# Tricomi's composition formula and the analysis of multiwavelet approximation methods for boundary integral equations

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**Abstract**

The present paper is mainly concerned with the convergence analysis of Galerkin–Petrov methods for the numerical solution of periodic pseudodifferential equations using wavelets and multiwavelets as trial functions and test functionals. Section 2 gives an overview on the symbol calculus of multidimensional singular integrals using Tricomi's composition formula. In Section 3 we formulate necessary and sufficient stability conditions in terms of the so–called numerical symbols and demonstrate applications to the Dirchlet problem for the Laplace equation.

# 1 Introduction

The first significant results on *multidimensional singular integral equations* go back to F. Tricomi (1926-1928). A singular integral in the Euclidean space $\mathbb{R}^n$ has the form

$$Au(x) := a(x)u(x) + \lim_{\epsilon \longrightarrow 0} \int\limits_{|x-y|>\epsilon} k(x, x-y)u(y)dy \;, \tag{0}$$

where $k$ satisfies

$$k(x,tz) = t^{-n}k(x,z) \;\text{ for } t > 0 \;, \int\limits_{|z|=1} k(x,z)dz = 0$$

and, in addition, is subject to certain integrability or smoothness conditions. Tricomi considered the case $n = 2$ and assumed that the coefficient $a$ and the kernel $k(x, z)$ are independent of $x$. He established a formula for the composition of two singular integrals, by means of which he then reduced the solution of the equation $Au = f$ to the solution of a certain one–dimensional singular equation. The next important work on multidimensional singular integrals is due to G. Giraud (1934), who studied singular integrals on closed Lyapunov manifolds of arbitrary dimension. Assuming that the kernel satisfies some (quite strong) conditions, Giraud extended the Plemelj–Privalov theorem on the boundedness of $A$ in the Hölder space $C^\alpha(\Gamma)(0 < \alpha < 1)$ to higher dimensions, and he also constructed a regularizer for $A$. In the work of both Tricomi and Giraud, there did not figure the so–called *ellipticity condition*, which is a necessary and sufficient condition for the singular operator to admit regularization or, which is equivalent, to be a Fredholm operator. This condition first appeared in the papers of S. G. Mikhlin (1936) who studied general singular integrals of the form (0) for $n = 2$ essentially using and generalizing Tricomi's results.

Mikhlin was also the first to introduce the concept of the *symbol* of the operator $A$ (in the case $n = 2$) which plays a crucial role in the advanced theory of singular integral and pseudodifferential operators. The symbol $\sigma_A$ of $A$ was constructed by means of the expansion of the kernel $k$ into the Fourier series with respect to spherical functions (resp. trigonometric polynomials), and in terms of the symbol $\sigma_A$ the celebrated ellipticity condition takes the simple form $\inf |\sigma_A| > 0$. This concept of the symbol was then extended by G. Giraud (1936) to the case $n \geq 3$. The symbol effects an isomorphism between an algebra of functions and an algebra of operators generated by the singular operators of the form (0) and all linear compact operators (in the corresponding function space). Moreover, the symbol allows to formulate conditions for the boundedness of the operators (0) in Lebesgue spaces.

It was A. P. Calderón and A. Zygmund (1952, 1956) who first applied Fourier transform techniques to singular integrals. Calderón and Zygmund as well as Mikhlin (1956) set up the extremely important formulas

$$\sigma_A(x,\xi) = a(x) + \hat{k}(x,\xi) \ , \quad A = F^{-1}_{\xi \longrightarrow x} \sigma_A(x,\xi) F_{x \longrightarrow \xi} \ ,$$

where $F$ refers to the Fourier transform and $\hat{k}(x,\xi)$ denotes the Fourier transform of the kernel $k(x,z)$ with respect to the variable $z$. These formulas became the foundation of further numerous investigations devoted to multidimensional singular integrals and integral equations. Moreover, the systematic use of the apparatus of the Fourier transform promoted a further synthesis of multidimensional singular integral equations and partial differential equations, which eventually culminated in the theory of *pseudodifferential operators* (see the pioneering papers of L. Hörmander (1965), J. J. Kohn and L. Nirenberg (1965), R. T. Seeley (1965)).

In the modern numerical analysis of pseudodifferential equations, in particular, in the convergence analysis of a wide class of approximation methods for the numerical solution of boundary integral equations, a crucial role is played by the notion of the *numerical symbol*. It turns out that to various approximation methods (e.g. Galerkin method, collocation method, quadrature methods using global polynomials or splines) for singular integral equations or Mellin convolution equations one can associate a certain operator valued function (depending on the approximation method under consideration) which owns many features of the symbol of singular integral operators. The (loosely) summarised result is that the approximation method for the operator $A$ is stable if and only if $A$ itself and all values of the aforementioned operator valued functions are invertible (see the monographs [26], [11]).

Recently, in [7], [24] - [25] the authors succeeded in developing a stability analysis of rather general Galerkin–Petrov methods for periodic pseudodifferential equations using wavelets and multiwavelets as trial functions and test functionals, e.g. collocation methods using splines with defect. Roughly speaking, it turns out that these methods are stable if and only if the (appropriately defined) numerical symbol is of elliptic type. It should be mentioned that when using wavelet type bases and applying compression techniques to the resulting stiffness matrices then the order of the overall computational work which is needed to realize a certain accuracy is of the form $O(N(\log N)^b)$ where $N$ is the number of unknowns and $b \geq 1$ is some real number (see [8] for more details).

The present paper is organized as follows. Section 1 gives an overview on the symbol calculus of multidimensional singular integral operators. As a new result we show that Tricomi's composition formula can be used in order to derive, in a simple manner, the symbol representation of a two–dimensional singular operator in the case when the *characteristic* $k(\frac{x-y}{|x-y|})$ is a trigonometric polynomial. The latter observation is the key property in the symbol calculus. Section 2 presents a rough survey of recent results of the convergence analysis of Galerkin–Petrov methods for periodic pseudodifferential equations using wavelets and multiwavelets. In particular, we formulate necessary and sufficient stability conditions in terms of the numerical symbols and demonstrate some applications to the Dirichlet problem for the Laplace equation.

