On the Stability of Piecewise Linear Wavelet Collocation and the Solution of the Double Layer Equation over Polygonal Curves

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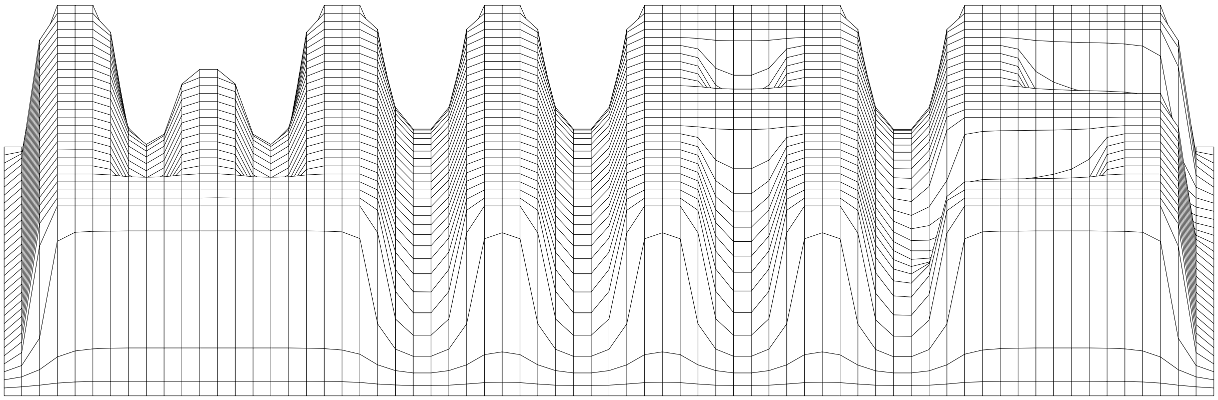

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Abstract

In this paper we consider a piecewise linear collocation method for the solution of strongly elliptic operator equations over closed curves. The trial space is a subspace of the space of all piecewise linear functions defined over a uniform grid. This space is spanned by an arbitrary subset of the biorthogonal wavelet basis. To the subspace in the trial space there corresponds a natural subspace in the space of test functionals. This subspace is spanned by certain linear combinations of the Dirac delta functionals taken at the uniformly distributed grid points. For the resulting wavelet collocation method and a strongly elliptic operator equation, we prove stability and convergence. In particular, this general result applies to the double layer equation over a polygonal curve. We show that the wavelet collocation method with piecewise linear trial functions over a uniform grid converges with order $O(n^{-2})$, where n is the number of degrees of freedom. Note that the step size of the underlying uniform partition is $n^{-\alpha}$, $\alpha \geq 1$. The stiffness matrix for the wavelet collocation method can be compressed to a matrix containing no more than $O(n \log n)$ non-zero entries such that the asymptotic convergence order is not effected.

1 Introduction

The stability and convergence of piecewise linear collocation for the numerical solution of operator equations over curves has been established in the work by Prößdorf, Schmidt [27, 32, 33], Arnold, Wendland [2, 3], Saranen [31], Costabel, Stephan [12, 13], Amini, Sloan [1], Chandler, Graham [7], and Elschner [20, 21] (cf. also the book by Prößdorf and Silbermann [28]). Wavelet algorithms for collocation methods have been considered by Dahmen, Prößdorf, Schneider [17, 18, 34], Harten, Yad-Shalom [22], and the author [29] (compare also the fundamental paper on wavelet algorithms for the numerical solution of integral equations by Beylkin, Coifman, and Rokhlin [5]). However, one of the problems in designing effective wavelet algorithms is that the solutions of the operator equations have a degree of smoothness which is local, i.e., it depends on the point of the curve. One way to take care of this locality is to introduce a transformation of the curve such that the solution of the resulting equation has a uniform degree of smoothness. This approach has been studied by the author in [29]. Though this transformation technique is very popular in the numerical solution of integral equation, the natural approach of the wavelet theory is a different one. Specialists in wavelet compression suggest to take a spline space over a uniform grid with a very small step size, but to restrict the wavelet basis of this space to a subset of wavelet basis functions for which the wavelet coefficients of the solution are larger than a certain small threshold. In other words, the maximal level of the wavelet basis should depend on the local degree of smoothness. For the Galerkin method and several types of potential equations, this approach has been considered by v.Petersdorff and Schwab [26]. Moreover, adaptive Galerkin methods in the same spirit have been analyzed by Dahlke, Dahmen, Hochmuth, and Schneider [14, 23].

The main topic of the present paper is to analyze the stability for the collocation method if the maximal level of the wavelet basis in the trial space depends on the local point of the curve. To the subspace in the trial space there corresponds a natural subspace in the space of test functionals. This subspace is spanned by certain linear combinations of the Dirac delta functionals at the uniformly distributed grid points. If $Ax = y$ is the operator equation over the one dimensional closed curve Γ, if ∂ denotes the operator of differentiation with respect to the arc length parametrization, and if ∂^{-1} denotes the inverse of ∂ over the space orthogonal to constant functions, then the strong ellipticity of $\partial A \partial^{-1}$ implies the stability of the piecewise linear collocation method defined by the restricted wavelet basis in the trial space and by the corresponding test space. In particular, the stability result applies to the numerical solution of the double layer potential equation over polygonal curves. It turns out that the wavelet collocation method with piecewise linear trial functions over a uniform grid converges with order $O(n^{-2})$, where n is the number of degrees of freedom. Note that the step size of the underlying uniform partition is $n^{-\alpha}$, $\alpha \geq 1$. The stiffness matrix for the wavelet collocation method can be compressed to a matrix containing no more than $O(n \log n)$ non-zero entries such that the asymptotic convergence order is not effected. Thus, if the entries of the matrix are computed by analytic formulae and if the matrix equation is solved by a cascadic iterative method (cf. e.g. the GMRes method by Saad and Schultz [30]), then only $O(n \log n)$ arithmetic operations are required to solve the integral equation up to an error of $O(n^{-2})$. Remark that, using further compression techniques (cf. the compression of matrix entries with overlapping supports of test and trial functions due to Schneider [34]), a reduction to $O(n)$ operations seems to be possible.

The plan of the paper is as follows: In Section 2 we will recall some facts on biorthogonal wavelets. This general setting will be applied to the construction of a wavelet basis for the piecewise linear trial space in Section 3 and to the definition of a wavelet basis for the test space of Dirac delta distributions in Section 4. In Section 5 we will set up the wavelet collocation method and prove its stability for the case of "strongly elliptic" operators A, i.e., of operators A such that $\partial A \partial^{-1}$ is strongly elliptic. We will show in Section 6 that the double layer operator defined over polygonal curves satisfies the stability assumption. Moreover, we will define a subspace of the wavelet basis in the trial space which is convenient for the optimal approximation of the solution x. In Section 7 we will introduce a compression algorithm for the stiffness matrix and prove that the compressed matrix contains no more than $O(n \log n)$ non-zero entries. Finally, we show the stability of the compressed wavelet collocation and derive the asymptotic error estimate $O(n^{-2})$ in Section 8.

2 Biorthogonal Wavelets

Biorthogonal wavelets have been introduced in the fundamental paper [9] and, for additional properties, we refer to e.g. [15, 16, 34]. First we introduce the wavelets over the real axis $\mathbb{R}$. We consider two hierarchical sequences of function spaces over $\mathbb{R}$

$$\ldots \subset V_{j-1} \subset V_j \subset V_{j+1} \ldots, \tag{2.1}$$

$$\ldots \subset \tilde{V}_{j-1} \subset \tilde{V}_j \subset \tilde{V}_{j+1} \ldots \tag{2.2}$$

such that f belongs to V_j if and only if $t \mapsto f(2t)$ is contained in V_{j+1} and similarly for the $\tilde{V}_j$. In other words, the functions of V_{j+1} can be obtained from those of V_j by a dilation in the argument with scaling factor two. Moreover, we suppose that the spaces V_0 and $\tilde{V}_0$ are spanned by the integer shifts $t \mapsto \varphi(t-k)$ and $t \mapsto \tilde{\varphi}(t-k)$ of the so called scaling functions φ and $\tilde{\varphi}$, respectively. This means that V_0 and $\tilde{V}_0$ are considered to be subspaces of L^2 or of a Sobolev space H^s, $s \in I\!R$ and that the set of finite linear combinations of the integer shifts is dense in V_0 and $\tilde{V}_0$, respectively. Together with the dilation property we arrive at

$$V_j := \text{cl span}\,\{\varphi_k^j : k \in \mathbb{Z}\}, \quad \varphi_k^j(t) := 2^{j/2}\varphi(2^j t - k), \tag{2.3}$$
$$\tilde{V}_j := \text{cl span}\,\{\tilde{\varphi}_k^j : k \in \mathbb{Z}\}, \quad \tilde{\varphi}_k^j(t) := 2^{j/2}\tilde{\varphi}(2^j t - k). \tag{2.4}$$

The sequence $(V_j)_{j\in\mathbb{Z}}$ is called **multi-resolution analysis** if $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\cup_{j\in\mathbb{Z}} V_j = L^2$ and if $\{\varphi_k^j : k \in \mathbb{Z}\}$ is a Riesz basis of $V_j \subseteq L^2$. Recall that $\{\varphi_k^j : k \in \mathbb{Z}\}$ is called a Riesz basis of V_j if the linear span of $\{\varphi_k^j : k \in \mathbb{Z}\}$ is dense in $V_j \subseteq L^2$ and if there exists a positive constant[1] C with

$$\frac{1}{C}\sqrt{\sum_{k\in\mathbb{Z}} |\lambda_k|^2} \le \left\| \sum_{k\in\mathbb{Z}} \lambda_k \varphi_k^j \right\|_{L^2} \le C\sqrt{\sum_{k\in\mathbb{Z}} |\lambda_k|^2}. \tag{2.5}$$

For the biorthogonal setting, we require that the spaces are dual with respect to the L^2 scalar product and that[2]

$$\langle \varphi_k^j, \tilde{\varphi}_{k'}^j \rangle = \delta_{k,k'}, \quad k, k' \in \mathbb{Z} \tag{2.6}$$

holds for any integer j. The **biorthogonal wavelets** are defined as a special hierarchical basis, i.e., we introduce certain complement spaces W_j and $\tilde{W}_j$ such that $V_{j+1} = V_j \oplus W_j$ and $\tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j$ and that

$$W_j := \text{cl span}\,\{\psi_k^j : k \in \mathbb{Z}\}, \quad \psi_k^j(t) := 2^{j/2}\psi(2^j t - k), \tag{2.7}$$
$$\tilde{W}_j := \text{cl span}\,\{\tilde{\psi}_k^j : k \in \mathbb{Z}\}, \quad \tilde{\psi}_k^j(t) := 2^{j/2}\tilde{\psi}(2^j t - k), \tag{2.8}$$
$$\langle \psi_k^j, \tilde{\psi}_{k'}^{j'} \rangle = \delta_{k,k'}\delta_{j,j'}, \quad j, j', k, k' \in \mathbb{Z}. \tag{2.9}$$

The basis functions ψ_k^j and $\tilde{\psi}_k^j$ are called wavelets and the generating functions ψ and $\tilde{\psi}$ mother wavelets. We will always suppose that the scaling functions and mother wavelets are real valued.

The starting point in the construction of biorthogonal wavelets is the definition of the scaling functions by their mask coefficients. Indeed, from (2.1) and (2.2), we observe that the scaling functions satisfy the so called **refinement equations**

$$\varphi(t) = \sum_{k\in\mathbb{Z}} h_k \sqrt{2}\varphi(2t - k), \tag{2.10}$$
$$\tilde{\varphi}(t) = \sum_{k\in\mathbb{Z}} \tilde{h}_k \sqrt{2}\tilde{\varphi}(2t - k). \tag{2.11}$$

The numbers h_k and $\tilde{h}_k$ are called **mask coefficients**. For real valued φ and $\tilde{\varphi}$, the h_k and $\tilde{h}_k$ are real. Under mild assumptions the scaling functions can be reconstructed from

[1]From now on we use the letter C to denote a general positive constant the value of which varies from instance to instance.

[2]Throughout the present paper the bracket $\langle \cdot, \cdot \rangle$ stands for the L^2 scalar product or for its extension to a duality pairing between the Sobolev space H^s and its dual H^{-s}.

the coefficients h_k and $\tilde{h}_k$ of their refinement equations by the formulae

$$\mathcal{F}\varphi(\xi) = \prod_{l=1}^{\infty} \left[\frac{1}{\sqrt{2}} h(e^{i2\pi\xi 2^{-l}})\right], \quad h(z) := \sum_{k\in\mathbb{Z}} h_k z^k, \tag{2.12}$$

$$\mathcal{F}\tilde{\varphi}(\xi) = \prod_{l=1}^{\infty} \left[\frac{1}{\sqrt{2}} \tilde{h}(e^{i2\pi\xi 2^{-l}})\right], \quad \tilde{h}(z) := \sum_{k\in\mathbb{Z}} \tilde{h}_k z^k. \tag{2.13}$$

Here $\mathcal{F}\varphi$ stands for the Fourier transform

$$\mathcal{F}\varphi(\xi) := \int_{-\infty}^{+\infty} \varphi(t) e^{i2\pi t\xi} \mathrm{d}t. \tag{2.14}$$

Sufficient for the representations (2.12) and (2.13) is that the functions h and $\tilde{h}$ admit factorizations

$$h(e^{i2\pi\xi}) = \sqrt{2}\,[\cos(\pi\xi)]^L\, F(\xi), \quad \xi \in \mathbb{R}, \tag{2.15}$$

$$\tilde{h}(e^{i2\pi\xi}) = \sqrt{2}\,[\cos(\pi\xi)]^{\tilde{L}}\, \tilde{F}(\xi), \quad \xi \in \mathbb{R}, \tag{2.16}$$

where

$$B_K := \sup_{\xi\in\mathbb{R}} \left|F(\xi)F(2\xi)\dots F(2^{K-1}\xi)\right|^{1/K} < 2^{L-1/2}, \tag{2.17}$$

$$\tilde{B}_{\tilde{K}} := \sup_{\xi\in\mathbb{R}} \left|\tilde{F}(\xi)\tilde{F}(2\xi)\dots \tilde{F}(2^{\tilde{K}-1}\xi)\right|^{1/\tilde{K}} < 2^{\tilde{L}-1/2}, \tag{2.18}$$

and K, $\tilde{K}$ are fixed positive integers. Moreover, it is not hard to see that even the corresponding mother wavelets are determined by the mask coefficients h_k and $\tilde{h}_k$. If the scaling functions satisfy (2.10) and (2.11), then the mother wavelets are necessarily defined by

$$\psi(t) = \sum_{k\in\mathbb{Z}} (-1)^k \tilde{h}_{1-k} \sqrt{2}\varphi(2t-k), \tag{2.19}$$

$$\tilde{\psi}(t) = \sum_{k\in\mathbb{Z}} (-1)^k h_{1-k} \sqrt{2}\tilde{\varphi}(2t-k). \tag{2.20}$$

