A NOTE ON THE DECAY OF CORRELATIONS UNDER δ–PINNING

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ABSTRACT. We prove that for a class of massless $\nabla\phi$ interface models on $\mathbb{Z}^2$ an introduction of an arbitrary small pinning self-potential leads to exponential decay of correlation, or, in other words, to creation of mass.

In this note we study a family of effective interface models over $\mathbb{Z}^2$ with the formal Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H}(\phi) \;=\; \sum_{i\sim j} V(\phi_i - \phi_j), \tag{1}$$

where the summation is over all nearest neighbours $i \sim j$ of $\mathbb{Z}^2$, and the following two assumptions are made on the interaction potential V:

- V is even and smooth
- There exists a constant $c_V \geq 1$, such that

$$\frac{1}{c_V} \;\leq\; V''(t) \;\leq\; c_V \qquad\qquad \forall t \in \mathbb{R}. \tag{2}$$

Remark 1. No further assumptions on c_V are made, and, in fact, we expect that the results of the paper remain true if only the lower bound in (2) is assumed. Also, though we do not stipulate it explicitly at each particular instance, the values of **all** the positive constants we use below depend on c_V.

Given a set $A \subset \mathbb{Z}^2$ with a finite complement $A^c \;\overset{\Delta}{=}\; \mathbb{Z}^2 \setminus A$, we use $\mathbb{P}_A$ to denote the finite volume Gibbs measure on $\Omega_A \;\overset{\Delta}{=}\; \mathbb{R}^{A^c}$ with the Hamiltonian $\mathcal{H}$ and zero boundary conditions on A;

$$\mathbb{P}_A(\mathrm{d}\phi) \;=\; \frac{1}{\mathbf{Z}(A)} \mathrm{e}^{-\mathcal{H}(\phi)} \prod_{i\in A^c} \mathrm{d}h_i \prod_{j\in A} \delta_0(\mathrm{d}h_j). \tag{3}$$

It is well known that $\mathbb{P}_A$ delocalizes as $A^c \nearrow \mathbb{Z}^2$; maybe the easiest way to see this is to use the reverse Brascamp-Lieb inequality [DGI] which implies that the variance of ϕ_0 under $\mathbb{P}_A$ dominates the corresponding Gaussian variance. If, however, an, essentially arbitrary small, pinning self-potential is added to $\mathcal{H}$, then the situations radically changes, and the infinite volume Gibbs state exists in the usual sense. This phenomenon has been first worked out in the Gaussian case ($c_V = 1$) in [DMRR]. Our main reference [DV] contains a proof of the localization for a fairly general class of interactions and self-potentials. In this note we prove that in the case of the family of random interfaces as in (1), the delocalization/localization transition is sharp in the sense that it always comes together with the exponential decay of correlations, or, using the language of a more physically oriented literature, with the creation of mass.

For simplicity, but also in order to give a cleaner exposition of otherwise more general renormalization ideas behind the proof, we consider here only the case of the so called δ-pinning, thereby generalizing recent results of [BB] on purely Gaussian fields (that is again $c_V = 1$):

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Given a box $\Lambda_N \triangleq [-N,...,N]^2 \subset \mathbb{Z}^2$ and a number $J \in \mathbb{R}$ (which characterizes the strength of the pinning) we define the following measure $\hat{\mathbb{P}}_N$ on $\mathbb{R}^{\Lambda_N}$:

$$\hat{\mathbb{P}}_N(\mathrm{d}\phi) \;=\; \frac{1}{\hat{\mathbf{Z}}_N}\mathrm{e}^{-\mathcal{H}(\phi)}\prod_{i\in\Lambda_N}\left(\mathrm{d}\phi_i + \mathrm{e}^J\delta_0(\mathrm{d}\phi_i)\right)\prod_{j\in\mathbb{Z}^2\setminus\Lambda_N}\delta_0(\mathrm{d}\phi_j). \tag{4}$$

Notice that the case $J = -\infty$ corresponds to the original measure on $\mathbb{R}^{\Lambda_N}$ with the Hamiltonian (1), which delocalizes as $N \to \infty$.