# 2 Tricomi's formula and the symbol calculus of singular integrals

## 2.1 Tricomi's formula

The first important work on multidimensional singular integral equations is due to F. Tricomi. In his papers [32], [33] he investigated double singular integrals of the form

$$Au(x) := \int\limits_{\mathbb{R}^2} \frac{f(\theta)}{r^2} u(y)dy, \tag{1}$$

where $r$ and $\theta$ are the polar coordinates of the point $y \in \mathbb{R}^2$ with respect to the point $x \in \mathbb{R}^2$, i.e.,

$$r = |y-x| \ , \quad \theta = (y-x)/r.$$

Tricomi called the function $f$ the *characteristic* of the integral (1) which has to be interpreted in the Cauchy principal value sense, i.e.,

$$Au(x) = \lim_{\epsilon \longrightarrow 0} \int\limits_{r>\epsilon} \frac{f(\theta)}{r^2} u(y)dy.$$

Tricomi established the following necessary and sufficient condition for the *existence* of the integral (1) (at least in the case when the density function $u$ satisfies a Hölder condition):

$$\int\limits_{-\pi}^{\pi} f(\theta) = 0. \tag{2}$$

One of the most important results of Tricomi [32], [33] is his formula for the *composition* of two double singular integrals, in other words, the rule of multiplication of singular operators of the form (1). Let

$$A_j u = \int\limits_{\mathbb{R}^2} \frac{f_j(\theta)}{r^2} u(y)dy \ , \quad j = 1, 2.$$

Then

$$A_1 A_2 u = \alpha u(x) + \int\limits_{\mathbb{R}^2} \frac{f(\theta)}{r^2} u(y) dy, \tag{3}$$

where

$$\alpha = 2\pi \int\limits_{-\pi}^{\pi} \tilde{f}_1(\theta) \tilde{f}_2(\theta + \pi) d\theta , \tag{4}$$

$$f(\theta) = \frac{d}{d\theta} \int\limits_{-\pi}^{\pi} \left\{ \tilde{f}_1(\psi)\tilde{f}_2(\theta) + \tilde{f}_1(\theta)\tilde{f}_2(\psi) \; - \tilde{f}_1(\psi)\tilde{f}_2(\psi + \pi) \right\} \cot(\psi - \theta) d\psi. \tag{5}$$

In these formulas $\tilde{f}_j$ $(j = 1, 2)$ denotes the primitive function of $f_j$ the integral of which is vanishing:

$$\frac{d}{d\theta} \tilde{f}_j(\theta) = f_j(\theta), \quad \int\limits_{-\pi}^{\pi} \tilde{f}_j(\theta) d\theta = 0.$$

To prove (3) Tricomi derived the formula for the *differentiation* of double integrals with weak singularity

$$\frac{\partial}{\partial x_k} \int\limits_{\mathbb{R}^2} \frac{\varphi(\theta)}{r} u(y) dy = \int\limits_{\mathbb{R}^2} \frac{\partial}{\partial x_k} \left[ \frac{\varphi(\theta)}{r} \right] u(y) dy + u(x) \int\limits_{-\pi}^{\pi} \varphi(\theta) \frac{\partial r}{\partial x_k} d\theta.$$

The latter formula is also of independent interest.

As a simple consequence of formula (3), one obtains a representation of the singular operator

$$A_n u(x) := \int\limits_{\mathbb{R}^2} \frac{e^{in\theta}}{r^2} u(y) dy \quad (n \in \mathbb{Z}, n \neq 0)$$

in the form $c_n A_1^n$ with some constants $c_n$ depending on $n$. Indeed, setting

$$f_1(\theta) = e^{i\theta} \; , \quad f_2(\theta) = e^{in\theta} \; ,$$

we get

$$\tilde{f}_1(\theta) = \frac{1}{i} e^{i\theta} \; , \quad \tilde{f}_2(\theta) = \frac{1}{in} e^{in\theta} \; ,$$

and by (4)

$$\alpha = -\frac{(-1)^n}{n} \int\limits_{-\pi}^{\pi} e^{i(n+1)\theta} d\theta = 0.$$

Using (5) and the well–known formula[1]

$$\frac{1}{2\pi} \int\limits_0^{2\pi} e^{im\psi} \cot \frac{\psi - \theta}{2} d\psi = \begin{cases} e^{im\theta} \; , & m > 0 \\ -e^{im\theta} \; , & m < 0 \; , \end{cases}$$

[1]This follows from Cauchy's integral and residue theorems.

we obtain

$$f(\theta) = \frac{d}{d\theta}\frac{2\pi}{in}e^{i(n+1)\theta} = 2\pi\frac{n+1}{n}e^{i(n+1)\theta} \ .$$

Hence, we have

$$A_1 A_n = 2\pi\frac{n+1}{n}A_{n+1} \quad \text{for } n > 0. \tag{6}$$

Similarly, we find

$$A_1 A_n = -2\pi\frac{n+1}{n}A_{n+1} \quad \text{for } n < 0 \ . \tag{7}$$

Putting $n = 2, 3, \ldots$ in (6), we get

$$A_n = \frac{2\pi i^n}{n}h^n \ , \quad n > 0 \ , \tag{8}$$

where

$$hu = \frac{1}{2\pi i}A_1 u = \frac{1}{2\pi i}\int\limits_{\mathbb{R}^2}\frac{e^{i\theta}}{r^2}u(y)dy \ .$$

Further, formula (3) implies that

$$A_1 A_{-1} u = A_{-1} A_1 u = -(2\pi)^2 u,$$

i.e.,

$$h^{-1} = (2\pi i)^{-1} A_{-1} \ . \tag{9}$$

Putting $n = -2, -3, \ldots$ in (7) and using (9), we get

$$A_{-n} = \frac{2\pi i^n}{n}h^{-n} \ , \quad n < 0 \ . \tag{10}$$

Both formulae (8) and (10) can be combined into the one

$$A_n u = \int\limits_{\mathbb{R}^2}\frac{e^{in\theta}}{r^2}u(y)dy = \frac{2\pi i^{|n|}}{|n|}h^n u \quad \text{for all } n \in \mathbb{Z} \ , \quad n \neq 0 \ . \tag{11}$$

Formula (11) has been one of the cornerstones for the symbol calculus of general singular integral operators established by S. G. Mikhlin in [17].

