

Smooth density field of catalytic super-Brownian motion

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Abstract

Given an (ordinary) super-Brownian motion (SBM) ϱ on \mathbb{R}^d of dimension $d = 2, 3$, we consider a (catalytic) SBM X^e on \mathbb{R}^d with “local branching rates” $\varrho_s(d\mathbf{x})$.

We show that X_t^e is absolutely continuous with a density function ξ_t^e , say. Moreover, there exists a version of the map $(t, z) \mapsto \xi_t^e(z)$ which is C^∞ and solves the heat equation off the catalyst ϱ , more precisely, off the (zero set of) closed support of the time-space measure $ds \varrho_s(d\mathbf{x})$.

Using self-similarity, we apply this result to answer the question of the long-term behavior of X^e in dimension $d = 2$: If ϱ and X^e start with a Lebesgue measure, then X_T^e converges (persistently) as $T \rightarrow \infty$ towards a random multiple of Lebesgue measure.

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1 Introduction

1.1 Motivation and sketch of results

Consider continuous super-Brownian motion $\varrho = (\varrho_t)_{t \geq 0}$ in \mathbb{R}^d with a constant branching rate. Roughly speaking, the *catalytic super-Brownian motion* $X^\ell = (X_t^\ell)_{t \geq 0}$ is a continuous super-Brownian motion in \mathbb{R}^d with local branching rate “proportional to” ϱ . A rigorous construction can be found in [DF97].

In [DF97] also the study of longtime behavior of X^ℓ was initiated, and then continued in [DF96] and [EF96]. From these papers it is known that if both initial states ϱ_0 and X_0^ℓ are Lebesgue measures ℓ_c and ℓ_r , respectively, then X^ℓ is *persistent* in all three dimensions $d \leq 3$ of its non-trivial existence. (In $d = 3$ the catalyst process ϱ was actually started from its steady state rather than from ℓ_c at time zero; this simplification is of course not possible in lower dimensions where ϱ clusters in the longtime limit, hence dies out locally.)

Here persistence means that all weak limit points of X_T^ℓ as $T \rightarrow \infty$ have the full intensity measure ℓ_r again. In dimensions one and three the stronger result of persistent *convergence* has been shown in ([DF97, DF96]). For dimension $d = 2$ persistence of X^ℓ was proved in ([EF96]). The approach of [EF96] was to show the relative compactness of the set of laws of random second moments by p.d.e. methods. However, uniqueness of the limit point, and hence convergence, remained open.

In dimension $d = 2$, the process X^ℓ has a self-similarity property that connects the long-term behavior of X^ℓ with local properties at a fixed time. Thus, as noted in [DF96, Remark 14], persistent convergence of X_T^ℓ as $T \rightarrow \infty$ is equivalent to the existence of the limit $\xi_1^\ell(0)$ of $(2\varepsilon)^{-2} X_1^\ell((-\varepsilon, \varepsilon)^2)$ as $\varepsilon \downarrow 0$, with full expectation, and hence to the absolute continuity of X_1^ℓ . Our main objective in this paper is to show that X_1^ℓ is absolutely continuous.

It is well-known that the (continuous) SBM with constant branching rate has *absolutely continuous states* only in dimension one. In $d = 1$ actually “every” catalytic SBM has densities ([DFR91]). [DF95] construct *higher-dimensional* catalytic SBM (with finite variance branching) with absolutely continuous states where the branching rate is given by a certain class of additive functionals of Brownian motion. This class includes catalysts concentrated on hyperplanes. They show absolute continuity via constructing fundamental solutions of the related cumulant equation.

Recently Delmas [Del96] considered a class of time-independent catalysts in \mathbb{R}^d with carrying Hausdorff dimension greater than $d - 2$. He shows that the reactant has a smooth density off the catalyst. His technique is a refinement of the *method of Brownian excursions*, introduced by [FL95] for a single point-catalytic model in $d = 1$. The procedure in those two papers is first to determine the (singular) occupation density measures λ , say, on the (time-independent) catalyst, and then to represent the SBM by means of Brownian excursion densities off the catalyst (supported by a Lebesgue zero set) starting with random

masses according to λ . Clearly, excursion densities are smooth and satisfy the heat equation. At least at a heuristic level, this makes clear that in these cases a smooth density field exists.

Our strategy is to first show in $d = 2, 3$ that X^ℓ has densities in an L^2 -sense on the complement of the support of ϱ . Next we use a modification of Delmas' representation of catalytic SBM "off the catalyst" on a local level to derive our *main result*. Namely, we show that off the catalyst, X^ℓ has a *smooth density field* ξ^ℓ that solves the heat equation (Theorem 1 at page 6).

Finally we use this result to answer the open question mentioned above: In two dimensions, if we start ϱ and X^ℓ in Lebesgue measures, then X_T^ℓ converges in law to the random multiple $\xi_1^\ell(0)$ of the (normed) Lebesgue measure ℓ (Corollary 2 (b)).

1.2 Informal description of the model

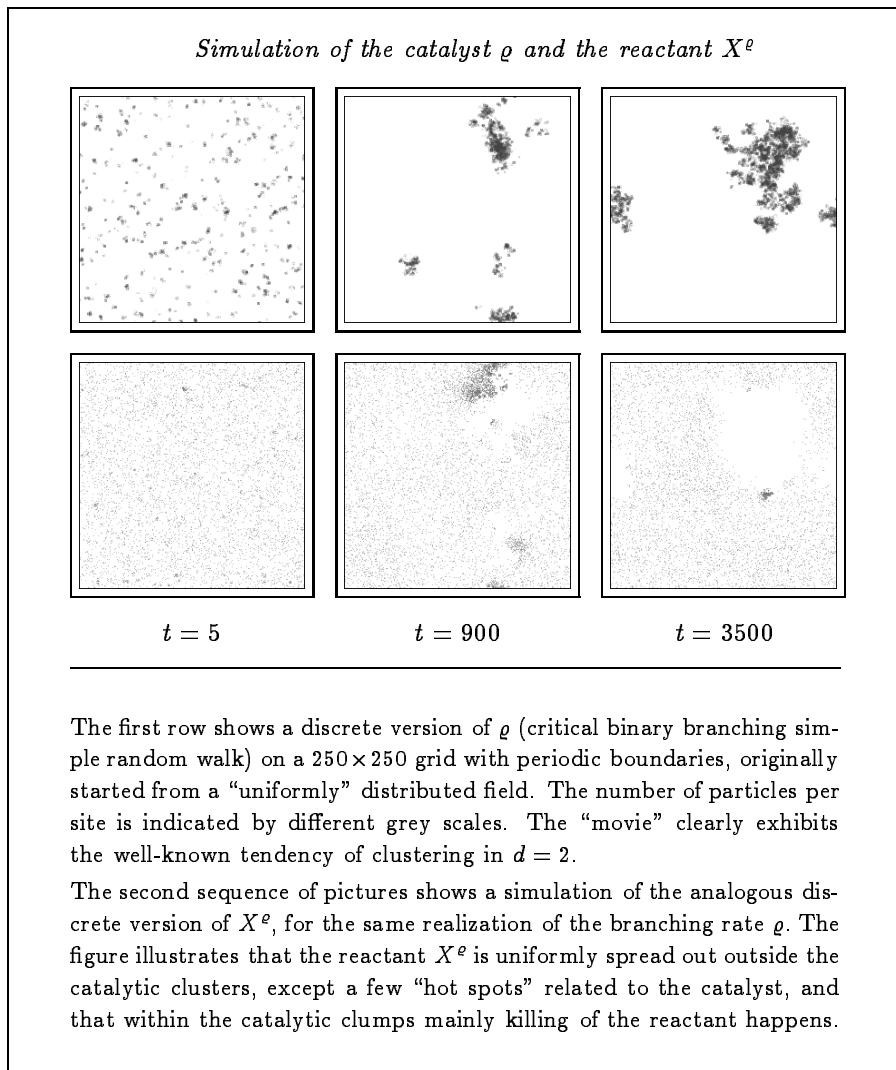
We consider a stochastic model for a *chemical (or biological) diffusion-reaction system* of two substances (or species) C and R, say. While C evolves independently of R, the *reaction* of R is *catalyzed* by C, that is takes place locally only in the presence of C but without affecting C.