Of course, not every pair of mask sequences $(h_k)_k$ and $(\tilde{h}_k)_k$ can be used for the construction of biorthogonal wavelets. Indeed, the duality relation (2.6) and the two-scale relations (2.10) and (2.11) imply the two equivalent relations

$$h(z)\overline{\tilde{h}(z)} + h(-z)\overline{\tilde{h}(-z)} = 2, \tag{2.21}$$

$$\sum_{k\in\mathbb{Z}} h_k \tilde{h}_{k+2j} = \delta_{j,0}. \tag{2.22}$$

These relations, however, are sufficient in the following sense (cf. [9]):

Theorem 2.1 *Suppose we are given two real sequences $(h_k)_{k\in\mathbb{Z}}$ and $(\tilde{h}_k)_{k\in\mathbb{Z}}$ which decay faster for $|k| \longrightarrow \infty$ than the sequence $(|k|^{-2})_{k\in\mathbb{Z}}$. Moreover, suppose these sequences* *satisfy (2.22) as well as (2.15)-(2.18) for certain positive integers L, K, $\tilde{L}$, and $\tilde{K}$. Then the functions $\mathcal{F}\varphi$ and $\mathcal{F}\tilde{\varphi}$ defined by (2.12) and (2.13), respectively, fulfil*

$$|\mathcal{F}\varphi(\xi)| \leq C(1+|\xi|)^{-1/2-\varepsilon}, \quad \varepsilon := L - \frac{1}{2} - \frac{\log B_K}{\log 2} > 0, \tag{2.23}$$

$$|\mathcal{F}\tilde{\varphi}(\xi)| \leq C(1+|\xi|)^{-1/2-\tilde{\varepsilon}}, \quad \tilde{\varepsilon} := \tilde{L} - \frac{1}{2} - \frac{\log \tilde{B}_{\tilde{K}}}{\log 2} > 0, \tag{2.24}$$

i.e., the functions φ and $\tilde{\varphi}$ belong to the Lebesgue space L^2. These functions satisfy the refinement equations (2.10) and (2.11), the duality relations (2.6), and, if the closures of the linear spans of $\{\varphi^j_k : k \in \mathbb{Z}\}$ and $\{\tilde{\varphi}^j_k : k \in \mathbb{Z}\}$ are denoted by V_j and $\tilde{V}_j$, respectively, then we arrive at two multi-resolution analyses generated by the scaling functions φ and $\tilde{\varphi}$, respectively. Finally, if we introduce the mother wavelets and the basis wavelet functions by (2.19), (2.20), (2.7), and (2.8), then the duality relation (2.9) holds and the systems $\{\psi^l_k : l,k \in \mathbb{Z}\}$ and $\{\tilde{\psi}^l_k : l,k \in \mathbb{Z}\}$ are Riesz bases of the space L^2.

Sometimes the mother wavelets ψ and $\tilde{\psi}$ have not the desired properties. Then one can try to replace ψ and $\tilde{\psi}$ by a **modified mother wavelet** which is a linear combination of integer shifts:

$$\psi^+(t) := \sum_{k\in\mathbb{Z}} g_k \psi_k(t-k), \tag{2.25}$$

$$\tilde{\psi}^+(t) := \sum_{k\in\mathbb{Z}} \tilde{g}_k \tilde{\psi}_k(t-k). \tag{2.26}$$

We suppose that the coefficients g_k and $\tilde{g}_k$ are real. The representations (2.25) and (2.26) lead to the new two-scale relations

$$\psi^+(t) = \sum_{k\in\mathbb{Z}} h^+_k \sqrt{2}\varphi(2t-k), \tag{2.27}$$

$$\tilde{\psi}^+(t) = \sum_{k\in\mathbb{Z}} \tilde{h}^+_k \sqrt{2}\tilde{\varphi}(2t-k), \tag{2.28}$$

where the coefficients are determined by

$$h^+(z) := \sum_{k\in\mathbb{Z}} h^+_k z^k = -z\tilde{h}(-z^{-1})g(z^2), \quad g(z) := \sum_{k\in\mathbb{Z}} g_k z^k \tag{2.29}$$

$$\tilde{h}^+(z) := \sum_{k\in\mathbb{Z}} \tilde{h}^+_k z^k = -zh(-z^{-1})\tilde{g}(z^2), \quad \tilde{g}(z) := \sum_{k\in\mathbb{Z}} \tilde{g}_k z^k. \tag{2.30}$$

Now the following criterion for the Riesz basis property is easy to prove.

Lemma 2.1 *Suppose that the assumptions of Theorem 2.1 are satisfied and that g is a continuous function on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The new basis $\{(\psi^+)^l_k : l,k \in \mathbb{Z}\}$ is a Riesz basis of L^2 if and only if g does not vanish over $\mathbb{T}$. The bases $\{(\psi^+)^l_k\}$ and $\{(\tilde{\psi}^+)^l_k\}$ are dual in the sense of (2.9) if $\tilde{g}(z) = 1/\overline{g(z^{-1})}$.*

Clearly, if $\{\psi^l_k : l,k \in \mathbb{Z}\}$ is a Riesz basis, then also $\{\varphi^0_k : k \in \mathbb{Z}\} \cup \{\psi^l_k : l = 0,\dots,j-1, k \in \mathbb{Z}\}$ is a Riesz basis of L^2. Since we will not use the wavelet functions ψ^l_k

for negative l, we introduce the notation $\psi_k^{-1} := \varphi_k^0$ and $\tilde{\psi}_k^{-1} := \tilde{\varphi}_k^0$. With this notation a Riesz basis of L^2 is given by $\{\psi_k^l : \; k \in \mathbb{Z}, \; l = -1, \dots\}$ and by $\{\tilde{\psi}_k^l : \; k \in \mathbb{Z}, \; l = -1, \dots\}$. The projection Q_j from L^2 onto V_j parallel to the space $W_j \oplus W_{j+1} \oplus \dots$ and its adjoint is given by

$$Q_j f = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\varphi}_k^j \rangle \varphi_k^j \; = \; \sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_k^l \rangle \psi_k^l, \tag{2.31}$$

$$Q_j^* f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_k^j \rangle \tilde{\varphi}_k^j \; = \; \sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} \langle f, \psi_k^l \rangle \tilde{\psi}_k^l. \tag{2.32}$$

Note that Q_j^* projects L^2 onto $\tilde{V}_j$ parallel to the space $\tilde{W}_j \oplus \tilde{W}_{j+1} \oplus \dots$. The projections Q_j, Q_j^* and the spaces V_j, $\tilde{V}_j$ satisfy the following **approximation**[3] **and inverse**[4] **properties**.

Lemma 2.2 *Approximation Property (Jackson type theorem): Suppose that the assumptions of Theorem 2.1 are satisfied and that the l-th derivative of φ and $\tilde{l}$-th derivative of $\tilde{\varphi}$ decay faster at infinity than $|t|^{-\max\{L,\tilde{L}\}-3}$ for any $|l| \le M$ and $|\tilde{l}| \le \tilde{M}$, where M and $\tilde{M}$ are fixed positive integers less or equal to L and $\tilde{L}$, respectively. Then there holds:*

$$\|f - Q_j f\|_{H^s} \le C \left[2^{-j}\right]^{r-s} \|f\|_{H^r}, \quad -\tilde{L} \le s \le r \le L, \; s \le M, \; -\tilde{M} \le r, \tag{2.33}$$

$$\|f - Q_j^* f\|_{H^s} \le C \left[2^{-j}\right]^{r-s} \|f\|_{H^r}, \quad -L \le s \le r \le \tilde{L}, \; s \le \tilde{M}, \; -M \le r. \tag{2.34}$$

Lemma 2.3 *Inverse Property (Bernstein inequality): Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for any $v_j \in V_j$ and for any $\tilde{v}_j \in \tilde{V}_j$, there holds*

$$\|v_j\|_{H^r} \le C \left[2^{-j}\right]^{s-r} \|v_j\|_{H^s}, \quad s \le r < \varepsilon, \; s \le L, \tag{2.35}$$

$$\|\tilde{v}_j\|_{H^r} \le C \left[2^{-j}\right]^{s-r} \|\tilde{v}_j\|_{H^s}, \quad s \le r < \tilde{\varepsilon}, \; s \le \tilde{L}. \tag{2.36}$$

The approximation and inverse property imply the following **discrete norm equivalences** (cf. e.g. [15, 16, 34]).

Corollary 2.1 *Suppose that the assumptions of Theorem 2.1 and Lemmas 2.1 and 2.2 are satisfied. If $-\min\{\tilde{\varepsilon}, \tilde{M}\} < s < \min\{\varepsilon, M\}$ and $-\min\{\varepsilon, M\} < \tilde{s} < \min\{\tilde{\varepsilon}, \tilde{M}\}$, then there exists a positive constant C such that, for all sequences $(\lambda_k^l)_{k,l}$,*

$$\frac{1}{C} \sqrt{\sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} 2^{2sl} |\lambda_k^l|^2} \; \le \; \left\| \sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} \lambda_k^l \psi_k^l \right\|_{H^s} \; \le \; C \sqrt{\sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} 2^{2sl} |\lambda_k^l|^2}, \tag{2.37}$$

$$\frac{1}{C} \sqrt{\sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} 2^{2\tilde{s}l} |\lambda_k^l|^2} \; \le \; \left\| \sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} \lambda_k^l \tilde{\psi}_k^l \right\|_{H^{\tilde{s}}} \; \le \; C \sqrt{\sum_{l=-1}^{j-1} \sum_{k \in \mathbb{Z}} 2^{2\tilde{s}l} |\lambda_k^l|^2}. \tag{2.38}$$

[3]Note that, in view of (2.12)-(2.16), the scaling functions satisfy the Strang-Fix conditions. The approximation property follows in the usual way (cf. e.g. [8, 4, 25, 28]).

[4]Choosing the bases $\{\varphi_k^j\}$ and $\{\tilde{\varphi}_k^j\}$ and using the definition of the Sobolev norms via Fourier transform, the inverse property is an easy consequence of (2.12)-(2.16), (2.23), and (2.24). For proofs in several special cases, see e.g. [8, 25, 16, 28].

In the next sections we have to approximate functions over a closed one-dimensional curve Γ. Using a one periodic parametrization $\gamma : \mathbb{R} \longrightarrow \Gamma$, we identify the function f over Γ with the one periodic function $f = f \circ \gamma$. To approximate the last by wavelet functions, we need the periodic version of the wavelet setting. We denote the operator of **periodization** by *Per*, i.e.,

$$Per\, f(t) := \sum_{k \in \mathbb{Z}} f(t-k). \tag{2.39}$$

Now the wavelet functions ψ_k^l and $\tilde{\psi}_k^l$ can be replaced by $Per\, \psi_k^l$ and $Per\, \tilde{\psi}_k^l$, respectively. Since we will not use the original non-periodic wavelets anymore, we simply write ψ_k^l and $\tilde{\psi}_k^l$ for $Per\, \psi_k^l$ and $Per\, \tilde{\psi}_k^l$, respectively. Similarly, we consider the functions φ_k^j and $\tilde{\varphi}_k^j$ to be periodized. Clearly, the periodic functions ψ_k^l and $\psi_{k \pm 2^l}^l$ coincide and the periodic wavelet spaces take the form

$$\begin{aligned} W_l &:= \operatorname{span}\{\psi_k^l : k = 0,1,\dots,n_l\}, \quad & V_j &:= \operatorname{span}\{\varphi_k^j : k = 0,1,\dots,n_j\}, & (2.40) \\ \tilde{W}_l &:= \operatorname{span}\{\tilde{\psi}_k^l : k = 0,1,\dots,n_l\}, \quad & \tilde{V}_j &:= \operatorname{span}\{\tilde{\varphi}_k^j : k = 0,1,\dots,n_j\}, & (2.41) \end{aligned}$$

where $n_{-1} := 0$ and $n_l := 2^l - 1$, $l = 0, 1, \dots$. In particular, it is not hard to see that $\psi_0^{-1} = \varphi_0^0$ is constant and $V_0 = W_{-1}$ is the space of constant functions. All the results of the present section formulated for the functions and spaces over $\mathbb{R}$ remain valid for the functions and spaces over the periodic interval if the summations over $k \in \mathbb{Z}$ are replaced by summations over $k = 0, 1, \dots, n_l$ and if the spaces L^2 and H^s over $\mathbb{R}$ are replaced by the corresponding spaces L^2 and H^s over the periodic interval $[0, 1]$.

3 A Piecewise Linear Wavelet Basis

In the present section we introduce the biorthogonal setting for the piecewise linear trial space. To distinguish the spaces and wavelet functions from those defined for the space of test functionals in the next section, we add a left upper index[5] A to all the objects of the trial space and a left upper index T to all objects from the test space.

Now the trial space is the space ${}^A V_j$ of piecewise linear functions over the uniform grid $\{k 2^{-j}\}$ or a subspace of ${}^A V_j$. If the scaling function is the well-known hat function ${}^A\varphi$ defined by ${}^A\varphi(t) := \max\{0, 1 - |t|\}$, then ${}^A V_j$ is the span of $\{{}^A\varphi_k^j : k = 0, \dots, n_j\}$. To find a wavelet basis function, we seek a linear combination (2.19) of the shifts $\sqrt{2}\, {}^A\varphi(2 \cdot -k)$ with a minimal support and two vanishing moments, i.e. orthogonal to linear functions. Note that the minimal support is important for the fast computation of the stiffness matrix with respect to the wavelet basis and the vanishing moments are essential for the compression of the stiffness matrix. The solution is (cf. Figure 1)

$${}^A\psi(t) \;:=\; \frac{1}{2\sqrt{2}} \sqrt{2}\, {}^A\varphi(2t) - \frac{1}{\sqrt{2}} \sqrt{2}\, {}^A\varphi(2t-1) + \frac{1}{2\sqrt{2}} \sqrt{2}\, {}^A\varphi(2t-2) \tag{3.1}$$

[5] A from the German word "Ansatzraum" for the English "trial space"!