**Remark 1.** Mikhlin [17] gave another proof of formulae (3) and (11) where he corrected a slip made by Tricomi [33] in developing formula (4). Mikhlin's proof is based on a generalization of Tricomi's approach. Notice, however, that Tricomi presented in [32] the true formula (4) without proof.

## 2.2 The symbol of a singular integral operator

On the basis of Tricomi's investigations [32], [33], Mikhlin[17] studied *general singular operators* of the form

$$Au = a_0(x)u(x) + \int\limits_{\mathbb{R}^2} \frac{f(x,\theta)}{r^2} u(y)dy + Tu \ , \tag{12}$$

where $u \in L_p(\mathbb{R}^2)$ for a certain $p$, $1 < p < \infty$, and $T$ is a linear compact operator in $L_p(\mathbb{R}^2)$. In this case the characteristic $f$ can be represented by the Fourier series

$$f(x,\theta) = \sum_{n=-\infty}^{\infty}{}' \, b_n(x)e^{in\theta} \ ;$$

the dash means the omission of the term for $n = 0$ (see (2)). Using formula (11), one can represent the singular integral of (12) in the form

$$\int\limits_{\mathbb{R}^2} \frac{f(x,\theta)}{r^2} u(y)dy = \sum_{n=-\infty}^{\infty}{}' \, a_n(x)h^n u \ ,$$

where

$$a_n(x) = \frac{2\pi i^{|n|}}{|n|} b_n(x) \quad (n = \pm 1, \pm 2, \ldots) \ .$$

Hence, the singular operator (12) can be represented in the form of a *series of powers of the operator h* (modulo a compact perturbation $T$):

$$Au = \sum_{n=-\infty}^{\infty} a_n(x)h^n u + Tu \ . \tag{13}$$

In his symbol calculus Mikhlin [17] assumed that $a_0$ and $f$ satisfy the following conditions of Hölder type:

$$\begin{array}{lll} \text{(A)} & |a_0(y) - a_0(x)| & \leq c_0 r^\lambda [(1+x^2)(1+y^2)]^{-\lambda/2} \\ \text{(B)} & |f(y,\theta) - f(x,\theta)| & \leq c_1 r^\mu [(1+x^2)(1+y^2)]^{-\mu/2} \end{array}$$

for some $\lambda, \mu > 0$.

He called the function

$$\sigma_A(x,\theta) = \sum_{n=-\infty}^{\infty} a_n(x)e^{in\theta} \ , \quad -\pi \leq \theta \leq \pi \ ,$$

the *symbol* of the operator $A$ represented in the form (13). Obviously, when the symbol is given the singular operator can be recovered to within a compact term.

The symbol calculus is essentially based on Mikhlin's important observation that the commutator $[h,a]u = hau - ahu$ is compact in $L_p(\mathbb{R}^2)$ for any function $a$ satisfying condition (A). Thus all operators $T_n u := h^n(au) - ah^n u \quad (n \in \mathbb{N})$ are compact.

Now, if

$$\tilde{A}u = \sum_{n=-\infty}^{\infty} \tilde{a}_n(x)h^n u + \tilde{T}u$$

is another singular operator of the form (13), then from the observation above it follows that

$$\sigma_{A+\tilde{A}} = \sigma_A + \sigma_{\tilde{A}} \; , \; \sigma_{A\tilde{A}} = \sigma_A \sigma_{\tilde{A}} \; , \tag{14}$$

since

$$A\tilde{A}u = \sum_{k=-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} a_{k-n}\tilde{a}_n \right) h^k u + T_0 u \; ,$$

where $T_0$ is compact.

In particular, when $|\sigma_A(x,\theta)| \geq \mathrm{const} > 0$ the singular operator $B$ with symbol $\sigma_B(x,\theta) = [\sigma_A(x,\theta)]^{-1}$, called a *regularizer* of $A$, has the properties

$$BA = I + T_1 \; , \quad AB = I + T_2 \; ,$$

where $T_1$ and $T_2$ are compact operators. Thus, $A$ is a Fredholm operator.

**Remark 2.** Using formula (3), Tricomi [33] has solved the regularization problem for the singular operator (12) in the case when $a_0$ and $f$ do not depend on $x$.

Generalizing the aforementioned methods, G. Giraud [9] introduced the symbol of a singular integral operator of the form

$$Au(x) = a_0(x)u(x) + \int_{\mathbb{R}^n} \frac{f(x,\theta)}{r^n} u(y)dy + Tu \quad (x \in \mathbb{R}^n) \tag{15}$$

according to the following rule. If the characteristic $f$ can be expanded in a series of $n$–dimensional spherical harmonics of order $k$

$$f(x,\theta) = \sum_{k=1}^{\infty} Y_{k,n}(x,\theta)$$

then the *symbol* of the operator (15) is defined by

$$\sigma_A(x,\theta) := a_0(x) + \sum_{k=1}^{\infty} \gamma_{k,n} Y_{k,n}(x,\theta) \tag{16}$$

where

$$\gamma_{k,n} = \frac{i^k \pi^{\frac{n}{2}} \Gamma(\frac{k}{2})}{\Gamma(\frac{n+k}{2})} .$$

**Remark 3.** Giraud investigated only the case when $a_0$ and $f$ are independent of $x$. He published formula (16) without proof of the rules (14) which was given afterwards by Mikhlin (see [18]) and the references there).