Lemma 2. *For every $J \in \mathbb{R}$ there exists an exponent (mass) $m = m(J) > 0$ and a constant $c_1 = c_1(J) < \infty$, such that*

$$\mathbb{C}\mathrm{ov}_{\hat{\mathbb{P}}_N}\left(\phi_i;\phi_j\right) \;\leq\; c_1\,\mathrm{e}^{-m\|i-j\|} \tag{5}$$

uniformly in N and in $i,j \in \mathbb{Z}^2$.

Of course, there is nothing to prove if either i or j lies outside of Λ_N. In fact, the sub-index N is superfluous - all the estimates we use and obtain simply do not depend on a particular Λ_N, and the only reason we need it is to make the definitions mathematically meaningful. From now on we shall drop the sub-index N from the notation.

A right way to think about (4) is as of the joint distribution of the field of random interface heights $\{\phi_i\}_{i\in\mathbb{Z}^2}$ and the random "dry" set $\mathcal{A}$;

$$\mathcal{A} \;\triangleq\; \left\{i \in \mathbb{Z}^2 : \phi_i = 0\right\}.$$

Integrating out all the height variables ϕ in (4) we arrive to the following probability distribution for $\mathcal{A}$;

$$\hat{\mathbb{P}}\left(\mathcal{A} = A\right) \;\triangleq\; \rho(A) \;=\; \frac{1}{\hat{\mathbf{Z}}}\mathrm{e}^{J|A|}\mathbf{Z}(A) \;=\; \frac{\mathrm{e}^{J|A|}\mathbf{Z}(A)}{\sum_D \mathrm{e}^{J|D|}\mathbf{Z}(D)}, \tag{6}$$

where the partition function $\mathbf{Z}(A)$ is the same as in (3).

Using the probabilistic weights $\{\rho(A)\}$ one can rewrite $\hat{\mathbb{P}}$ as the convex combination,

$$\hat{\mathbb{P}}(\cdot) \;=\; \sum_A \rho(A)\mathbb{P}_A(\cdot). \tag{7}$$

Since under each $\mathbb{P}_A$ the distribution of ϕ_i is symmetric for every $i \in \mathbb{Z}^2$, this gives rise to the following decomposition of the covariances:

$$\mathbb{C}\mathrm{ov}_{\hat{\mathbb{P}}}\left(\phi_i;\phi_j\right) \;=\; \sum_A \rho(A)\langle\phi_i;\phi_j\rangle_A. \tag{8}$$

At this point we shall utilize the random walk representation of $\langle\phi_i;\phi_j\rangle_A$ which has been first developed in the PDE context in [HS]. We follow the approach of [DGI], where the Helffer-Sjöstrand representation was put on the probabilistic tracks:

One constructs a stochastic process $\left(\Phi(t), X(t)\right)$, where:

- $\Phi(\cdot)$ is a diffusion on $\mathbb{R}^{A^c}$ with the invariant measure $\mathbb{P}_A$.
- Given a trajectory $\phi(\cdot)$ of the process Φ, $X(t)$ is an, in general inhomogeneous, transient random walk on $A^c \cup \partial A^c \subset \mathbb{Z}^2$ with the life-time

$$\tau_A \;\triangleq\; \inf\{t :\; X(t) \in A\},$$

and the time-dependent jump rates

$$a(i,j;t) \;=\; \begin{cases} V''\left(\phi_i(t) - \phi_j(t)\right), & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Let us use $\mathcal{E}^A_{i,\phi}$ to denote the law of $\big(X(t),\Phi(t)\big)$ starting from the point $(i,\phi)\in A^c\times\mathbb{R}^{A^c}$. Then ([HS],[DGI]),

$$\langle \phi_i,\phi_j\rangle_A \;=\; \left\langle \mathcal{E}^A_{i,\phi}\int_0^{\tau_A}\mathbb{I}_{\{X(s)=j\}}\mathrm{d}s\right\rangle_A . \tag{10}$$

Substituting the latter expression into (8),

$$\mathbb{C}\mathrm{ov}_{\hat{\mathbb{P}}}\big(\phi_i;\phi_j\big) \;=\; \sum_A \rho(A)\left\langle \mathcal{E}^A_{i,\phi}\int_0^{\tau_A}\mathbb{I}_{\{X(s)=j\}}\mathrm{d}s\right\rangle_A . \tag{11}$$

