5] when fragmentation is not taken into account. Also a related system of ordinary differential equations arises in the modelling of hydrolysis and polymerisation of silicon alkoxides in the presence of ammonia [6].

Our interest in this paper is the behaviour of some approximations of (1.2)-(1.3) by finite systems of ordinary differential equations when the number of equations increases to infinity. More precisely, given $N \geq 3$ and $\delta \geq 0$ we denote by $c^N = \left(c_i^N\right)_{1 \leq i \leq N}$ the solution to

$$\begin{cases} \dfrac{dc_1^N}{dt} = -a_1 \left(c_1^N\right)^2 - \displaystyle\sum_{i=1}^{N-1} a_i c_1^N c_i^N - \delta a_N c_1^N c_N^N, \\ \\ \dfrac{dc_i^N}{dt} = a_{i-1} c_1^N c_{i-1}^N - a_i c_1^N c_i^N, \quad 2 \leq i \leq N-1, \\ \\ \dfrac{dc_N^N}{dt} = a_{N-1} c_1^N c_{N-1}^N + \dfrac{\delta}{N} a_N c_1^N c_N^N, \\ \\ c_i^N(0) = c_i^0, \quad 1 \leq i \leq N. \end{cases} \tag{1.4}$$

Assuming that

$$c_i^0 \geq 0 \quad \text{for} \quad i \geq 1 \quad \text{and} \quad \sum_{i=1}^{\infty} i c_i^0 < \infty, \tag{1.5}$$

we infer from [5, Theorem 2.2] that, if

$$\sup_{i \geq 1} \frac{a_i}{i} < \infty,$$

there is a subsequence of $\left(c^N\right)_{N \geq 3}$ which converges as $N \to +\infty$ towards a solution to (1.2)-(1.3) in the sense of Definition 2.4 below (in fact, only the case $\delta = 0$ is considered in [5] but their proof easily extends to the case $\delta > 0$). A similar result does not hold if

$$\lim_{i \to +\infty} \frac{a_i}{i} = +\infty. \tag{1.6}$$

Indeed if (1.6) holds there are initial data fulfilling (1.5) for which (1.2)-(1.3) has no solution in the sense of Definition 2.4 (even locally in time) [5, Theorem 2.7]. In fact we prove in this paper that for a large class of coagulation rates $(a_i)_{i\ge 1}$ satisfying (1.6) and for any initial data with $c_1^0 \neq 0$ fulfilling (1.5) the system (1.2)-(1.3) has no solution (see Proposition 2.5 below for a precise statement). However the main result of this paper is that we are able to prove that the sequence $\left(c^N\right)_{N\ge 3}$ still converges as $N \to +\infty$ under the assumption (1.6) and to identify its limit as well, namely

$$\lim_{N\to+\infty} c_1^N(t) = 0 \quad \text{for a.e.} \quad t \in (0,+\infty),$$
$$\lim_{N\to+\infty} c_i^N(t) = c_i^0 \quad \text{for} \quad t \in [0,+\infty) \quad \text{and} \quad i \ge 2.$$

Clearly when $c_1^0 \neq 0$ the limit $\left(c^N\right)_{N\ge 3}$ is not a solution to (1.2)-(1.3) in the sense of Definition 2.4 below but it is exactly the post-gel solution to (1.2)-(1.3) obtained by Brilliantov and Krapivsky [7] for coagulation rates $a_i = i^\alpha$, $\alpha > 1$, using formal arguments along the lines of van Dongen [8]. Our result thus shows that though (1.2)-(1.3) has no solution when the coagulation rates satisfies (1.6) the occurrence of instantaneous gelation in this model may be seen in the limiting behaviour of a sequence of approximating finite systems.

# 2 Main results

Before stating precisely our results we recall some notations we will use throughout the paper and the definition of a solution to (1.2) as well. Define

$$X = \left\{ c = (c_i)_{i\ge 1}, \quad \sum_{i=1}^{\infty} i|c_i| < \infty \right\},$$

which is a Banach space when endowed with the norm

$$\|c\| = \sum_{i=1}^{\infty} i|c_i|, \quad c \in X.$$

We denote by $X^+$ the positive cone of $X$

$$X^+ = \{ c = (c_i)_{i\ge 1} \in X, \quad c_i \ge 0 \quad \text{for each} \quad i \ge 1 \}.$$

Our main results then read as follows.