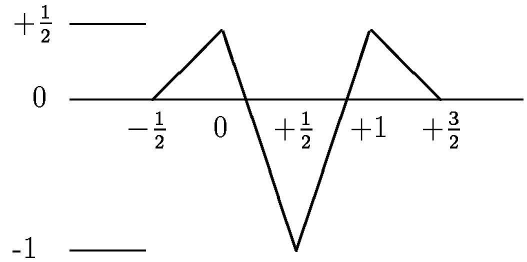

Figure 1: Mother wavelet ${}^A\psi$.

$$= \begin{cases} [t+\frac{1}{2}] & \text{if } -\frac{1}{2} \le t \le 0 \\ -3[t-0]+\frac{1}{2} & \text{if } 0 \le t \le \frac{1}{2} \\ 3[t-\frac{1}{2}]-1 & \text{if } \frac{1}{2} \le t \le 1 \\ -[t-1]+\frac{1}{2} & \text{if } 1 \le t \le \frac{3}{2} \\ 0 & \text{else .} \end{cases}$$

This fits into the biorthogonal setting of the previous section if

$$ {}^A h(e^{i2\pi\xi}) := \sqrt{2}[\cos(\pi\xi)]^2, \tag{3.2}$$

$$ {}^A \tilde{h}(e^{i2\pi\xi}) := \sqrt{2}[\cos(\pi\xi)]^2 \frac{4}{3+\cos(4\pi\xi)}, \tag{3.3}$$

and if we apply Lemma 2.1 with ${}^A g(e^{i2\pi\xi}) := [3+\cos(2\pi\xi)]/4$. Indeed, the hat function satisfies the refinement equation

$$ {}^A\varphi(t) := \frac{1}{2\sqrt{2}}\sqrt{2}\,{}^A\varphi(2t+1) + \frac{1}{\sqrt{2}}\sqrt{2}\,{}^A\varphi(2t) + \frac{1}{2\sqrt{2}}\sqrt{2}\,{}^A\varphi(2t-1) \tag{3.4}$$

which leads to the definition of ${}^A h$. The definition of ${}^A\tilde{h}$ and ${}^A g$ results in (3.1) (cf. (2.27) and (2.29)). It is not hard to prove, that the assumptions of Theorem 2.1 are fulfilled with $L = \tilde{L} = 2$, $K = 1$, $B_1 = 1$, $\tilde{K} = 2$, and $\tilde{B}_2 = 1.65767\ldots$. Using the arguments of the proof to Theorem 2.1, we even conclude that (2.24) holds in a strip around $\mathbb{R} \subseteq \mathbb{C}$. Hence, the dual scaling function and by (2.20) also the dual mother wavelet decays exponentially for $t \longrightarrow \pm\infty$. This exponential decay remains true also for the dual mother wavelet ${}^A\tilde{\psi}(t)$ modified according to Lemma 2.1 with ${}^A\tilde{g}(z) := 1/\,\overline{{}^A g(z)}$ since the Fourier coefficients ${}^A\tilde{g}_k$ of the analytic function ${}^A\tilde{g}$ decay exponentially, too. In other words, the assumptions of the Lemmas 2.2 and 2.3 are satisfied and we conclude:

Lemma 3.1 *The linear functions fulfil:*

i) If the projection ${}^A Q_j$ is defined as

$$ {}^A Q_j f := \sum_{k=0}^{n_j} \langle f, {}^A\tilde{\varphi}_k^j \rangle\, {}^A\varphi_k^j = \sum_{l=-1}^{j-1} \sum_{k=0}^{n_l} \langle f, {}^A\tilde{\psi}_k^l \rangle\, {}^A\psi_k^l \tag{3.5}$$

and if[6] $-2 \le s \le r \le 2,\ -0.7708\ldots < r,\ s < 3/2$, *then*

$$\|f - {}^{A}Q_j f\|_{H^s} \le C\left[2^{-j}\right]^{r-s} \|f\|_{H^r}. \tag{3.6}$$

ii) For $v_j \in {}^{A}V_j$ *and* $\tilde{v}_j \in {}^{A}\tilde{V}_j$, *there holds:*

$$\|v_j\|_{H^r} \le C\left[2^{-j}\right]^{s-r} \|v_j\|_{H^s}, \quad s \le r < 3/2, \tag{3.7}$$

$$\|\tilde{v}_j\|_{H^r} \le C\left[2^{-j}\right]^{s-r} \|\tilde{v}_j\|_{H^s}, \quad s \le r < 0.7708\ldots\ . \tag{3.8}$$

iii) If[7] $-0.7708\ldots < s < 3/2$, *then there exists a positive constant* C *such that, for all sequences* $(\lambda_k^l)_{k,l}$,

$$\frac{1}{C}\sqrt{\sum_{l=-1}^{j-1}\sum_{k=0}^{n_l} 2^{2sl}|\lambda_k^l|^2} \le \left\|\sum_{l=-1}^{j-1}\sum_{k\in\mathbb{Z}} \lambda_k^l\, {}^{A}\psi_k^l\right\|_{H^s} \le C\sqrt{\sum_{l=-1}^{j-1}\sum_{k=0}^{n_l} 2^{2sl}|\lambda_k^l|^2}. \tag{3.9}$$

iv) The wavelet functions ${}^{A}\psi_k^l$, $l \ge 0$ *have two vanishing moments, i.e. they are orthogonal to all linear functions.*

Remark that the dual scaling function ${}^{A}\tilde{\varphi}$ and the dual wavelet functions ${}^{A}\tilde{\psi}_k^l$ will not appear in the implementation of the wavelet collocation algorithm. However, they will play an important role in the theoretical analysis of the algorithm.

To define the trial space, we choose a $j_0 \le j$ and an arbitrary index set Λ such that

$$\{(l,k):\ l = -1,\ldots,j_0-1,\ k = 0,\ldots,n_l\} \subseteq \Lambda, \tag{3.10}$$

$$\Lambda \subseteq \{(l,k):\ l = -1,\ldots,j-1,\ k = 0,\ldots,n_l\}. \tag{3.11}$$

We introduce the **trial space** ${}^{A}V$ as the span of all the functions ${}^{A}\psi_k^l$ with $(l,k) \in \Lambda$. Hence, we get ${}^{A}V_{j_0} \subseteq {}^{A}V \subseteq {}^{A}V_j$. The inclusion (3.10) will guarantee that smooth functions can be approximated with high order, whereas (3.11) will enable us to approximate functions with singularities if, for high level l and for a support of ${}^{A}\psi_k^l$ close to the singularity points, we take (l,k) into Λ.

4 A Wavelet Basis in the Test Space

The space of test functionals is the space ${}^{T}\tilde{V}_j$ of Dirac delta distributions (functionals of function evaluation) at the points of the uniform grid $\{k2^{-j}\}$ or a subspace of ${}^{T}\tilde{V}_j$. Thus the dual scaling function is the well-known Dirac delta distribution ${}^{T}\tilde{\varphi}$ defined by ${}^{T}\tilde{\varphi}(f) := \langle\delta_0, f\rangle = f(0)$ and the space ${}^{T}\tilde{V}_j$ is the span of $\{{}^{T}\tilde{\varphi}_k^j = 2^{-j/2}\delta_{k2^{-j}}:\ k = 0,\ldots,n_j\}$. Note

[6] From Lemma 2.2 we get the conditions $s \le 1$ and $0 \le r$. However, in view of the well-known approximation property for piecewise linear functions the bound $s \le 1$ can easily be improved to $s < 3/2$. Using special techniques for non-integer orders, the assumption $0 \le r$ can be improved to $-{}^{A}\tilde{\varepsilon} = -0.7708\ldots < r$.

[7] From Lemma 2.2 and Corollary 2.1 we get the condition $0 \le s < 1$. Arguments similar to the last footnote lead to $-0.7708\ldots < s < 3/2$.

that the normalization factor $2^{-j/2}$ is in accordance with Section 2. Indeed, it corresponds to

$$\left\langle {}^T\tilde{\varphi}^j_k, f\right\rangle = 2^{-j/2}\int_{\mathbb{R}} f\left([\tau+k]2^{-j}\right)\, {}^T\tilde{\varphi}(\tau)\mathrm{d}\tau \tag{4.1}$$

which coincides with $\tilde{\varphi}^j_k(t) := 2^{j/2}\tilde{\varphi}(2^jt-k)$ for integrable distributions $\tilde{\varphi}$.

This time $\tilde{\psi}^{-1}_0 = \tilde{\varphi}^0_0 = \delta_0$ and the refinement equation turns into ${}^T\tilde{\varphi} = \sqrt{2}\,2^{-1/2}\tilde{\varphi}$, i.e., the function ${}^T\tilde{h}$ is equal to the constant $\sqrt{2}$. We choose the corresponding wavelet functions as

$$\begin{aligned} {}^T\tilde{\psi} &:= -\frac{1}{16}\delta_{-\frac{1}{2}} + \frac{1}{4}\delta_0 - \frac{3}{8}\delta_{\frac{1}{2}} + \frac{1}{4}\delta_1 - \frac{1}{16}\delta_{\frac{3}{2}}, & (4.2)\\ {}^T\tilde{\psi}^l_k &:= 2^{-l/2}\left\{-\frac{1}{16}\delta_{[k-\frac{1}{2}]2^{-l}} + \frac{1}{4}\delta_{k2^{-l}} - \frac{3}{8}\delta_{[k+\frac{1}{2}]2^{-l}} + \frac{1}{4}\delta_{[k+1]2^{-l}} - \frac{1}{16}\delta_{[k+\frac{3}{2}]2^{-l}}\right\}. & (4.3)\end{aligned}$$

The motivation for this choice is the following lemma which is a simple version of the so-called **Arnold-Wendland lemma** (compare [2]) and which is fundamental for the stability analysis in Section 5.

Lemma 4.1 *Let ∂ stand for the operator of differentiation. For any continuously differentiable function f, we get*

$$\left\langle \partial f, \partial\, {}^A\psi^l_k\right\rangle = \int_0^1 \frac{d}{dt}f(t)\frac{d}{dt}[{}^A\psi^l_k](t)\mathrm{d}t = 16\cdot 2^{2l}\cdot\left\langle {}^T\tilde{\psi}^l_k, f\right\rangle. \tag{4.4}$$

Clearly, this kind of wavelet does not fit perfectly into the biorthogonal setting of Section 2. Nevertheless, the wavelet nature of these functionals is well known (cf. e.g. [17, 18, 22]) and we can use the techniques of Section 2 to establish the predual system. From (4.2) and (2.28), we have ${}^T\tilde{h}^+(e^{i2\pi\xi}) = -\sqrt{2}e^{i2\pi\xi}[\sin(\pi\xi)]^4$. In view of (2.21), of Lemma 2.1, and of the relation (2.30), we set

$$\begin{aligned} {}^Th(e^{i2\pi\xi}) &:= \sqrt{2}[\cos(\pi\xi)]^4\frac{4}{3+\cos(4\pi\xi)}, & (4.5)\\ {}^T\tilde{g}(e^{i2\pi\xi}) &:= \frac{3+\cos(2\pi\xi)}{4}. & (4.6)\end{aligned}$$

Following the techniques of the proof to Theorem 2.1 (including $L=4$, $K=2$, $B_2 = 1.65767\ldots$), we observe that the scaling function corresponding to the mask coefficients generated by function Th satisfies

$$|\mathcal{F}\,{}^T\varphi(\xi)| \le C(1+|\xi|)^{-3.27081\ldots}. \tag{4.7}$$

Hence, the second derivative of ${}^T\varphi$ is continuous. Again following the proof of Theorem 2.1, we get the orthogonality relations (2.6) and (2.9). Analogously to the Lemmas 2.2 and 2.3 we arrive at:

Lemma 4.2 *The Dirac delta test functionals and their duals fulfil:*

i) If the interpolation projection ${}^TQ_j$ *is defined as*

$$ {}^TQ_jf := \sum_{k=0}^{n_j} f\left(\frac{k}{2^j}\right) 2^{-j/2}\, {}^T\varphi_k^j \;=\; \sum_{k=0}^{n_j} \langle f,\, {}^T\tilde{\varphi}_k^j\rangle\, {}^T\varphi_k^j \;=\; \sum_{l=-1}^{j-1}\sum_{k=0}^{n_l} \langle f,\, {}^T\tilde{\psi}_k^l\rangle\, {}^T\psi_k^l \quad (4.8) $$

and if[8] $0 \le s \le r \le 4,\ 1/2 < r,\ s < 2.7708\ldots$, *then*

$$ \|f - {}^TQ_jf\|_{H^s} \le C\left[2^{-j}\right]^{r-s} \|f\|_{H^r}. \quad (4.9) $$

ii) For $v_j \in {}^TV_j$ *and* $\tilde{v}_j \in {}^T\tilde{V}_j$*, there holds:*

$$ \|v_j\|_{H^r} \;\le\; C\left[2^{-j}\right]^{s-r} \|v_j\|_{H^s}, \quad s \le r < 2.7708\ldots\ , \quad (4.10) $$

$$ \|\tilde{v}_j\|_{H^r} \;\le\; C\left[2^{-j}\right]^{s-r} \|\tilde{v}_j\|_{H^s}, \quad s \le r < -1/2. \quad (4.11) $$

iii) If[9] $1/2 < s < 2.7708\ldots$, *then there exists a positive constant* C *such that, for all sequences* $(\lambda_k^l)_{k,l}$*,*

$$ \frac{1}{C}\sqrt{\sum_{l=-1}^{j-1}\sum_{k=0}^{n_l} 2^{2sl}|\lambda_k^l|^2} \;\le\; \left\|\sum_{l=-1}^{j-1}\sum_{k\in\mathbb{Z}} \lambda_k^l\, {}^T\psi_k^l\right\|_{H^s} \;\le\; C\sqrt{\sum_{l=-1}^{j-1}\sum_{k=0}^{n_l} 2^{2sl}|\lambda_k^l|^2}. \quad (4.12) $$

iv) The wavelet functionals ${}^T\tilde{\psi}_k^l$*,* $l \ge 0$ *have four vanishing moments, i.e. they are orthogonal to all cubic functions.*

Remark that the predual scaling function ${}^T\varphi$ and the predual wavelet functions ${}^T\psi_k^l$ will not appear in the implementation of the wavelet collocation algorithm. However, they will play an important role in the theoretical analysis of the algorithm.

We introduce the **test space** ${}^TV$ as the span of all the functions ${}^T\tilde{\psi}_k^l$ with $(l,k) \in \Lambda$.

5 The Wavelet Collocation Algorithm

Now we consider the linear operator equation $Ax = y$ with the operator A mapping the Sobolev space $H^{\mathbf{r}/2+1}$ over the one-periodic interval into the space $H^{-\mathbf{r}/2+1}$. Here $\mathbf{r}$ is the order of the operator. The **wavelet collocation method** seeks an approximate solution $x_\Lambda \in {}^AV$ for the exact solution x of $Ax = y$ such that

$$ \langle Ax_\Lambda,\, {}^T\tilde{\psi}_k^l\rangle = \langle y,\, {}^T\tilde{\psi}_k^l\rangle, \quad (l,k) \in \Lambda. \quad (5.1) $$

Note that the last system is equivalent to the classical collocation method $Ax_\Lambda(k2^{-j}) = y(k2^{-j})$, $k = 0,\ldots,2^j-1$ if in (3.11) the equality sign is true. We can write (5.1) in form of the operator equation $A_\Lambda x_\Lambda = {}^TQ_\Lambda y$ with

$$ A_\Lambda \;:=\; {}^TQ_\Lambda A|_{{}^AV} : \operatorname{im}{}^AQ_\Lambda = \left({}^AV, \|\cdot\|_{H^{\mathbf{r}/2+1}}\right) \longrightarrow \operatorname{im}{}^TQ_\Lambda = \left({}^TV, \|\cdot\|_{H^{-\mathbf{r}/2+1}}\right) \quad (5.2) $$

[8] Though Lemma 2.2 would suggest the condition $s \le 2$, special techniques for non-integer orders lead to $s < {}^T\varepsilon = 2.7708\ldots$.