It is very easy now to explain the logic behind the proof of Lemma 2: The expression

$$\mathcal{E}^A_{i,\phi}\int_0^{\tau_A}\mathbb{I}_{\{X(s)=j\}}\mathrm{d}s$$

describes the time spent by the random walk $X(\cdot)$ starting at i in the site j before being killed upon entering the dry set A which, for the purpose, could be considered as a random killing obstacle. In order to prove that this time is exponentially (in $\|i-j\|$) small one needs an appropriate density estimate on A and a certain path-wise control on the exit distributions of $X(\cdot)$. In the Gaussian case considered in [BB], $X(\cdot)$ happens to be just the simple random walk on $\mathbb{Z}^2$ which is completely decoupled from the diffusion part $\Phi(\cdot)$, and, thus, behaving independently of A and the initial condition $\phi\in\mathbb{R}^{A^c}$. This lead in [BB] to a resummation argument, which substantially facilitated the matter. One of the main difficulties in the non-Gaussian case we consider here is the dependence of the distribution of $X(\cdot)$ on the realization of the dry set A and on the sample path of the diffusion Φ. We still have very little to say about this dependence. However, due to the basic assumption (2) on the interaction potential V, the jump rates $a(i,j;t)$ in (9) are uniformly bounded above and below:

$$\frac{1}{c_V} \le a(i,j;t) \;\le\; c_V . \tag{12}$$

In particular one always has a rough control over probabilities of hitting distributions. For example, if the random walk X enters a box $\mathbf{B}_l$ of linear size l which is known to contain a dry site; it would be convenient to call such a box "dirty", then the probability that X hits this site (and consequently dies there) before leaving $\mathbf{B}_l$ should be bounded below by some positive number $p=p(l)>0$. Thus if the realisation A of the random dry set $\mathcal{A}$ is such, that on its way from i to j the walk X cannot avoid visiting less than $\epsilon\|i-j\|$ disjoint dirty l-boxes, the probability that it eventually reaches j before being killed should be bounded above by something like

$$\big(1-p(l)\big)^{\epsilon\|i-j\|} .$$

Proposition 5 below makes this computation precise.

The crux of the matter, however, is to ensure that on a certain finite l-scale the density of the dirty l-boxes is so high, that only with exponentially small probabilities the realization A of $\mathcal{A}$ enables an ϵ-clean passage from i to j. A statement of this sort is given in Proposition 4.

Once the renormalization approach sketched above is accepted as the strategy of the proof, the first drive of an associative thinking is to try to compare the distribution of $\mathcal{A}$ on different l-scales with, say, independent Bernoulli percolation or other known models with controllable decay of connectivities. This we have tried and failed, and, at least in the case of $\mathbb{Z}^2$, such a comparison is unlikely.

The relevant statistical properties of the random dry set $\mathcal{A}$ on various finite length scales are captured in the following estimate which generalizes the key Proposition 5.1 in [DV]

Theorem 3. *For each $J \in \mathbb{R}$ there exists a number $R = R(J) < \infty$ and exponent $\nu = \nu(J) > 0$, such that whenever a finite set $B \subset \mathbb{Z}^2$ admits a decomposition*

$$B = \bigvee_{l=1}^{n} B_l \tag{13}$$

into connected disjoint components $B_1, ..., B_n$ with

$$diam(B_l) \geq R; \qquad l = 1, ..., n, \tag{14}$$

the following exponential upper bound on having all of B "clean of dry points" holds:

$$\sum_{A \cap B = \emptyset} \rho(A) \leq e^{-\nu|B|}. \tag{15}$$

We relegate the proof of Theorem 3 to the end of the paper, and, assuming for the moment its validity, directly proceed to the proof of the mass-generation claim of Lemma 2.