**Theorem 2.1** *Assume that the coagulation rates $(a_i)_{i\ge 1}$ fulfil*

$$\lim_{i\to+\infty} \frac{a_i}{i} = +\infty, \tag{2.1}$$

*and put*

$$\gamma_m = \min_{i\ge m} \frac{a_i}{i}, \quad m \ge 1. \tag{2.2}$$

*Assume also that*

$$c^0 = \left(c_i^0\right)_{i\geq 1} \in X^+ \quad \textit{and} \quad \lim_{m\to+\infty} \gamma_m \sum_{i=m}^{\infty} i c_i^0 = +\infty. \tag{2.3}$$

*Finally let $\delta$ be a nonnegative real number and for $N \geq 3$ we denote by $c^N = \left(c_i^N\right)_{1\leq i\leq N}$ the solution to (1.4). For each $i \geq 1$ the sequence $(c_i^N)_{N\geq 3}$ has a limit as $N \to +\infty$ and*

$$\lim_{N\to+\infty} c_1^N(t) = 0 \quad \textit{for a.e.} \quad t \in (0,+\infty), \tag{2.4}$$

$$\lim_{N\to+\infty} c_i^N(t) = c_i^0 \quad \textit{for} \quad t \in (0,+\infty) \quad \textit{and} \quad i \geq 2. \tag{2.5}$$

Note that the above result is only valid for initial data whose components increase sufficiently fast as $i \to +\infty$. In order to be able to state a similar result valid for general initial data in $X^+$ we need to strengthen the assumptions on the coagulation rates and to assume that $\delta > 0$. More precisely, we have the following result.

**Theorem 2.2** *Assume that the coagulation rates $(a_i)_{i\geq 1}$ satisfy*

$$\lim_{i\to+\infty} \frac{a_i}{i\ \ln(1+a_i)} = +\infty \quad \textit{and} \quad a_{i+1} \geq a_i \geq a_1 > 0, \quad i \geq 1, \tag{2.6}$$

$$a_i \geq K\ i\ \left(\ln(1+i)\right)^\alpha, \quad i \geq 1, \tag{2.7}$$

*for some $\alpha > 1$ and $K > 0$. Assume further that*

$$c^0 = \left(c_i^0\right)_{i\geq 1} \in X^+ \quad \textit{and} \quad c_1^0 \neq 0. \tag{2.8}$$

*Finally let $\delta$ be a positive real number and $c^N = \left(c_i^N\right)_{1\leq i\leq N}$ be the solution to (1.4) for $N \geq 3$. For each $i \geq 1$ the sequence $(c_i^N)_{N\geq 3}$ has a limit as $N \to +\infty$ and (2.4)-(2.5) hold.*

**Remark 2.3** *1. We actually prove a stronger result than (2.5), namely that the convergence (2.5) holds uniformly on compact subsets of $[0,+\infty)$.*

*2. It is straightforward to check that $a_i = i^\beta\ \left(\ln(1+i)\right)^\alpha$ satisfies (2.6)-(2.7) when $\beta = 1$ and $\alpha > 1$ and when $\beta > 1$ and $\alpha \geq 0$. Also, $a_i = e^i$ satisfies (2.6)-(2.7).*

*3. It is clear that if $c_1^0 = 0$ then $c^N = (0, c_2^0, \ldots, c_N^N)$ and the convergences (2.4)-(2.5) are still valid.*

In order to prove Theorem 2.2, we shall show that the addition model (1.2) has no solution with a non-zero first component when the coagulation rates satisfy (2.6)-(2.7). We first recall the definition of a solution to (1.2).

