Singularly perturbed boundary value problems in case of exchange of stablities

V.F. Butuzov*
Moscow State University
Faculty of Physics
Department of Mathematics
119899 Moscow, Russia

N.N. Nefedov†
Moscow State University
Faculty of Physics
Department of Mathematics
119899 Moscow, Russia

K.R. Schneider‡
Weierstraß–Institut für Angewandte
Analysis und Stochastik,
Mohrenstraße 39,
D–10117 Berlin, Germany

Abstract

We consider a mixed boundary value problem for a system of two second order nonlinear differential equations where one equation is singularly perturbed. We assume that the associated equation has two intersecting families of equilibria. This property excludes the application of standard results. By means of the method of upper and lower solutions we prove the existence of a solution of the boundary value problem and determine its asymptotic behavior with respect to the small parameter. The results can be used to study differential systems modelling bimolecular reactions with fast reaction rates.

MSC: 34B15, 34E15

Key words: boundary value problem, singularly perturbed system, exchange of stability

*e-mail: butuzov@mt384.phys.msu.su

†e–mail: nefedov@mt384.phys.msu.su

‡e–mail: schneider@wias-berlin.de

1 Introduction.

In this paper we study boundary value problems for singularly perturbed systems of the form

$$\begin{aligned} \varepsilon^2 \frac{d^2u}{dx^2} &= g(u,v,x,\varepsilon), \\ \frac{d^2v}{dx^2} &= f(u,v,x,\varepsilon) \end{aligned} \tag{1.1}$$

where u and v are scalars, $0 < x < 1$. Systems of this type describe steady state solutions of the reaction–diffusion system

$$\begin{aligned} -\frac{\partial u}{\partial t} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} &= g(u,v,x,\varepsilon), \\ -\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} &= f(u,v,x,\varepsilon). \end{aligned} \tag{1.2}$$

We consider (1.1) under the assumption that the corresponding degenerate equation

$$g(u,v,x,0) = 0 \tag{1.3}$$

has two solutions $u = \varphi_1(v,x)$ and $u = \varphi_2(v,x)$ which intersect transversally. This can be interpreted as an exchange of stability of the two branches of equilibria $u = \varphi_1(v,x)$ and $u = \varphi_2(v,x)$ consisting of a saddle point $(g_u(.) < 0)$ or a center $(g_u(.) > 0)$ of the associated equation

$$\frac{d^2u}{d\xi^2} = g(u,v,x,0)$$

where v and x are considered as parameters. Under the assumption that (1.3) has two intersecting solutions the standard theory for singularly perturbed systems (see [1, 2, 3]) can not be applied to (1.1). Our goal is to extend results of the authors concerning the solution of initial value problems for singularly perturbed systems of the form [4]

$$\begin{aligned} \varepsilon \frac{du}{dt} &= g(u,v,t,\varepsilon), \\ \frac{dv}{dt} &= f(u,v,t,\varepsilon) \end{aligned} \tag{1.4}$$

and of boundary value problems for scalar equations [5] in case of exchange of stability.

To motivate our investigations we consider the following differential system modelling a bimolecular reaction with fast bimolecular reaction rate $r(\bar{u},\bar{v})/\varepsilon^2$, slow monomolecular reaction rates $g_1(\bar{u})$ and $g_2(\bar{v})$, and inputs $I_a(x)$ and $I_b(x)$ depending only on the space variable x

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} &= \frac{\partial^2 \bar{u}}{\partial x^2} + I_a(x) - g_1(\bar{u}) - \frac{r(\bar{u},\bar{v})}{\varepsilon^2},\\ \frac{\partial \bar{v}}{\partial t} &= \frac{\partial^2 \bar{v}}{\partial x^2} + I_b(x) - g_2(\bar{v}) - \frac{r(\bar{u},\bar{v})}{\varepsilon^2}.\end{aligned} \tag{1.5}$$

A stationary solution of (1.5) satisfies

$$\begin{aligned}\varepsilon^2 \frac{d^2 \bar{u}}{dx^2} &= -\varepsilon^2 (I_a(x) - g_1(\bar{u})) + r(\bar{u},\bar{v}),\\ \varepsilon^2 \frac{d^2 \bar{v}}{dx^2} &= -\varepsilon^2 (I_b(x) - g_2(\bar{v})) + r(\bar{u},\bar{v}).\end{aligned} \tag{1.6}$$

After the coordinate transformation $u = \bar{u}, v = \bar{u} - \bar{v}$ system (1.6) can be rewritten as

$$\begin{aligned}\varepsilon^2 \frac{d^2 u}{dx^2} &= -\varepsilon^2 (I_a(x) - g_1(u)) + r(u, u-v),\\ \frac{d^2 v}{dx^2} &= I_b(x) - I_a(x) - g_2(u-v) + g_1(u)\end{aligned} \tag{1.7}$$

which has the form (1.1).

The main results of this paper concern the existence and the asymptotic behavior in ε of the solution of some boundary value problem related to system (1.1) in case of exchange of stability.

The paper is organized as follows. In section 2 we formulate the boundary value problem under consideration and introduce our assumptions. Section 3 contains our main result concerning the existence and asymptotic behavior of the solution of the boundary value problem. In the final section we illustrate our result by considering two examples modelling the stationary concentrations of fast bimolecular reactions.

2 Notation. Formulation of the problem. Assumptions

Let I_ω be the interval defined by $I_\omega := \{x \in R : |x| < \omega\}$, $\omega > 0$, let $I_\omega^+ := I_\omega \cap \{x \in R : x > 0\}$. We introduce the sets G_0 and D_0 by $G_0 := I_{v_0} \times (0,1)$, $D_0 := I_{u_0} \times G_0$ where u_0 and v_0 are positive numbers.

In what follows we study the singularly perturbed nonlinear boundary value problem

$$\begin{aligned}
\varepsilon^2 \frac{d^2u}{dx^2} &= g(u,v,x,\varepsilon),\\
\frac{d^2v}{dx^2} &= f(u,v,x,\varepsilon), \quad x\in(0,1),\\
u'(0) &= u'(1)=0, \quad v(0)=v^0, \quad v(1)=v^1
\end{aligned} \tag{2.1}$$

under the following assumptions:

(A$_1$) Let ε_0 be a small positive number. The functions f and g are twice continuously differentiable in $D_0 \times I^+_{\varepsilon_0}$ where all derivatives are continuous in the closure of $D_0 \times I^+_{\varepsilon_0}$.