Proof of Lemma 2: The number $R = R(J)$ which appears in the basic Theorem 3 sets up the stage for the finite scale renormalization analysis of the random dry set $\mathcal{A}$. Let us pick a number $l > R$; $l \in \mathbb{N}$, and define the renormalized lattice

$$\mathbb{Z}_l^2 \triangleq (2l+1)\mathbb{Z}^2.$$

To distinguish between the sets on the original lattice $\mathbb{Z}^2$ and those on the renormalized one $\mathbb{Z}_l^2$ we shall always mark the latter by the super-index l. For example $\mathbf{B}^l(x,r)$ stands for the $\mathbb{Z}_l^2$ lattice box centered at $x \in \mathbb{Z}_l^2$;

$$\mathbf{B}^l(x,r) \triangleq \left\{ y \in \mathbb{Z}_l^2 : \ \|x-y\| \leq lr \right\}.$$

Let us define $\Gamma^l(r)$ as the set of all $\mathbb{Z}_l^2$-nearest neighbour lattice paths leading from the origin to the boundary $\partial \mathbf{B}^l(x,r)$. With each $\gamma^l \in \Gamma^l(r)$ we associate a connected chain $\tilde{\gamma}^l$ of l-blocks on the original lattice $\mathbb{Z}^2$;

$$\tilde{\gamma}^l \triangleq \bigcup_{x \in \gamma^l} \mathbf{B}(x,l).$$

Let us fix a number $\epsilon \in (0,1)$. We say that a path $\gamma^l \in \Gamma^l$ is (r,ϵ)-clean in $A \subset \mathbb{Z}^2$, if

$$\# \left\{ x \in \gamma^l : \ \mathbf{B}(x,l) \cap A \neq \emptyset \right\} < \epsilon r.$$

Similarly, we say that a set $A \subset \mathbb{Z}^2$ is (r,ϵ)-clean if there exists a path $\gamma^l \in \Gamma^l(r)$ which is (r,ϵ)-clean for A. Otherwise, we shall call A $(r,\epsilon)-$dirty.

Proposition 4. *For each $\epsilon \in (0,1)$ there exist a number $l_0 = l_0(\epsilon, J) < \infty$ and a radius $r_0 = r_0(\epsilon)$, such that for every choice of $l \geq l_0$;*

$$\sum_{A \ is \ (r,\epsilon)-clean} \rho(A) \leq e^{-c_2(\epsilon,l)r},$$

uniformly in $r \geq r_0$, where $c_2(\epsilon,l)$ diverges (as l^2) with l.

Proof: The condition on $r_0(\epsilon)$ is a semantic one - the only thing we want is to ensure that $r > [\epsilon r]$.

Let us estimate the probability of the event $\{A \text{ is } (r,\epsilon) - \text{clean}\}$ as follows:

$$\sum_{A \text{ is } (r,\epsilon)-\text{clean}} \rho(A) \leq \sum_{k=r}^{\infty} \sum_{\gamma^l \in \Gamma^l : |\gamma^l| = k} \sum_{A : \gamma_l \text{ is } (r,\epsilon)-\text{clean in } A} \rho(A). \tag{16}$$

Each path $\gamma^l = (0, x_1, ..., x_k)$; $\gamma^l \in \Gamma^l$, which is (r,ϵ)-clean in A contains at most $[\epsilon r]$ vertices $x_{i_1}, ..., x_{i_M}$; $M \leq [\epsilon r]$, such that the corresponding $l-$blocks have a non-empty intersection with A;

$$\mathbf{B}(x_i, l) \cap A \neq \emptyset; \qquad i = 1, ..., M.$$

Whatever happens, for a path γ^l of length k there are at most 2^k (in fact much less due to the restriction $M \leq [\epsilon r]$) possible ways to choose a sub-family $\tilde{\gamma}^l_{\text{dirty}}$;

$$\tilde{\gamma}^l_{\text{dirty}} \triangleq \bigcup_{i=1}^{M} \mathbf{B}(x_i, l),$$

of "dirty" block along $\tilde{\gamma}^l$. On the other hand, fixing both $\tilde{\gamma}^l$ and its "dirty part" $\tilde{\gamma}^l_{\text{dirty}}$, we can use Theorem 3 to obtain

$$\sum_{A \cap \tilde{\gamma}^l \setminus \tilde{\gamma}^l_{\text{dirty}} = \emptyset} \rho(A) \leq \exp\{-\nu|\tilde{\gamma}^l \setminus \tilde{\gamma}^l_{\text{dirty}}|\} \leq \mathrm{e}^{-\nu(k-[\epsilon r])l^2} \tag{17}$$