[9] Similarly to the last footnote, the assumption $1/2 < s < 2$ with an integer upper bound 2 can be improved to $1/2 < s < 2.7708\ldots$.

and the projections

$$ {}^TQ_\Lambda f := \sum_{(l',k')\in\Lambda} \left\langle f, {}^T\tilde\psi^{l'}_{k'}\right\rangle {}^T\psi^{l'}_{k'}, \quad {}^AQ_\Lambda f := \sum_{(l,k)\in\Lambda} \left\langle f, {}^A\tilde\psi^{l}_{k}\right\rangle {}^A\psi^{l}_{k}. \tag{5.3} $$

Moreover, the system (5.1) is equivalent to the matrix equation

$$ \mathcal{A}\xi \;=\; \eta, \tag{5.4} $$

where $\mathcal{A}$ is the matrix of A_Λ with respect to the bases $\{{}^A\psi^l_k\}$ and $\{{}^T\psi^l_k\}$,

$$ \mathcal{A} \;:=\; \left(\alpha_{(l',k'),(l,k)}\right)_{(l',k'),(l,k)\in\Lambda}, \quad \alpha_{(l',k'),(l,k)} \;:=\; \left\langle A\,{}^A\psi^l_k, {}^T\tilde\psi^{l'}_{k'}\right\rangle, $$

and the discretized right-hand side η as well as the unknown vector ξ is defined as

$$ \begin{aligned} \eta &:= (\eta^{l'}_{k'})_{(l',k')\in\Lambda}, & \eta^{l'}_{k'} &:= \left\langle y, {}^T\tilde\psi^{l'}_{k'}\right\rangle, \\ \xi &:= (\xi^l_k)_{(l,k)\in\Lambda}, & x_\Lambda &= \sum_{(l,k)\in\Lambda} \xi^l_k\,{}^A\psi^l_k. \end{aligned} $$

In order to present a theorem on the convergence of the wavelet collocation, we recall some definition. The wavelet collocation method is called **stable** if there exists a j_0' such that the approximate operators A_Λ are invertible for any Λ with $j_0 \geq j_0'$ and if the norms of their inverses are uniformly bounded. Note that, if the method is stable, then the approximate solutions x_Λ tend in $H^{\mathbf{r}/2+1}$ to the exact solution x for any right-hand side $y \in H^{-\mathbf{r}/2+1}$ (take into account (5.2), Lemmas 3.1 and 4.2 and cf. e.g. [28]). We call an operator $B:\ H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$ **strongly elliptic** if there exists a compact operator $T:\ H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$ such that the Gårding inequality

$$ \mathrm{Re}\,\langle (B-T)f, f\rangle \;\geq\; \frac{1}{C}\|f\|_{H^{\mathbf{r}/2}} \tag{5.5} $$

holds for any $f \in H^{\mathbf{r}/2}$. Finally, we denote the operator of differentiation by ∂ and its one-sided inverse by ∂^{-1}. For any $k \in \mathbb{Z}$, this operator ∂^{-1} maps the function $t \mapsto e^{i2\pi kt}$ to the function $t \mapsto e^{i2\pi kt}/[i2\pi k]$ if $k \neq 0$ and to the function $t \mapsto 1$ if $k = 0$.

Theorem 5.1 *Suppose that the operator $A:\ H^{\mathbf{r}/2+1} \longrightarrow H^{-\mathbf{r}/2+1}$ is invertible, that the order $\mathbf{r}$ satisfies $-3.542\ldots < \mathbf{r} < 1$ and that $B := \partial A \partial^{-1}:\ H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$ is strongly elliptic. Then:*

i) The wavelet collocation method is stable.

ii) For $\mathbf{r}/2+1 \leq s < 3/2$, we get the error estimate (cf. Lemma 3.1 i))

$$ \|x - x_\Lambda\|_{H^s} \;\leq\; C\left(2^{-j}\right)^{\mathbf{r}/2+1-s}\|x - {}^AQ_\Lambda x\|_{H^{\mathbf{r}/2+1}} \;+\; C\|x - {}^AQ_\Lambda x\|_{H^s}. \tag{5.6} $$

iii) If $\mathbf{r} \leq s \leq \mathbf{r}/2+1$ and if $A:\ H^s \longrightarrow H^{s-\mathbf{r}}$ is bounded and invertible, then

$$ \|x - x_\Lambda\|_{H^s} \leq C\left(2^{-j_0}\right)^{\mathbf{r}/2+1-s}\|x - {}^AQ_\Lambda x\|_{H^{\mathbf{r}/2+1}}. \tag{5.7} $$

Proof. i) By assumption $B := \partial A \partial^{-1}$ is strongly elliptic. Hence, there exists T such that (5.5) holds. We split $A = A' + T'$, where $A' := \Pi + [A - \partial^{-1} T \partial](I - \Pi)$, $T' := -\Pi + A\Pi + \partial^{-1} T \partial (I - \Pi)$ and where Π is the one-dimensional projection which maps a function f to the constant function equal to the zero-th Fourier coefficient $f_0 = \int f$. Clearly, T' is compact and, by well-known perturbation arguments from the theory of projection methods (cf. e.g. [28]), it suffices to prove stability for operator A'. However, the space of constants $\operatorname{im} \Pi = \operatorname{span}\{{}^A\psi_0^{-1}\} = \operatorname{span}\{{}^T\psi_0^{-1}\}$ is an invariant subspace of A'. Therefore, we can restrict our stability analysis to the restriction of operator ${}^T Q_\Lambda A'|_{{}^A V}$ to the spaces $\operatorname{span}\{{}^A\psi_k^l : (l,k) \in \Lambda, \ (l,k) \neq (-1,0)\}$ and $\operatorname{span}\{{}^T\psi_k^l : (l,k) \in \Lambda, \ (l,k) \neq (-1,0)\}$. We introduce the l^2- scalar product

$$[\xi, \eta] := \sum_{(l,k) \in \Lambda} \xi_k^l \overline{\eta_k^l}.$$

The assumption $-3.542\ldots < \mathbf{r} < 1$ is equivalent to $-0.7708\ldots < \mathbf{r}/2 + 1 < 3/2$ and to $1/2 < -\mathbf{r}/2 + 1 < 2.7708\ldots$. In view of Lemma 3.1 iii) and Lemma 4.2 iii) we only have to prove

$$\operatorname{Re} \left[\mathcal{A}' \left(2^{-(\mathbf{r}/2+1)l} \lambda_k^l\right)_{(l,k)\in\Lambda} , \left(2^{(-\mathbf{r}/2+1)l'} \lambda_{k'}^{l'}\right)_{(l',k')\in\Lambda} \right] \geq \frac{1}{C} \sqrt{\sum_{(l,k)\in\Lambda} \left|\lambda_k^l\right|^2} \tag{5.8}$$

where $\mathcal{A}'$ is defined for A' as $\mathcal{A}$ for A and where $\lambda_0^{-1} = 0$. Taking Lemma 4.1 into account and using the definition of A' and (5.5), we conclude

$$\begin{aligned}
& \operatorname{Re} \left[\mathcal{A}' \left(2^{-(\mathbf{r}/2+1)l} \lambda_k^l\right)_{(l,k)\in\Lambda} , \left(2^{(-\mathbf{r}/2+1)l'} \lambda_{k'}^{l'}\right)_{(l',k')\in\Lambda} \right] \qquad (5.9) \\
= \ & \operatorname{Re} \left\langle A' \left\{ \sum_{(l,k)\in\Lambda} 2^{-(\mathbf{r}/2+1)l} \lambda_k^l \, {}^A\psi_k^l \right\}, \left\{ \sum_{(l',k')\in\Lambda} 2^{(1-\mathbf{r}/2)l'} \lambda_{k'}^{l'} \, {}^T\tilde{\psi}_{k'}^{l'} \right\} \right\rangle \\
= \ & \frac{1}{16} \operatorname{Re} \left\langle \partial [A - \partial^{-1} T \partial] \left\{ \sum_{(l,k)\in\Lambda} 2^{-(\mathbf{r}/2+1)l} \lambda_k^l \, {}^A\psi_k^l \right\}, \partial \left\{ \sum_{(l',k')\in\Lambda} 2^{-(\mathbf{r}/2+1)l'} \lambda_{k'}^{l'} \, {}^A\tilde{\psi}_{k'}^{l'} \right\} \right\rangle \\
= \ & \frac{1}{16} \operatorname{Re} \left\langle [B - T] \partial \left\{ \sum_{(l,k)\in\Lambda} 2^{-(\mathbf{r}/2+1)l} \lambda_k^l \, {}^A\psi_k^l \right\}, \partial \left\{ \sum_{(l',k')\in\Lambda} 2^{-(\mathbf{r}/2+1)l'} \lambda_{k'}^{l'} \, {}^A\tilde{\psi}_{k'}^{l'} \right\} \right\rangle \\
\geq \ & \frac{1}{C} \left\| \partial \left\{ \sum_{(l,k)\in\Lambda} 2^{-(\mathbf{r}/2+1)l} \lambda_k^l \, {}^A\psi_k^l \right\} \right\|_{H^{\mathbf{r}/2}}^2 .
\end{aligned}$$

Since the space $\operatorname{span}\{{}^A\psi_k^l : (l,k) \in \Lambda, \ (l,k) \neq (-1,0)\}$ is orthogonal to

$$\operatorname{span}\{{}^A\tilde{\psi}_0^{-1}\} = \operatorname{span}\{{}^A\psi_0^{-1}\} = \operatorname{span}\{1\}$$

and since ∂ is invertible on the orthogonal complement of this one-dimensional space, we continue (5.9) by

$$\frac{1}{C} \left\| \partial \left\{ \sum_{(l,k)\in\Lambda} 2^{-(\mathbf{r}/2+1)l} \lambda_k^l \, {}^A\psi_k^l \right\} \right\|_{H^{\mathbf{r}/2}}^2 \geq \frac{1}{C} \left\| \sum_{(l,k)\in\Lambda} 2^{-(\mathbf{r}/2+1)l} \lambda_k^l \, {}^A\psi_k^l \right\|_{H^{\mathbf{r}/2+1}}^2 \tag{5.10}$$

which together with (3.9) yields (5.8).

ii) First we consider the case $s = \mathbf{r}/2 + 1$ and conclude

$$\begin{aligned} x - x_\Lambda &= x - {}^A Q_\Lambda x + A_\Lambda^{-1} \left\{ A_\Lambda {}^A Q_\Lambda x - A_\Lambda x_\Lambda \right\} \qquad (5.11) \\ &= x - {}^A Q_\Lambda x + A_\Lambda^{-1} \left\{ {}^T Q_\Lambda A {}^A Q_\Lambda x - {}^T Q_\Lambda A x \right\}, \\ \|x - x_\Lambda\|_{H^{\mathbf{r}/2+1}} &\leq \left\{ 1 + \|A_\Lambda^{-1}\|_{\mathcal{L}(H^{-\mathbf{r}/2+1}, H^{\mathbf{r}/2+1})} \|{}^T Q_\Lambda\|_{\mathcal{L}(H^{-\mathbf{r}/2+1})} \|A\|_{\mathcal{L}(H^{\mathbf{r}/2+1}, H^{-\mathbf{r}/2+1})} \right\} \\ &\quad \times \|x - {}^A Q_\Lambda x\|_{H^{\mathbf{r}/2+1}}. \qquad (5.12) \end{aligned}$$

This, the stability, and the uniform boundedness of the projections (cf. Lemma 3.1 i)) prove the error estimate. The general case $\mathbf{r}/2+1 \leq s < 3/2$ follows from the just treated case and from the inverse property of Lemma 3.1 ii).

iii) The last estimate follows by the well-known Nitsche trick. For $\mathbf{r} \leq s < \mathbf{r}/2 + 1$, we get

$$\begin{aligned} \|x - x_\Lambda\|_{H^s} &\leq C \sup_{\|v\|_{H^{-s}} \leq 1} |\langle x - x_\Lambda, v \rangle| \leq C \sup_{\|v\|_{H^{\mathbf{r}-s}} \leq 1} |\langle x - x_\Lambda, A^* v \rangle| \qquad (5.13) \\ &\leq C \sup_{\|v\|_{H^{\mathbf{r}-s}} \leq 1} |\langle y - A x_\Lambda, v \rangle| = C \sup_{\|v\|_{H^{\mathbf{r}-s}} \leq 1} \left| \left\langle y - A x_\Lambda, v - {}^T Q_\Lambda^* v \right\rangle \right| \\ &\leq C \|y - A x_\Lambda\|_{H^{-\mathbf{r}/2+1}} \sup_{\|v\|_{H^{\mathbf{r}-s}} \leq 1} \|v - {}^T Q_\Lambda^* v\|_{H^{\mathbf{r}/2-1}}. \end{aligned}$$

Using the invertibility of $A: \ H^{\mathbf{r}/2+1} \longrightarrow H^{-\mathbf{r}/2+1}$ and the approximation property Lemma 4.2 i) in its adjoint form, we arrive at the assertion iii) of our theorem.

$\diamond$

Remark 5.1 *Theorem 5.1 is a generalization of a result in [2]. In that paper only the case with equality in (3.11) is treated. However, in [2] general non-uniform meshes are allowed.*

Remark 5.2 *Theorem 5.1 remains true if we do not suppose the strong ellipticity of* $B := \partial A \partial^{-1}: \ H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$, *but assume the strong ellipticity of* $B := \partial^2 A: \ H^{\mathbf{r}/2+1} \longrightarrow H^{-\mathbf{r}/2-1}$ *or that of* $B := A\partial^{-2}: \ H^{\mathbf{r}/2-1} \longrightarrow H^{-\mathbf{r}/2+1}$. *The proof is the same.*

In order to give another sufficient condition for the stability, we introduce several operators. The operator ${}^A D: \ H^s \longrightarrow H^{s-1}$ for $-0.7708\ldots < s - 1 < s < 3/2$ is defined by

$${}^A D f := \sum_{l=-1}^{\infty} \sum_{k=0}^{n_l} 2^{\max\{0,l\}} \left\langle f, {}^A \tilde{\psi}_k^l \right\rangle {}^A \psi_k^l \qquad (5.14)$$

and the operator ${}^T D: \ H^s \longrightarrow H^{s-1}$ for $1/2 < s - 1 < s < 2.7708\ldots$ similarly. Clearly, these operators are invertible and

$${}^T D^{-1} f := \sum_{l=-1}^{\infty} \sum_{k=0}^{n_l} 2^{-\max\{0,l\}} \left\langle f, {}^T \tilde{\psi}_k^l \right\rangle {}^T \psi_k^l. \qquad (5.15)$$

It is not hard to see that the compositions $[\partial^T D^{-1}]$ and $[{}^A D \partial^{-1}]$ take the form

$$[\partial^T D^{-1}] f = \sum_{l=0}^{\infty} \sum_{k=0}^{n_l} \left\langle f, {}^T\tilde{\psi}_k^l \right\rangle [\partial^T \psi]_k^l, \tag{5.16}$$

$$[{}^A D \partial^{-1}] f = \sum_{l=-1}^{\infty} \sum_{k=0}^{n_l} \left\langle f, [\partial^{-1} {}^A\tilde{\psi}]_k^l \right\rangle {}^A\psi_k^l, \tag{5.17}$$

and that (cf. Lemma 4.1)

$$[{}^A D \partial^{-1}][\partial^T D^{-1}]^{*\,-1} f = \frac{1}{16} \sum_{l=0}^{\infty} \sum_{k=0}^{n_l} \left\langle f, {}^T\psi_k^l \right\rangle {}^A\psi_k^l, \tag{5.18}$$

$$[{}^A D \partial^{-1}][\partial^T D^{-1}]^{*\,-1} : \quad {}^T\tilde{\psi}_k^l \mapsto {}^A\psi_k^l, \quad l \geq 0. \tag{5.19}$$