**Remark 4.** I. Gohberg [10] proved that the symbol (16) realizes an isomorphism between the quotient algebra $\mathcal{A}/\mathcal{K}$ and the algebra of functions of the form (16) which coincides with the Gelfand homomorphism of the commutative algebra $\mathcal{A}/\mathcal{K}$. Hereby $\mathcal{A}$ is the algebra of all operators of the form (15) defined on $L_2(\mathbb{R}^n)$ and $\mathcal{K}$ is the ideal of all linear compact operators on $L_2(\mathbb{R}^n)$. In particular, if (15) is a Fredholm operator then the symbol does not vanish. (For corresponding results in the case of $L_p$ spaces, $1 < p < \infty$, see e.g. [19] and the references there.)

A. P. Calderón and A. Zygmund [5, 6] were the first who used the Fourier transform $F$ in the investigation of multidimensional singular integral operators. From their results it follows that the singular operator of the form (15) (with $T = 0$) can be represented in the form

$$Au = F^{-1}_{\xi \longrightarrow x} \sigma_A(x, \xi) F_{x \longrightarrow \xi} u \, . \tag{17}$$

(For more details see also [18], [19].)

It is possible to consider operators of the form (17) where $\sigma_A(x, \xi)$ is not necessarily a positively homogeneous function of degree 0 with respect to $\xi$. Under sufficiently general assumptions on the function $\sigma_A$ this leads to the concept of *pseudodifferential operators* (see the pioneering papers [12], [15], [30] as well as monographs on this topic, e.g., [31],[14], [13]).

## 2.3 Periodic pseudodifferential operators

In the remainder of this paper, we will be concerned with periodic pseudodifferential equations and with Galerkin–Petrov methods for their numerical solution. For a 1–periodic function $u$, a periodic pseudodifferential operator (PPDO) is defined by formula (17) with $F$ replaced by the discrete Fourier transform.

More precisely, for given $r \in \mathbb{R}$ and $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ let $S^r(\mathbb{T}^n)$ (the so–called Hörmander class) denote the set of all functions $\sigma \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ satisfying

$$|\partial_x^\beta \Delta_\xi^\alpha \sigma(x, \xi)| \le C_{\alpha,\beta} (1 + |\xi|)^{r - |\alpha|}$$

for all $x \in \mathbb{T}^n, \xi \in \mathbb{Z}^n$ and for all multiindices $\alpha, \beta$, where $\partial_x$ means the operator of partial differentiation with respect to $x$ and $\Delta_\xi$ the partial forward difference operator with respect to $\xi$. The function $\sigma \in S^r(\mathbb{T}^n)$ is called a *global symbol* of order $r$ on the $n$–dimensional torus $\mathbb{T}^n$.

For $\sigma \in S^r(\mathbb{T}^n)$ we define the (global) PPDO $\sigma(x,D)$ by

$$\sigma(x,D)u(x) = \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i \langle \xi, x \rangle} \sigma(x,\xi) \tilde{u}(\xi)$$

with $\tilde{u}$ being the *discrete Fourier transform*

$$\tilde{u}(\xi) = \int_{\mathbb{T}^n} e^{-2\pi i \langle \xi, x \rangle} u(x) dx, \quad u \in C^\infty(\mathbb{T}^n) \; .$$

The symbol of a PPDO is uniquely defined up to a smooth function belonging to $S^{-\infty}(\mathbb{T}^n) = \cap_{r \in \mathbb{R}} S^r(\mathbb{T}^n)$.

In what follows we restrict ourselves to the subclass of $S^r(\mathbb{T}^n)$ consisting of all symbols $\sigma \in S^r(\mathbb{T}^n)$ which admit a decomposition $\sigma = \sigma_0 + \sigma_1$ (for $\xi \neq 0$), where $\sigma_1 \in S^{r_1}(\mathbb{T}^n)$ with some $r_1 < r$ and $\sigma_0 \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n \backslash \{0\})$. Here we will assume that $\sigma_0(x,0) = 1$ (without loss of generality) and that $\sigma_0$ is positively homogeneous of degree $r$, i.e.,

$$\sigma_0(x, \lambda\xi) = \lambda^r \sigma_0(x,\xi) \quad \text{for} \quad \lambda > 0, \xi \neq 0.$$

For $s \in \mathbb{R}$, let $H^s(\mathbb{T}^n)$ denote the *periodic Sobolev space* of order $s$ equipped with the norm

$$\|u\|_s := \left( \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\tilde{u}(\xi)|^2 \right)^{1/2} ,$$

where the function $\xi \mapsto \langle \xi \rangle, \xi \in \mathbb{Z}^n$, is defined by $\langle \xi \rangle = |\xi|$ if $\xi \neq 0$ and $\langle \xi \rangle = 1$ if $\xi = 0$. It is easy to see that the operator $\sigma(x,D)$ maps

$$\sigma(x,D) : H^s(\mathbb{T}^n) \longrightarrow H^{s-r}(\mathbb{T}^n), \quad s \in \mathbb{R} \, ,$$

boundedly. Moreover, this operator is Fredholm if and only if the principal symbol $\sigma_0$ is *elliptic*, i.e., if there exist $C, R > 0$ such that

$$|\xi|^{-r} |\sigma_0(x,\xi)| \geq C \quad \text{for} \quad |\xi| > R, x \in \mathbb{T}^n.$$

# 3 Galerkin–Petrov methods for periodic pseudodifferential equations and their symbols

## 3.1 The case of one scaling function

Our central objective now is to solve the pseudodifferential equation

$$Au = f \tag{18}$$

on $\mathbb{T}^n$ for $u \in H^s(\mathbb{T}^n)$. Here $f \in H^{s-r}(\mathbb{T}^n)$ is given and $A : H^s(\mathbb{T}^n) \longrightarrow H^{s-r}(\mathbb{T}^n)$ is assumed to be an invertible operator of the form $A = \sigma(x, D) + K$ with $\sigma \in S^r(\mathbb{T}^n)$ satisfying the conditions of Sect. 2.3 and $K$ being a smoothing integral operator given by $Ku(x) = \int_{\mathbb{T}^n} k(x,y)u(y)dy$ where $k \in C^\infty(\mathbb{T}^n \times \mathbb{T}^n)$. For solving equation (18), we consider generalized Galerkin–Petrov methods in the framework of *multiresolution* (or *multiscale analysis*) as it typically arises in connection with the construction of wavelets and multiwavelets.