The boundary value problem

$$\begin{aligned}
\frac{d^2v}{dx^2} &= f(u,v,x,0), \quad x\in(0,1)\\
0 &= g(u,v,x,0),\\
v(0) &= v^0, \quad v(1)=v^1
\end{aligned} \tag{2.2}$$

is called the degenerate problem to (2.1).

In case that $g(u,v,x,0)=0$ has an isolated solution $u=\varphi(v,x)$ in $\overline{D_0}$, the degenerate problem (2.2) can be written in the form

$$\begin{aligned}
\frac{d^2v}{dx^2} &= f(\varphi(v,x),v,x,0), \quad x\in(0,1),\\
v(0) &= v^0, \quad v(1)=v^1,
\end{aligned}$$

and under some additional assumptions the standard theory [1, 2, 3] can be applied to (2.1).

In the sequel we study (2.1) in the non-standard case by assuming

(A$_2$) Equation (1.3) has two twice continuously differentiable solutions $u=\varphi_1(v,x)$ and $u=\varphi_2(v,x)$ in $\overline{D_0}$.

(A$_3$) There exists a continuous function $k:[0,1]\to R$ such that $\varphi_1(k(x),x)\equiv\varphi_2(k(x),x)$ $\forall x\in[0,1]$.

Assumption (A_3) says that the surfaces $u=\varphi_1(v,x)$ and $u=\varphi_2(v,x)$ intersect at a curve whose projection into the region G_0 is described by $v=k(x)$.

(A$_4$) There is a point $x_0\in(0,1)$ such that the boundary value problems

$$\begin{aligned}
\frac{d^2v}{dx^2} &= f(\varphi_1(v,x),v,x,0), \quad 0<x<x_0,\\
v(0) &= v^0, \quad v(x_0)=k(x_0)
\end{aligned}$$

and

$$\begin{aligned} \frac{d^2v}{dx^2} &= f(\varphi_2(v,x),v,x,0), \quad x_0 < x < 1, \\ v(x_0) &= k(x_0),\ v(1) = v^1 \end{aligned}$$

have solutions $v_1(x)$ and $v_2(x)$ defined on $[0, x_0]$ and $[x_0, 1]$ respectively and satisfying

$$v_1'(x_0) = v_2'(x_0).$$

We define the function $\hat{v}(x)$ by

$$\hat{v}(x) = \begin{cases} v_1(x) & \text{for} \quad 0 \le x \le x_0, \\ v_2(x) & \text{for} \quad x_0 \le x \le 1. \end{cases}$$

It is easy to see that $\hat{v}(x)$ is twice continuously differentiable and represents a solution of the degenerate problem

$$\begin{aligned} \frac{d^2v}{dx^2} &= f(\hat{\varphi}(v,x),v,x,0), \\ v(0) &= v^0, \quad v(1) = v^1, \quad 0 < x < 1 \end{aligned} \tag{2.3}$$

where $\hat{\varphi}(v,x)$ is defined by

$$\hat{\varphi}(v,x) := \begin{cases} \varphi_1(v,x) & \text{for} \quad 0 \le x \le x_0, \\ \varphi_2(v,x) & \text{for} \quad x_0 \le x \le 1. \end{cases} \tag{2.4}$$

By means of $\hat{v}(x)$ we introduce the functions

$$\psi_1(x) := \varphi_1(\hat{v}(x),x), \quad \psi_2(x) := \varphi_2(\hat{v}(x),x) \tag{2.5}$$

which are twice continuously differentiable solutions of the degenerate equation (1.3) with $v = \hat{v}(x)$ and satisfy

$$\psi_1(x_0) = \psi_2(x_0). \tag{2.6}$$

Concerning the relative position of the curves $u = \psi_1(x)$ and $u = \psi_2(x)$ we assume

(A_5) (i) $\psi_1(x) > \psi_2(x)$ for $0 \le x < x_0$,
(ii) $\psi_1(x) < \psi_2(x)$ for $x_0 < x \le 1$.

From (A_1) and (A_5) it follows

$$\psi_2'(x_0) \ge \psi_1'(x_0). \tag{2.7}$$

We define the function $\hat{u}(x)$ by

$$\hat{u}(x) = \begin{cases} \psi_1(x) & \text{for} \quad 0 \leq x \leq x_0, \\ \psi_2(x) & \text{for} \quad x_0 \leq x \leq 1. \end{cases} \tag{2.8}$$

We note that $\hat{u}(x)$ is not necessarily differentiable at $x = x_0$.
To motivate the following assumptions we introduce the associated equation to system (1.2)

$$\frac{d^2u}{d\xi^2} = g(u, v, x, 0) \tag{2.9}$$

in which v, x are considered as parameters. From hypothesis (A_2) it follows that (2.9) has two intersecting families of equilibria the stability of which is determined by the sign of g_u at these families. The following asssumption describes the behavior of the sign of g_u at these families as a function of x and characterizes an exchange of stabilities of these families.

(A_6)

$$\begin{aligned} g_u(\psi_1(x), \hat{v}(x), x, 0) > 0 \quad & \text{for} \quad 0 \leq x < x_0, \\ g_u(\psi_1(x), \hat{v}(x), x, 0) < 0 \quad & \text{for} \quad x_0 < x \leq 1, \\ g_u(\psi_2(x), \hat{v}(x), x, 0) < 0 \quad & \text{for} \quad 0 \leq x < x_0, \\ g_u(\psi_2(x), \hat{v}(x), x, 0) > 0 \quad & \text{for} \quad x_0 < x \leq 1. \end{aligned}$$

From this assumption it follows

$$g_u(\psi_i(x_0), \hat{v}(x_0), x_0, 0) = 0, \quad i = 1, 2.$$

Definition 2.1 *Under the assumptions* (A_1) - (A_6) *the vector function* $(\hat{u}(x), \hat{v}(x))$ *is referred to as the composed stable solution of (2.2).*

For the sequel it is convenient to introduce the following notation: the hat-sign "ˆ" over g and f or some derivatives of g and f denotes that we have to consider the arguments (u, v, ε) at $(\hat{u}(x), \hat{v}(x), 0)$.