**Definition 2.4** *[5] Let $T \in (0,+\infty]$. A solution $c=(c_i)_{i\geq 1}$ to the addition model (1.2) on [0,T) is a function $c : [0,T) \to X$ such that*

*(i) $c_i(t) \geq 0$ for all $t \in [0,T)$ and $i \geq 1$,*

*(ii) $c_i \in \mathcal{C}([0,T))$ for each $i \geq 1$ and $\sup_{t\in[0,T)} \|c(t)\| < \infty$,*

*(iii) $\sum_{i=1}^{\infty} a_i c_i \in L^1(0,t)$ for each $t \in (0,T)$,*

*(iv) and for each $t \in [0,T)$*

$$
\begin{aligned}
c_1(t) &= c_1(0) - \int_0^t \left( a_1 c_1(s) + \sum_{i=1}^{\infty} a_i c_i(s) \right) c_1(s)\ ds, \\
c_i(t) &= c_i(0) + \int_0^t \left( a_{i-1} c_{i-1}(s) - a_i c_i(s) \right) c_1(s)\ ds, \quad i \geq 2.
\end{aligned}
$$

Our final result extends [5, Theorem 2.7] for coagulation rates satisfying (2.6)-(2.7) and reads as follows.

**Proposition 2.5** *Assume that the coagulation rates $(a_i)_{i\geq 1}$ fulfil (2.6)-(2.7) and let $c$ be a solution to (1.2) on $[0,T)$ (in the sense of Definition 2.4) for some $T>0$. Then there is a sequence $(r_i)_{i\geq 1}$ in $X^+$ such that $r_1 = 0$ and*

$$
c_1 \equiv 0 \quad \textit{and} \quad c_i \equiv r_i \quad \textit{for} \quad i \geq 2.
$$

The proof of Proposition 2.5 follows the lines of van Dongen [8] and Carr and da Costa [9]. Let us mention at this point that the (local) existence of a solution to (1.2)-(1.3) for the monodisperse initial datum $c_1^0 = 1$ and $c_i^0 = 0$, $i \geq 2$ seems to be still open for the coagulation rates $a_i = i\ (\ln(1+i))^{\alpha}$ with $\alpha \in (0,1]$.

# 3 Proofs of Theorems 2.1 & 2.2

A straightforward computation first yields the following result.

**Lemma 3.1** *Let $N \geq 3$ and $(g_i)_{1\leq i\leq N}$ be $N$ nonnegative real numbers. For $t \in [0,+\infty)$ and $\tau \in [0,t]$ there holds*

$$
\begin{aligned}
\sum_{i=1}^{N} g_i \left( c_i^N(t) - c_i^N(\tau) \right) &= \int_\tau^t \sum_{i=1}^{N-1} (g_{i+1} - g_i - g_1) a_i c_1^N(s) c_i^N(s)\ ds \\
&+ \delta \left( \frac{g_N}{N} - g_1 \right) \int_\tau^t a_N c_1^N(s) c_N^N(s)\ ds, \qquad (3.1) \\
\sum_{i=1}^{N} i c_i^N(t) &= \sum_{i=1}^{N} i c_i^0. \qquad (3.2)
\end{aligned}
$$

We fix $T \in (0,+\infty)$.

**Lemma 3.2** *The sequence $\left(c_1^N\right)_{N\geq 3}$ is a sequence of non-increasing functions which is bounded in $L^\infty(0,T)\cap W^{1,1}(0,T)$. For $i\geq 2$, the sequence $\left(c_i^N\right)_{N\geq 3}$ is bounded in $W^{1,\infty}(0,T)$.*

*Proof.* Let $i\geq 1$. Since $\left(c_i^N\right)_{N\geq 3}$ is a sequence of non-negative functions, the boundedness of $\left(c_i^N\right)_{N\geq 3}$ in $L^\infty(0,T)$ follows at once from (3.2) and either the first part of (2.3) or (2.8).
If $i\geq 2$, we infer from (1.4) and (3.2) that

$$\left|\frac{dc_i^N}{dt}\right| \leq (a_{i-1}+a_i)\|c^0\|^2,$$

hence the boundedness of $\left(c_i^N\right)_{N\geq 3}$ in $W^{1,\infty}(0,T)$.
Finally by (1.4) $c_1^N$ is a non-increasing function on $[0,T]$ and

$$\int_0^T \left|\frac{dc_1^N}{dt}(s)\right|\ ds \leq c_1^0.$$

The proof of the lemma is thus complete. □

**Lemma 3.3** *There is a function $c=(c_i)_{i\geq 1}\ :\ [0,T]\to X^+$ and a subsequence of $\left(c^N\right)_{N\geq 3}$ (not relabeled) such that*

$$c_1^N(t) \longrightarrow c_1(t) \quad \textit{for each} \quad t\in[0,T], \tag{3.3}$$
$$c_i^N \longrightarrow c_i \quad \textit{in} \quad \mathcal{C}([0,T]) \quad \textit{for} \quad i\geq 2. \tag{3.4}$$