Remark 5.3 *If the stability is considered for $A : H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$ and if $-1.441\ldots < \mathbf{r} < 1$, then Theorem 5.1 remains true if we replace the strong ellipticity of $B := \partial A \partial^{-1} : H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$ by that of $B := [\partial^T D^{-1}] A [{}^A D \partial^{-1}] : H^{\mathbf{r}/2} \longrightarrow H^{-\mathbf{r}/2}$ and if we replace the threshold $\mathbf{r}/2+1$ in the error estimates by $\mathbf{r}/2$. The proof is the same. Unfortunately, the strong ellipticity of B seems to be hard to verify. Note that B is strongly elliptic if and only if $A[{}^A D \partial^{-1}][\partial^T D^{-1}]^{*\,-1}$ is strongly elliptic.*

The advantage of the the wavelet collocation in comparison to conventional piecewise linear collocation is that, due to the moment conditions of the trial and test functionals, the stiffness matrix $\mathcal{A}$ contains a lot of very small entries. For special operators A, one can give a set[10] $CA \subseteq \Lambda \times \Lambda$ such that the **compressed matrix**

$$\mathcal{A}^C := \left(\alpha^C_{(l',k'),(l,k)} \right)_{(l',k'),(l,k) \in \Lambda}, \tag{5.20}$$

$$\alpha^C_{(l',k'),(l,k)} := \begin{cases} \alpha_{(l',k'),(l,k)} & \text{if } ((l',k'),(l,k)) \in CA \\ 0 & \text{else} \end{cases}$$

is close to $\mathcal{A}$. The replacement of $\mathcal{A}$ by $\mathcal{A}^C$ in (5.4) leads to an additional compression error less than the expected asymptotic error $O(n^{-2+\mathbf{r}})$ of the piecewise linear collocation. Here n stands for the number of degrees of freedom, i.e. for the cardinality of Λ. The number of non-zero entries in $\mathcal{A}^C$, i.e., the numbers of pairs of indices in CA, is[11] $O(n \log n)$. Hence, the following algorithm leads to an approximate solution of accuracy $O(n^{-2+\mathbf{r}})$ with $O(n \log n)$ arithmetic operations and a storage requirement of about $O(n \log n)$ numbers.

Compressed Wavelet Algorithm:

i) Determine the sparsity pattern CA of the compressed stiffness matrix.

ii) Set up the compressed stiffness matrix $(2^{(-\mathbf{r}/2+1)l'} \alpha^C_{(l',k'),(l,k)} 2^{(-\mathbf{r}/2-1)l})$ of operator A_Λ taken with respect to the bases $\{2^{(-\mathbf{r}/2-1)l}\, {}^A\psi_k^l\}$ and $\{2^{(\mathbf{r}/2-1)l}\, {}^T\psi_k^l\}$. Using analytic formulas, for instance, each entry should be computed with a finite amount of operations.

[10]The letters CA stand for compression algorithm.

[11]Further reductions up to $O(n)$ non-zero entries seem to be possible if the technique of [34] for the compression of entries corresponding to overlapping trial and test functionals is adopted.

iii) Compute the right-hand side of the equation, i.e., the vector $\eta = (\langle y, {}^T\tilde{\psi}^{l'}_{k'}\rangle)_{(l',k')}$.

iv) Solve the matrix equation

$$\left(2^{(-\mathbf{r}/2+1)l'}\alpha^C_{(l',k'),(l,k)}2^{(-\mathbf{r}/2-1)l}\right)_{(l',k'),(l,k)\in\Lambda}\left(2^{(\mathbf{r}/2+1)l}\xi^l_k\right)_{(l,k)} = \left(2^{(-\mathbf{r}/2+1)l'}\eta^{l'}_{k'}\right)_{(l',k')}$$

by an iterative method (cf. e.g. [30]).

Note that the replacement of the stiffness matrix $\mathcal{A}^C$ by $(2^{(-\mathbf{r}/2+1)l'}\alpha^C_{(l',k'),(l,k)}2^{(-\mathbf{r}/2-1)l})$ leads to a matrix with a a bounded condition number (cf. Theorem 5.1, Lemma 3.1 iii), and Lemma 4.2 iii)). For bounded condition numbers, the CG or GMRes solvers require $O(\log n)$ iteration steps and $O(n[\log n]^2)$ arithmetic operations. If a cascadic CG or GMRes solver is used, i.e., if the wavelet method is considered over a sequence of grids and if the initial solutions for the iteration on each level is just the final solution from the coarser level, then the linear system can be solved with only $O(n \log n)$ arithmetic operations.

6 The Algorithm for the Double Layer Equation

Now we turn to the double layer equation over the polygonal boundary and apply the method of Section 5. Let Ω be a bounded simply connected polygon, and let Γ denote its boundary. The Dirichlet problem for Laplace's equation

$$\begin{aligned} \triangle U(t) &= 0, \quad t \in \Omega, \\ U|_\Gamma &= g \end{aligned} \tag{6.1}$$

with a smooth function g, can be reduced to the **double layer potential equation** (cf. e.g. [24, 6])

$$(I - 2W)x = -2g =: y, \tag{6.2}$$

$$(Wx)(t) := -\frac{1}{2}\chi(t)x(t) + \frac{1}{2\pi}\int_\Gamma \frac{\nu(s)\cdot(t-s)}{|t-s|^2}x(s)\mathrm{d}_s\Gamma, \quad t \in \Gamma, \tag{6.3}$$

where $\nu(s)$ is the exterior normal of Ω at $s \in \Gamma = \partial\Omega$ and $\chi(s) \in (-1,1)$ is chosen such that $[1+\chi(s)]\pi$ is the exterior angle between the tangents to Γ at t as $t \to s\pm$. Especially, $\chi(s) = 0$ if s is not a corner point of Γ.

As mentioned in Section 2 we take a one-periodic parametrization $\gamma : \mathbb{R} \longrightarrow \Gamma$ (e.g. the normalized arc-length parametrization) and identify the functions x, y, and χ over Γ with the one-periodic functions $x \circ \gamma$, $y \circ \gamma$, and $\chi \circ \gamma$, respectively. In this sense (6.2) takes the form $Ax = y$, where

$$(Ax)(t) := [1 - \chi(t)]\,x(t) + \int_0^1 \mathcal{K}(t,\tau)x(\tau)\mathrm{d}\tau = y(t), \tag{6.4}$$

$$\mathcal{K}(t,\tau) := \frac{1}{\pi}\frac{\nu(\gamma(\tau))\cdot(\gamma(t)-\gamma(\tau))}{|\gamma(t)-\gamma(s)|^2}|\gamma'(\tau)|. \tag{6.5}$$

The operator A has an order $\mathbf{r} = 0$. It maps H^s continuously into H^s for $-1/2 < s < 3/2$, and is invertible at least for $0 \le s \le 1$ (cf. e.g. [10]). We suppose that the right-hand side

y is smooth, i.e., that y is continuous and that y is infinitely differentiable over each side of the polygonal boundary Γ. Then, for t_0 the parameter value of a corner point $\gamma(t_0)$, the **asymptotic behaviour of the solution** x is given by (cf. e.g. [11, 19, 24])

$$x(t) \sim C_0 + C_1(t-t_0)^{\kappa(t_0)} + \dots, t \longrightarrow t_0, \quad \kappa(t_0) := \frac{1}{\max\{1+\chi(t_0), 1-\chi(t_0)\}}. \tag{6.6}$$

In particular, the exponent $\kappa := \kappa(t_0)$ is a number between 1/2 and 1, the solution x is Hölder continuous of order κ in the neighbourhood of t_0, and the m-th derivative of x is bounded by $C|t-t_0|^{\kappa-m}$. We get the following **decay property for the wavelet coefficients of the solution** x:

Lemma 6.1 *Consider a wavelet function* ${}^A\psi^l_k$. *By* t_0 *we denote the parameter of the corner which is the nearest to the support of* ${}^A\psi^l_k$. *Then the coefficient* $\langle x, {}^A\tilde{\psi}^l_k\rangle$ *of* ${}^A\psi^l_k$ *in the representation of* x *with respect to the basis* $\{{}^A\psi^{l'}_{k'}\}$ *satisfies*

$$\left|\left\langle x, {}^A\tilde{\psi}^l_k\right\rangle\right| \;\le\; C\begin{cases} 2^{-\frac{5}{2}l}\left|t_0-\frac{k}{2^l}\right|^{\kappa(t_0)-2} & \text{if } \left|t_0-\frac{k}{2^l}\right|>0 \\ 2^{-\left(\kappa(t_0)+\frac{1}{2}\right)l} & \text{else .}\end{cases} \tag{6.7}$$

Proof. Without loss of generality we suppose $t_0 = 0$, $l \ge 0$, $0 \le k < 2^{l/2}$, and that x and the wavelets are given over the real axes. First we consider the case $|t_0 - k/2^l| > 0$, i.e., $k > 0$. Using the fact that ${}^A\tilde{\psi}^l_k$ is orthogonal to linear functions (note that linear functions belong to V_{l-1}), we get

$$\begin{aligned}
\left\langle x, {}^A\tilde{\psi}^l_k\right\rangle &= \int_{-\infty}^{+\infty}\left[x(t)-x\left(\frac{k}{2^l}\right)-x'\left(\frac{k}{2^l}\right)\left(t-\frac{k}{2^l}\right)\right]{}^A\tilde{\psi}^l_k(t)\mathrm{d}t, \\
\left|\left\langle x, {}^A\tilde{\psi}^l_k\right\rangle\right| &\le \int_{-\infty}^{+\infty}\left|\left[x(t)-x\left(\frac{k}{2^l}\right)-x'\left(\frac{k}{2^l}\right)\left(t-\frac{k}{2^l}\right)\right]{}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t \\
&\le \int_{-\infty}^{2^{-l}} C\left|\left(t-\frac{k}{2^l}\right)^{\kappa}{}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t + \int_{1/2}^{-\infty} C\left|\left(t-\frac{k}{2^l}\right)^{\kappa}{}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t \\
&\quad + C\left(\frac{k}{2^l}\right)^{\kappa-1}\left\{\int_{-\infty}^{2^{-l}}\left|\left(t-\frac{k}{2^l}\right){}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t + \int_{1/2}^{-\infty}\left|\left(t-\frac{k}{2^l}\right){}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t\right\} \\
&\quad + \int_{2^{-l}}^{1/2}\left|\left[x(t)-x\left(\frac{k}{2^l}\right)-x'\left(\frac{k}{2^l}\right)\left(t-\frac{k}{2^l}\right)\right]{}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t.
\end{aligned} \tag{6.8}$$

Now we observe that ${}^A\tilde{\psi}$ decays exponentially, i.e., for a suitable constant C and a small $\epsilon > 0$, we have

$$|{}^A\tilde{\psi}(t)| \le Ce^{-\epsilon|t|}. \tag{6.9}$$

Thus

$$\begin{aligned}
\int_{-\infty}^{2^{-l}}\left|\left(t-\frac{k}{2^l}\right)^{\kappa}{}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t &\le C\int_{-\infty}^{(1-k)2^{-l}}|t|^{\kappa}2^{l/2}e^{-\epsilon|t2^l|}\mathrm{d}t \\
&\le C2^{-\left(\frac{1}{2}+\kappa\right)l}\int_{-\infty}^{(1-k)}|\tau|^{\kappa}e^{-\epsilon|\tau|}\mathrm{d}\tau \;\le\; C2^{-\left(\frac{1}{2}+\kappa\right)l}e^{-\epsilon k/2},
\end{aligned} \tag{6.10}$$

$$\int_{1/2}^{-\infty}\left|\left(t-\frac{k}{2^l}\right)^{\kappa}{}^A\tilde{\psi}^l_k(t)\right|\mathrm{d}t \le C2^{-\left(\frac{1}{2}+\kappa\right)l}e^{-\epsilon k/2}. \tag{6.11}$$

Similar arguments lead to

$$\left(\frac{k}{2^l}\right)^{\kappa-1}\left\{\int_{-\infty}^{2^{-l}}\left|\left(t-\frac{k}{2^l}\right)\ {}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t+\int_{1/2}^{-\infty}\left|\left(t-\frac{k}{2^l}\right)\ {}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t\right\} \tag{6.12}$$
$$\leq C2^{-\left(\frac{1}{2}+\kappa\right)l}k^{\kappa-1}e^{-\epsilon k/2}.$$

For the last term on the right-hand side of (6.8), we conclude from the integral expression of the remainder term of the Taylor series expansion

$$\int_{2^{-l}}^{1/2}\left|\left[x(t)-x\left(\frac{k}{2^l}\right)-x'\left(\frac{k}{2^l}\right)\left(t-\frac{k}{2^l}\right)\right]\ {}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t \tag{6.13}$$
$$\leq\ C\int_{2^{-l}}^{k2^{-l}}\int_t^{k2^{-l}}|\tau-t|\,|\tau|^{\kappa-2}\mathrm{d}\tau\left|{}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t+C\int_{k2^{-l}}^{1/2}\int_{k2^{-l}}^{t}|t-\tau|\,|\tau|^{\kappa-2}\mathrm{d}\tau\left|{}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t.$$

Changing the order of integration and using (6.9), we get

$$\int_{2^{-l}}^{k2^{-l}}\int_t^{k2^{-l}}|\tau-t|\,|\tau|^{\kappa-2}\mathrm{d}\tau\left|{}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t \tag{6.14}$$
$$=\ C\int_{2^{-l}}^{k2^{-l}}\int_{2^{-l}}^{\tau}|\tau-t|\,2^{l/2}e^{-\epsilon|2^lt-k|}\mathrm{d}t|\tau|^{\kappa-2}\mathrm{d}\tau$$
$$\leq\ C\int_{2^{-l}}^{k2^{-l}}2^{l/2}\left\{\int_{2^{-l}}^{\max\{2^{-l},\tau-(k2^{-l}-\tau)\}}|k2^{-l}-t|\,e^{-\epsilon|2^lt-k|}\mathrm{d}t+\right.$$
$$\left.|k2^{-l}-\tau|e^{-\epsilon|2^l\tau-k|}\int_{\max\{2^{-l},\tau-(k2^{-l}-\tau)\}}^{\tau}\mathrm{d}t\right\}|\tau|^{\kappa-2}\mathrm{d}\tau$$
$$\leq\ C\int_{2^{-l}}^{k2^{-l}}2^{l/2}\left\{\left|k2^{-l}-\tau\right|^2e^{-\epsilon|2^l\tau-k|}\right\}|\tau|^{\kappa-2}\mathrm{d}\tau.$$

This can be estimated by

$$C2^{-(\kappa+1/2)l}\int_0^{k-1}t^2e^{-\epsilon t}|t-k|^{\kappa-2}\mathrm{d}t \tag{6.15}$$
$$\leq\ C2^{-(\kappa+1/2)l}\left\{\int_0^{k/2}t^2e^{-\epsilon t}\mathrm{d}tk^{\kappa-2}+k^2e^{-\epsilon k}\int_{k/2}^{k-1}|t-k|^{\kappa-2}\mathrm{d}t\right\}\ \leq\ C2^{-(\kappa+1/2)l}k^{\kappa-2}.$$

Analogously, we obtain

$$\int_{k2^{-l}}^{1/2}\int_{k2^{-l}}^{t}|t-\tau|\,|\tau|^{\kappa-2}\mathrm{d}\tau\left|{}^A\tilde{\psi}_k^l(t)\right|\mathrm{d}t\leq C2^{-(\kappa+1/2)l}k^{\kappa-2}. \tag{6.16}$$

The estimate (6.8) and (6.10)-(6.16) together prove (6.7) for $|t_0-k/2^l|>0$.