The composed stable solution satisfies

$$\begin{aligned} g(\hat{u}(x), \hat{v}(x), x, 0) =: \hat{g}(x) \quad & \equiv \quad 0 \text{ for } x \in [0, 1], \\ \hat{g}_u(x) > 0 \text{ for } x \neq x_0, \quad & \quad \hat{g}_u(x_0) = 0. \end{aligned} \tag{2.10}$$

In the sequel, the concept of ordered lower and upper solutions to the boundary value problem (2.1) plays a central role in our approach.

Definition 2.2 *The functions* $\alpha(x,\varepsilon) = (\alpha^u(x,\varepsilon), \alpha^v(x,\varepsilon)), \beta(x,\varepsilon) = (\beta^u(x,\varepsilon),\ \beta^v(x,\varepsilon))$ *which are continuous on* $[0,1] \times I^+_{\varepsilon_0}$ *and piecewise twice continuously differentiable with respect to* x *in* $(0,1)$ $\forall \varepsilon \in I^+_{\varepsilon_0}$ *are called lower and upper solutions of (2.1) respectively for* $\varepsilon \in I^+_{\varepsilon_0}$ *if they satisfy the following inequalitites*

$$\varepsilon^2 \frac{d^2\alpha^u}{dx^2} - g(\alpha^u, \alpha^v, x, \varepsilon) \geq \ 0 \ \geq \varepsilon^2 \frac{d^2\beta^u}{dx^2} - g(\beta^u, \beta^v, x, \varepsilon), \quad \text{for} \quad 0 < x < 1 \tag{2.11}$$

$$\frac{d^2\alpha^v}{dx^2} - f(\alpha^u, \alpha^v, x, \varepsilon) \geq \ 0 \ \geq \frac{d^2\beta^v}{dx^2} - f(\beta^u, \beta^v, x, \varepsilon), \quad \text{for} \quad 0 < x < 1 \tag{2.12}$$

$$\frac{d\beta^u}{dx}(0,\varepsilon) \leq 0 \leq \frac{d\alpha^u}{dx}(0,\varepsilon) \quad , \quad \frac{d\beta^u}{dx}(1,\varepsilon) \geq 0 \geq \frac{d\alpha^u}{dx}(1,\varepsilon), \tag{2.13}$$

$$\alpha^v(0,\varepsilon) \leq v^0 \leq \beta^v(0,\varepsilon) \quad , \quad \alpha^v(1,\varepsilon) \leq v^1 \leq \beta^v(1,\varepsilon). \tag{2.14}$$

At any point $x^* \in (0,1)$ *where* α *or* β *is not differentiable with respect to* x *we require*

$$\alpha'(x^* - 0) \leq \alpha'(x^* + 0), \quad \beta'(x^* - 0) \geq \beta'(x^* + 0). \tag{2.15}$$

They are called ordered lower and upper solutions if they additionally satisfy

$$\alpha(x,\varepsilon) \ \leq \ \beta(x,\varepsilon). \tag{2.16}$$

The following assumptions are used to construct ordered lower and upper solutions near the composed stable solution.

(A_7) There are positive constants μ, d_1, d_2 and ϱ such that for $x \in [0,1]$

- (i) $\mu - \hat{\varphi}_v(\hat{v}(x), x) \geq d_1$.
- (ii) $\hat{g}_{uu}(x_0)\mu^2 + 2\hat{g}_{uv}(x_0)\mu + \hat{g}_{vv}(x_0) \geq d_2$.
- (iii) $\hat{f}_u(x)\mu + \hat{f}_v(x) \geq -\pi^2 + \varrho$.

(A_8)

$$\hat{g}_\varepsilon(x_0) < 0. \tag{2.17}$$

By a general theorem (see for example [6], page 406) the existence of ordered lower and upper solutions to (2.1) for $\varepsilon \in I^+_{\varepsilon_0}$ implies the existence of at least one solution of (2.1) provided (g,f) has the property of quasimonotonicity.

Definition 2.3 *We call* $(g(u,v,x,\varepsilon), f(u,v,x,\varepsilon))$ *quasimonotone nonincreasing with respect to* (u,v) *in* $[\underline{u}(x,\varepsilon) \leq u \leq \overline{u}(x,\varepsilon)] \times [\underline{v}(x,\varepsilon) \leq v \leq \overline{v}(x,\varepsilon)]$ *for* $(x,\varepsilon) \in (0,1) \times I^+_\varepsilon$ *iff* $g(u,v,x,\varepsilon)$ *is nonincreasing in* v *for* $v \in [\underline{v}(x,\varepsilon), \overline{v}(x,\varepsilon)]$ *for any* $u \in [\underline{u}(x,\varepsilon), \overline{u}(x,\varepsilon)]$, *and* $f(u,v,x,\varepsilon)$ *is nonincreasing in* u *for* $u \in [\underline{u}(x,\varepsilon), \overline{u}(x,\varepsilon)]$ *for any* $v \in [\underline{v}(x,\varepsilon), \overline{v}(x,\varepsilon)]$.

Hence, we finally assume

(A_9) The vector function $(g(u,v,x,\varepsilon), f(u,v,x,\varepsilon))$ is quasimonotone nonincreasing in (u,v) in some neighborhood of the composed stable solution.

3 Existence and asymptotic behavior of the solution.

In this section we show that the boundary value problem (2.1) has a solution which is close to the composed stable solution for sufficiently ε.