*Moreover, $c_1$ is a non-increasing function on $[0,T]$,*

$$\sum_{i=1}^\infty a_i c_1 c_i \in L^1(0,T), \tag{3.5}$$

*and for $i\geq 2$ and $t\in[0,T]$ there holds*

$$c_i(t) = c_i^0 + \int_0^t \left(a_{i-1}c_{i-1}(s) - a_i c_i(s)\right) c_1(s)\ ds. \tag{3.6}$$

*Finally we have*

$$\|c(t)\| \leq \|c^0\| \quad \textit{for} \quad t\in[0,T]. \tag{3.7}$$

*Proof.* Since $\left(c_1^N\right)_{N\geq 3}$ is bounded in $L^\infty(0,T)\cap W^{1,1}(0,T)$ the everywhere convergence of a subsequence of $\left(c_1^N\right)_{N\geq 3}$ follows from the Helly selection principle [10, p. 372–374] and $c_1$ is a non-increasing function as a limit of non-increasing functions. Owing to

Lemma 3.2 we may apply the Arzela-Ascoli theorem to the sequence $\left(c_i^N\right)_{N\geq 3}$ for $i \geq 2$ and obtain (3.4) by a diagonal procedure. Letting then $N \to +\infty$ in (3.2) yields (3.7).

We next integrate the first equation of (1.4) over $(0, T)$ ; this gives

$$\int_0^T \sum_{i=1}^{N-1} a_i c_1^N(s) c_i^N(s)\ ds \leq c_1^0.$$

Fix $M \geq 2$. For $N \geq M+1$ the above inequality entails

$$\int_0^T \sum_{i=1}^{M} a_i c_1^N(s) c_i^N(s)\ ds \leq c_1^0.$$

We may then let $N \to +\infty$ in the above inequality and use (3.3), (3.4) and the Fatou lemma to conclude that

$$\int_0^T \sum_{i=1}^{M} a_i c_1(s) c_i(s)\ ds \leq c_1^0.$$

As $M$ is arbitrary, we have proved (3.5). Finally (3.6) follows from (3.3), (3.4), (3.2) and the Lebesgue dominated convergence theorem by letting $N \to +\infty$ in (1.4). $\square$

**Lemma 3.4** *Let $m \geq 1$ and $t \in [0, T]$. The sequence $c = (c_i)_{i\geq 1}$ defined in Lemma 3.3 satisfies*

$$\sum_{i=m+1}^{\infty} i c_i(t) = \sum_{i=m+1}^{\infty} i c_i^0 + \int_0^t \left( \sum_{i=m+1}^{\infty} a_i c_1(s) c_i(s) + (m+1) a_m c_1(s) c_m(s) \right)\ ds. \quad (3.8)$$

*Proof.* As $c = (c_i)_{i\geq 1}$ satisfies (3.6) which is nothing but the addition model without the first equation, the proof of Lemma 3.4 is similar to that of [5, Theorem 2.5] to which we refer. $\square$

*Proof of Theorem 2.1* Let $t \in [0, T]$ and $m \geq 1$. By (3.8) $s \mapsto \sum_{i=m+1}^{\infty} i c_i(s)$ is a non-decreasing function on $[0, T]$ while $c_1$ is a non-increasing function by Lemma 3.3. Therefore

$$\gamma_m t c_1(t) \sum_{i=m+1}^{\infty} i c_i^0 \leq \gamma_m \int_0^t \sum_{i=m+1}^{\infty} i c_1(s) c_i(s)\ ds \leq \left| \sum_{i=1}^{\infty} a_i c_1 c_i \right|_{L^1(0,T)}. \quad (3.9)$$

By (3.5) the right hand side of (3.9) is finite. We then let $m \to +\infty$ in the left hand side of (3.9) and infer from (2.3) that

$$t c_1(t) = 0 \quad \text{for each} \quad t \in [0, T].$$

Thus, $c_1(t) = 0$ for each $t \in (0, T]$ which together with (3.6) entails that $c_i(t) = c_i^0$ for $t \in [0, T]$ and $i \geq 2$.