Now consider the case $k=0$. Since ${}^A\tilde{\psi}_k^l$ is orthogonal to the constant functions, we get

$$\left\langle x,\ {}^A\tilde{\psi}_k^l\right\rangle\ =\ \int_{-\infty}^{+\infty}[x(t)-x(0)]\ {}^A\tilde{\psi}_0^l(t)\mathrm{d}t. \tag{6.17}$$

Using the exponential decay (6.9) and the Hölder continuity, we arrive at

$$\left|\left\langle x,\ {}^A\tilde{\psi}_k^l\right\rangle\right|\ \leq\ C\int_{-\infty}^{+\infty}|t|^\kappa 2^{l/2}e^{-\epsilon\left|2^lt\right|}\mathrm{d}t \tag{6.18}$$
$$\leq\ C2^{-\left(\kappa+\frac{1}{2}\right)l}\int_{-\infty}^{+\infty}|\tau|^\kappa e^{-\epsilon|\tau|}\mathrm{d}\tau\ =\ C2^{-\left(\kappa+\frac{1}{2}\right)l}.$$

Thus (6.7) follows for $k = 0$, too.

$\diamond$

From now on, by κ we denote the minimum of the exponents $\kappa(t_0)$ with t_0 taken over the finite set Γ_c of all t_0 such that $\gamma(t_0)$ is a corner point of Γ. Lemma 6.1 enables us to define the set of indices Λ such that we get the usual **order of approximation for the solution** x in the Sobolev space H^1 (cf. the error estimates in Theorem 5.1).

Lemma 6.2 *Fix a positive integer j_0 and set j equal to the smallest integer greater or equal to $j_0/(\kappa - 1/2)$. By κ' denote a number $\kappa - \varepsilon$ which is close to κ and satisfies $1/2 < \kappa' < \kappa$. Finally, choose Λ to be the set of all (l, k) with $-1 \le l \le j-1$ and*

$$\min_{t_0 \in \Gamma_c} \left| k2^{-l} - t_0 \right| \;\le\; \rho_l \;:=\; \begin{cases} 1 & \text{if } l < j_0 \\ 2^{-\frac{l-j_0}{3/2-\kappa'}} & \text{if } j_0 \le l \le j-1 \\ 0 & \text{if } j \le l\,. \end{cases} \tag{6.19}$$

Then the number of index pairs in Λ is less than $C\,2^{j_0}$. If x is the solution of the double layer equation (6.4) with a smooth right-hand side y. Then we get the approximation order

$$\|x - {}^{A}Q_\Lambda x\|_{H^1} \le C\,2^{-j_0}. \tag{6.20}$$

Proof. Without loss of generality we suppose that $\Gamma_c = \{0\}$. Then the number of index pairs is less

$$2^{j_0} + C\sum_{l=j_0}^{j-1} \rho_l/2^{-l} \;\le\; 2^{j_0} + C2^{\frac{j_0}{3/2-\kappa'}} \sum_{l=j_0}^{j-1} 2^{\left(1-\frac{1}{3/2-\kappa'}\right)l} \;\le\; C\,2^{j_0}. \tag{6.21}$$

To estimate the approximation error we utilize the Lemmas 3.1 iii) and 6.1 and conclude

$$\begin{aligned}
\|x - {}^{A}Q_\Lambda x\|_{H^1} \;&\le\; C\sqrt{\sum_{l=-1}^{\infty} \sum_{k=\rho_l 2^l}^{n_l} 2^{2l} \left|\left\langle x,\, {}^{A}\tilde{\psi}^l_k \right\rangle\right|^2} \\
&\le\; C\sqrt{\sum_{l=j_0}^{\infty} 2^{2(1/2-\kappa)l} \sum_{k=\rho_l 2^l}^{n_l} \min\{1, k^{2(\kappa-2)}\}} \\
&\le\; C\sqrt{\sum_{l=j_0}^{\infty} 2^{2(1/2-\kappa)l} (\rho_l 2^l + 1)^{2(\kappa-3/2)}} \\
&\le\; C\sqrt{2^{-j_0 2\frac{3/2-\kappa}{3/2-\kappa'}} \sum_{l=j_0}^{j-1} 2^{2l\frac{\kappa'-\kappa}{3/2-\kappa'}} + \sum_{l=j}^{\infty} 2^{2(1/2-\kappa)l} C} \;\le\; C2^{-j_0}.
\end{aligned}$$

$\diamond$

Remark 6.1 *If $1 \le s < \kappa + 1/2$ and if the assumptions of Lemma 6.2 are satisfied, then the same proof yields*

$$\|x - {}^{A}Q_\Lambda x\|_{H^s} \le C\,2^{j(s-1)-j_0} \;=\; C2^{[(s-1)/(\kappa-1/2)-1]j_0}. \tag{6.22}$$

Theorem 6.1 *Suppose that Λ is defined as in Lemma 6.2 and that the right-hand side y of the double layer equation (6.4) is smooth. Then:*

i) The wavelet collocation method (5.1) applied to the double layer equation is stable in the space H^1.

ii) For $1 \leq s < \kappa + 1/2$, we get the error estimate

$$\|x - x_\Lambda\|_{H^s} \leq C\, 2^{[(s-1)/(\kappa-1/2)-1]j_0}. \tag{6.23}$$

iii) If $0 \leq s \leq 1$, then

$$\|x - x_\Lambda\|_{H^s} \leq C\, 2^{-(2-s)j_0}. \tag{6.24}$$

Proof. The theorem is a direct consequence of Theorem 5.1, Lemma 6.2, and Remark 6.1. The only thing to be checked is the strong ellipticity of $B = \partial A \partial^{-1} : \ L^2 \longrightarrow L^2$. This, however, is done in Lemma 2.1 of [12].

$\diamond$

Remark 6.2 *Theorem 6.1 is a generalization of a result in [12]. In that paper only the case with equality in (3.11) is treated. However, in [12] special non-uniform meshes are allowed.*

In the next section we will introduce a compression algorithm (5.20) for the double layer equation. The final result on the sparsity, stability, and convergence of this compressed wavelet collocation method will be collected in Theorem 8.1 in the end of Section 8.

7 The Compression Scheme

Now we shall see that the majority of the matrix entries $\alpha_{(l',k'),(l,k)}$ in the stiffness matrix of the double layer operator is very small. To formulate the corresponding **decay estimate for the matrix entries** we need some notation. First we introduce the metric ϱ over the periodic interval $[0,1]$ by setting $\varrho(t,\tau) := \min\{|t-\tau|,\ |t-\tau+1|,\ |t-\tau-1|\}$. We denote the support of the wavelet ${}^A\psi^l_k$ by Ψ^l_k. Note that $\Psi^{l'}_{k'}$ is roughly speaking the convex hull of the support of ${}^T\tilde{\psi}^{l'}_{k'}$. The distance of the two sets Ψ^l_k and $\Psi^{l'}_{k'}$ is defined as

$$dist \ := \ \operatorname{dist}\left(\Psi^l_k,\, \Psi^{l'}_{k'}\right) \ := \ \inf_{t\in\Psi^l_k,\ \tau\in\Psi^{l'}_{k'}} |\varrho(t,\tau)|. \tag{7.1}$$

Lemma 7.1 *For those entries $\alpha_{(l',k'),(l,k)} = \langle A\, {}^A\psi^l_k,\ {}^T\tilde{\psi}^{l'}_{k'}\rangle$ in the stiffness matrix of the double layer operator A for which the supports Ψ^l_k and $\Psi^{l'}_{k'}$ of the trial and test functionals are disjoint to the set of parameters Γ_c of corner points, we get the estimate*

$$|\langle\, A\, {}^A\psi^l_k,\ {}^T\tilde{\psi}^{l'}_{k'}\,\rangle| \ \leq \ C\, 2^{-(1/2+2)l}\, 2^{-(1/2+4)l'}\, dist^{-7}. \tag{7.2}$$

A proof of this lemma can be found e.g. in [18, 34]. The essential assumptions leading to (7.2) is the well-known Calderón-Zygmund estimate for the kernel function (6.5)

$$|\partial_t^m \partial_\tau^{m'} \mathcal{K}(t,\tau)| \le C \varrho(t,\tau)^{-1-m-m'}, \quad 0 \le m \in \mathbb{Z}, \quad 0 \le m' \in \mathbb{Z} \tag{7.3}$$

and the vanishing moment conditions of Lemma 3.1 iv) and Lemma 4.2 iv).

In view of the decay property (7.2), we can neglect small entries in the stiffness matrix $\mathcal{A}$. For fixed l and l', the large entries are located around the diagonal, i.e., the entries $\alpha_{(l',k'),(l,k)}$ with small distance between Ψ_k^l and $\Psi_{k'}^{l'}$ cannot be neglected. Moreover, since the wavelet coefficients of the solution x for wavelets close to the corner points are large, we must not neglect the matrix entries corresponding to trial and test functionals close to the corners either. Hence, we introduce the **compressed stiffness matrix** $\mathcal{A}^C$ (cf. (5.20)) as follows: Define di to be the distance between $\Psi_{k'}^{l'}$ and the corner set Γ_c. Then CA is defined as the set of all pairs $((l',k'),(l,k)) \in \Lambda \times \Lambda$ such that one of the following conditions is true

$$\begin{aligned}
\Gamma_c \cap \Psi_{k'}^{l'} &\neq \emptyset, \\
\operatorname{dist}\left(\Psi_k^l, \Psi_{k'}^{l'}\right) &\le \max\left\{2^{-l},\ 2^{-l'},\ a2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')}\left[2^{-l'} + di\right]^{-\delta}\right\}, \\
\operatorname{dist}\left(\Psi_k^l, \Gamma_c\right) &\le \min\left\{a2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')}\left[2^{-l'} + di\right]^{-\delta},\right. \\
&\qquad \left.\left[a2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')}\right]^{1/\delta}\left[2^{-l'} + di\right]^{-1/\delta}\right\}.
\end{aligned} \tag{7.4}$$

Here a is a real constant greater or equal to one and, for a small ε with $0 < \varepsilon < 1/12$, we set

$$\delta := \frac{3/2-\kappa}{6} + \varepsilon, \qquad \beta_l := \gamma_l := \begin{cases} \beta^+ := \gamma^+ := 2/3+\varepsilon & \text{if } l < j_0 \\ \beta^- := \gamma^- := 1/2-\varepsilon & \text{if } l \ge j_0, \end{cases} \tag{7.5}$$

Recall that $\kappa' = \kappa - \varepsilon$ (cf. the definition of Λ in Lemma 6.2).

Lemma 7.2 *Suppose that the minimum κ of the exponents of the corner singularities (cf. (6.6)) satisfies $9/14 < \kappa$ and that ε is less than the minimum of $1/12$ and $\kappa \cdot 7/36 - 1/8$. If Λ is chosen according to Lemma 6.2 and the compression algorithm is chosen according to (7.4), then the number of non-zero entries in the compressed stiffness matrix $\mathcal{A}^C$ is less than $Caj_0 2^{j_0}$.*

Proof. i) First we count the entries with $dist \le \max\{2^{-l}, 2^{-l'}\}$. Without loss of generality we suppose $l' \ge l$. Clearly, the number of such entries for a fixed test functional ${}^T\tilde{\psi}_{k'}^{l'}$ and a fixed level $l \le l'$ is less than seven. Summing up over l', k', and k, we arrive at the bound

$$\sum_{l'=-1}^{j-1} \sum_{k'} \sum_{l=-1}^{l'} 7. \tag{7.6}$$

We observe that the maximal number of k' is less than the number of corners times $\rho_{l'}2^{l'}$. Thus the last bound can further be estimated by

$$C\sum_{l'=-1}^{j-1}\rho_{l'}2^{l'}\sum_{l=-1}^{l'}1 \;\leq\; C\left\{\sum_{l'=-1}^{j_0-1}2^{l'}l' + 2^{\frac{1}{3/2-\kappa'}j_0}\sum_{l'=j_0}^{j-1}2^{\left(1-\frac{1}{3/2-\kappa'}\right)l'}l'\right\} \;\leq\; Cj_0 2^{j_0}. \qquad (7.7)$$

ii) From now on, for the sake of simplicity, we suppose $\Gamma_c = \{0\}$ and, without loss of generality, we restrict our consideration to entries $\alpha_{(l',k'),(l,k)}$ with $k \leq 2^{l/2}$ and $k' \leq 2^{l'/2}$. Next we consider the non-zero entries such that $l' \geq l$ and $\max\{2^{-l}, 2^{-l'}\} < dist \leq \mathcal{G}$ with (cf. (7.1) and (7.4))

$$\mathcal{G} := a2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')}\left[2^{-l'} + di\right]^{-\delta}. \qquad (7.8)$$

For the number of these entries, we get the upper bound

$$C\sum_{l'=-1}^{j-1}\sum_{k'}\sum_{l=-1}^{l'}\mathcal{G}/2^{-l} \qquad (7.9)$$

$$\begin{aligned}
&\leq \sum_{l',k'}a2^{-j_0+\gamma_{l'}(j_0-l')}\left[2^{-l'} + di\right]^{-\delta}\left\{2^{\beta^+j_0}\sum_{l=-1}^{j_0-1}2^{(1-\beta^+)l} + 2^{\beta^-j_0}\sum_{l=j_0}^{l'}2^{(1-\beta^-)l}\right\}\\
&\leq Ca2^{-j_0}\left\{2^{j_0}\sum_{l'=-1}^{j-1}2^{\gamma_{l'}j_0}2^{(1-\gamma_{l'})l'}2^{-l'}\sum_{k'=0}^{\rho_{l'}2^{l'}}\left[2^{-l'} + di\right]^{-\delta}\right.\\
&\qquad\left.+2^{j_0\beta^-}\sum_{l'=-1}^{j-1}2^{\gamma_{l'}j_0}2^{(2-\gamma_{l'}-\beta^-)l'}2^{-l'}\sum_{k'=0}^{\rho_{l'}2^{l'}}\left[2^{-l'} + di\right]^{-\delta}\right\}.
\end{aligned}$$