Theorem 3.1 *Assume hypotheses* (A_1) - (A_9) *to be valid. Then there exists a sufficiently small* ε_1 *such that for* $0 < \varepsilon \le \varepsilon_1$ *the boundary value problem* (2.1) *has a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *satisfying for* $x \in [0,1]$

$$\begin{aligned} \lim_{\varepsilon\to 0} u(x,\varepsilon) &= \hat{u}(x), \\ \lim_{\varepsilon\to 0} v(x,\varepsilon) &= \hat{v}(x). \end{aligned} \tag{3.1}$$

Moreover we have for $x \in [0,1]$

$$\begin{aligned} u(x,\varepsilon) &= \hat{u}(x) + O(\sqrt{\varepsilon}), \\ v(x,\varepsilon) &= \hat{v}(x) + O(\sqrt{\varepsilon}). \end{aligned}$$

Proof. To prove our theorem we use the technique of lower and upper solutions. As already mentioned above, under our hypotheses the existence of ordered lower and upper solutions implies that there exists at least one solution $(u(x,\varepsilon), v(x,\varepsilon))$ of (2.1) satisfying

$$\alpha^u(x,\varepsilon) \le u(x,\varepsilon) \le \beta^u(x,\varepsilon),\ \alpha^v(x,\varepsilon) \le v(x,\varepsilon) \le \beta^v(x,\varepsilon).$$

For the construction of lower and upper solutions we will use the composed stable solution $(\hat{u}(x), \hat{v}(x))$. From the definition of $\hat{u}(x)$ in (2.8) and from (2.7) it follows that $\hat{u}(x)$ satisfies the inequality $\hat{u}'(x_0 - 0) \le \hat{u}'(x_0 + 0)$. Thus, $\hat{u}(x)$ fulfils the condition (2.15) at $x = x_0$ for the lower solution, but in the case $\hat{u}'(x_0 - 0) < \hat{u}'(x_0 + 0)$ it does not fulfil the condition (2.15) at $x = x_0$ for the upper solution. Therefore, to construct an upper solution we smooth $\hat{u}(x)$ by means of a known procedure (see, for example, [7]).

Using the function

$$\omega(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi} \exp(-s^2)ds \tag{3.2}$$

where ξ is defined by

$$\xi = (x - x_0)/\varepsilon \tag{3.3}$$

we introduce the smooth function

$$\tilde{u}(x,\varepsilon) := \psi_1(x)\omega(-\xi) + \psi_2(x)\omega(\xi) \tag{3.4}$$

satisfying $\tilde{u}'(x-0,\varepsilon) = \tilde{u}'(x+0,\varepsilon)$ for all x. It is easy to show [7] that

$$\tilde{u}(x,\varepsilon) = \hat{u}(x) + O(\varepsilon). \tag{3.5}$$

Now we construct an upper solution $(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon))$ of (2.1) by using the smooth function $\tilde{u}(x,\varepsilon)$ in the following form

$$\begin{aligned} \beta^u(x,\varepsilon) &= \tilde{u}(x,\varepsilon) + \sqrt{\varepsilon}\gamma\mu h(x) + \varepsilon(e^{-\frac{\kappa x}{\varepsilon}} + e^{-\frac{\kappa(1-x)}{\varepsilon}}), \\ \beta^v(x,\varepsilon) &= \hat{v}(x) + \sqrt{\varepsilon}\gamma h(x) \end{aligned} \tag{3.6}$$

where

$$h(x) := \sin\frac{\pi(x+\delta)}{1+2\delta}, \quad \delta > 0 \tag{3.7}$$

is positive for $x \in [0,1]$, μ is defined in assumption (A_7), and the positive constants γ , κ and δ will be chosen in an appropriate way later.
From $\hat{v}(0) = v_0, \hat{v}(1) = v_1$ and from (3.6) it follows that the inequality (2.14) for β^v is fulfilled.
By (3.2) - (3.4) we have

$$\frac{d\tilde{u}}{dx} = \psi_1'(x)\omega(-\xi) + \psi_2'(x)\omega(\xi) + \frac{1}{\varepsilon\sqrt{\pi}}\exp{(-\xi^2)}(\psi_2(x) - \psi_1(x)). \tag{3.8}$$

According to (2.5) and (2.6) there is a positive constant $\tilde{c}$ such that for $x \in [0,1]$

$$\begin{aligned} |\psi_2(x) &- \psi_1(x)| = |\psi_2(x_0) + \psi_2'(x_0 + \theta_1(x-x_0))(x-x_0) \\ &- \psi_1(x_0) - \psi_1'(x_0 + \theta_2(x-x_0))(x-x_0)| \le \tilde{c}|x-x_0|. \end{aligned} \tag{3.9}$$

Hence, by (3.8), (3.9) and (3.3) there exists a positive constant c_1 such that

$$\left|\frac{d\tilde{u}}{dx}\right| \le c_1 + \frac{\tilde{c}}{\sqrt{\pi}}|\xi|\exp{(-\xi^2)} \ \le c_1 + \tilde{c}. \tag{3.10}$$

If we differentiate β^u with respect to x at $x=0$ and $x=1$ respectively we get

$$\frac{d\beta^u}{dx}(0,\varepsilon) = \frac{d\tilde{u}}{dx}(0,\varepsilon) - \kappa + O(\sqrt{\varepsilon}), \quad \frac{d\beta^u}{dx}(1,\varepsilon) = \frac{d\tilde{u}}{dx}(1,\varepsilon) + \kappa + O(\sqrt{\varepsilon}).$$

Consequently, by (3.10), $\frac{d\beta^u}{dx}(x,\varepsilon)$ is negative at $x=0$ and positive at $x=1$, respectively for κ sufficiently large and ε sufficiently small, i.e. the inequalities for β^u in (2.13) are fulfilled.
Now we check that β^u and β^v satisfy the inequalities (2.11). From (3.6) - (3.9) we obtain that there is a positive constant c_2 such that for $\varepsilon \in I_{\varepsilon_0}$

$$
\begin{aligned}
\varepsilon^2 \frac{d^2\beta^u}{dx^2} &= \varepsilon^2\Big\{[\psi_1''(x)\omega(-\xi) + \psi_2''(x)\omega(\xi)] + \\
&+ \varepsilon^{-1}\frac{2}{\sqrt{\pi}}(\psi_2'(x) - \psi_1'(x))\exp(-\xi^2) + \\
&- \varepsilon^{-2}\frac{2}{\sqrt{\pi}}(\psi_2(x) - \psi_1(x))\xi\exp(-\xi^2) + \\
&+ \sqrt{\varepsilon}\mu\gamma h''(x) + \kappa^2\varepsilon^{-1}(\exp(-\frac{\kappa x}{\varepsilon}) + \exp(-\frac{\kappa(1-x)}{\varepsilon}))\Big\} \le c_2(1+\varepsilon^{3/2}\gamma)\varepsilon.
\end{aligned} \tag{3.11}
$$