By Lemma 3.2 the sequence $\left(c_1^N\right)_{N\geq 3}$ is relatively compact in $L^1(0,T)$ while the sequence $\left(c_i^N\right)_{N\geq 3}$ is relatively compact in $\mathcal{C}([0,T])$ for each $i\geq 2$. Since $\left(c^N\right)_{N\geq 3}$ has one and only one cluster point $(0,c_2^0,\ldots,c_i^0,\ldots)$ as $N\to+\infty$ we conclude that the whole sequence $\left(c_1^N\right)_{N\geq 3}$ converges to zero in $L^1(0,T)$ and the whole sequence $\left(c_i^N\right)_{N\geq 3}$ converges to $c_i^0$ in $\mathcal{C}([0,T])$ for $i\geq 2$. As $T$ was arbitrary, the proofs of Theorem 2.1 and Remark 2.3 are complete. □

*Proof of Theorem 2.2* Without loss of generality we assume that $\delta=1$.
Step 1. we first claim that for a.e. $t\in(0,T)$ there holds

$$c_1(t)\left(\|c(t)\|-\|c^0\|\right)=0. \tag{3.10}$$

Indeed, on the one hand it follows from (3.2) and (3.3) that

$$\lim_{N\to+\infty} c_1^N(t)\sum_{i=1}^N ic_i^N(t)=\|c^0\|c_1(t) \quad \text{for each} \quad t\in[0,T]. \tag{3.11}$$

On the other hand integration of the first equation of (1.4) over $(0,T)$ entails

$$\int_0^T \sum_{i=1}^N a_i c_1^N(s)c_i^N(s)\ ds\leq c_1^0 \tag{3.12}$$

since $\delta>0$. We fix $M\geq 2$. For $N\geq M+1$ we infer from (3.5) and (3.12) that

$$\begin{aligned}
&\int_0^T \left|\sum_{i=1}^N ic_1^N(s)c_i^N(s)-c_1(s)\|c(s)\|\right|\ ds\leq \sum_{i=1}^M i\left|c_1^Nc_i^N-c_1c_i\right|_{L^1(0,T)}\\
&+\int_0^T\sum_{i=M+1}^N ic_1^N(s)c_i^N(s)\ ds+\int_0^T\sum_{i=M+1}^\infty ic_1(s)c_i(s)\ ds\\
&\leq \sum_{i=1}^M i\left|c_1^Nc_i^N-c_1c_i\right|_{L^1(0,T)}\\
&+\frac{1}{\gamma_M}\left(\left|\sum_{i=M+1}^N a_ic_1^Nc_i^N\right|_{L^1(0,T)}+\left|\sum_{i=M+1}^\infty a_ic_1c_i\right|_{L^1(0,T)}\right)\\
&\leq \sum_{i=1}^M i\left|c_1^Nc_i^N-c_1c_i\right|_{L^1(0,T)}+\frac{1}{\gamma_M}\left(c_1^0+\left|\sum_{i=1}^\infty a_ic_1c_i\right|_{L^1(0,T)}\right).
\end{aligned}$$

Owing to (3.3), (3.4), (3.2) and the Lebesgue dominated convergence theorem we may let $N\to+\infty$ in the above inequality and obtain

$$\limsup_{N\to+\infty}\int_0^T\left|\sum_{i=1}^N ic_1^N(s)c_i^N(s)-c_1(s)\|c(s)\|\right|\ ds\leq\frac{1}{\gamma_M}\left(c_1^0+\left|\sum_{i=1}^\infty a_ic_1c_i\right|_{L^1(0,T)}\right).$$

As $M$ is arbitrary it follows from (2.7) that

$$\sum_{i=1}^{N} i c_1^N c_i^N \longrightarrow c_1 \|c\| \quad \text{in} \quad L^1(0,T). \tag{3.13}$$