The mathematical model that we choose for the catalyst is the so-called *super-Brownian motion* (SBM) ϱ . It arises as the high density short lifetime limit of *branching Brownian motion*. The latter is an (infinite) particle system, where the particles move around in \mathbb{R}^d according to independent Brownian motions. The catalyst particles die with a constant rate γ , say, and are replaced at the location of their death by zero or two offspring, each possibility occurring with probability $\frac{1}{2}$ (critical binary branching). The offspring continue to evolve in the same manner as their parent. Now assign the mass $\varepsilon > 0$ to each particle and replace the branching rate γ by γ/ε . Then (see, e.g., [Daw93, Section 4.4]) SBM ϱ is the limiting process if we let $\varepsilon \downarrow 0$ (provided that the initial states converge). Summarizing, the *catalyst* ϱ arises as a diffusion approximation to a critical binary branching Brownian motion with constant branching rate. For background on SBM we recommend [Daw93].

The mathematical model X^ℓ for the *reactant* is also SBM, however the branching rate of an "infinitesimal reactant particle" is the local concentration of catalytic matter. Consequently, the heuristic picture is the same except that the reactant particles die only when they are in contact with the catalyst. The catalyst itself varies in time and space and concentrates in some localized regions.

The model is interesting only in dimensions $d \leq 3$. Roughly speaking, the catalyst is a $(d \wedge 2)$ -dimensional object in \mathbb{R}^d , thus a reactant particle (which performs Brownian motion) cannot meet the catalyst if $d \geq 4$. Hence, in $d \geq 4$, the "reactant" X^ℓ is only the deterministic heat flow.

A mathematical approach to this "*one-way interaction*" model is possible by means of Dynkin's additive functional approach to superprocesses ([Dyn91]).



In fact, given the medium ϱ , an intrinsic X^ϱ -particle (reactant) following a Brownian path W branches according to the clock given by the *collision local time*, $L_{[W, \varrho]}(ds)$, of W with ϱ ([BEP91]). Somewhat more formally:

$$L_{[W, \varrho]}(ds) = ds \int \varrho_s(dy) \delta_y(W_s). \quad (1)$$

For sufficiently nice initial states of ϱ , these collision local times $L_{[W, \varrho]}$ make sense non-trivially in dimensions $d \leq 3$ ([EP94]), although the measures $\varrho_s(dy)$ are singular for $d \geq 2$ ([DH79]). For this reason, in dimensions $d \leq 3$ the *cat-*

alytic SBM X^ℓ could be constructed in [DF97] as a continuous measure-valued (time-inhomogeneous) Markov process $(X^\ell, P_{r,m}^\ell)$, given the catalyst process ϱ (*quenched approach*). By standard notation, $P_{r,m}^\ell$ denotes the law of the process X^ℓ (for ϱ fixed) if at time r we start X^ℓ in the measure m . The laws of the catalyst process ϱ will be denoted by \mathbb{P}_μ if $\varrho_0 = \mu$.

Averaging the random laws $P_{0,m}^\ell$ by means of \mathbb{P}_μ gives the *annealed* distribution

$$\mathcal{P}_{\mu,m} := \mathbb{P}_\mu P_{0,m}^\ell \quad (2)$$

of X^ℓ .

Of particular interest is the case

$$\mu = \ell_c := i_c \ell, \quad m = \ell_r := i_r \ell \quad (3)$$

for some constants $i_c, i_r > 0$.

Consider for the moment the *critical dimension* $d = 2$ and initial states $(\varrho_0, X_0^\ell) = (\ell_c, \ell_r)$. Here the catalyst ϱ_T dies out locally in probability as $T \rightarrow \infty$. In the large regions without catalyst only the smoothing heat flow acts on the reactant X^ℓ . On the other hand, a finite window of observation will be visited by increasingly large catalytic clumps at arbitrarily large times (recall that the time averaged two-dimensional catalyst ϱ has a proper random limit despite local extinction, see, e.g., [FG86]). These clumps lead locally to a great variability of the concentration of reactant: In contact with the catalyst, reactant mass piles up in relatively small areas, whereas large areas become vacant. But according to [EF96, Theorem 1], the smoothing effect in the large catalyst free regions wins this competition with the “*turbulence*” at the catalyst, leading to persistence: The intensity measure ℓ_r of X_T^ℓ is preserved also for all accumulation points (in law) as $T \rightarrow \infty$.

A formal description of the pair (ϱ, X^ℓ) will be given in § 2.1.

1.3 Notation and regularity assumption

Let p denote the standard heat kernel:

$$p_s(x) := (2\pi s)^{-d/2} \exp\left[-\frac{|x|^2}{2s}\right], \quad s > 0, \quad x \in \mathbb{R}^d, \quad (4)$$

and set

$$\mu * p_s(y) := \int \mu(dx) p_s(y-x). \quad (5)$$

Introduce the spatial *shift* operators θ_z defined on functions φ on \mathbb{R}^d :

$$\theta_z \varphi(y) := \varphi(y-z), \quad y, z \in \mathbb{R}^d, \quad (6)$$

and write $\langle \nu, \varphi \rangle$ for integral expressions as $\int \nu(dy) \varphi(y)$.

The construction of our processes actually needs an integrability condition for the initial states μ and m . Namely, we will assume that $\mu, m \in \mathcal{M}_p$ for some $p > d$. Here \mathcal{M}_p is the set of measures μ on \mathbb{R}^d such that $\langle \mu, \phi_p \rangle < \infty$, where

$$\phi_p(x) := \frac{1}{(1 + |x|^2)^{p/2}}, \quad x \in \mathbb{R}^d. \quad (7)$$

\mathcal{M}_p is endowed with the coarsest topology such that the map $\mu \mapsto \langle \mu, \varphi \rangle$ is continuous for $\varphi = \phi_p$ and for each φ in the cone C_+^{comp} of all non-negative continuous functions on \mathbb{R}^d with compact support.

In addition we impose a hypothesis on the local structure of the initial states μ of the catalyst ϱ .

Notation A measure $\mu \in \mathcal{M}_p$ is called **strongly diffusive** if there exists an $\eta \in (0, \frac{1}{4})$ such that the map

$$(t, z) \mapsto \int_0^t ds \mu * p_s(z), \quad (t, z) \in [0, \infty) \times \mathbb{R}^d, \quad (8)$$

is locally Hölder continuous of order η . \diamond

Example Some important examples for strongly diffusive measures are:

- $\mu = i_c \ell$, for some constant $i_c > 0$.
- μ absolutely continuous with a locally bounded density function.
- In $d \leq 3$, almost all samples of (ordinary) super Brownian motion at a (strictly) positive time (in particular almost all samples of a steady state in $d = 3$) are strongly diffusive (see Lemma 14 in the Appendix).

Note that in $d \geq 2$ a measure μ is *not* strongly diffusive if it has an atom. \diamond

1.4 Results

The *key* to our main result (Theorem 1) is the fact that in dimensions two and three the *closed support* S^ℓ of the locally finite measure $ds \varrho_s(dx)$ on $(0, \infty) \times \mathbb{R}^d$ is an $\ell^+ \times \ell$ -zero set (Corollary 8). Here ℓ^+ denotes the (normed) Lebesgue measure on $(0, \infty)$. Write $Z^\ell \subset (0, \infty) \times \mathbb{R}^d$ for the *complement* of S^ℓ in $(0, \infty) \times \mathbb{R}^d$. In Z^ℓ only the heat flow acts on X^ℓ . This suggests that here X^ℓ has densities satisfying the heat equation.

Theorem 1 Let $d = 2$ or 3 , assume $r \geq 0$, $\mu, m \in \mathcal{M}_p$, and that μ is strongly diffusive. For \mathbb{P}_μ -almost all ϱ the following statements hold.

- (a) **(absolutely continuous states)** With $P_{r,m}^\varrho$ -probability one, X_t^ϱ is absolutely continuous with respect to Lebesgue measure, for all $t > r$.