By assumption (A_1) and taking into account (3.5) we have the representation

$$
\begin{aligned}
g(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) &= g(\hat{u}(x) + \sqrt{\varepsilon}\mu\gamma h(x) + O(\varepsilon), \hat{v}(x) + \sqrt{\varepsilon}\gamma h(x), x, \varepsilon) = \\
&= g(\hat{u}(x) + \sqrt{\varepsilon}\mu\gamma h(x), \hat{v}(x) + \sqrt{\varepsilon}\gamma h(x), x, 0) + r(x,\varepsilon)
\end{aligned}
$$

where $r(x,\varepsilon)$ satisfies $|r(x,\varepsilon)| \le c_3\varepsilon$ with some constant c_3. Using this representation and (2.10) we get

$$
\begin{aligned}
g(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) &= g(\hat{u}, \hat{v}, x, 0) + [\hat{g}_u(x)\mu + \hat{g}_v(x)]\sqrt{\varepsilon}\gamma h(x) + \\
&+\frac{1}{2}\varepsilon\gamma^2[\hat{g}_{uu}(x)\mu^2 + 2\hat{g}_{uv}(x)\mu + \hat{g}_{vv}(x)]h^2(x) + \\
&+r(x,\varepsilon) + o(\varepsilon) = \\
&= \hat{g}_u(x)(\mu - \hat{\varphi}_v(x))\gamma h(x)\sqrt{\varepsilon} + \\
&+\frac{\gamma^2}{2}[\hat{g}_{uu}(x)\mu^2 + \ 2\hat{g}_{uv}(x)\mu + \hat{g}_{vv}(x)]h^2(x)\varepsilon + r(x,\varepsilon) + o(\varepsilon).
\end{aligned} \tag{3.12}
$$

From (3.7) it follows that there is a positive constant $c_4 = c_4(\delta)$ such that

$$
h(x) \ge c_4(\delta) \quad \text{for} \quad x \in [0,1]. \tag{3.13}
$$

Let ν be any positive number such that $I_\nu := (x_0 - \nu, x_0 + \nu) \subset (0,1)$. By assumption $(A_7) - -(ii)$ for sufficiently small ν there is a positive constant $c_5 = c_5(\nu)$ such that

$$
\hat{g}_{uu}(x)\mu^2 + 2\hat{g}_{uv}(x)\mu + \hat{g}_{vv}(x) \ge c_5(\nu) \quad \text{for} \quad x \in I_\nu. \tag{3.14}
$$

Therefore, taking into account that $\hat{g}_u(x) \ge 0$ (see (2.10)), $\mu - \hat{\varphi}_v(x) > 0$ (see (A_7)–(i)) in the interval I_ν, we have

$$
g(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) \ge (c_5(\nu)c_4^2(\delta)\gamma^2/2 - c_3)\varepsilon + o(\varepsilon). \tag{3.15}
$$

From (3.11) and (3.15) we get for $x \in I_\nu$

$$
\varepsilon^2 \frac{d^2\beta^u}{dx^2} - g(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) \le (c_2(1+\varepsilon^{3/2}\gamma) + c_3 - c_4^2(\delta)c_5(\nu)\gamma^2/2)\varepsilon + o(\varepsilon) \tag{3.16}
$$

If we choose γ so large that

$$
c_1(1+\varepsilon^{3/2}\gamma) + c_3 - c_4^2(\delta)c_5(\nu)\gamma^2/2 < 0
$$

then (β^u, β^v) satisfies the inequality (2.11) for $x \in I_\nu$ and sufficiently small ε.
From the assumptions (A_1) and $(A_7) - -(i)$ and from the inequality $\hat{g}_u(x) > 0$ for $x \neq x_0$ it follows that there is a positive constant $c_6 = c_6(\nu)$ such that

$$\hat{g}_u(x)\mu + \hat{g}_v(x) = \hat{g}_u(x)(\mu - \hat{\varphi}_v) \geq c_6(\nu) \quad \text{for} \quad x \in I_\nu^c := [0, x_0 - \nu] \cup [x_0 + \nu, 1]. \tag{3.17}$$

Then, by (3.12) and (3.17), we have for $x \in I_\nu^c$

$$g(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) \geq c_6(\nu)c_4(\delta)\gamma\sqrt{\varepsilon} + o(\sqrt{\varepsilon}). \tag{3.18}$$

Therefore we get

$$\varepsilon^2 \frac{d^2\beta^u}{dx^2} - g(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) \leq -c_6(\nu)c_4(\delta)\gamma\sqrt{\varepsilon} + o(\sqrt{\varepsilon}) \leq 0 \tag{3.19}$$

for sufficiently small ε. Consequently, (β^u, β^v) satisfies inequality (2.11) for $x \in (0,1)$.

Concerning $\beta^v(x,\varepsilon)$ we obtain from (3.6), (3.7), and (2.3)

$$\begin{aligned}
&\frac{d^2\beta^v}{dx^2} - f(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) = \\
&= \frac{d^2\hat{v}}{dx^2} + \sqrt{\varepsilon}\gamma h''(x) - \{f(\hat{u}(x), \hat{v}(x), x, 0) + (\hat{f}_u(x)\mu + \hat{f}_v(x))\sqrt{\varepsilon}\gamma h(x) + o(\sqrt{\varepsilon})\} = \\
&= \gamma\sqrt{\varepsilon}\{h''(x) - [\hat{f}_u(x)\mu + \hat{f}_v(x)]h(x)\} + o(\sqrt{\varepsilon}) = \\
&= \gamma\sqrt{\varepsilon}\left[-\left(\frac{\pi}{1+2\delta}\right)^2 - (\hat{f}_u(x)\mu + \hat{f}_v(x))\right] \sin\frac{\pi(x+\delta)}{1+2\delta} + o(\sqrt{\varepsilon}).
\end{aligned}$$

It follows from assumption (A$_7$)–(iii) that we can choose δ sufficiently small to satisfy

$$-\left(\frac{\pi}{1+2\delta}\right)^2 - (\hat{f}_u(x)\mu + \hat{f}_v(x)) \leq -\rho_1, \quad \rho_1 > 0. \tag{3.20}$$

Consequently, we get the inequality

$$\frac{d^2\beta^v}{dx^2} - f(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon), x, \varepsilon) \leq 0 \quad \text{for} \quad x \in (0,1)$$

and this implies that $(\beta^u(x,\varepsilon), \beta^v(x,\varepsilon)))$ satisfies the inequalities (2.11) - (2.16).