Combining (3.11) and (3.13) then yields the claim (3.10).
Step 2. In order to prove that $c_1$ vanishes identically on $(0,T]$ we argue by contradiction. Assume thus that

$$c_1(t_0) > 0 \quad \text{for some} \quad t_0 \in (0,T]. \tag{3.14}$$

As $c_1$ is a non-increasing function on $[0,T]$ we have in fact

$$c_1(t) \geq c_1(t_0) > 0 \quad \text{for each} \quad t \in [0,t_0]. \tag{3.15}$$

We next introduce a function $\Gamma = (\Gamma_i)_{i \geq 1} \ : [0,t_0] \to X^+$ defined by

$$\Gamma_1(t) \;=\; c_1^0 - \int_0^t \left( a_1 c_1(s) + \sum_{i=1}^{\infty} a_i c_i(s) \right) c_1(s)\, ds \quad \text{for} \quad t \in [0,t_0], \tag{3.16}$$

$$\Gamma_i(t) \;=\; c_i(t) \quad \text{for} \quad t \in [0,t_0] \quad \text{and} \quad i \geq 2. \tag{3.17}$$

By (3.16), (3.4), (3.5) and (3.7) we have

$$\Gamma_i \in \mathcal{C}([0,t_0]) \quad \text{for} \quad i \geq 1 \quad \text{and} \quad \sup_{t \in [0,t_0]} \|\Gamma(t)\| \leq \|c^0\|. \tag{3.18}$$

We then infer from (3.10), (3.15) and (3.8) that for almost every $t \in (0,t_0)$ there holds

$$c_1(t) = \|c^0\| - \sum_{i=2}^{\infty} i c_i(t) = c_1^0 - \int_0^t \sum_{i=2}^{\infty} a_i c_1(s) c_i(s)\, ds - 2 \int_0^t a_1 c_1(s)^2\, ds,$$

hence

$$c_1(t) = \Gamma_1(t) \quad \text{for a.e.} \quad t \in (0,t_0). \tag{3.19}$$

Owing to (3.19) and (3.17), (3.16) and (3.6) now read

$$\Gamma_1(t) \;=\; c_1^0 - \int_0^t \left( a_1 \Gamma_1(s) + \sum_{i=1}^{\infty} a_i \Gamma_i(s) \right) \Gamma_1(s)\, ds \quad \text{for} \quad t \in [0,t_0],$$

$$\Gamma_i(t) \;=\; c_i^0 + \int_0^t \left( a_{i-1} \Gamma_{i-1}(s) - a_i \Gamma_i(s) \right) \Gamma_1(s)\, ds \quad \text{for} \quad t \in [0,t_0] \quad \text{and} \quad i \geq 2,$$

while (3.5), (3.19) and (3.15) yield $\sum_{i=1}^{\infty} a_i \Gamma_i \in L^1(0,t_0)$. Recalling (3.18) we have thus shown that $\Gamma$ is a solution to the addition model (1.2) on $[0,t_0)$ in the sense of Definition 2.4. As the coagulation rates satisfy (2.6)-(2.7) we infer from Proposition 2.5 that $\Gamma_1 \equiv 0$, hence a contradiction since $\Gamma_1(0) = c_1^0 \neq 0$ by (2.8).

Consequently, $c_1(t) = 0$ for each $t \in (0,T]$. We now proceed as in the proof of Theorem 2.1 to conclude. $\square$

## 4 Non-existence of solutions

This section is devoted to the proof of Proposition 2.5. As already mentioned, the approach we shall use follows the lines of van Dongen [8] and Carr and da Costa [9].

From now on we assume that the coagulation rates $(a_i)_{i\geq 1}$ fulfil (2.6)-(2.7) and that $c = (c_i)_{i\geq 1}$ is a solution to (1.2) on $[0,T)$ in the sense of Definition 2.4 for some $T \in (0,+\infty)$. If $c_1(0) = 0$ then $c_1 \equiv 0$ and there is nothing to prove. We therefore assume that

$$c_1(0) \neq 0. \tag{4.1}$$

A similar proof to that of [5, Theorem 4.6] yields that

$$c_i(t) > 0 \quad \text{for} \quad t \in (0,T) \quad \text{and} \quad i \geq 1, \tag{4.2}$$