(b) **(smooth density field ξ^ℓ)** Denoting by $\xi^\ell = \{\xi_t^\ell(z) : t > r, z \in \mathbb{R}^d\}$ the density field of X^ℓ , there is a version of ξ^ℓ such that $P_{r,m}^\ell$ -a.s. the mapping $(t, z) \mapsto \xi_t^\ell(z)$, $(t, z) \in \mathbf{Z}^\ell$, $t > r$, is of class C^∞ and solves the heat equation:

$$\frac{\partial}{\partial t} \xi_t^\ell(z) = \frac{1}{2} \Delta \xi_t^\ell(z), \quad (t, z) \in \mathbf{Z}^\ell, \quad t > r. \quad (9)$$

(c) **(moments)** The $\xi_t^\ell(z)$ belong to $L^2 = L^2(P_{r,m}^\ell)$, have expectation

$$P_{r,m}^\ell \xi_t^\ell(z) = m * p_{t-r}(z) \quad (10)$$

and covariances

$$\begin{aligned} & \text{Cov}_{r,m}^\ell [\xi_{t_1}^\ell(z_1), \xi_{t_2}^\ell(z_2)] \\ &= 2 \int_r^{t_1 \wedge t_2} ds \left\langle \varrho_s, (m * p_{s-r})(\theta_{z_1} p_{t_1-s})(\theta_{z_2} p_{t_2-s}) \right\rangle \geq 0, \end{aligned} \quad (11)$$

$(t_i, z_i) \in \mathbf{Z}^\ell$, $t_i > r$, $i = 1, 2$.

(d) **(local L^2 -Lipschitz continuity)** The field $\{\xi_t^\ell(z) : (t, z) \in \mathbf{Z}^\ell, t > r\}$ is locally $L^2(P_{r,m}^\ell)$ -Lipschitz continuous: For every compact subset C of $\mathbf{Z}^\ell \cap ((r, \infty) \times \mathbb{R}^d)$ there is a constant $c = c(\varrho, C)$ such that

$$\|\xi_{t_1}^\ell(z_1) - \xi_{t_2}^\ell(z_2)\|_2 \leq c |(t_1, z_1) - (t_2, z_2)|, \quad (12)$$

$(t_1, z_1), (t_2, z_2) \in C$.

Note that in the case $m = \ell_r$ the expectation and covariance formulas reduce to

$$P_{r,\ell_r}^\ell \xi_t^\ell(z) \equiv i_r > 0, \quad (13)$$

and

$$\begin{aligned} & \text{Cov}_{r,\ell_r}^\ell [\xi_{t_1}^\ell(z_1), \xi_{t_2}^\ell(z_2)] \\ &= 2 i_r \int_r^{t_1 \wedge t_2} ds \left\langle \varrho_s, (\theta_{z_1} p_{t_1-s})(\theta_{z_2} p_{t_2-s}) \right\rangle. \end{aligned} \quad (14)$$

Formula (11) has the following *geneological interpretation*. The covariance measures the probability of two infinitesimal particles at (t_1, z_1) and (t_2, z_2) to have a common ancestor. On the other hand, the integrand at the r.h.s. is the “distribution” of the time-space location (s, x) of a possible latest common ancestor of these infinitesimal particles.

Remark Since a given point $(t, z) \in (r, \infty) \times \mathbb{R}^d$ belongs to \mathbf{Z}^ℓ with \mathbb{P}_{ℓ_c} -probability one, $\xi_t^\ell(z)$ is a well-defined $P_{r,m}^\ell$ -random variable, \mathbb{P}_μ -a.s. \diamond

Remark (annealed model) Statement (a) of Theorem 1 implies that also with respect to the *annealed* law $\mathcal{P}_{\mu,m}$ the catalytic SBM X^ℓ lives on the set of *absolutely continuous* measures. Clearly, (11) and (10) yield that the $\mathcal{P}_{\mu,m}$ -*covariances* of ξ^ℓ are given by

$$\begin{aligned} & \text{Cov}_{\mu,m} [\xi_{t_1}^\ell(z_1), \xi_{t_2}^\ell(z_2)] \\ &= 2 \int_0^{t_1 \wedge t_2} ds \left\langle \ell, (\mu * p_s)(m * p_s)(\theta_{z_1 p_{t_1-s}})(\theta_{z_2 p_{t_2-s}}) \right\rangle < \infty \end{aligned} \quad (15)$$

$(t_i, z_i) \in (0, \infty) \times \mathbb{R}^d$, $i = 1, 2$, $(t_1, z_1) \neq (t_2, z_2)$. Hence (if $\mu, m \neq 0$), the covariance tends to infinity if $(t_2, z_2) \rightarrow (t_1, z_1)$. In particular,

$$\text{Var}_{\mu,m} \xi_t^\ell(z) \equiv \infty, \quad (t, z) \in (0, \infty) \times \mathbb{R}^d. \quad \diamond$$

Now we come back to the limiting behavior of X_T^ℓ as $T \uparrow \infty$ in $d = 2$ with $(\varrho_0, X_0^\ell) = (\ell_c, \ell_r)$. In this dimension, the long-term behavior of X^ℓ is connected to local properties (such as absolute continuity of states) by a *self-similarity* property. Proposition 13 in [DF96] states that

$$X_T^\ell \stackrel{\mathcal{L}}{=} K^{-1} X_{KT}^\ell(K^{1/2} \cdot), \quad T, K > 0, \quad (16)$$

with respect to the random laws P_{0,ℓ_r}^ℓ . Here coincidence w.r.t. the random laws P_{0,ℓ_r}^ℓ formally means that

$$\mathbb{P}_{\ell_c} \left[P_{0,\ell_r}^\ell [X_T^\ell \in (\cdot)] \in (\cdot) \right] = \mathbb{P}_{\ell_c} \left[P_{0,\ell_r}^\ell \left[K^{-1} X_{KT}^\ell(K^{1/2} \cdot) \in (\cdot) \right] \in (\cdot) \right]. \quad (17)$$

From this discussion the following corollary of Theorem 1 is immediate.

Corollary 2 *In dimension two, with respect to the random laws P_{0,ℓ_r}^ℓ (with ϱ distributed according to \mathbb{P}_{ℓ_c}) the following two statements hold:*

(a) **(self-similarity)**

$$\xi_T^\ell \stackrel{\mathcal{L}}{=} \xi_{KT}^\ell(K^{1/2} \cdot), \quad T, K > 0. \quad (18)$$

(b) **(persistent convergence)** X_T^ℓ converges in distribution to a random multiple of Lebesgue measure:

$$X_T^\ell \xrightarrow{T \uparrow \infty} \xi_1^\ell(0) \ell. \quad (19)$$

Coincidence in law in statement (a) is understood in the same way as in (17). Similarly, the assertion in (b) has the following formal meaning. Given

ϱ , let Q_T^ϱ and Q_∞^ϱ denote the laws of the random measures X_T^ϱ and $\xi_1^\varrho(0)\ell$, respectively. Set

$$\mathbf{Q}_T := \mathbb{P}_{\ell_c} [Q_T^\varrho \in (\cdot)], \quad \mathbf{Q}_\infty := \mathbb{P}_{\ell_c} [Q_\infty^\varrho \in (\cdot)]. \quad (20)$$

Then the formal expression for the claim in (b) is

$$\mathbf{Q}_T \text{ converges } \textit{weakly} \text{ to } \mathbf{Q}_\infty \text{ as } T \rightarrow \infty. \quad (21)$$

Note that for fixed medium ϱ one cannot expect convergence since ϱ_T itself does *not converge a.s.* as $T \rightarrow \infty$.

It is known from [DF97, Theorem 51] that in dimension *one*

$$X_T^\varrho \xrightarrow[T \uparrow \infty]{} \ell_r, \quad \text{in } P_{0,\ell_r}^\varrho\text{-probability, for } \mathbb{P}_{\ell_c}\text{-almost all } \varrho. \quad (22)$$

(It is still open whether this statement is true P_{0,ℓ_r}^ϱ -a.s.) The reason for this behavior is that in $d = 1$ the catalyst dies out locally *almost surely*. In contrast, in $d = 2$ the catalyst goes to local extinction only in \mathbb{P}_{ℓ_c} -probability. Hence, the reactant meets the catalyst at arbitrarily large times. The randomness in the limit in (19) reflects the random medium as experienced by the reactant at large times. In particular, X^ϱ does not converge almost surely. The almost sure properties of Theorem 1 get lost on the way to Corollary 2 (b) by using the self-similarity that holds only in distribution.