A lower solution $(\alpha^u(x,\varepsilon), \alpha^v(x,\varepsilon))$ will be constructed in the form

$$\begin{aligned}
\alpha^u(x,\varepsilon) &:= \hat{u}(x) - \varepsilon\mu\gamma h(x) - \varepsilon(e^{-\frac{\kappa x}{\varepsilon}} + e^{-\frac{\kappa(1-x)}{\varepsilon}}), \\
\alpha^v(x,\varepsilon) &:= \hat{v}(x) - \varepsilon\gamma h(x)
\end{aligned} \tag{3.21}$$

where $h(x)$ is the same function as in (3.7), μ is the constant from assumption (A_7), κ and γ are some positive numbers to be chosen in an appropriate way. It is obvious that the inequalities (2.14) for α^v are fulfilled.

If we differentiate α^u with respect to x at $x = 0$ and $x = 1$ respectively we get

$$\frac{d\alpha^u}{dx}(0,\varepsilon) = \frac{d\hat{u}}{dx}(0,\varepsilon) + \kappa + O(\varepsilon), \quad \frac{d\alpha^u}{dx}(1,\varepsilon) = \frac{d\hat{u}}{dx}(0,\varepsilon) - \kappa + O(\varepsilon).$$

Consequently, for κ sufficiently large and ε sufficiently small, $\frac{d\alpha^u}{dx}(x,\varepsilon)$ is positive at $x=0$ and negative at $x=1$, respectively, i.e. the inequalities (2.13) for α^u are satisfied.
In order to check that $(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon))$ defined by (3.21) satisfies conditions (2.11) and (2.12) we substitute (3.21) into (2.11). We get

$$\begin{aligned}&\varepsilon^2\ \frac{d^2\alpha^u}{dx^2}-g(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon),x,\varepsilon)=\varepsilon^2\ \frac{d^2\hat{u}}{dx^2}-g(\hat{u},\hat{v},x,0)-\varepsilon\hat{g}_\varepsilon(x)+\\&+\varepsilon\gamma h(x)\{\hat{g}_u(x)\mu+\hat{g}_v(x)\}+\\&\varepsilon e^{-\frac{\kappa x}{\varepsilon}}(\hat{g}_u(x)-\kappa^2)+\varepsilon e^{-\frac{\kappa(1-x}{\varepsilon}}(\hat{g}_u(x)-\kappa^2)+O(\varepsilon^2).\end{aligned}\tag{3.22}$$

Taking into account (2.10), (3.13), (3.17) and the inequality $|\hat{g}_\varepsilon(x)|\le c_7$ where c_7 is some positive constant, we have for $x\in I_\nu^c$

$$\begin{aligned}&\varepsilon^2\ \frac{d^2\alpha^u}{dx^2}-g(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon),x,\varepsilon)\ge\\&\ge\varepsilon[\gamma c_6(\nu)c_4(\delta)+(e^{-\frac{\kappa x}{\varepsilon}}+e^{-\frac{\kappa(1-x)}{\varepsilon}})(\hat{g}_u(x)-\kappa^2)-c_7]+O(\varepsilon^2).\end{aligned}$$

From the boundedness of $\hat{g}_u(x)$ in [0,1] we get that we can choose γ suffciently large such that the expression in the brackets is positive, that is for suffiently small ε we have

$$\varepsilon^2\ \frac{d^2\alpha^u}{dx^2}-g(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon),x,\varepsilon)\ge 0\quad\text{for}\quad x\in I_\nu^c$$

In order to check (2.11) for $x\in I_\nu$ we note that by assumption (A_7)–(i) $\hat{g}_u(x)\mu+\hat{g}_v(x)\ge 0$ and that the exponential terms in (3.22) are less than any order of ε. Therefore, we get from (3.22)

$$\varepsilon^2\ \frac{d^2\alpha^u}{dx^2}-g(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon),x,\varepsilon)\ge-\varepsilon\hat{g}_\varepsilon(x)+O(\varepsilon^2).$$

From (2.17) it follows for sufficiently small ν that there is a positive constant $c_8=c_8(\nu)$ such that $-\hat{g}_\varepsilon(x)\ge c_8$ for $x\in I_\nu$. Hence, $-\varepsilon\hat{g}_\varepsilon(x)+O(\varepsilon^2)\ge 0$ for sufficiently small ε and for $x\in I_\nu$. Thus, the validity of the inequalities (2.11) for α^u in I_ν follows from (A_8).

Substituting (3.21) into (2.12) we get

$$\begin{aligned}&\frac{d^2\alpha^v}{dx^2}-f(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon),x,\varepsilon)=\frac{d^2\hat{v}}{dx^2}-\hat{f}(x)-\varepsilon\hat{f}_\varepsilon(x)+\\&-\varepsilon\gamma\{h''(x)-(\hat{f}_u(x)\mu+\hat{f}_v(x))h(x)\}+\\&+\varepsilon\hat{f}_u(x)(e^{-\frac{\kappa x}{\varepsilon}}+e^{-\frac{\kappa(1-x)}{\varepsilon}})+O(\varepsilon^2).\end{aligned}$$

By the boundedness of $\hat{f}_\varepsilon(x)$ and by (3.20) we can choose γ sufficiently large such that we have

$$\frac{d^2\alpha^v}{dx^2}-f(\alpha^u(x,\varepsilon),\alpha^v(x,\varepsilon),x,\varepsilon)\ge 0.$$