In general, the main ingredient of such a multiresolution approximation is a so called *refinable function* (or *scaling function*) $\varphi \in L_2(\mathbb{R}^n)$. By this we mean that $\varphi$ satisfies a *refinement* (or *scaling*) *equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}^n} a_k \varphi(2x - k) , \quad x \in \mathbb{R}^n , \tag{19}$$

where the *mask* $\{a_k\}_{k \in \mathbb{Z}^n}$ is some fixed sequence in $\ell_1(\mathbb{Z}^n)$. For many purposes it suffices to assume that $\varphi$ belongs to the space (see [7])

$$\mathcal{L}_2 := \left\{ f \in L_2(\mathbb{R}^n) : \sum_{k \in \mathbb{Z}^n} |f(\cdot - k)| \in L_2([0,1]^n) \right\} .$$

In what follows we are primarily interested in refinable functions with the following properties:

- $\varphi$ is refinable, has compact support and belongs to $C^d(\mathbb{R}^n)$.
- The integer shifts of $\varphi$ form a Riesz basis in $L_2(\mathbb{R}^n)$.

We now turn to an appropriate *periodic setting* based on the above notions. Identifying one–periodic functions with functions on $\mathbb{T}^n$, the periodization operator

$$[f](x) := \sum_{k \in \mathbb{Z}^n} f(x + k)$$

maps $\mathcal{L}_2$ into $L_2(\mathbb{T}^n)$. For any function $\phi \in \mathcal{L}_2$ we define now

$$\Phi_k^j := 2^{jn/2}[\Phi(2^j \cdot -k)] , \quad k \in \mathbb{Z}^n . \tag{19a}$$

Specifically, for any refinable function $\varphi \in \mathcal{L}_2$ we define the *finite dimensional spaces*

$$V^j := \mathrm{span}\{\varphi_k^j : k \in \mathbb{Z}^{n,j}\} , \quad \mathbb{Z}^{n,j} := \mathbb{Z}^n / (2^j \mathbb{Z}^n) .$$

From (19) we conclude that

$$V^0 \subset V^1 \subset \ldots \subset V^j \subset V^{j+1} \subset \ldots \subset L_2(\mathbb{T}^n) . \tag{19b}$$

Moreover, one can show that, under the above assumption, the union over all $V^j$ $(j = 0, 1, \ldots)$ is dense in $L_2(\mathbb{T}^n)$.

We will consider a rather general class of numerical schemes for the solution of (18) based on a fixed compactly supported distribution $\eta \in H^{-s'}(\mathbb{R}^n)$, where $s' \geq 0$ satisfies $AV^j \subset H^{s'}(\mathbb{T}^n)$. Thus, defining for $g \in H^{s'}(\mathbb{R}^n)$

$$\eta_k^j(g) := 2^{-jn/2}\eta\Big(g\big(2^{-j}(\cdot + k)\big)\Big) , \tag{19c}$$

the space $X^j := \operatorname{span}\{\eta_k^j : k \in \mathbb{Z}^{n,j}\}$ is contained in $(AV^j)'$, the dual of $AV^j$. The corresponding *Galerkin–Petrov method* reads then as follows: Determine $u^j \in V^j$ from

$$\eta_k^j(Au^j) = \eta_k^j(f) , \quad k \in \mathbb{Z}^{n,j} . \tag{20}$$

Specifically, the choice $\eta = \delta(\cdot - \epsilon)$ with $0 \leq \epsilon < 1$, i.e., $\eta(g) := g(\epsilon)$ gives rise to the *collocation scheme* with collocation points $2^{-j}(\epsilon + k)$, while $\eta = \varphi$, i.e.,

$$\eta(g) := \int_{\mathbb{R}^n} g(x)\varphi(x)dx$$

corresponds to the standard *Galerkin scheme.*

One of the central problems of the convergence analysis of the Galerkin–Petrov methods (20) is to find conditions ensuring the stability of (20), i.e., (20) admits a unique solution and the approximation operators $A^j : V^j \longrightarrow \mathbb{C}^L, L = \dim V^j$, defined by (20) have uniformly bounded inverses with respect to appropriate norms. It turns out that the method (20) is stable if and only if the function $\alpha(x,\xi)$ $(x, \xi \in \mathbb{T}^n)$ defined by

$$\alpha(x,\xi) := \sum_{\ell \in \mathbb{Z}^n} \sigma_0(x, \xi + \ell)\hat{\varphi}(\xi + \ell)\overline{\hat{\eta}(\xi + \ell)} \tag{21}$$

is of elliptic type, where the Fourier transform of the functional $\eta$ is given by $\hat{\eta}(y) = \overline{\eta(e^{2\pi i y \cdot})}$. Thus, the function $\alpha$ plays a similar role in the stability analysis of the Galerkin–Petrov methods as the principal symbol $\sigma_0$ does in the Fredholm theory of pseudodifferential operators. The function $\alpha$ is frequently called the *numerical symbol* of the method (20).

More precisely, under some additional assumptions, the following holds.

**Theorem 1([7]).** *Assume that* (21) *converges absolutely for each* $x \in \mathbb{T}^n$ *and* $\xi \in \mathbb{T}^n$. *Then the Galerkin-Petrov method* (20) *is stable if and only if*

$$|\xi|^{-r}|\alpha(x,\xi)| \geq C > 0 \tag{22}$$

*holds for any* $x \in \mathbb{T}^n$ *and* $|\xi| > 0$.