Note that the two-dimensional reactant X^ϱ exhibits the following interesting phenomenon: Though started in a (spatially) ergodic state, the limit is *not* ergodic.

Remark 3 (annealed model) The self-similarity (16) holds also with respect to the annealed law $\mathcal{P}_{\ell_c,\ell_r}$ ([DF96, Proposition 13]). Hence, (18) and (19) are true also w.r.t. the annealed law. In other words, we have the following weak convergence of averaged distributions:

$$\mathbb{P}_{\ell_c} Q_T^\varrho \xrightarrow[T \uparrow \infty]{} \mathbb{P}_{\ell_c} Q_\infty^\varrho. \quad \diamond$$

Remark 4 (lattice model) In the model of two-dimensional simple branching random walk in the simple branching random medium, one can show a statement analogous to Corollary 2 (b): Here the reactant converges to a mixed Poisson system (homogeneous Poisson point process) with random intensity ξ_1^ϱ ([GKW97, Theorem 1.3]). The proof of this statement is based on our Theorem 1. However, since there is no scaling property in the lattice model, things become rather complicated. \diamond

The rest of the *paper is laid out* as follows. In Section 2 we recall the formal characterization of the catalytic SBM X^ϱ . We establish the fact that around $\ell^+ \times \ell$ -almost all time-space points (t, z) there is no catalytic mass. The key step in Section 2 is to show that at those (t, z) an asymptotic spatial $L^2(P_{r,m}^\varrho)$ -density $\xi_t^\varrho(z)$ of X_t^ϱ exists. Our theorem is proved in Section 3.

2 Preparations

2.1 Formal description of catalytic SBM

First we want to recall the formal characterization of the catalytic SBM X^ℓ in terms of its Laplace transition functional.

Recall that $p > d$ with d the dimension of \mathbb{R}^d , and that ϕ_p is the reference function of (7). Write \mathcal{B}^p for the set of all functions φ on \mathbb{R}^d such that $|\varphi| \leq c_\varphi \phi_p$ for some (finite) constant c_φ , and \mathcal{B}_+^p for the subset of its non-negative members.

Fix a constant $\gamma > 0$. By definition, the *catalyst process* $\varrho = (\varrho_t)_{t \geq 0}$ is a continuous (critical) SBM with branching rate γ . This is the continuous \mathcal{M}_p -valued time-homogeneous Markov process $(\varrho, \mathbb{P}_\mu)$ with Laplace transition functional

$$\mathbb{P}_\mu \exp \langle \varrho_t, -\varphi \rangle = \exp \langle \mu, -u(t) \rangle, \quad t \geq 0, \quad \mu \in \mathcal{M}_p, \quad \varphi \in \mathcal{B}_+^p. \quad (23)$$

Here $u = \{u(t) : t \geq 0\} = \{u(t, x) : t \geq 0, x \in \mathbb{R}^d\}$ is the unique non-negative solution to the *basic cumulant equation*

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u - \gamma u^2 \quad \text{on } (0, \infty) \times \mathbb{R}^d \quad (24)$$

with *initial condition* $u(0, x) = \varphi(x)$, $x \in \mathbb{R}^d$. (Where needed, ‘solution’ has to be understood in a *mild* sense.)

The process ϱ serves as a random medium for a catalytic SBM X^ℓ . In order to characterize X^ℓ , roughly speaking, we have to replace the constant rate γ in (24) by the (randomly) varying rate $\varrho_t(x)$, where $\varrho_t(x)$ is the *generalized derivative* $\frac{\varrho_t(d\mathbf{x})}{d\mathbf{x}}(x)$ of the measure $\varrho_t(d\mathbf{x})$. Our aim is to define X^ℓ via its log-Laplace transition functionals v_t^ℓ that solve a certain cumulant equation. We do so by first making precise sense of this equation.

Because of time-inhomogeneity, it is convenient to write the formal cumulant equation in a *backward setting*:

$$-\frac{\partial}{\partial r} v_t^\ell(r, \mathbf{x}) = \frac{1}{2} \Delta v_t^\ell(r, \mathbf{x}) - \varrho_r(\mathbf{x}) v_t^\ell(r, \mathbf{x})^2, \quad (25)$$

$0 \leq r \leq t$, $\mathbf{x} \in \mathbb{R}^d$. Note that the initial condition has become a *terminal condition*: $v_t^\ell(t) = \varphi$. After a formal integration, we can rewrite (25) rigorously and probabilistically as

$$v_t^\ell(r, \mathbf{x}) = \Pi_{r, \mathbf{x}} \left[\varphi(W_t) - \gamma \int_r^t L_{[W, \varrho]}(ds) v_t^\ell(s, W_s)^2 \right], \quad (26)$$

$0 \leq r \leq t$, $\mathbf{x} \in \mathbb{R}^d$, where $\Pi_{r, \mathbf{x}}$ is the law of (standard) Brownian motion W starting at time r from \mathbf{x} , and $L_{[W, \varrho]}$ denotes the *collision local time* of W with ϱ , formally introduced in (1). For $d \leq 3$ and finite μ with some regularity,

[EP94] show that for \mathbb{P}_μ -a.a. ϱ the collision local time $L_{[W, \varrho]}$ makes sense as a non-trivial additive functional of W . (In $d \geq 4$, actually $L_{[W, \varrho]} = 0$.) As pointed out in [DF97, Theorem 42], this is still true for strongly diffusive $\mu \in \mathcal{M}_p$ (recall the notation at page 6). Moreover ([DF97, Theorem 42 and Proposition 6]), for \mathbb{P}_μ -almost all ϱ and for t, φ fixed, there is a unique non-negative solution v_t^ϱ to (26). Finally ([DF97, § 5.4]), for \mathbb{P}_μ -a.a. ϱ , there exists a continuous \mathcal{M}_p -valued time-inhomogeneous Markov process $(X^\varrho, P_{r,m}^\varrho)$ with Laplace transition functional

$$P_{r,m}^\varrho \exp \langle X_t^\varrho, -\varphi \rangle = \exp \langle m, -v_t^\varrho(r) \rangle, \quad (27)$$

$0 \leq r \leq t$, $m \in \mathcal{M}_p$, $\varphi \in \mathcal{B}_+^p$, and v_t^ϱ the solution to (26). This is the *catalytic SBM* X^ϱ with catalyst ϱ , which was intuitively introduced in § 1.2.

From now on we adopt the *convention*, that we consider only initial states $\varrho_0 = \mu \in \mathcal{M}_p$ strongly diffusive and those samples ϱ of the catalyst process such that the (continuous) catalytic SBM X^ϱ exists.

Since the branching mechanism is critical, X^ϱ has *expectation measure*

$$P_{r,m}^\varrho X_t^\varrho \equiv S_{t-r} m, \quad (28)$$

independent of the catalytic medium ϱ . Here

$$S = \{S_t : t \geq 0\} \quad (29)$$

is the semigroup of Brownian motion. In particular,

$$P_{r,\ell_r}^\varrho X_t^\varrho \equiv \ell_r, \quad (30)$$

independent of time. The *covariances* (given ϱ) related to (28) can be written as

$$\begin{aligned} & \text{Cov}_{r,m}^\varrho \left[\langle X_{t_1}^\varrho, \psi_1 \rangle, \langle X_{t_2}^\varrho, \psi_2 \rangle \right] \\ &= 2 \int_r^{t_1 \wedge t_2} ds \left\langle \varrho_s, (m * p_{s-r})(S_{t_1-s} \psi)(S_{t_2-s} \psi_2) \right\rangle, \end{aligned} \quad (31)$$

$0 \leq r \leq t_1, t_2$, $m \in \mathcal{M}_p$, $\psi_1, \psi_2 \in \mathcal{B}^p$; see [DF97, formula (95)]. In particular,

$$\begin{aligned} & \text{Cov}_{r,\ell_r}^\varrho \left[\langle X_{t_1}^\varrho, \psi_1 \rangle, \langle X_{t_2}^\varrho, \psi_2 \rangle \right] \\ &= 2 i_r \int_r^{t_1 \wedge t_2} ds \left\langle \varrho_s, (S_{t_1-s} \psi_1)(S_{t_2-s} \psi_2) \right\rangle < \infty. \end{aligned} \quad (32)$$

2.2 Catalyst free regions

Denote by $B_\delta(z)$ the open ball in \mathbb{R}^d of radius δ centered at z . The starting point for our development is the following observation.