Thus, we have constructed a lower (3.21) and an upper (3.6) solution to (2.1).
From these relations it follows that the inequality (2.16) is satisfied, i.e. α and β are ordered lower and upper solutions to (2.1). Therefore, we can conclude that there exists a solution $(u(x,\varepsilon), v(x,\varepsilon)$ of (2.1) satisfying

$$\alpha^u(x,\varepsilon) \le u(x,\varepsilon) \le \beta^u(x,\varepsilon), \alpha^v(x,\varepsilon) \le v(x,\varepsilon) \le \beta^v(x,\varepsilon). \tag{3.23}$$

The relations (3.6) and (3.21) show that

$$\begin{aligned} \alpha^u(x,\varepsilon) = \hat{u}(x) - r_-^u(x,\varepsilon); \beta^u(x,\varepsilon) = \hat{u}(x) + r_+^u(x,\varepsilon), \\ \alpha^v(x,\varepsilon) = \hat{v}(x) - r_-^v(x,\varepsilon); \alpha^v(x,\varepsilon) = \hat{v}(x) - r_-^v(x,\varepsilon) \end{aligned} \tag{3.24}$$

where r_-^u and r_-^v are positive functions satisfying $r_-^{u,v} = O(\varepsilon)$, while r_+^u and r_+^v are positive functions satisfying $r_+^{u,v} = O(\sqrt{\varepsilon})$. Obviously, we get from (3.23) and (3.24)

$$u(x,\varepsilon) = \hat{u}(x) + O(\sqrt{\varepsilon}), v(x,\varepsilon) = \hat{v}(x) + O(\sqrt{\varepsilon})$$

and consequently the relations (3.1) hold. This completes the proof of Theorem 3.1.

Remark 3.2 *If we assume that the functions $\psi_1(x)$ and $\psi_2(x)$ intersect for $x = x_0$ with the same slope then the proof can be simplified because we do not need the smoothing procedure.*

Remark 3.3 *Concerning the assumption (A_9) it should be noted that the property of quasimonotonicity of the vector function (g, f) in (u, v) is required only in the region bounded by lower and upper solutions. We use this fact in the next section.*

4 Examples.

Example 4.1. We study the boundary value problem

$$\begin{aligned} \varepsilon^2 \frac{d^2u}{dx^2} &= u(u-v) - \varepsilon I(x), \\ \frac{d^2v}{dx^2} &= \delta, \qquad x \in (0,1) \\ u'(0) &= u'(1) = 0, \quad v(0) = -1, \; v(1) = 1 \end{aligned} \tag{4.1}$$

where $I : [0,1] \to R^+$ is positive and continuous. System (4.1) follows from (1.7) if we set $g_1(u) \equiv 0$, $g_2(u-v) \equiv 0$, $r(u, u-v) \equiv u(u-v)$, $I_a(x) = I(x)/\varepsilon$, $I_b(x) = I_a(x) + \delta$, where δ is positive. Consequently, system (4.1) describe the steady-state behavior of a reaction-diffusion system with pure fast bimolecular reactions.

Taking into account that the boundary value problem

$$\begin{aligned} \frac{d^2v}{dx^2} &= \delta, \quad x \in (0,1) \\ v(0) &= -1, \ v(1) = 1 \end{aligned}$$

has the solution

$$\hat{v}(x) \equiv \frac{\delta}{2}x^2 + \frac{4-\delta}{2}x - 1 \tag{4.2}$$

then we can reduce the boundary value problem (4.1) to the following one

$$\begin{aligned} \varepsilon^2 \frac{d^2u}{dx^2} &= u(u - \hat{v}(x)) - \varepsilon I(x) \\ u'(0) &= u'(1) = 0. \end{aligned} \tag{4.3}$$

It is clear that Theorem 3.1 can be applied also to the boundary value problem (4.3) containing only fast variables. In that case the functions φ_1 and φ_2 coincide with the functions ψ_1 and ψ_2, the conditions (A_4) and (A_9) are trivially fulfilled, conditions (A_7)–(i) and (A_7)–(ii) can be satisfied by choosing $\mu \geq d_1, \ \varrho < \pi^2$.
The degenerate equation to (4.3) has the solutions $u = \varphi_1(x) \equiv 0, \ u = \varphi_2(x) \equiv \hat{v}(x)$ intersecting at $x = x_0 := (-4 + \delta + \sqrt{16 + \delta^2})/2\delta \in (0,1)$. The conditions (A_5) and (A_6) can be easily verified. The composed stable solution $\hat{u}(x)$ is defined by

$$\hat{u}(x) \ := \begin{cases} 0 & \text{for} \quad 0 \leq x \leq x_0, \\ \hat{v}(x) & \text{for} \quad x_0 \leq x \leq 1. \end{cases} \tag{4.4}$$

Assumption (A_7)–(ii) reads $2\mu^2 \geq d_2$. (A_8) is satisfied since $I(x) > 0$.

By Theorem 3.1 we get that (4.3) has a solution $u(x, \varepsilon)$ satisfying

$$u(x, \varepsilon) = \hat{u}(x) + O(\sqrt{\varepsilon}).$$

Example 4.2. We consider the boundary value problem

$$\begin{aligned} \varepsilon^2 \frac{d^2u}{dx^2} &= u(u - v) - \varepsilon I(x) - 3\varepsilon^2 u, \\ \frac{d^2v}{dx^2} &= -3u, \qquad\qquad x \in (0,1), \\ u'(0) &= u'(1) = 0, \ v(0) = -1, \ v(1) = 1 \end{aligned} \tag{4.5}$$

where $I(x)$ is a positive continuous function. System (4.5) follows from (1.7) if we set $g_1(u) \equiv -3u, g_2(v) \equiv 0, \ r(u, u - v) \equiv u(u - v), \ I_a(x) \equiv I_b(x) = \frac{1}{\varepsilon}I(x)$. The degenerate equation $u(u-v)=0$ has exactly two solutions $u=\varphi_1(v,x)\equiv 0$ and $u=\varphi_2(v,x)\equiv v$ intersecting at the x-axis in the u,v,x-space, i.e. $k(x)\equiv 0$. In order to construct the composed stable solution we consider the following boundary value problems.