while [5, Corollary 2.6] entails

$$\|c(t)\| = \|c(0)\| \quad \text{for} \quad t \in [0,T). \tag{4.3}$$

Owing to (4.1), (4.2) and the continuity of $c_1$ on $[0,T/2]$ there is a positive real number $\mu$ such that

$$c_1(t) \geq \mu > 0 \quad \text{for} \quad t \in [0,T/2]. \tag{4.4}$$

**Lemma 4.1** *For each integer $p \geq 1$ we have*

$$\sup_{t\in[0,T/4]} \sum_{i=1}^{\infty} i^p a_i c_i(t) < \infty. \tag{4.5}$$

*Proof.* By [5, Theorem 2.5] and (4.4) we have for $m \geq 2$ and $0 \leq t_1 \leq t_2 \leq T/2$

$$\begin{aligned} \sum_{i=m}^{\infty} i c_i(t_2) &= \sum_{i=m}^{\infty} i c_i(t_1) + \int_{t_1}^{t_2} \sum_{i=m}^{\infty} a_i c_1(s) c_i(s)\, ds \\ &\quad + m \int_{t_1}^{t_2} a_{m-1} c_1(s) c_{m-1}(s)\, ds \\ &\geq \sum_{i=m}^{\infty} i c_i(t_1) + \gamma_m \mu \int_{t_1}^{t_2} \sum_{i=m}^{\infty} i c_i(s)\, ds \end{aligned}$$

where

$$\gamma_m = \min_{i\geq m} \frac{a_i}{i}.$$

The Gronwall lemma and (4.3) then yield

$$\sum_{i=m}^{\infty} i c_i(t) \leq \|c(0)\| \exp\left(\gamma_m \mu (t - T/2)\right), \quad t \in [0,T/2].$$

Consequently, for $t \in [0,T/4]$ and $m \geq 2$ we have

$$mc_m(t) \leq \sum_{i=m}^{\infty} ic_i(t) \leq \|c(0)\| \, \exp\left(-\gamma_m \mu T/4\right). \tag{4.6}$$

Now let $p \geq 1$ be an integer and $t \in [0,T/4]$. We infer from (4.6) that

$$\sum_{i=2}^{\infty} i^p a_i c_i(t) \leq \|c(0)\| \, \sum_{i=2}^{\infty} \, \exp\left((p-1)\ln i + \ln(1+a_i) - \gamma_i \mu T/4\right), \tag{4.7}$$

and the right hand side of (4.7) is finite by (2.6). Indeed, it follows from (2.6) that for $i$ large enough

$$\frac{\gamma_i}{\ln i} \geq \frac{\gamma_i}{\ln(1+a_i)} \geq \min_{k\geq i} \frac{a_k}{k\ln(1+a_k)} \longrightarrow +\infty,$$

and the series on the right hand side of (4.7) is convergent. □

**Remark 4.2** *The proof of Lemma 4.1 does not make use of (2.7).*

**Lemma 4.3** *For each integer $p \geq 2$ and $t \in [0,T/4]$ we have*

$$\sum_{i=1}^{\infty} i^p c_i(t) - \sum_{i=1}^{\infty} i^p c_i(0) = \int_0^t \sum_{i=1}^{\infty} \left((i+1)^p - i^p - 1\right) a_i c_1(s) c_i(s) \; ds. \tag{4.8}$$

*Proof.* Let $p \geq 2$. Owing to Lemma 4.1 we have

$$\int_0^t \sum_{i=1}^{\infty} \left((i+1)^p - i^p\right) a_i c_i(s) \; ds < \infty$$

and

$$\sum_{i=1}^{\infty} i^p c_i(t) \, , \, \sum_{i=1}^{\infty} i^p c_i(0) < \infty.$$

We then infer from [5, Theorem 2.5] that

$$\sum_{i=2}^{\infty} i^p c_i(t) - \sum_{i=2}^{\infty} i^p c_i(0) = \int_0^t \sum_{i=2}^{\infty} \left((i+1)^p - i^p\right) a_i c_1(s) c_i(s) \; ds + 2^p \int_0^t a_1 c_1(s)^2 \; ds.$$