We, thus, conclude, that for any $k \geq r$ and for each $\gamma^l \in \Gamma^l$ with $|\gamma^l| = k$,

$$\sum_{A:\gamma_l \text{ is } (r,\epsilon)-\text{clean in } A} \rho(A) \leq \mathrm{e}^{-\nu(J)l^2(k-[\epsilon r])+k\log 2}.$$

Using the above estimate together with the trivial bound;

$$\#\left\{\gamma^l \in \Gamma^l : |\gamma^l| = k\right\} \leq 4^k,$$

to perform the summation in (16) we arrive at the claim of Proposition 4. $\square$

Nothing in the above argument depends on the fact that the box $\mathbf{B}(0, rl)$ is centered at the origin. Without any loss of generality we shall prove (5) only for the case $i = 0$.

Let us fix l and ϵ as in the statement of Proposition 4. For each j with $\|j\| > rl$ we use (11) and estimate:

$$\begin{aligned} \mathbb{C}\mathrm{ov}\left(\phi_0; \phi_j\right) \leq & \sum_{A \text{ is } (r,\epsilon)-\text{clean}} \rho(A) \\ & + \sum_{A \text{ is } (r,\epsilon)-\text{dirty}} \rho(A) \max_{\phi} \mathcal{E}^A_{0,\phi} \int_0^{\tau_A} \mathbb{I}_{\{X(s)=j\}} \mathrm{d}s. \end{aligned} \tag{18}$$

The first term in (18) has been just estimated in Proposition 4. Let us use τ_{rl} to denote the exit time from $\mathbf{B}(0, rl)$. The second term in (18) could be further bounded above as

$$\max_{A \text{ is } (r,\epsilon)-\text{dirty}} \max_{\phi} \mathcal{E}^A_{0,\phi}\left(\tau_A > \tau_{rl}\right) \sum_B \rho(B) \max_{\psi} \mathcal{E}^B_{j,\psi} \int_0^{\tau_B} \mathbb{I}_{\{X(s)=j\}} \mathrm{d}s. \tag{19}$$

It is convenient to estimate the above expression in a complete generality of time dependent random walks with bounded jump rates $a(i,j;t)$:

Let $X(t)$ be the time-inhomogeneous Markov process with the transition rates as in (12). It is always possible to homogenize it, and to consider

$$\tilde{X}(t) \triangleq (X(t), t).$$

We shall use $\tilde{\mathbb{E}}_{(i,t)}$ to denote the law of $\tilde{X}$ with the space-time starting point $(i,t) \in \mathbb{Z}^2 \times \mathbb{R}$.

The $\mathbf{B}(0, rl)$ box is decomposed to the disjoint union of sub-blocks on the l-scale as:

$$\mathbf{B}(0, rl) = \cup_{x \in \mathbf{B}^l(0,r)} \mathbf{B}(x, l).$$

To a generic point $i \in \mathbf{B}(0, rl)$ we associate an l-block $\mathbf{B}_l(i)$ according to the following rule:

$$\mathbf{B}_l(i) = \mathbf{B}(x, l) \text{ if } i \in \mathbf{B}(x, l) \text{ for some } x \in \mathbb{Z}^2_l.$$

Given a (r,ϵ)-dirty set $A \subset \mathbb{Z}^2$, let us call a block $\mathbf{B}(x,l);\ x \in \mathbb{Z}_l^2$, dirty if

$$\mathbf{B}(x,l) \cap A \neq \emptyset.$$

We introduce now the following family of stopping times for the process $\tilde{X}(t)$:

$$T_1 = \inf_{t\geq 0}\{\mathbf{B}_l(X(t)) \text{ is dirty}\}.$$

$$S_1 = \inf_{t\geq T_1}\{\mathbf{B}_l(X(t)) \neq \mathbf{B}_l(X(T_1))\}.$$

$$T_2 = \inf_{t\geq S_1}\{\mathbf{B}_l(X(t)) \ \text{ is dirty}\}$$

....

$$S_n = \inf_{t\geq T_n}\{\mathbf{B}_l(X(t)) \neq \mathbf{B}_l(X(T_n))\}.$$

....