Using

$$2^{-l'}\sum_{k'=0}^{\rho_{l'}2^{l'}}\left[2^{-l'} + di\right]^{-\delta} \;\leq\; \int_{2^{-l'}}^{\rho_{l'}}t^{-\delta}\mathrm{d}t \;\leq\; C\rho_{l'}^{1-\delta} \qquad (7.10)$$

and (6.19), we continue the estimation from (7.9) by

$$\begin{aligned}
&Ca2^{-j_0}\left\{2^{(1+\gamma^+)j_0}\sum_{l'=-1}^{j_0-1}2^{(1-\gamma^+)l'} + 2^{\left(1+\gamma^-+\frac{1-\delta}{3/2-\kappa'}\right)j_0}\sum_{l'=j_0}^{j-1}2^{\left(1-\gamma^--\frac{1-\delta}{3/2-\kappa'}\right)l'} + \right. \qquad (7.11)\\
&\qquad\left.2^{(\beta^-+\gamma^+)j_0}\sum_{l'=-1}^{j_0-1}2^{(2-\gamma^+-\beta^-)l'} + 2^{\left(\beta^-+\gamma^-+\frac{1-\delta}{3/2-\kappa'}\right)j_0}\sum_{l'=j_0}^{j-1}2^{\left(2-\beta^--\gamma^--\frac{1-\delta}{3/2-\kappa'}\right)l'}\right\}\\
&\qquad\qquad\leq\; Ca2^{j_0}.
\end{aligned}$$

Note that we have used $1-\gamma^+ > 0$, $2-\gamma^+-\beta^- > 0$ as well as $1-\gamma^--(1-\delta)/(3/2-\kappa') < 0$ and $2-\beta^--\gamma^--(1-\delta)/(3/2-\kappa') < 0$. The last relation leads us to the assumptions on κ and ε.

iii) Now we count the non-zero entries such that $l > l'$ and $\max\{2^{-l}, 2^{-l'}\} < dist \leq \mathcal{G}$. Analogously to (7.9), we get the bound

$$C\sum_{l=-1}^{j-1}\sum_{k}\sum_{l'=-1}^{l}\mathcal{H}/2^{-l'}, \qquad (7.12)$$

where $\mathcal{H}$ is the measure of the set

$$\left\{t \in [0,1] :\ 2^{-l} \ <\ \mathrm{dist}(\Psi^l_k, t) \ \le\ a 2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')} \left[2^{-l'} \ + \mathrm{dist}(t, \Gamma_c)\right]^{-\delta}\right\}. \quad (7.13)$$

In part iv) of the present proof we will show

$$\mathcal{H} \ \le\ a 2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')} \left[\mathrm{dist}(k2^{-l}, \Gamma_c)\right]^{-\delta}. \quad (7.14)$$

If this is done, then (7.12) can be estimated by

$$C \sum_{l=-1}^{j-1} \sum_k \sum_{l'=-1}^{l} \mathcal{G}'/2^{-l'} \ +\ C \sum_{l=-1}^{j-1} \ \sum_{k:\, \Gamma_c \cap \Psi^l_k \neq \emptyset} \ \sum_{(l',k')} C \quad (7.15)$$

with

$$\mathcal{G}' \ :=\ a 2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')} \left[2^{-l} + \mathrm{dist}(k2^{-l}, \Gamma_c)\right]^{-\delta}. \quad (7.16)$$

Now the first sum in (7.15) can be treated as (7.9) and is less than $Ca2^{j_0}$. The second sum is less than the product of the number of trial functions ${}^A\psi^l_k$ with $\Gamma_c \cap \Psi^l_k \neq \emptyset$ times the number of indices in Λ. Thus the upper bound in (7.15) is less than $Ca2^{j_0} + Cj_0 2^{j_0}$.

iv) It remains to prove (7.14). More general, we only have to prove that, for positive constants D and $\delta < 1$, the measure of the set (recall the simplicity assumption $\Gamma_c = \{0\}$)

$$\mathcal{M} := \left\{t \in [0,1] :\ |t-\tau| \le Dt^{-\delta}\right\} \quad (7.17)$$

is less than $CD\tau^{-\delta}$. We split $\mathcal{M}$ into $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ with $\mathcal{M}_1 := \mathcal{M} \cap [0, \tau/2)$, $\mathcal{M}_2 := \mathcal{M} \cap [\tau/2, 2\tau]$, and $\mathcal{M}_3 := \mathcal{M} \cap [2\tau, 1]$.

For $t \in \mathcal{M}_2$, we get $\tau/2 \le t \le 2\tau$ and $|t-\tau| \le 2^\delta D\tau^{-\delta}$. Consequently, $\mathcal{M}_2$ is contained in the interval $[\tau - 2^\delta D\tau^{-\delta}, \tau + 2^\delta D\tau^{-\delta}]$. Thus the measure of $\mathcal{M}_2$ is less than $2 \cdot 2^\delta D\tau^{-\delta}$.
For $t \in \mathcal{M}_3$, we get $Dt^{-\delta} \ge |t-\tau| = t - \tau \ge t/2$ and

$$\begin{aligned} t \ &\le\ 2Dt^{-\delta}, \\ t \ &\le\ 2^{\frac{1}{1+\delta}} D^{\frac{1}{1+\delta}} \ \le\ 2^{\frac{1}{1+\delta}} D \left[D^{\frac{1}{1+\delta}}\right]^{-\delta}. \end{aligned} \quad (7.18)$$

Using $\tau < t/2 < 2^{1/(1+\delta)-1} D^{1/(1+\delta)}$, we arrive at

$$t \ \le\ 2^{\frac{1}{1+\delta}} D \left[\tau 2^{1-\frac{1}{1+\delta}}\right]^{-\delta} \ \le\ CD\tau^{-\delta}. \quad (7.19)$$

For $t \in \mathcal{M}_1$, we get $|t-\tau| = \tau - t \ge \tau/2$ and

$$\begin{aligned} \tau \ &\le\ 2|t-\tau| \le 2Dt^{-\delta}, \\ t \ &\le\ 2^{1/\delta} D^{1/\delta} \tau^{-1/\delta}. \end{aligned} \quad (7.20)$$

We distinguish two cases. First suppose the last bound in (7.20) is less than $\tau/2$. Then $2D^{1/(1+\delta)} < \tau$ and (7.20) implies

$$t \ \le\ 2^{1/\delta} D^{1/\delta} \tau^{\delta-1/\delta} \tau^{-\delta} \ \le\ 2^{1/\delta} D^{1/\delta} \left[2D^{\frac{1}{1+\delta}}\right]^{\delta-1/\delta} \tau^{-\delta} \ \le\ CD\tau^{-\delta}. \quad (7.21)$$

In the case that $2^{1/\delta}D^{1/\delta}\tau^{-1/\delta}$ is greater or equal to $\tau/2$, we have $2D^{1/(1+\delta)} \geq \tau$ and

$$t \;\leq\; \tau/2 \;\leq\; 2^{-1}\tau^{1+\delta}\tau^{-\delta} \;\leq\; 2^{-1}\left[2D^{\frac{1}{1+\delta}}\right]^{1+\delta}\tau^{-\delta} \;\leq\; CD\tau^{-\delta}. \tag{7.22}$$

v) Now we count the entries for which $l' \geq l$ and for which the last condition of (7.4) holds. Obviously, (7.9) is an upper bound for the number of these entries, too. Hence, the arguments of part ii) prove that the number is less than $Ca2^{j_0}$. Finally, we consider the entries for which $l' < l$ and for which the last condition of (7.4) holds. Then (7.12) is an upper bound for the number of these entries if $\mathcal{H}$ is the measure of the set

$$\begin{aligned} &\left\{t \in [0,1] :\; \mathrm{dist}(\Psi_k^l, \{0\}) \leq \left[a2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')}\right]^{1/\delta}\left[2^{-l'} + \mathrm{dist}(t,\{0\})\right]^{-1/\delta}\right\} \\ &\subseteq \left\{t \in [0,1] :\; t \leq a2^{-j_0+\beta_l(j_0-l)+\gamma_{l'}(j_0-l')}\left[\mathrm{dist}(\Psi_k^l,\{0\})\right]^{-\delta}\right\}. \end{aligned} \tag{7.23}$$

Clearly, $\mathcal{H}$ fulfills (7.14) and, analogously to part iii), we conclude that the number of entries is less than $Ca2^{j_0} + Cj_0 2^{j_0}$.

$\diamond$

Remark 7.1 *The condition* $9/14 < \kappa$ *is equivalent to the requirement that the angle* $\pi(1+\chi(t_0))$ *of the polygon at the corner* $\gamma(t_0)$ *satisfies* $\pi 4/9 < \pi(1+\chi(t_0)) < \pi 14/9$. *In particular, the condition is satisfied for rectangular polygons. For smaller or larger angles, the moment conditions must be improved. Indeed, if the mother wavelet* ${}^A\psi$ *has* $d > 0$ *vanishing moments instead of two (cf. Lemma 3.1 iv)), then* ${}^T\tilde{\psi}$ *has* $d+2$ *vanishing moments and the condition* $9/14 < \kappa$ *turns into* $(2d+5)/(4d+6) < \kappa$.

8 Stability and Error Estimates for the Compressed Wavelet Collocation

Recall (cf. (5.2)) that A_Λ is the operator

$$A_\Lambda \;\;:=\;\; {}^TQ_\Lambda A|_{{}^AV} :\; \left({}^AV, \|\cdot\|_{H^1}\right) \longrightarrow \left({}^TV, \|\cdot\|_{H^1}\right) \tag{8.1}$$

and its matrix with respect to the bases $\{{}^A\psi_k^l\}$ and $\{{}^T\psi_{k'}^{l'}\}$ is $\mathcal{A}$. We denote the operator mapping in the same spaces as A_Λ but corresponding to the matrix $\mathcal{A}^C$ by A_Λ^C. This is the approximate operator of the compressed wavelet collocation applied to A. To prove the **stability** of A_Λ^C, we need a variant of the well-known Schur lemma.

Lemma 8.1 *The norm of* A_Λ, *i.e. the Euclidean matrix norm of* $(2^{l'}\alpha_{(l',k'),(l,k)}2^{-l})_{(l',k'),(l,k)}$ *(cf. the discrete norm equivalences in Lemma 3.1 iii) and Lemma 4.1 iii)), is less than* $C\sqrt{\Sigma_1}\sqrt{\Sigma_2}$, *where*

$$\Sigma_1 \;\;:=\;\; \sup_{l',k'}\left|\, 2^{l'/2}\sum_{l,k}\left[2^{l'}\alpha_{(l',k'),(l,k)}2^{-l}\right]2^{-l/2}\,\right|, \tag{8.2}$$

$$\Sigma_2 \;\;:=\;\; \sup_{l,k}\left|\, 2^{l/2}\sum_{l',k'}2^{-l'/2}\left[2^{l'}\alpha_{(l',k'),(l,k)}2^{-l}\right]\,\right|.$$

Lemma 8.2 *If Λ is as in Lemma 6.2 and if A^C_Λ is defined by (7.4), then the compressed wavelet collocation applied to the double layer operator A is stable for sufficiently large a (cf. (7.4)).*

Proof. Since the operator sequence A_Λ is stable by Theorem 5.1, we only need to prove that the difference $A^C_\Lambda - A_\Lambda$ is small in norm for sufficiently large a. To show this we apply Lemma 8.1 to the difference $A^C_\Lambda - A_\Lambda$. Using Lemma 7.1, (7.4), and (7.8), we get

$$\begin{aligned}\Sigma_1 &\leq C \sup_{l',k'} \left| 2^{l'/2} \sum_{l,k:\, dist>\max\{2^{-l},\mathcal{G}\}} \left[2^{l'} 2^{-(1/2+4)l'} dist^{-7} 2^{-(1/2+2)l} 2^{-l}\right] 2^{-l/2} \right| \\ &\leq C \sup_{l',k'} 2^{-3l'} \sum_l 2^{-3l}\, 2^{-l} \sum_{k:\, dist>\max\{2^{-l},\mathcal{G}\}} dist^{-7}.\end{aligned} \tag{8.3}$$

We observe

$$2^{-l} \sum_{k:\, dist>\max\{2^{-l},\mathcal{G}\}} dist^{-7} \;\leq\; C \int_{\mathcal{G}}^1 \tau^{-7} \mathrm{d}\tau \;\leq\; C\mathcal{G}^{-6} \tag{8.4}$$

and continue

$$\begin{aligned}\Sigma_1 &\leq C \sup_{l',k'} 2^{-3l'} \sum_l 2^{-3l} \left[a 2^{-j_0} 2^{\beta_l (j_0-l)} 2^{\gamma_{l'}(j_0-l')}\right]^{-6} \\ &\leq C \sup_{l'} a^{-6} 2^{(6-6\gamma_{l'})j_0} 2^{(-3+6\gamma_{l'})l'} \left\{ 2^{-6\beta^+ j_0} \sum_{l=-1}^{j_0-1} 2^{(-3+6\beta^+)l} + 2^{-6\beta^- j_0} \sum_{l=j_0}^{j-1} 2^{(-3+6\beta^-)l} \right\} \\ &\leq C \sup_{l'} a^{-6} 2^{(3-6\gamma_{l'})j_0} 2^{(-3+6\gamma_{l'})l'} \;\leq\; C a^{-6}.\end{aligned} \tag{8.5}$$

Note that we have used $-3+6\beta^+ > 0$, $\;-3+6\beta^- < 0$, $\;-3+6\gamma^+ > 0$, and $-3+6\gamma^- < 0$. Similarly, we obtain $\Sigma_2 \leq Ca^{-6}$. By Lemma 8.1 we conclude $\|A^C_\Lambda - A_\Lambda\| \leq Ca^{-6}$.

◇

Now we turn to the **error estimates**.