Since

$$c_1(t) = c_1(0) - 2\int_0^t a_1 c_1(s)^2 \; ds - \int_0^t \sum_{i=2}^{\infty} a_i c_1(s) c_i(s) \; ds$$

by Definition 2.4 we obtain (4.8) after summing the above two identities. □

*Proof of Proposition 2.5* Let $p \geq 2$ be an integer and put (recall (4.3))

$$M_p(t) = \frac{1}{\|c(t)\|} \sum_{i=1}^{\infty} i^p c_i(t) = \frac{1}{\|c(0)\|} \sum_{i=1}^{\infty} i^p c_i(t), \quad t \in [0, T/4].$$

Let $t \in [0, T/4]$ and $s \in [0, t)$. Since $(i+1)^p - i^p - 1 \geq p i^{p-1}$ for $i \geq 1$ it follows from (4.8), (4.4) and (2.7) that

$$M_p(t) \geq M_p(s) + Kp\mu \int_s^t \sum_{i=1}^{\infty} i^{p-1} \left(\ln(1+i)\right)^{\alpha} \frac{i c_i(\sigma)}{\|c(0)\|}\, d\sigma. \tag{4.9}$$

As $1/(p-1) \in (0,1]$ we have for $i \geq 1$

$$\begin{aligned} i^{p-1} \left(\ln(1+i)\right)^{\alpha} &\geq \frac{1+i^{p-1}}{2} \left(\ln\left(\left(1+i^{p-1}\right)^{1/(p-1)}\right)\right)^{\alpha} \\ &\geq \frac{1}{2(p-1)^{\alpha}} \left(1+i^{p-1}\right) \left(\ln\left(1+i^{p-1}\right)\right)^{\alpha}. \end{aligned}$$

Recalling (4.3) it follows from (4.9) and the above inequality that

$$M_p(t) \geq M_p(s) + \int_s^t \sum_{i=1}^{\infty} \varphi_p\left(i^{p-1}\right) \frac{i c_i(\sigma)}{\|c(\sigma)\|}\, d\sigma, \tag{4.10}$$

where

$$\varphi_p(x) = \frac{Kp\mu}{2(p-1)^{\alpha}} (1+x) \left(\ln(1+x)\right)^{\alpha}, \quad x \in [0, +\infty).$$

As $\varphi_p$ is a convex function the Jensen inequality and (4.10) entail

$$M_p(t) \geq M_p(s) + \int_s^t \varphi_p\left(M_p(\sigma)\right)\, d\sigma, \quad 0 \leq s < t \leq T/4. \tag{4.11}$$

Combining (4.11) and the following lemma ensure that $T$ cannot exceed some upper bound depending on $p$.

**Lemma 4.4** *Let $\vartheta : (0, +\infty) \to (0, +\infty)$ be a positive and non-decreasing continuous function such that*

$$\int_1^{\infty} \frac{dx}{\vartheta(x)} < \infty.$$

*We next consider a positive and non-decreasing continuous function $f$ defined on the interval $[0, \tau]$ for some $\tau > 0$ and satisfying*

$$f(t) \geq f(0) + \int_0^t \vartheta(f(s))\, ds \quad \text{for} \quad t \in [0, \tau].$$

*Then necessarily*

$$\tau \leq \int_{f(0)}^{\infty} \frac{dx}{\vartheta(x)}.$$

By Definition 2.4 (ii) and Lemma 4.1 $M_p(. + T/8) \in \mathcal{C}([0, T/8])$ and Lemma 4.4 and (4.11) entail

$$T/8 \leq \int_{M_p(T/8)}^{\infty} \frac{dx}{\varphi_p(x)} \ ,$$

hence

$$T \leq \frac{16}{(\alpha-1)K\mu} \left( \ln \left( (1 + M_p(T/8))^{1/p} \right) \right)^{1-\alpha} . \tag{4.12}$$

We then infer from (4.2) and [9, Lemma 2.2] that

$$\lim_{p \to +\infty} (1 + M_p(T/8))^{1/p} = +\infty.$$

Since (4.12) is valid for each integer $p \geq 2$ we may let $p \to +\infty$ in (4.12) and conclude that $T = 0$, hence a contradiction. Consequently we have necessarily $c_1(0) = 0$ and thus $c_1 \equiv 0$ on $[0, T]$. The proof of Proposition 2.5 is then complete. $\square$


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