The condition of A being (r,ϵ)-dirty is readily translatable under $\mathbb{P}_A$ to the sure event

$$\{\tau_{rl} \ > \ T_{\epsilon r}\}\,.$$

Consequently, if, as before, we use τ_A to denote the hitting time of the set A ,

$$\tilde{\mathbb{P}}_{(0,0)}(\tau_A > \tau_{rl}) \ \leq \ \tilde{\mathbb{P}}_{(0,0)}(\tau_A > T_{\epsilon r}) \ = \ \tilde{\mathbb{E}}_{(0,0)}\tilde{\mathbb{E}}_{\tilde{X}(T_1)}\mathbb{I}_{\tau_A > S_1}\ldots\tilde{\mathbb{E}}_{\tilde{X}(T_{\epsilon r})}\mathbb{I}_{\tau_A > S_{\epsilon r}}\,.$$

We claim that each of the ϵr terms in the above product admits an upper bound of the form

$$1 \ - \ \left(\frac{1}{3c_V^2+1}\right)^{2l}. \tag{20}$$

uniformly in all Markov chains with bounded rates condition (12) and (which is the same) in all possible values of above stopping times.

Indeed let $\mathbf{B}_l$ be a box of side length l, and $i,k \in \mathbf{B}_l$. Then one strategy for a random walk starting at i to hit k before leaving $\mathbf{B}_l$ is to march to k directly along some prescribed unambiguous trajectory, say first horizontally and then vertically. Clearly if one pulls down the rates along such a trajectory to the minimum value $1/c_V$ and pushes the rates leading out of this trajectory to the maximal value c_V, then the probability to follow the trajectory itself only decreases, but to an exactly computable value

$$\left(\frac{1}{3c_V^2+1}\right)^{\|i-k\|},$$

where the power $\|i-k\|$, of course, corresponds to the number of steps along the trajectory. Hence (20).

As a result:

Proposition 5. *Uniformly in r and in (r,ϵ)-dirty sets A,*

$$\max_{\phi}\mathcal{E}^A_{0,\phi}\,(\tau_A > \tau_{rl}) \ \leq \ e^{-c_3 rl}.$$

□

Finally,

$$\begin{aligned}\sum_B \rho(B)\max_{\phi}\mathcal{E}^B_{j,\phi}\int_0^{\tau_B}\mathbb{I}_{\{X(s)=j\}}\mathrm{d}s \\ = \sum_{k=1}^{\infty}\ \sum_{B:\mathrm{d}(j,B)=k} \rho(B)\max_{\phi}\mathcal{E}^B_{j,\phi}\int_0^{\tau_B}\mathbb{I}_{\{X(s)=j\}}\mathrm{d}s,\end{aligned} \tag{21}$$

where $\mathrm{d}(j,B) \triangleq \inf\{\|j-i\| : i \in B\}$.

Proceeding as in the proof of Proposition 5, we readily obtain that there exists a number $M = M(c_V) < \infty$, such that;

$$\max_{\phi} \mathcal{E}^B_{j,\phi} \int_0^{\tau_B} \mathbb{I}_{\{X(s)=j\}} \mathrm{d}s \le M^k,$$

whenever $\mathrm{d}(j,B) = k$. On the other hand, by Theorem 3,

$$\sum_{B:\,\mathrm{d}(j,B)=k} \rho(B) \le \mathrm{e}^{-\nu k^2},$$

as soon as $k > R$. Therefore, the sum in (21) converges, and the proof of Lemma 2 is, thereby, concluded

□

Proof of Theorem 3: Let us start by introducing some additional notation: Given a finite set $B \subset \mathbb{Z}^2$ with the decomposition (13) into the disjoint union of connected components $B_1, ..., B_n$ we say that another set A is a dry neighbour of B; $A \in \mathcal{D}_B$, if

$$A \cap B = \emptyset \qquad \text{but} \qquad D \cup \partial B_l \neq \emptyset;\ l = 1, ..., n.$$