**Corollary.** *Suppose the operator* $A$ *is strongly elliptic, i.e.,* $Re|\xi|^{-r}\sigma_0(x,\xi) \geq C_0 > 0$. *Then condition* (22) *is satisfied for both the break point collocation method* ($\epsilon = 0$) *using tensor products of smoothest splines of even order* $d$ *and the Galerkin method.*

These results are due to [22], [1], [2], [27] for one–dimensional equations (see also [26] for an overview of the various univariate results) and to [4], [23] for the multivariate case.

Indeed, in the case of the aforementioned collocation method, $\eta = \delta$, i.e. $\hat{\eta} = 1$, and

$$\hat{\varphi}(\xi) = \prod_{k=1}^{n} \left( \frac{\sin \pi \xi_k}{\pi \xi_k} \right)^d \geq c > 0 \; .$$

For the Galerkin method we have $\eta = \varphi$ and

$$\sum_{\ell \in \mathbb{Z}^n} |\hat{\varphi}(\xi + \ell)|^2 \geq c > 0 \; .$$

Next we will explain the *main ingredients of the proof* to Theorem 1 (see [7] for more details). Let us first consider the case when $\sigma_0(x, \xi) = \sigma(\xi)$ does not depend on $x$. Then $A = \sigma(D)$ is a translation invariant operator. This property together with

$$\tilde{\varphi}_0^j(\xi) = 2^{-jn/2} \hat{\varphi}(2^{-j}\xi) \; , \quad \xi \in \mathbb{Z}^n \; ,$$

implies

$$A\varphi_k^j(x) = A\varphi_0^j(x - 2^{-j}k) \; .$$

From this it follows immediately that the stiffness matrix of the method (20) is a circulant matrix, i.e.,

$$\boldsymbol{A}^j := (a_{k,m}^j)_{k,m \in \mathbb{Z}^{n,j}} = (b_{[k-m]}^j)_{k,m \in \mathbb{Z}^{n,j}}$$

with $[k-m] := k - m \bmod 2^j$, where

$$a_{k,m}^j := \eta_k^j(A\varphi_m^j) \; , \quad b_k^j := \eta_k^j(A\varphi_0^j) \; . \tag{23}$$

It is well-known that the finite Fourier transform $\boldsymbol{F}$ diagonalizes $\boldsymbol{A}^j$, i.e.,

$$\boldsymbol{F}^{-1} \boldsymbol{A}^j \boldsymbol{F} = (\alpha_k \delta_{k,m})_{k,m \in \mathbb{Z}^{n,j}} \; ,$$

where

$$\alpha_k := \sum_{m \in \mathbb{Z}^{n,j}} b_m^j e^{-2\pi i 2^{-j} <k,m>} \; .$$

Using this formula, (23) and the homogeneity of $\sigma$ yield that the *eigenvalues* $\alpha_k$ are given by

$$\alpha_0 = 1 \; , \quad \alpha_k = 2^{jr} \alpha(2^{-j}k) \; , \quad k \in \mathbb{Z}^{n,j} \; , \quad j \in \mathbb{N}_0 \; , \tag{24}$$

where $\alpha$ is defined by (21). Hence, in the case $\sigma_0(x, \xi) = \sigma(\xi)$ it is easily seen that Theorem 1 is a consequence of (24) when using a well–known equivalence between the norms $\|\cdot\|_s$ and corresponding discrete Sobolev norms. The case of symbols $\sigma_0$ depending on $x$ can be reduced to the aforementioned case $\sigma_0(x,\xi) = \sigma(\xi)$ by using a *local principle* [21] (such local principles can be viewed as a numerical counterpart to the principle of freezing coefficients in the theory of partial differential equations). The main ingredient for applying localization techniques are certain *superapproximation* results for the projections $P_j : L_2 \longrightarrow V_j$ defining the numerical scheme (20). These results are sometimes referred to as *discrete commutator properties* and have the form

$$\|(1-P_j)gP_ju\|_s \leq c2^{-j(t-s+\rho)}\|u\|_t \tag{25}$$

as well as

$$\|P_jg(1-P_j)u\|_s \leq c\rho_j 2^{-j(t-s)}\|u\|_t \ , \tag{26}$$

where $c$ and $\rho$ are positive constants independent of $u \in H^t(\mathbb{T}^n)$ (but depending on $s$ and $t$, in general). The orders $t$ and $s$ satisfy $t \geq s$ and are restricted by the choice of the projections $P_j$ (e.g. orthogonal or interpolation projections). Here $g$ is a sufficiently smooth periodic function and $0 < \rho_j \longrightarrow 0$ as $j \longrightarrow \infty$. (See e.g. [22], [21], [2], [26], [23], [4], [7], [16], [25] for particular cases of properties (25), (26).)

## 3.2 The case of multiscaling functions

In the remainder of this paper we consider the case when the shift–invariant spaces $V^j$ are generated by a finite number of scaling functions, say,

$$\varphi := (\varphi_\ell)_{\ell\in\Lambda_M} \ , \quad \Lambda_M := \mathbb{Z} \cap [-\frac{M}{2}, \frac{M}{2}) \ , \quad M \in \mathbb{N} \ ,$$

which satisfy the conditions of Sect. 3.1. (Notice that such spaces are frequently used as trial spaces in numerical procedures for engineering applications, see e.g. [16].) For simplicity, assume that $n = 1$. Under some additional assumptions $\varphi$ is called a *multiscaling* function. We define

$$V^j := \text{span}\ \{\varphi^j_{\ell,k} : k \in \mathbb{Z}^{1,j}\} \ ,$$

where $\varphi^j_{\ell,k} = (\varphi_\ell)^j_k$ is given by (19a). Suppose that (19b) holds and that $\cup_{j\in\mathbb{N}_0} V^j$ is dense in $L_2(\mathbb{T})$.