Proposition 5 (catalyst free regions close to time-space points)

Consider $d \geq 2$. Assume that $\mu \in \mathcal{M}_p$. For $t > 0$, denote by \mathbf{Z}_t^ϱ the open set of all those $z \in \mathbb{R}^d$ such that there exists a $\delta = \delta(\varrho, t, z) \in (0, t)$ with

$$\sup_{s \in [t-\delta, t+\delta]} \varrho_s(B_\delta(z)) = 0. \quad (33)$$

(a) (full measure) Then

$$\sup_{t > 0} \ell(\mathbb{R}^d \setminus \mathbf{Z}_t^\varrho) = 0, \quad \mathbb{P}_\mu\text{-a.s.} \quad (34)$$

(b) (absence at a given point) In particular, for fixed $t > 0$ and $z \in \mathbb{R}^d$, there is a $\delta = \delta(\varrho, t, z)$ such that (33) holds for \mathbb{P}_μ -almost all ϱ .

Remark 6 (polarity of points) Statement (b) has been known in the case of finite initial measures μ ([Dyn93, Theorem 11.2]). In potential-theoretical language, Dynkin shows that (in $d \geq 2$) a given time-space point (t, z) is polar for the graph of SBM ϱ . \diamond

Proof of Proposition 5 1° (reduction) By a well-known scaling property of SBM (see, e.g., [FK94, Lemma 4.5.1]), w.l.o.g. we may take $\gamma = \frac{1}{2}$ for the catalyst's branching rate. It suffices to show that

$$\mathbb{P}_\mu \left(\ell \left((\mathbf{Z}_t^\varrho)^c \cap B_1(x) \right) = 0, \quad 0 < t \leq T \right) = 1, \quad (35)$$

for all $x \in \mathbb{R}^d$ and $T > 0$. Without loss of generality we will show this only for $x = 0$. We may reformulate (35) as

$$\mathbb{P}_\mu \left(\ell \left(|z| < 1 : \sup_{s \in [t-\delta, t+\delta]} \varrho_s(B_\delta(z)) > 0 \quad \forall \delta \in (0, t) \right) > 0, \quad 0 < t \leq T \right) = 0.$$

We want to distinguish between the contributions from different initial regions.

2° (decomposition) For $n, N \geq 1$, write

$$A^{n,N} := \left\{ x \in \mathbb{R}^d : N(n-1) \leq |x| < Nn \right\}, \quad (36)$$

and $\mu^{n,N}$ for the 'restriction' $1_{A^{n,N}} \mu$ of μ to the 'ring' $A^{n,N}$. Then, for N fixed, ϱ can be represented as the sum of independent SBM $\varrho^{n,N}$, $n \geq 1$, on \mathbb{R}^d , where $\varrho^{n,N}$ starts from the finite measure $\mu^{n,N}$.

3° (negligible contribution from outside) First we will show that for $T, \eta > 0$,

$$\sum_{n \geq 2} \mathbb{P}_{\mu^{n,N}} \left(\sup_{s \leq T} \varrho_s(B_\eta(0)) > 0 \right) \xrightarrow{N \uparrow \infty} 0. \quad (37)$$

For this purpose, we may assume that $\eta > 2\sqrt{T}$. Applying [DIP89, Theorem 3.3 (a)] (with R there replaced by η), we find a constant $c = c(d, T)$ such that for $n \geq 2$ and $N > (\eta + 2)$,

$$\begin{aligned} & \mathbb{P}_{\mu^{n,N}} \left(\sup_{s \leq T} \varrho_s(B_\eta(0)) > 0 \right) \\ & \leq c \int \mu^{n,N}(d\mathbf{x}) (|\mathbf{x}| - (\eta + 1))^{d-2} \exp \left[- \frac{(|\mathbf{x}| - (\eta + 1))^2}{2T} \right]. \end{aligned} \quad (38)$$

Hence, for the sum in (37) we get the upper bound

$$c \int_{|\mathbf{x}| \geq N} \mu(d\mathbf{x}) (|\mathbf{x}| - (\eta + 1))^{d-2} \exp \left[- \frac{(|\mathbf{x}| - (\eta + 1))^2}{2T} \right] \xrightarrow[N \uparrow \infty]{} 0,$$

proving (37).

4° (*main term*) By the previous step, it suffices to show that for $N \geq 1$ fixed, statement (34) and (b) hold with \mathbb{P}_μ replaced by $\mathbb{P}_{\mu^{1,N}}$ (finite initial measure). Since (b) is simpler (and proved, e.g., in [Dyn93, Theorem 11.2]), we only show (34).

For $t > 0$, introduce the *closed support* $S_t = S_t^c$ of the measure ϱ_t . From Corollary 1.3 in [Per89] we know that with $\mathbb{P}_{\mu^{1,N}}$ -probability one, S_t is a Lebesgue *zero set* for all $t > 0$. Recall also that the process $t \mapsto S_t$, $t > 0$, is *càdlàg* (with respect to the Hausdorff metric on compact sets) with $\mathbb{P}_{\mu^{1,N}}$ -probability one; see [Per90, Theorem 1.4]. Moreover, by the same theorem, with probability one, the left-hand limits S_{t-} satisfy

$$S_{t-} \supseteq S_t \quad \text{and} \quad S_{t-} \setminus S_t \text{ is empty or a singleton, for all } t > 0. \quad (39)$$

(In each singleton, an isolated subpopulation of X^ℓ becomes extinct.) We claim that

$$\mathbf{Z}_t^\ell = S_{t-}^c, \quad t > 0, \quad \mathbb{P}_{\mu^{1,N}}\text{-a.s.} \quad (40)$$

In fact, first let t be a continuity point of the closed support process S . If $z \in S_t^c$, then by continuity a time-space box around (t, z) exists with no catalytic mass in the sense of (33), whereas for $z \in S_t$ it does not. On the other hand, if t is a discontinuity point, then $z \in \mathbf{Z}_t^\ell$ if and only if $z \in S_{t-}^c$, since S_{t-} is closed (together with S_t).

To finish the proof, now it remains to note that each S_{t-} is a Lebesgue zero set, by (39) since S_t is. \blacksquare

Remark 7 (dimension one) Properties as in Proposition 5 are *not* valid in $d = 1$ since there ϱ has a jointly continuous density field on $(0, \infty) \times \mathbb{R}$ (see, e.g., [KS88]). \diamond

Recall that $\mathbf{Z}^\ell \subset (0, \infty) \times \mathbb{R}^d$ denotes the complement in $(0, \infty) \times \mathbb{R}^d$ of the closed support \mathbf{S}^ℓ of the measure $ds \varrho_s(dx)$. The following corollary is immediate from Proposition 5.

Corollary 8 (thin time-space support) *Let $d \geq 2$. We can write \mathbf{Z}^ℓ in terms of the \mathbf{Z}_t^ℓ from Proposition 5:*

$$\mathbf{Z}^\ell = \left\{ (t, z) : t > 0, z \in \mathbf{Z}_t^\ell \right\}. \quad (41)$$

Hence,

$$\ell^+ \times \ell \left((0, \infty) \times \mathbb{R}^d \setminus \mathbf{Z}^\ell \right) = 0, \quad \mathbb{P}_\mu\text{-a.s.} \quad (42)$$

2.3 Asymptotic L^2 -densities of the reactant

Recall the definition of the reference function ϕ_p from (7). We state the following trivial heat kernel estimates without proof.