Proposition 6. *There exists a constant $c_4 = c_4(J)$, such that for every finite $B \subset \mathbb{Z}^2$,*

$$\sum_{A \in \mathcal{D}_B} \rho(A) \le e^{-c_4 |B|}. \tag{22}$$

The proof of Proposition 6 relies on the following two basic estimates which have been proven in [DV]:

1. There exists a number $M = M(J)$ and a constant $c_5 = c_5(J)$, such that,

$$\inf_{A \in \mathcal{D}_B} \sum_{C \subset B} \mathrm{e}^{J|C|} \frac{\mathbf{Z}(A \cup C)}{\mathbf{Z}(A)} \ge \mathrm{e}^{c_5 |B|}, \tag{23}$$

   whenever B is connected and $\mathrm{diam}(B) \ge M$.
2. Let $A \neq \emptyset$ and $i \in \mathbb{Z}^2 \setminus A$. Then,

$$\frac{\mathbf{Z}(A \cup \{i\})}{\mathbf{Z}(A)} \ge \frac{c_6(J)}{\sqrt{\mathrm{d}(i,A)}}. \tag{24}$$

The above estimates are linked to the claim of Proposition 6 in the following way:

$$\sum_{A \in \mathcal{D}_B} \rho(A) \le \left(\inf_{A \in \mathcal{D}_B} \sum_{C_1 \subset B_1} \cdots \sum_{C_n \subset B_n} \frac{\mathbf{Z}(A \cup_1^n C_l)}{\mathbf{Z}(A)} \mathrm{e}^{J \sum_1^n |C_l|} \right)^{-1}.$$

If, for some $m \in [1, ..., n-1]$, we regroup B as

$$B = B^+ \cup B^- \triangleq \{B_1, ..., B_m\} \bigcup \{B_{m+1}, ..., B_n\},$$

then, since $A \cup_1^m C_l$ always belongs to $\mathcal{D}_{\cup_{m+1}^n B_l}$, we obtain the following decoupling estimate:

$$\begin{aligned} &\inf_{A \in \mathcal{D}_B} \sum_{C_1 \subset B_1} \cdots \sum_{C_n \subset B_n} \frac{\mathbf{Z}(A \cup_1^n C_l)}{\mathbf{Z}(A)} \mathrm{e}^{J \sum_1^n |C_l|} \\ &\quad \ge \inf_{A \in \mathcal{D}_B^+} \sum_{C_1 \subset B_1} \cdots \sum_{C_m \subset B_m} \frac{\mathbf{Z}(A \cup_1^m C_l)}{\mathbf{Z}(A)} \mathrm{e}^{J \sum_1^m |C_l|} \\ &\qquad \times \inf_{A \in \mathcal{D}_B^-} \sum_{C_{m+1} \subset B_{m+1}} \cdots \sum_{C_n \subset B_n} \frac{\mathbf{Z}(A \cup_{m+1}^n C_l)}{\mathbf{Z}(A)} \mathrm{e}^{J \sum_{m+1}^n |C_l|}. \end{aligned} \tag{25}$$

In particular, the claim (22) directly follows from the estimate (23) whenever $\mathrm{diam}(B_l) > M$ for each $l = 1, ..., n$. In fact, in view of (23) and (25), it remains to study only the case when all connected components of B are small; $\mathrm{diam}(B_l) < M$; $l = 1, ..., n$.

In the latter situation, however, we can use (24) and estimate;

$$\frac{\mathbf{Z}(A \cup C_l)}{\mathbf{Z}(A)} \geq \left(\frac{c_6}{\sqrt{2M+1}}\right)^{|C_l|},$$

for every l; $A \in \mathcal{D}_{B_l}$ and $C_l \subset B_l$. Therefore,

$$\inf_{A \in \mathcal{D}_B} \sum_{C_1 \subset B_1} \cdots \sum_{C_n \subset B_n} \frac{\mathbf{Z}(A \cup_1^n C_l)}{\mathbf{Z}(A)} \mathrm{e}^{J \sum_{l=1}^n |C_l|} \geq \prod_1^n \left(1 + \frac{c_6}{\sqrt{2M+1}}\right)^{|B_l|},$$

and (22) follows.