Lemma 8.3 *If Λ is as in Lemma 6.2, if the compressed wavelet collocation defined by (7.4) and applied to the double layer operator A is stable, and if the right-hand side y is smooth, then the estimate (6.23) holds for the compressed wavelet collocation, too.*

Proof. As in the proof to Theorem 5.1 ii) we may suppose $s = 1$. Moreover, for the sake of simplicity, we suppose $\Gamma_c = \{0\}$ and $x \equiv 0$ on $[1/2, 1]$. Analogously to (5.11) and (5.12), we get

$$\begin{aligned}x - x_\Lambda &= x - {}^AQ_\Lambda x + \left[A^C_\Lambda\right]^{-1} \left\{ {}^TQ_\Lambda A\, {}^AQ_\Lambda x - {}^TQ_\Lambda A x \right. \\ &\qquad \left. + \left[A^C_\Lambda - A_\Lambda\right] {}^AQ_\Lambda x \right\},\end{aligned} \tag{8.6}$$

$$\|x - x_\Lambda\|_{H^1} \;\leq\; C \left\| x - {}^AQ_\Lambda x \right\|_{H^1} + C \left\| \left[A^C_\Lambda - A_\Lambda\right] {}^AQ_\Lambda x \right\|_{H^1}. \tag{8.7}$$

In view of Lemma 6.2 it remains to estimate the second term on the right-hand side. We write

$$\left\| \left[A_\Lambda^C - A_\Lambda\right] {}^A Q_\Lambda x \right\|_{H^1} \leq \sum_{l,l'} \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'}\right] \left[A_\Lambda^C - A_\Lambda\right] \left[{}^A Q_{l+1} - {}^A Q_l\right] {}^A Q_\Lambda x \right\|_{H^1} \quad (8.8)$$

and, from Lemma 4.2 iii) and the Lemmas 6.1 and 7.1 we conclude

$$\begin{aligned} & \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'}\right] \left[A_\Lambda^C - A_\Lambda\right] \left[{}^A Q_{l+1} - {}^A Q_l\right] {}^A Q_\Lambda x \right\|_{H^1}^2 \quad (8.9) \\ \leq\ & C \sum_{k'} \left[\sum_k 2^{l'} 2^{(-1/2-4)l'} dist^{-7} 2^{(-1/2-2)l} 2^{-5/2\,l} (2^{-l} + |k2^{-l}|)^{\kappa-2} \right]^2 \\ \leq\ & C \sum_{k'} \left[2^{-3.5l'} 2^{-4l} 2^{-l} \sum_k dist^{-7} (2^{-l} + |k2^{-l}|)^{\kappa-2} \right]^2 . \end{aligned}$$

Here the summation over k runs for all $k = 0, 1, \dots, 2^l - 1$ such that the second and third condition of (7.4) are violated. Setting $\tau = k' 2^{-l'}$ and

$$\mathcal{M} := \left\{ t \in [0,1] :\ |t - \tau| > \max\{\mathcal{G},\, 2^{-l},\, 2^{-l'}\},\ t > \min\{\mathcal{G},\, \mathcal{G}^{1/\delta}[2^{-l'} + \tau]^{1-1/\delta}\} \right\}, \quad (8.10)$$

we observe

$$\begin{aligned} 2^{-l} \sum_k dist^{-7} (2^{-l} + |k2^{-l}|)^{\kappa-2} \ \leq\ & C \int_{\mathcal{M}} |t-\tau|^{-7}\, |t + 2^{-l}|^{\kappa-2} \mathrm{d}t \quad (8.11) \\ \leq\ & C\tau^{-7} \int_{\mathcal{M} \cap [0,\tau/2]} |t + 2^{-l}|^{\kappa-2} \mathrm{d}t \\ & + C \int_{\mathcal{M} \cap [\tau/2, 2\tau]} |t-\tau|^{-7} \mathrm{d}t\, |\tau + 2^{-l'}|^{\kappa-2} \\ & + C \int_{\mathcal{M} \cap [2\tau, 1/2]} t^{-7}\, t^{\kappa-2} \mathrm{d}t \\ \leq\ & C\tau^{-7} \min\left\{ \mathcal{G},\, \mathcal{G}^{1/\delta}[2^{-l'} + \tau]^{1-1/\delta} \right\}^{\kappa-1} \\ & + C \left[2^{-l'} + \tau\right]^{\kappa-2} \mathcal{G}^{-6}. \end{aligned}$$

Note that the first term on the right-hand side appears only if $\mathcal{M} \cap [0, \tau/2] \neq \emptyset$, i.e., if $\tau - \max\{2^{-l},\, 2^{-l'},\, \mathcal{G}\} > 0$. This implies $\tau > 2^{-l'}$ and $\tau > \mathcal{G}$. We get

$$2^{-l} \sum_k dist^{-7} (2^{-l} + |k2^{-l}|)^{\kappa-2} \ \leq\ C \left[2^{-l'} + \tau\right]^{\kappa-2} \mathcal{G}^{-6}. \quad (8.12)$$

Substituting the last result into (8.9) and using $2(6\delta + \kappa - 2) > -1$, we arrive at

$$\begin{aligned} & \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'}\right] \left[A_\Lambda^C - A_\Lambda\right] \left[{}^A Q_{l+1} - {}^A Q_l\right] {}^A Q_\Lambda x \right\|_{H^1}^2 \quad (8.13) \\ \leq\ & \left[C 2^{-3l'} 2^{-4l} \left[a 2^{-j_0} 2^{\beta_l (j_0 - l)} 2^{\gamma_{l'} (j_0 - l')} \right]^{-6} \right]^2 2^{-l'} \sum_{k'} \left[2^{-l'} + di \right]^{2(6\delta + \kappa - 2)} \\ \leq\ & \left[\dots \right]^2 \int_0^1 t^{2(6\delta+\kappa-2)} \mathrm{d}t \ \leq\ \left[\dots \right]^2 C \\ \leq\ & \left[C a^{-6} 2^{6j_0} 2^{-6\beta_l j_0} 2^{-6\gamma_{l'} j_0} 2^{(-4+6\beta_l)l} 2^{(-3+6\gamma_{l'})l'} \right]^2 . \end{aligned}$$

This together with (8.8) yields

$$\left\| \left[A_\Lambda^C - A_\Lambda \right] \, {}^A Q_\Lambda x \right\|_{H^1} \le C a^{-6} 2^{6 j_0} \left\{ \sum_{l=-1}^{j-1} 2^{-6\beta_l j_0} 2^{(-4+6\beta_l) l} \right\} \left\{ \sum_{l'=-1}^{j-1} 2^{-6\gamma_{l'} j_0} 2^{(-3+6\gamma_{l'}) l'} \right\}$$

$$\le C a^{-6} 2^{6 j_0} 2^{-4 j_0} 2^{-3 j_0} \le C a^{-6} 2^{-j_0}. \tag{8.14}$$

Note that we have used $-4+6\beta^+ > 0$, $-4+6\beta^- < 0$, $-3+6\gamma^+ > 0$, and $-3+6\gamma^- < 0$.

$\diamond$

Lemma 8.4 *If Λ is as in Lemma 6.2, if the compressed wavelet collocation defined by (7.4) and applied to the double layer operator A is stable, and if the right-hand side y is smooth, then the estimate (6.24) holds for the compressed wavelet collocation, too.*

Proof. Again, for the sake of simplicity, we suppose $\Gamma_c = \{0\}$ and that $x \equiv 0$ on $[1/2, 1]$. Analogously to (5.13), we get

$$\|x - x_\Lambda\|_{H^s} \le C \sup_{\|v\|_{H^{-s}} \le 1} |\langle y - A x_\Lambda, v \rangle| \tag{8.15}$$

$$= C \sup_{\|v\|_{H^{-s}} \le 1} \left| \left\langle y - A x_\Lambda, v - {}^T Q_\Lambda^* v \right\rangle \right| + C \sup_{\|v\|_{H^{-s}} \le 1} \left| \left\langle y - A x_\Lambda, {}^T Q_\Lambda^* v \right\rangle \right|.$$

The first term on the right-hand side can be estimated as in the proof to Theorem 5.1 iii). The second can be written as

$$\sup_{\|v\|_{H^{-s}} \le 1} \left| \left\langle \left[A_\Lambda^C - A_\Lambda \right] x_\Lambda, {}^T Q_\Lambda^* v \right\rangle \right| = \mathcal{T}_1 + \mathcal{T}_2, \tag{8.16}$$

$$\mathcal{T}_1 := \sup_{\|v\|_{H^{-s}} \le 1} \left| \left\langle \left[A_\Lambda^C - A_\Lambda \right] \left[x_\Lambda - {}^A Q_\Lambda x \right], {}^T Q_\Lambda^* v \right\rangle \right|,$$

$$\mathcal{T}_2 := \sup_{\|v\|_{H^{-s}} \le 1} \left| \left\langle \left[A_\Lambda^C - A_\Lambda \right] \, {}^A Q_\Lambda x, {}^T Q_\Lambda^* v \right\rangle \right|.$$

For the second term, we get

$$|\mathcal{T}_2| \le \sup_{\|v\|_{H^{-s}} \le 1} \sum_{l'} \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'} \right] \left[A_\Lambda^C - A_\Lambda \right] {}^A Q_\Lambda x \right\|_{H^1} \left\| \left[{}^T Q_{l'+1}^* - {}^T Q_{l'}^* \right] {}^T Q_\Lambda^* v \right\|_{H^{-1}}$$

$$\le \sup_{\|v\|_{H^{-s}} \le 1} \sum_{l'} \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'} \right] \left[A_\Lambda^C - A_\Lambda \right] {}^A Q_\Lambda x \right\|_{H^1} C \left(2^{-l'} \right)^{-s-[-1]} \|v\|_{H^{-s}}$$

$$\le C \sum_{l'} \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'} \right] \left[A_\Lambda^C - A_\Lambda \right] {}^A Q_\Lambda x \right\|_{H^1} 2^{-(1-s) l'} \tag{8.17}$$

$$\le C \sum_{l',l} \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'} \right] \left[A_\Lambda^C - A_\Lambda \right] \left[{}^A Q_{l+1} - {}^A Q_l \right] {}^A Q_\Lambda x \right\|_{H^1} 2^{-(1-s) l'}.$$

If we treat the last bound analogously to the estimation of the right-hand side in (8.8) and if we use $-4+s+6\gamma^+ > 0$, $-4+s+6\gamma^- < 0$ instead of $-3+6\gamma^+ > 0$, $-3+6\gamma^- < 0$, then we obtain $|\mathcal{T}_2| \le C a^{-6} 2^{-(2-s) j_0}$. For the term $\mathcal{T}_1$, we conclude

$$|\mathcal{T}_1| \le \sup_{\|v\|_{H^{-s}} \le 1} \sum_{l',l} \left\{ \left\| \left[{}^T Q_{l'+1} - {}^T Q_{l'} \right] \left[A_\Lambda^C - A_\Lambda \right] \left[{}^A Q_{l+1} - {}^A Q_l \right] \right\|_{H^1} \times \right. \tag{8.18}$$

$$\left\| \left[{}^AQ_{l+1} - {}^AQ_l \right] \left[x_\Lambda - {}^AQ_\Lambda x \right] \right\|_{H^1} \left\| \left[{}^TQ^*_{l'+1} - {}^TQ^*_{l'} \right] {}^TQ^*_\Lambda v \right\|_{H^{-1}} \Bigg\}$$

$$\leq \sum_{l',l} \left\| \left[{}^TQ_{l'+1} - {}^TQ_{l'} \right] \left[A^C_\Lambda - A_\Lambda \right] \left[{}^AQ_{l+1} - {}^AQ_l \right] \right\|_{H^1} 2^{-j_0} 2^{-(1-s)l'},$$

where we have used (cf. Lemma 8.3)

$$\left\| \left[{}^AQ_{l+1} - {}^AQ_l \right] \left[x_\Lambda - {}^AQ_\Lambda x \right] \right\|_{H^1} \leq \left\| \left[x_\Lambda - {}^AQ_\Lambda x \right] \right\|_{H^1} \leq C 2^{-j_0}. \tag{8.19}$$

Now we estimate the H^1 operator norms on the right-hand side of (8.18) by the norm of the corresponding matrices. Using the discrete norm equivalences (cf. Lemma 3.1 iii) and Lemma 4.2 iii)), the norms can be reduced to Euclidean matrix norms. They can be estimated by a Schur lemma argument similar to Lemma 8.1, where the weight factors $2^{\pm l/2}$ and $2^{\pm l'/2}$ are dropped. From Lemma 7.1, (7.4), and (7.8), we infer

$$\begin{aligned} \Sigma_1 &\leq C \sup_{k'} \sum_{k:\, dist > \mathcal{G}} 2^{l'} 2^{-(1/2+4)l'} dist^{-7} 2^{-(1/2+2)l} 2^{-l} \\ &\leq C \sup_{k'} 2^{-3.5l'} 2^{-2.5l} 2^{-l} \sum_{k:\, dist > \mathcal{G}} dist^{-7} \leq C \sup_{k'} 2^{-3.5l'} 2^{-2.5l} \mathcal{G}^{-6} \\ &\leq C a^{-6} 2^{6j_0} 2^{-3.5l'} 2^{-2.5l} 2^{-6\beta_l(j_0-l)} 2^{-6\gamma_{l'}(j_0-l')}. \end{aligned} \tag{8.20}$$

Analogous arguments lead to

$$\Sigma_2 \leq C a^{-6} 2^{6j_0} 2^{-2.5l'} 2^{-3.5l} 2^{-6\beta_l(j_0-l)} 2^{-6\gamma_{l'}(j_0-l')}, \tag{8.21}$$

and (8.20) and (8.20) together imply

$$\begin{aligned} \left\| \left[{}^TQ_{l'+1} - {}^TQ_{l'} \right] \left[A^C_\Lambda - A_\Lambda \right] \left[{}^AQ_{l+1} - {}^AQ_l \right] \right\| &\leq C \sqrt{\Sigma_1 \Sigma_2} \\ &\leq C a^{-6} 2^{6j_0} 2^{-3l'} 2^{-3l} 2^{-6\beta_l(j_0-l)} 2^{-6\gamma_{l'}(j_0-l')}. \end{aligned} \tag{8.22}$$

Thus (8.18) can be continued as

$$\begin{aligned} |\mathcal{T}_1| &\leq \sum_{l',l} C a^{-6} 2^{6j_0} 2^{-3l'} 2^{-3l} 2^{-6\beta_l(j_0-l)} 2^{-6\gamma_{l'}(j_0-l')} 2^{-j_0} 2^{-(1-s)l'}, \\ &\leq C a^{-6} 2^{5j_0} \left\{ 2^{-6\beta^+ j_0} \sum_{l=-1}^{j_0-1} 2^{(-3+6\beta^+)l} + 2^{-6\beta^- j_0} \sum_{l=j_0}^{j-1} 2^{(-3+6\beta^-)l} \right\} \\ &\qquad \times \left\{ 2^{-6\gamma^+ j_0} \sum_{l=-1}^{j_0-1} 2^{(-4+s+6\gamma^+)l'} + 2^{-6\gamma^- j_0} \sum_{l=j_0}^{j-1} 2^{(-4+s+6\gamma^-)l'} \right\} \\ &\leq C a^{-6} 2^{-(2-s)j_0} . \end{aligned} \tag{8.23}$$

Note that we have used $-3+6\beta^+ > 0$, $-3+6\beta^- < 0$, $s-4+6\gamma^+ > 0$, and $s-4+6\gamma^- < 0$.

◇

Collecting the results of the Lemmas 7.2, 8.2-8.4 together, we get

Theorem 8.1 *Consider the double layer equation (6.4) for a right-hand side y which is continuous on Γ and infinitely differentiable over each side of the polygonal boundary. Suppose that the minimum κ of the exponents of the corner singularities satisfies $9/14 < \kappa$ (cf. (6.6) and compare Remark 7.1). Choose ε less than the minimum of $1/12$ and $\kappa \cdot 7/36 - 1/8$. For this ε, we suppose that Λ is chosen according to Lemma 6.2 and that the compression algorithm is chosen according to (7.4). Then:*

i) The compressed wavelet collocation method (cf. the end of Section 5) applied to the double layer equation (6.4) is stable in the space H^1 for sufficiently large a (cf. (7.4)).

ii) For $1 \leq s < \kappa + 1/2$ and the approximate solution x_Λ of the compressed wavelet collocation method, we get the error estimate

$$\|x - x_\Lambda\|_{H^s} \leq C\, 2^{[(s-1)/(\kappa-1/2)-1]j_0}. \tag{8.24}$$

iii) If $0 \leq s \leq 1$, then

$$\|x - x_\Lambda\|_{H^s} \leq C\, 2^{-(2-s)j_0}. \tag{8.25}$$

iv) The number of degrees of freedom is less than $C2^{j_0}$ and the number of non-zero entries in the compressed stiffness matrix $\mathcal{A}^C$ is less than $Caj_0 2^{j_0}$. Consequently, the compressed wavelet collocation algorithm (cf. the end of Section 5) requires $O(j_0 2^{j_0})$ arithmetic operations and a storage capacity for $O(j_0 2^{j_0})$ real numbers.

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