Lemma 9 (estimates for the heat kernel) *For $d \geq 1$, let $C \subset (0, \infty) \times \mathbb{R}^d$ be compact, and let $k, n \geq 1$. Choose $\delta > 0$ such that*

$$C^\delta := \bigcup_{(t, z) \in C} [t - \delta, t + \delta] \times B_\delta(z) \subset (0, \infty) \times \mathbb{R}^d. \quad (43)$$

Then there are constants $c_i = c_i(d, C, n, \delta)$, $i = 1, 2, 3$, such that for $(t, z) \in C$ and $(s, x) \in ((0, \infty) \times \mathbb{R}^d) \setminus C^\delta$ with $s \leq t$, the following three statements hold:

$$\left| \frac{\partial}{\partial r} \theta_z p_r(x) \Big|_{r=n(t-s)} \right| \leq c_1 \theta_z p_{2n(t-s)}(x), \quad (44)$$

$$\left| \frac{\partial}{\partial z} \theta_z p_{n(t-s)}(x) \right| \leq c_2 \theta_z p_{2n(t-s)}(x), \quad (45)$$

$$\theta_z p_{n(t-s)}(x) \leq c_3 \phi_p^k(x). \quad (46)$$

We will also need the following estimate that is due to Theorem 42 of [DF97].

Lemma 10 *Assume that $\mu \in \mathcal{M}_p$ is strongly diffusive. Then, for \mathbb{P}_μ -almost all ϱ , for every $t > 0$,*

$$\sup_{x \in \mathbb{R}^d} \frac{1}{\phi_p(x)} \int_0^t ds \int \varrho_s(dy) p_s(y-x) \phi_p^2(y) < \infty. \quad (47)$$

The following L^2 -result is the key of our development.

Proposition 11 (asymptotic L^2 -densities at points in \mathbf{Z}^ℓ) *Let $d = 2$ or 3 . Take $r \geq 0$, $\mu, m \in \mathcal{M}_p$ and assume that μ is strongly diffusive. For \mathbb{P}_μ -almost all ϱ the following hold.*

(a) **(existence on \mathbf{Z}^ℓ)** *For each $(t, z) \in \mathbf{Z}^\ell$, $t > r$, there is an element $\xi_t^\ell(z) \geq 0$ in the Lebesgue space $L^2 = L^2(P_{r,m}^\ell)$ such that the L^2 -convergence*

$$X_t^\ell * p_\varepsilon(z) \xrightarrow{\varepsilon \downarrow 0} \xi_t^\ell(z) \quad (48)$$

takes place.

(b) **(locally uniform convergence)** *This convergence is uniform if (t, z) runs in a compact set $C = C(\varrho) \subset \mathbf{Z}^\ell \cap ((r, \infty) \times \mathbb{R}^d)$.*

(c) **(moments)** *$\xi_t^\ell(z)$ has expectation $m * p_{t-r}(z)$, and the covariances are given by (11).*

(d) **(existence at a given point)** *In particular, for $t > r$ and $z \in \mathbb{R}^d$ fixed, $\xi_t^\ell(z)$ exists with those properties, for \mathbb{P}_μ -almost all ϱ .*

Proof Since ϱ_r is strongly diffusive \mathbb{P}_μ -a.s. (see Lemma 14 in the appendix), we may use the Markov property to conclude that we can assume that $r = 0$ without loss of generality. By the covariance formula (32),

$$\begin{aligned} & \left\| X_t^\ell * p_{\varepsilon_1}(z) - X_t^\ell * p_{\varepsilon_2}(z) \right\|_2^2 \\ &= \left[m * p_{t+\varepsilon_1}(z) - m * p_{t+\varepsilon_2}(z) \right]^2 \\ & \quad + 2 \int_0^t ds \left\langle \varrho_s, (m * p_s) \left[\theta_z p_{\varepsilon_1+t-s} - \theta_z p_{\varepsilon_2+t-s} \right]^2 \right\rangle, \end{aligned} \quad (49)$$

$t \geq 0$, $z \in \mathbb{R}^d$, $\varepsilon_1, \varepsilon_2 > 0$.

Fix a compact set $C \subset \mathbf{Z}^\ell$ and $\delta > 0$ such that $C^\delta \subset \mathbf{Z}^\ell$ (recall notation (43)). Further let $\tau := \sup\{(t, z) \in C\}$: Clearly, the first summand on the r.h.s. of (49) goes to 0 as $\varepsilon_1, \varepsilon_2 \rightarrow 0$, uniformly in $(t, z) \in C$.

We use the convention $p_t := 0$ if $t < 0$. By (46) in Lemma 9, there exists a constant $c_3 < \infty$ such that

$$\begin{aligned} & \int_0^\infty ds \left\langle \varrho_s, (m * p_s) \sup_{\substack{(t_1, z_1), (t_2, z_2) \in C \\ 0 < \varepsilon_1, \varepsilon_2 < \delta/2}} (\theta_{z_1} p_{\varepsilon_1+t_1-s}) (\theta_{z_2} p_{\varepsilon_2+t_2-s}) \right\rangle \\ & \leq c_3^2 \int_0^{\tau+\delta} ds \left\langle \varrho_s, (m * p_s) \phi_p^2 \right\rangle. \end{aligned} \quad (50)$$

By Lemma 10 the latter quantity is bounded by $c_4 \langle m, \phi_p \rangle < \infty$. Note that for all $(s, x) \in (C^\delta)^c$,

$$\sup_{(t,z) \in C} \left| \theta_z (p_{\varepsilon_1+t-s} - p_{\varepsilon_2+t-s})(x) \right| \rightarrow 0 \quad \text{as } \varepsilon_1, \varepsilon_2 \downarrow 0. \quad (51)$$

If we combine (49), (50), and (51), the dominated convergence theorem yields that $(X_t^\varepsilon * p_\varepsilon(z))_{\varepsilon>0}$ is Cauchy in $L^2(P_{0,m}^\varepsilon)$ as $\varepsilon \downarrow 0$, uniformly in $(t, z) \in C$. Hence, the L^2 -limit $\xi_t^\varepsilon(z)$, say, exists, and

$$\left\| \sup_{(t,z) \in C} X_t^\varepsilon * p_\varepsilon(z) - \xi_t^\varepsilon(z) \right\|_2 \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0. \quad (52)$$

This proves (a) and (b).

Since L^2 -convergence implies L^1 -convergence, $P_{0,m}^\varepsilon \xi_t^\varepsilon(z) = m * p_t(z)$ follows from (28). But then also the covariance formula (11) can be derived from (31) and domination according to (50).

Statement (d) is immediate (recall Proposition 5 (b)). ■

3 Proof of the theorem

In this section we prove Theorem 1. First we use Proposition 11 to show (a),(c), and (d). Next we proceed similarly as in [Del96] to get the smoothness of the density field. Delmas uses a representation of his catalytic SBM in terms of excursions started from his (time-independent) catalytic set. Our catalyst is *not* time-independent. However, it is not crucial in Delmas' argument to start the excursions from the catalyst. Our idea is to use a Delmas type representation of X^ε on a local level with an occupation density measure Γ^ε concentrated on a nice set outside the catalyst.

For notational simplicity we assume $r = 0$. This is no loss of generality (see Lemma 14 in the appendix).

(a) *Absolute continuity* of X^ε on \mathbf{Z}^ε is immediate by the uniform convergence statement in Proposition 11. (An alternative argument for this fact will be given in the proof of part (b).) Since $\xi_t^\varepsilon(z)$ has expectation $m * p_t(z)$ on \mathbf{Z}_t^ε , and $\ell((\mathbf{Z}_t^\varepsilon)^c) = 0$, we get the absolute continuity of X_t^ε on the whole space \mathbb{R}^d by an exhaustion argument.

(c) (*moments*) This is Proposition 11 (c).