The function $v_1(x)$ is defined by the boundary value problem

$$\begin{aligned} \frac{d^2v}{dx^2} &= 0, \quad x\in(0,x_0),\\ v(0) &= -1,\ v(x_0)=0 \end{aligned} \tag{4.6}$$

where x_0 is any point of the interval $(0,1)$. This problem has the solution

$$v_1(x)=\frac{x-x_0}{x_0}.$$

The function $v_2(x)$ is determined by the boundary value problem

$$\begin{aligned} \frac{d^2v}{dx^2} &= -3v, \quad x\in(x_0,1),\\ v(x_0) &= 0, \quad v(1)=1. \end{aligned} \tag{4.7}$$

It is easy to show that

$$v_2(x)=\frac{\sin(\sqrt{3}(x-x_0))}{\sin(\sqrt{3}(1-x_0))}$$

solves (4.7). In order to construct $\hat{v}(x)$ we must determine the point x_0 such that

$$v_1'(x_0)=v_2'(x_0).$$

It follows that x_0 satisfies the equation

$$\sqrt{3}x=\sin(\sqrt{3}(1-x))$$

which has a unique solution $x_0\in(0,1)$ where $x_0\approx 0.46$. Therefore, we have

$$\hat{v}(x):=\begin{cases} \frac{x-x_0}{x_0} & \text{for} \quad 0\le x\le x_0,\\ \frac{\sin(\sqrt{3}(x-x_0))}{\sin(\sqrt{3}(1-x_0))} & \text{for} \quad x_0\le x\le 1. \end{cases}$$

We note that $\hat{v}(x)$ is negative for $0\le x<x_0$ and positive for $x_0<x\le 1$. Hence, we have by (2.4) and (2.5)

$$\begin{aligned} \hat{\varphi}_v(x) \quad &:= \quad \begin{cases} 0 \text{ for } 0\le x<x_0,\\ 1 \text{ for } x_0<x\le 1, \end{cases}\\ \psi_1(x) \quad &\equiv 0 \quad , \quad \psi_2(x)\equiv\hat{v}(x). \end{aligned} \tag{4.8}$$

Thus, the assumptions (A_1) - (A_5) are satisfied. From (4.5) we get $g_u=2u-v$. Therefore, we have

$$g_u(\psi_1(x),\hat{v}(x),x,0)\equiv-\hat{v}(x),\ g_u(\psi_2(x),\hat{v}(x),x,0)\equiv\hat{v}(x),$$

that means, assumption (A_6) holds.

The function $\hat{u}(x)$ reads

$$\hat{u}(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le x_0, \\ \frac{\sqrt{3}\sin(\sqrt{3}(x-x_0))}{\sin(\sqrt{3}(1-x_0))} & \text{for} \quad x_0 \le x \le 1. \end{cases} \tag{4.9}$$

Taking into account (4.8) and

$$\hat{g}_{uu}(x_0)\mu^2 + 2\hat{g}_{uv}(x_0)\mu + \hat{g}_{vv}(x_0) \equiv 2\mu(\mu-1),$$

$$\hat{f}_u(x)\mu + \hat{f}_v(x) \equiv -3\mu$$

assumption (A_7) is satisfied if we set $\mu = 3 + d_1$ where $0 < d_1 < \frac{\pi^2-9}{3}$.
From the positivity of the function $I(x)$ it follows immediately that $\hat{g}_\varepsilon(x_0) = -I(x_0) < 0$, i.e. the assumption (A_8) is fulfilled. Finally, from (4.5) we get $g_v = -3u,\ f_u \equiv -3$, that is, the vector function (g, f) is quasimonotone nonincreasing only for $u \ge 0$. It is easy to check that we can take in our example $\alpha^u = \hat{u}(x),\ \alpha^v = \hat{v}(x)$, in particular we have

$$\frac{d\alpha^u}{dx}(0) = \hat{u}'(0) = 0, \frac{d\alpha^u}{dx}(1) = \hat{u}'(1) = \frac{3\cos(\sqrt{3}(1-x_0))}{\sin(\sqrt{3}(1-x_0))} < 0 \quad \text{for} \quad \frac{\pi}{2} < \sqrt{3}(1-x_0) < \pi$$

. Then (g, f) will be quasimonotone nonincreasing in the region bounded by lower and upper solutions as the function $\hat{u}(x)$ is obviously nonnegative (see (4.9)).
By Remark 3.3, we can apply Theorem 3.1 and get that the boundary value problem (4.5) has a solution $(u(x,\varepsilon), v(x,\varepsilon))$ satisfying

$$u(x,\varepsilon) = \begin{cases} 0 + O(\sqrt{\varepsilon}) & \text{for} \quad 0 \le x \le x_0, \\ \frac{\sqrt{3}\sin(\sqrt{3}(x-x_0))}{\sin(\sqrt{3}(1-x_0))} + O(\sqrt{\varepsilon}) & \text{for} \quad x_0 \le x \le 1, \end{cases}$$

$$v(x,\varepsilon) = \begin{cases} \frac{x-x_0}{x_0} + O(\sqrt{\varepsilon}) & \text{for} \quad 0 \le x \le x_0, \\ \frac{\sin(\sqrt{3}(x-x_0))}{\sin(\sqrt{3}(1-x_0))} + O(\sqrt{\varepsilon}) & \text{for} \quad x_0 \le x \le 1. \end{cases}$$

References

[1] A.B. Vasil'eva and V.F. Butuzov, Asymptotic Methods in the Theory of Singular Perturbation, (in russian), Vyshaja Shkola, Moscow, 1990.

[2] R.E. O'Malley, Jr., Singular Perturbation Methods for Ordinary Differential Equations, Springer-Verlag, New York, 1991.

[3] D.R. Smith, Singular Perturbation Theory. Cambridge University Press, Cambridge, 1985.

[4] N.N. Nefedov, K.R. Schneider and A. Schuppert, Jumping behavior in singularly perturbed systems modelling bimolecular reactions, Weierstraß–Institut für Angewandte Analysis und Stochastik, Berlin, Preprint No. 137, 1994.

[5] V.F. Butuzov, N.N. Nefedov, Singularly perturbed boundary value problems for second order equations in case of exchange of stability (in russian). Mat. Zam. (to appear)

[6] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York and London, 1992.

[7] V.F. Butuzov, A.V. Nesterov, On some singularly perturbed problem with non-smooth boundary functions (in russian). Dokl. Akad. Nauk SSSR, 263, 786-789 (1982).