□

Remark 7. One could hope to deduce from Proposition 6 the claim of Theorem 3 even without the additional assumption (14). We were not able to do so, and, moreover, even not sure that the corresponding statement would be true — the entropy cancelation forced by the condition (14) could well be essential for the validity of the claim. We would like to stress, however, that within the framework of the renormalization approach we try to develop there is absolutely no point to relax (14).

The rest of the proof is an adaptation of the ideas of [DV] to the case of multiply connected sets:

First of all, for any finite $D \subset \mathbb{Z}^2$ let us denote its k-enlargement $D^{(k)}$ as

$$D^{(k)} \triangleq \left\{ i \in \mathbb{Z}^2 : \mathrm{d}(i, D) \leq k \right\}.$$

Assume now that $B = \bigvee_1^n B_l$ is as in the assumptions of Theorem 3, that is the diameter of each connected component B_l of B is bounded below, $\mathrm{diam}(B_l) \geq R$; $i = 1, ..., n$.

We have to show that the bound (15) holds uniformly in such B-s as soon as R is chosen large enough.

Let us say that a tuple $\underline{k} = (k_1, ..., k_n)$ of n natural numbers is B-admissible if:

- $k_1 \in \mathbb{N}$ (no restriction).
- either $k_2 = 0$, or the sets $B_1^{(k_1)}$ and $B_2^{(k_2)}$ are disjoint.
- either $k_3 = 0$, or the set $B_3^{(k_3)}$ is disjoint from
$$B_1^{(k_1)} \cup B_2^{(k_2)}.$$
- ....
- either $k_n = 0$, or the set $B_n^{(k_n)}$ is disjoint from
$$\bigcup_1^{n-1} B_l^{(k_l)}.$$

For any B-admissible tuple $\underline{k}$ we set

$$B^{(\underline{k})} \triangleq \bigcup_1^n B_l^{(k_l)}.$$

This construction enjoys the following two properties:

1. For any $A \cap B = \emptyset$ there is the unique B-admissible tuple $\underline{k}$, such that,
$$A \in \mathcal{D}_{B^{(\underline{k})}}.$$

Indeed, this tuple $\underline{k} = (k_1, ..., k_n)$ can be constructed in the following way:

$$\begin{aligned} k_1 &= \max\left\{k: \ B_1^{(k)} \cap A = \emptyset\right\} \\ k_2 &= \max\left\{k > 0: \ B_2^{(k)} \cap (A \cup B_1^{(k_1)}) = \emptyset\right\} \\ &\cdot \\ &\cdot \\ &\cdot \\ k_n &= \max\left\{k > 0: \ B_n^{(k)} \cap (A \cup_1^{n-1} B_l^{(k_l)}) = \emptyset\right\} \end{aligned}$$

with the convention that the maximum over an empty set equals zero.

2. For any B-admissible tuple $\underline{k} = (k_1, ..., k_n)$;

$$\left|B^{(\underline{k})}\right| \ \geq \ |B| + \sum_1^n k_l.$$

This follows directly from the definition of the B-admissibility.

Using Proposition 6 we, thereby, obtain:

$$\begin{aligned} \sum_{A \cap B = \emptyset} \rho(A) &= \sum_{B-\text{admissible } \underline{k}} \ \sum_{A \in \mathcal{D}_B^{(\underline{k})}} \rho(A) \\ &\leq \sum_{B-\text{admissible } \underline{k}} \mathrm{e}^{-c_4(|B| + \sum k_l)} \\ &\leq \ \mathrm{e}^{-c_4|B|}\left(1 - \mathrm{e}^{-c_4}\right)^{-n}. \end{aligned}$$

By the assumption (14), $n \leq |B|/R$. Thus it remains to choose $R = R(J)$ so large that,

$$\nu(J) \ \triangleq \ c_4(J) + \frac{\log(1 - \mathrm{e}^{-c_4(J)})}{R} \ > \ 0,$$

and (15) follows. □

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