(d) (*local $L^2(P_{0,m}^\varepsilon)$ -Lipschitz continuity on \mathbf{Z}^ε*) We may assume that the compact set $C \subset \mathbf{Z}^\varepsilon$ is a closed box. Let $\delta > 0$ such that $C^\delta \subset \mathbf{Z}^\varepsilon$ (recall (43)). Set $\tau := \sup\{t : (t, z) \in C\}$, and let $(t_1, z_1), (t_2, z_2) \in C$ with $t_1 \leq t_2$. From the moment formulas (10) and (11) we get

$$\left\| \xi_{t_1}^\varepsilon(z_1) - \xi_{t_2}^\varepsilon(z_2) \right\|_2^2 = I_1 + I_2 + I_3, \quad (53)$$

where

$$\begin{aligned}
I_1 &:= \left[m * p_{t_1}(z_1) - m * p_{t_2}(z_2) \right]^2 \\
I_2 &:= 2 \int_0^{t_1} ds \left\langle \varrho_s, (m * p_s) [\theta_{z_1} p_{t_1-s} - \theta_{z_2} p_{t_2-s}]^2 \right\rangle \\
I_3 &:= 2 \int_{t_1}^{t_2} ds \left\langle \varrho_s, (m * p_s) [\theta_{z_2} p_{t_2-s}]^2 \right\rangle.
\end{aligned} \tag{54}$$

We use the bound of the partial derivatives of $\theta_z p_{t-s}$ in Lemma 9 to derive the existence of a constant c^1 (depending only on C and ϱ) such that

$$I_2 \leq \left| |t_1 - t_2| + |z_1 - z_2| \right|^2 c^1 \int_0^\tau ds \left\langle \varrho_s, (m * p_s) \left[\sup_{(t,z) \in C} \theta_z p_{2(t-s)}^2 \right] \right\rangle.$$

By (46) and Lemma 10, this inequality can be continued with constants c^2, c^3 :

$$\begin{aligned}
&\leq |(t_1, z_1) - (t_2, z_2)|^2 c^2 \int_0^\tau ds \left\langle \varrho_s, (m * p_s) \phi_p^2 \right\rangle \\
&\leq c^3 |(t_1, z_1) - (t_2, z_2)|^2.
\end{aligned}$$

Analogously we get the existence of c^4 such that

$$I_1 \leq c^4 |(t_1, z_1) - (t_2, z_2)|^2.$$

The estimate for I_3 is similar. Note that $p_r(x) \leq c^5 r^2$, $r > 0$, for $|x| > \delta$ and a constant c^5 depending only on δ . Hence (again by (46) and Lemma 10) there exist c^6 and c^7 such that

$$\begin{aligned}
I_3 &\leq c^5 |t_1 - t_2|^2 \int_0^{t_2} ds \left\langle \varrho_s, (m * p_s) \theta_{z_2} p_{t_2-s} \right\rangle \\
&\leq c^6 |t_1 - t_2|^2 \int_0^\tau ds \left\langle \varrho_s, (m * p_s) \phi_p^2 \right\rangle \\
&\leq c^7 |t_1 - t_2|^2.
\end{aligned}$$

(b) (smooth density field) For the final part of proof we have the following strategy. We fix a cylinder \mathcal{Z} contained in \mathbf{Z}^ℓ and use Proposition 11 to construct the occupation density measure Γ^ℓ of X^ℓ on the lateral area \mathcal{A} of the cylinder. Next we use Delmas' representation of catalytic SBM in terms of Brownian excursions starting from \mathcal{A} to derive the smoothness of ξ^ℓ in \mathcal{Z} .

Let ϱ be such that the assertions in Proposition 11 (a)–(c) and in Corollary 8 hold. Recall the characterization of \mathbf{Z}^ℓ from Corollary 8, and that \mathbf{Z}^ℓ is open in $(0, \infty) \times \mathbb{R}^d$ and satisfies $\ell^+ \times \ell((\mathbf{Z}^\ell)^c) = 0$. Write $\bar{\ell}$ for the $(d-1)$ -dimensional Lebesgue measure on the boundary $\partial B_r(z)$ of the open ball $B_r(z)$ around z of radius r .

Fix $(t, z) \in \mathbf{Z}^\ell$, and a $\delta > 0$ such that

$$[t - \delta, t + \delta] \times B_{2\delta}(z) \subset \mathbf{Z}^\ell. \quad (55)$$

Define a measure Γ^ℓ on $\mathcal{A} := [t - \delta, t + \delta] \times \partial B_\delta(z)$ by

$$\Gamma^\ell(du, dy) := du \xi_u^\ell(y) \bar{\ell}(dy), \quad (u, y) \in \mathcal{A}, \quad (56)$$

with ξ_u^ℓ from Proposition 11 (a).

First we show that Γ^ℓ is the *occupation density measure (super-local time)* of X^ℓ on \mathcal{A} . For this purpose, we define random measures Γ_ε^ℓ , $\varepsilon > 0$, on the cylinder lateral area \mathcal{A} , via their density functions

$$(u, y) \mapsto X_u^\ell * p_\varepsilon(y), \quad (u, y) \in \mathcal{A},$$

with respect to the measure $\ell^+ \times \bar{\ell}$. The formal meaning of the statement that Γ^ℓ is the occupation density measure is that

$$\Gamma_\varepsilon^\ell \text{ converges weakly to } \Gamma^\ell \text{ as } \varepsilon \downarrow 0, \quad P_{0,m}^\ell \text{-a.s.} \quad (57)$$

To prove (57) it suffices to show that

$$\langle \Gamma_\varepsilon^\ell, f \rangle \xrightarrow{\varepsilon \downarrow 0} \langle \Gamma^\ell, f \rangle, \quad P_{0,m}^\ell \text{-a.s.}, \quad (58)$$

if f is a continuous function on \mathcal{A} . From the uniform convergence in Proposition 11 (b) we know that in $L^2(P_{0,m}^\ell)$,

$$\begin{aligned} & \|\langle \Gamma_\varepsilon^\ell, f \rangle - \langle \Gamma^\ell, f \rangle\|_2 \\ & \leq \int_{t-\delta}^{t+\delta} du \int_{\partial B_\delta(z)} dy |f(u, y)| \left\| X_u^\ell * p_\varepsilon(y) - \xi_u^\ell(y) \right\|_2 \xrightarrow{\varepsilon \downarrow 0} 0, \end{aligned}$$

Hence, for every sequence $\varepsilon_n \downarrow 0$ as $n \uparrow \infty$, there exists a subsequence $\varepsilon_{n(k)}$ such that

$$\left\langle \Gamma_{\varepsilon_{n(k)}}^\ell, f \right\rangle \xrightarrow{k \uparrow \infty} \langle \Gamma^\ell, f \rangle, \quad P_{0,m}^\ell \text{-a.s.}$$

Since the mapping $\varepsilon \mapsto \langle \Gamma_\varepsilon^\ell, f \rangle$, $\varepsilon > 0$, is continuous, we have shown (58), and hence (57).

The aim is now to use Γ^ℓ to get a representation of ξ^ℓ as in Proposition 7.1 of [Del96] (see also Theorem 2 of [FL95]). This is Proposition 12 below. From this representation it is easily shown that ξ^ℓ is C^∞ and solves the heat equation (see Theorem 8.1 of [Del96]).

We start by introducing the ingredients of the representation formula. Recall that $(W, \Pi_{r,z})$ denotes the Brownian motion on \mathbb{R}^d . Define the exit time

$$\tau^B := \inf \{s > 0 : W_s \notin B\} \quad (59)$$

of the open ball $B = B_\delta(z)$, and the *exit density*

$$q^B = \left\{ q_t^B(x, y) : t > 0, x \in B, y \in \partial B \right\} \quad (60)$$

by

$$\Pi_{0,x} f(\tau^B, W_{\tau^B}) = \int_0^\infty dt \int_{\partial B} \bar{\mathbb{Z}}(dy) q_t^B(x, y) f(t, y), \quad (61)$$

$f \in \mathcal{C}_b((0, \infty) \times \partial B)$, (that is f bounded and continuous). Clearly, for $y \in \partial B$ fixed, $(t, x) \mapsto q_t^B(x, y)$ is of class C^∞ and solves the heat equation.