**Example 1** *Hermite quadratic splines* ($d = 3, M = 2$):

$$\varphi_{-1}(x) := \begin{cases} x & \text{for} \quad x \in [0,1] \\ 2-x & \text{for} \quad x \in [1,2] \\ 0 & \text{otherwise} \end{cases} \ ,$$

$$\varphi_0(x) := \begin{cases} x^2 & \text{for} \quad x \in [0,1] \\ (2-x)^2 & \text{for} \quad x \in [1,2] \\ 0 & \text{otherwise} \end{cases} \ .$$

**Example 2** *Hermite cubic splines* ($d = 4, M = 2$):

$$\varphi_{-1}(x) := \begin{cases} 3x^2 - 2x^3 & \text{for} \quad x \in [0,1] \\ 3(2-x)^2 - 2(2-x)^3 & \text{for} \quad x \in [1,2] \\ 0 & \text{otherwise} \end{cases} ,$$

$$\varphi_0(x) := \begin{cases} x^2 - x^3 & \text{for} \quad x \in [0,1] \\ -(2-x)^2 + 2(2-x)^3 & \text{for} \quad x \in [1,2] \\ 0 & \text{otherwise} . \end{cases}$$

Now we choose a family of distributions $\eta := (\eta_\ell)_{\ell \in \Lambda_M} \in (H^{-s'}(\mathbb{R}))^M$, $s' \geq 0$, with compact support to define the test functionals $\eta^j_{\ell,k} = (\eta_\ell)^j_k$ by (19c). The *Galerkin–Petrov method* corresponding to the aforementioned trial spaces and test functionals reads as follows:

Find an approximate solution $u^j \in V^j$ such that

$$\eta^j_{\ell,k}(Au^j) = \eta^j_{\ell,k}(f), \quad \ell \in \Lambda_M \, , k \in \mathbb{Z}^{1,j} \, . \tag{27}$$

The following three examples are particular realizations of the scheme (27).

**Example 3** *Collocation method*: Choose a strictly increasing sequence $(\epsilon_\ell)_{\ell \in \Lambda_M} \in [0,1)^M$ and define the test functionals by

$$\eta_\ell(f) := f(\epsilon_\ell)$$

for $\ell \in \Lambda_M$. So we have to find a solution $u^j \in V^j$ satisfying

$$Au^j(2^{-j}(\epsilon_\ell + k)) = f(2^{-j}(\epsilon_\ell + k)) \, , \quad \ell \in \Lambda_M, k \in \mathbb{Z}^{1,j} \, .$$

**Example 4** *Galerkin method*: Let $(\varphi_\ell)_{\ell \in \Lambda_M} \in \mathcal{L}_2^M$ be a family of compactly supported functions. Then the test functionals in (27) are defined by (19c) and

$$\eta_\ell(f) := \langle f, \varphi_\ell \rangle_{L_2(\mathbb{R})} \, , \quad \ell \in \Lambda_M,$$

for $f \in L_2(\mathbb{T})$.

**Example 5** *Biorthogonal Galerkin method*: Let $(\tilde{\eta}_\ell)_{\ell \in \Lambda_M} \in \mathcal{L}_2^M$ be a family of compactly supported functions biorthogonal to $(\varphi_\ell)_{\ell \in \Lambda_M}$, i.e.,

$$\langle \varphi_r, \tilde{\eta}_s(\cdot - k) \rangle_{L_2(\mathbb{R})} = \delta_{r,s}\delta_{0,k}$$

for $r, s \in \Lambda_M$ and $k \in \mathbb{Z}$. Then the test functionals in (27) are defined by (19c) and

$$\eta_\ell(f) := \langle f, \tilde{\eta}_\ell \rangle_{L_2(\mathbb{R})} \, , \quad \ell \in \Lambda_M \, ,$$

for $f \in L_2(\mathbb{T})$.

In the present case, the linear system to the scheme (27) gives rise to introducing the matrix function

$$\nu(x,\xi) := \sum_{\ell\in\mathbb{Z}} \left(\overline{\hat{\eta}_p(\xi+\ell)}\sigma_0(x,\xi+\ell)\hat{\varphi}_q(\xi+\ell)\right)_{p,q\in\Lambda_M}$$

for $x,\xi \in [-\frac{1}{2},\frac{1}{2}]$, which we call the *numerical symbol* of the scheme (27).

Using the notation

$$\begin{aligned}\hat{\eta}(m;\xi) &:= \left(\hat{\eta}_p(mM+\ell+\xi)\right)_{\ell,p\in\Lambda_M},\\ \hat{\varphi}(m;\xi) &:= \left(\hat{\varphi}_p(mM+\ell+\xi)\right)_{\ell,p\in\Lambda_M},\\ f(m;\xi) &:= \operatorname{diag}\left(f(mM+\ell+\xi)\right)_{\ell\in\Lambda_M}\end{aligned}$$

for $f:\mathbb{R}\longrightarrow\mathbb{C}$, the numerical symbol takes the simpler form

$$\nu(x,\xi) = \sum_{m\in\mathbb{Z}} \hat{\eta}(m;\xi)^*\ \sigma_0(x,\cdot)(m;\xi)\ \hat{\varphi}(m;\xi)\ .$$

Under some additional assumptions on the multiscaling function $\varphi$ and the test functional $\eta$ (which are satisfied, e.g., for the aforementioned three particular numerical schemes, see [24], [25] for more details) the following holds.

**Theorem 2** ([25]).*Assume that the operator $A: H^s(\mathbb{T}) \longrightarrow H^{s-r}$ is invertible. The numerical method* (27) *is stable if and only if*

$$\left\| \left(\langle 0;\xi\rangle^{s-r}(\hat{\eta}(0;\xi)^{-1})^*\nu(x,\xi)\hat{\varphi}(0;\xi)^{-1}\langle 0;\xi\rangle^{-s}\right)^{-1}\right\| \le C$$

*where the constant $C$ does not depend on $x,\xi\in[-\frac{1}{2},\frac{1}{2}]$.*

Here $\|\cdot\|$ means any matrix norm, and $\langle\xi\rangle$ denotes the function introduced in Sect. 2.3.