Fix $T > 0$ and a compact set $D \subset B$. By a simple induction argument we derive from Lemma 9 that the partial derivatives of all orders are bounded, uniformly in $x \in D$, $y \in \partial B$, $t \in (0, T]$. Hence, for every finite measure ν on $(0, \infty) \times \partial B$, also the mixture

$$\nu * q_t^B(x) := \int_0^t \int_{\partial B} \nu(du, dy) q_{t-u}^B(x, y), \quad t > 0, x \in B, \quad (62)$$

is of class C^∞ and solves the heat equation in $(0, \infty) \times B$.

Define the transition density $p^B = \{p_t^B(x, x') : t > 0, x \in B, x' \in \mathbb{R}^d\}$ of *Brownian motion killed on B^c* :

$$\Pi_{0,x} \mathbf{1}_{\tau > t} f(W_t) = \int_B dx' p_t^B(x, x') f(x'), \quad f \in \mathcal{C}_b(\mathbb{R}^d). \quad (63)$$

As above, $(t, x) \mapsto p_t^B(x, x')$ is C^∞ and solves the heat equation. Further, for $n \in \mathcal{M}_p$, also the mixture

$$n * p_t^B(x) := \int n(dx') p_t^B(x, x'), \quad t > 0, x \in B, \quad (64)$$

is C^∞ and solves the heat equation.

Since $\nu * q^B$ and $n * p^B$ are C^∞ and solve the heat equation, the same is true for ξ^ℓ by the following proposition. Hence the proof of the theorem is complete. \blacksquare

Let $r \geq 0$ and $(t, z) \in \mathbb{Z}^\ell$, $t > r$. Choose $\delta \in (0, t-r)$ such that (55) holds. Define the cylinder $\mathcal{Z} := (t-\delta, t+\delta) \times B_\delta(z)$.

Proposition 12 (representation by excursion densities) *For \mathbb{P}_μ -a.a. ϱ , with $P_{r,m}^\ell$ -probability one,*

$$\xi_s^\ell(x) = X_{t-\delta}^\ell * p_{s-(t-\delta)}^B(x) + \Gamma^\ell * q_{s-(t-\delta)}^B(x), \quad (s, x) \in \mathcal{Z}. \quad (65)$$

Proof As above we may assume $r = 0$. We want to show that the difference of both sides of (65) vanishes in $L^2(P_{0,m}^\varrho)$. Clearly,

$$P_{0,m}^\varrho \xi_s^\varrho(x) = m * p_t(x) = P_{0,m}^\varrho \left[X_{t-\delta}^\varrho * p_{s-(t-\delta)}^B(x) + \Gamma^\varrho * q_{s-(t-\delta)}^B(x) \right].$$

Hence, it suffices to prove that the variance of the difference equals 0. We use the covariance formulas (11) and (32) to deduce that

$$\begin{aligned} \text{Var}_{0,m}^\varrho \left[\xi_s^\varrho(x) - X_{t-\delta}^\varrho * p_{s-(t-\delta)}^B(x) - \Gamma^\varrho * q_{s-(t-\delta)}^B(x) \right] \\ = \int_0^s ds' \int \varrho_{s'}(dx') m * p(x') \left[\theta_x p_{s-s'} - \mathbf{1}_{s' < t-\delta} \left(S_{t-\delta-s'} p_{s-(t-\delta)}^B \right) (x') \right. \\ \left. - \int_{s' \vee (t-\delta)}^s du \int_{\partial B} \bar{\ell}(dy) p_{u-s'}(y - x') q_{s-u}^B(x, y) \right]^2. \end{aligned}$$

However, the integrand vanishes if $(s, x) \in \mathcal{Z}$ and $(s', x') \in \mathcal{Z}^c$. In fact, we distinguish between the two cases whether the backward Brownian motion path leaves the cylinder \mathcal{Z} at the base $\{t - \delta\} \times B_\delta(z)$, or at the lateral area \mathcal{A} . This shows (65), hence the proof is complete. \blacksquare

Appendix

As mentioned at the end of § 1.3, almost all samples of SBM ϱ at positive time are strongly diffusive. This follows from [DF97, Theorem 32] via an expectation calculation. But for methodological reasons we give an independent proof.

We begin with a lemma that shows that mass is locally not “too concentrated”:

Lemma 13 *Let $d \geq 2$. Fix $\zeta, \tau > 0$ and $\mu \in \mathcal{M}_p$. For \mathbb{P}_μ -almost all ϱ_τ ,*

$$\varepsilon^{\zeta-2} \varrho_\tau(B_\varepsilon(x)) \rightarrow 0 \quad \text{as } \varepsilon \downarrow 0, \quad (66)$$

uniformly for x in compact subsets of \mathbb{R}^d .

Proof For μ finite, this is implied by Corollary 4.8 of [BEP91]. To carry this over to general $\mu \in \mathcal{M}_p$, we use a decomposition as in step 2° of the proof of Proposition 5. \blacksquare

Lemma 14 *Let $d \leq 3$, and fix $\mu \in \mathcal{M}_p$ and $\tau > 0$. For \mathbb{P}_μ -almost all ϱ , the map*

$$(t, x) \mapsto g(t, x) := \int_0^t ds \varrho_\tau * p_s(x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \quad (67)$$

is locally Hölder continuous of order η , for all $\eta \in (0, \frac{1}{4})$. In particular, \mathbb{P}_μ -almost all ϱ_τ are strongly diffusive.

Proof If $d = 1$, then ϱ_τ has a locally bounded density, hence we are done.

Let now $d = 2, 3$. Fix $\eta \in (0, \frac{1}{4})$, choose $\alpha \in (\eta + \frac{1}{4}, \frac{1}{2})$ and ζ such that $0 < \zeta < 2 [1 - \alpha^{-1}(\eta + \frac{1}{4})]$, and fix ϱ_τ such that the assertion in Lemma 13 holds. Take $K > 0$ and $T > 0$. We consider $g(t, x)$ for $(t, x) \in [0, T] \times B_K(0)$. By c we denote a constant that changes from place to place but depends only on ϱ_τ, K , and T .

First we show that

$$|g(t, x) - g(s, x)| \leq c(t-s)^{2\eta}, \quad 0 \leq s \leq t \leq T, \quad x \in B_K(0). \quad (68)$$

Note that for $y \in \mathbb{R}^d$ and $u \in [0, T]$,

$$p_u(y) \mathbf{1}\{|y| \geq u^\alpha\} \leq c \phi_p(y) \quad (69)$$

and

$$p_u(y) \mathbf{1}\{|y| \leq u^\alpha\} \leq u^{-d/2}. \quad (70)$$

Hence,

$$\int_s^t du \varrho_\tau * p_u(x) \leq c(t-s) \langle \varrho_\tau, \phi_p \rangle + \int_s^t du u^{-d/2} \varrho_\tau(B_{u^\alpha}(x)). \quad (71)$$

By Lemma 13, the integral on the r.h.s. of (71) is dominated by

$$c \int_s^t du u^{-d/2} u^{\alpha(2-\zeta)} \leq c(t-s)^{2\eta}. \quad (72)$$

Hence, we have shown (68).

Next we prove

$$|g(t, x) - g(t, y)| \leq c|x-y|^\eta, \quad x, y \in B_K(0), \quad t \in [0, T]. \quad (73)$$

Let $t' := t \wedge |x-y|^{1/2}$. Then by the triangular inequality,

$$|g(t, x) - g(t, y)| \leq g(t', x) + g(t', y) + \int_{t'}^t ds \int \mu(dz) |\theta_z p_s(x) - \theta_z p_s(y)|.$$

Using (68) and the simple heat kernel estimate

$$|\theta_z p_s(x) - \theta_z p_s(y)| \leq c \frac{|x-y|}{s} p_{16s}(z+x), \quad t' < s \leq T,$$

we get

$$\begin{aligned} |g(t, x) - g(t, y)| &\leq c|x-y|^\eta + c|x-y|^{1/2} \int_0^t ds \int \mu(dz) p_{16s}(x+z) \\ &\leq c|x-y|^\eta + c|x-y|^{1/2} g(16t, x). \end{aligned}$$

Since $g(16t, x)$ is bounded, this implies (73), and the proof is complete. \blacksquare

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