Notice that the stability combined with approximation properties of the projections defining the numerical scheme yield error estimates via standard arguments. (For details, we refer the reader to [26] and [7]).

In what follows we consider the particular case of the *collocation method using splines with defect $M$ and order $d$* (i.e., degree $d-1$) as trial functions. In this case the stability condition of Theorem 2 can be described in a more explicit way by using the characterization of the splines in terms of their Fourier coefficients (for detailed discussion we refer the reader to the papers [16], [24], [29]). Moreover, the following error estimates hold.

**Theorem 3** ([16]). *Assume that the collocation method is stable and $r+M<d$. Let $s$ and $t$ be real number satisfying*

$$s < d-M+\frac{1}{2}\ ,\quad r+\frac{1}{2} < t\ ,\quad r\le s\le t\le d\ .$$

*If* $u \in H^t$ *then*

$$\|u - u^j\|_s \leq C 2^{-j(t-s)} \|u\|_t \ .$$

The following theorem shows, for $M = 2$, that the stability of the collocation method

$$Au^j\left(2^{-j}(\epsilon_\ell + k)\right) = f\left(2^{-j}(\epsilon_\ell + k)\right) , \quad \ell = -1, 0, \quad k \in \mathbb{Z}^{1,j} \tag{28}$$

where $0 \leq \epsilon_\ell < 1$ , $\ell = -1, 0$, essentially depends on the choice of $\epsilon_\ell$.

**Theorem 4** ([16]). *Assume* $M = 2$ *and* $\sigma_0(x, \xi) = \sigma_0(\xi)$ *is either even or odd. Choose the following collocation points:*

*(a)* $\epsilon_{-1} = 0, \epsilon_0 = \frac{1}{2}$

- *i) If* $d$ *and* $\sigma_0$ *have like parity, then the method* (28) *is unstable.*
- *ii) If* $d$ *and* $\sigma_0$ *have opposite parity, then the method* (28) *is stable.*

*(b)* $\epsilon_{-1} = \epsilon$ , $\epsilon_0 = 1 - \epsilon$ *with* $0 < \epsilon < \frac{1}{2}$

- *i) If* $d$ *and* $\sigma_0$ *have opposite parity, then the method* (28) *is unstable.*
- *ii) If* $d$ *and* $\sigma_0$ *have like parity, then the method* (28) *is stable.*

## 3.3 Application to the Dirichlet problem for the Laplace equation

Consider the *Dirichlet problem* for the Laplace equation,

$$\Delta W = 0 \quad \text{on } \Omega, \quad W = F \quad \text{on } \Gamma, \tag{29}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\Gamma$. We choose a 1–periodic, regular parametric representation $\gamma : \mathbb{R} \longrightarrow \Gamma$, and make the *single layer Ansatz*

$$W(X) = \int_0^1 u(y) \log \frac{\omega}{|X - \gamma(y)|} dy \quad \text{for } X \in \Omega \ ,$$

with any fixed $\omega >$ diameter of $\Omega$. To satisfy the boundary condition in (29), the function $u$ must be a solution of *Symm's integral equation* $Au = f$, in which

$$(Au)(x) = \int_0^1 u(y) \log \frac{\omega}{|\gamma(x) - \gamma(y)|} dy$$

and $f(x) = F[\gamma(x)]$. The presence of the parameter $\omega$ ensures that $A$ has a trivial null space, and it is well known that $A = A_0 + A_1$ where $A_0$ has the symbol $\sigma_0(\xi) = (2|\xi|)^{-1}$ and $A_1$ is a smoothing operator. Thus, $A$ is an invertible pseudodifferential operator of order $r = -1$.

**Theorem 5** ([16]). *Consider the collocation method* (28) *applied to Symm's equation.*

1. *If we use continuous, piecewise–quadratics* ($d = 3, M = 2$) *and collocate at breakpoints and midpoints* ($\epsilon_1 = 0, \epsilon_2 = 1/2$) *then the method is stable. Moreover, one obtains one additional order of convergence for* $s = -2$ *(in comparison with Theorem 3).*

2. *If we use Hermite cubics* ($d = 4, M = 2$) *and collocate at two symmetric points in each interval* ($\epsilon_0 = 1 - \epsilon_{-1}$), *then the method is stable for any choice of* $\epsilon_{-1} \in (0, 1/2)$. *Moreover we obtain the convergence rate 7 (i.e., two additional orders of convergence) with respect to the Sobolev norm* $s = -3$ *for the special choice of the parameter*

$$\epsilon_{-1} = 0.2451188417393384 =: \epsilon^* . \tag{30}$$

Let $W^j$ denote the single layer potential of $u^j$. Assuming $u \in H^{r+2}$, the error estimate above implies that

$$W^j(X) = W(X) + O(h^{r+1+b}) , \quad h := 2^{-j} ,$$

uniformly for $X$ in any compact subset of $\Omega$. Thus, both methods described in Theorem 5 achive $O(h^5)$ accuracy, and in the Hermite cubic case the convergence improves to $O(h^7)$ for the special choice of collocation points given by (30).

## 3.4 Compression

Compression techniques for the resulting stiffness matrices relative to wavelet type bases have been analyzed in [3] for operators of order zero and Galerkin methods and in [8] for periodic pseudodifferential operators of arbitrary real order and Galerkin–Petrov schemes. The key idea is that for appropriate bases of the trial and test spaces the vast majority of entries of corresponding stiffness matrices although being different from zero tend rapidly to zero as the discretization gets finer. This suggests discarding those entries, i.e., to approximate the initial stiffness matrix by a sparse matrix. It has been proved that the order of the overall computational work which is needed to realize a certain accuracy is of the form $O(N(\log N)^b)$ where $N$ is the number of unknown and $b$ is some positive number (see [8], [28], [20]). Moreover, this compression technique is preserving the convergence rate. Recently, in [29] corresponding compression techniques have been developed for the numerical schemes (27) using multiwavelet bases of the orthogonal complements $W_j = V^{j+1} \ominus V^j$.

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