On the acoustic waves in two-component linear poroelastic materials

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1. Preliminaries

The problem of propagation of acoustic waves in porous materials has been first considered for the purpose of evaluation of the seismic measurements of soils. The pioneering work of M. A. BIOT ([1956], [1962, 1,2]) has revealed the important new feature of such media that a small dynamical disturbance creates at least three acoustic waves: the usual shear wave and two, instead of one, longitudinal waves. The faster one of the latter corresponds to the classical longitudinal wave in elastic solids and it is called, in theories of porous materials, a P1-wave. The slower one is characteristic for multicomponent systems and it is called a P2-wave (also called the Biots´s wave). Below, we quote from T. BOURBIE, O. COUSSY and B. ZINSZNER [1987] (after the work of T. J. PLONA) a few examples of the velocities of those three waves in porous and granular materials, fully saturated with water.

	Porosity	U_{P1}	U_{shear}	U_{P2}
sintered glass	0.283	$4.05\times10^3\ ^m/_s$	$2.37\times10^3\ ^m/_s$	$1.04\times10^3\ ^m/_s$
porous steel	0.480	$2.74\times10^3\ ^m/_s$	$1.54\times10^3\ ^m/_s$	$0.92\times10^3\ ^m/_s$
porous titanium	0.410	$2.72\times10^3\ ^m/_s$	$1.79\times10^3\ ^m/_s$	$0.91\times10^3\ ^m/_s$
porous inconel	0.360	$2.12\times10^3\ ^m/_s$	$1.15\times10^3\ ^m/_s$	$0.93\times10^3\ ^m/_s$
coors (ceramics)	0.415	$3.95\times10^3\ ^m/_s$	$2.16\times10^3\ ^m/_s$	$0.96\times10^3\ ^m/_s$

M. A. Biot has obtained the above mentioned three waves within his linear viscoelastic model of porous materials. He has also found that the attenuation of P2-waves is much stronger than this of P1-waves. He has attributed this attenuation to the damping through the relative motion of components, and to the viscosity of the skeleton. The latter was needed to explain the attenuation in the case of empty pores.

Similar results were later obtained for various modifications of the Biot´s model (e.g. R. DZIECIELAK [1980], [1995], [1996], R. I. NIGMATULIN [1990], R. I. NIGMATULIN, A. A. GUBAIDULLIN [1992], A. A. GUBAIDULLIN, O. Y. KUCHUGURINA [1995]).

The high attenuation of P2-waves was indeed confirmed by experiments on porous and granular materials (e.g. T. J. PLONA [1980], D. L. JOHNSON, T. J. PLONA [1982], T. BOURBIE, O. COUSSY, B ZINSZNER [1987]). However, the amplitude of reflected P2-waves may be in some cases even higher than the amplitude of P1-waves. Simultaneously, as we see further, the P2-waves deliver much more data on the porous materials than the P1-waves do. For this reason it is important to know what sort of information one can expect to obtain by measuring such waves, and how to develop the software (in particular, for the purposes of the medical diagnosis, where the large deformations of biological tissues play an important role) which would be capable to sort out the relevant part of the signals from the useless noice.

In this work, we present some results for the propagation of sound waves in porous materials, described by a new multicomponent model. The most fundamental feature of this model is the balance equation for porosity. This equation describes the microscopic relaxation processes through the presence of the source term as well as the transport processes of porosity. The latter yields the description of important couplings between components which influence the speeds of propagation of sound waves. However, for the conditions of the propagation of monochromatic waves, it is even more important to have the former property of the model. This is due to the fact that the relaxation of porosity yields naturally the attenuation of waves. It is not necessary any more to introduce the viscosity of the skeleton which, in many cases of practical bearing - like sintered glass, seems to be rather artificial.

It should be mentioned that the new model does not contain any constraints which were typical for many earlier models of porous materials. It can be shown (e.g. K. WILMANSKI [1995, 2]) that such constraints eliminate at least one of the longitudinal waves - the set of field equations is not hyperbolic any more. The results for the sound waves in such models do not seem to correspond to any observed phenomena. For instance, the models based on the so-called incompressibility of real components, introduced by R. M. Bowen, yield the unnatural coupling of amplitudes and speeds of propagation (see: R. DE BOER, ZHANFANG LIU [1995]).

We organize the paper in the following manner. The next section and the third one contain the governing equations of the new model for isothermal processes in two-component poroelastic materials. In the fourth section, we present the propagation condition for the linear model. In the fifth section, we derive the dispersion relation for plane monochromatic waves. The sixth section contains the discussion of this relation. In particular, we discuss the influence of the variation of the new material parameters on the phase velocities and on the attenuation.

The references to an extensive literature are rather scarce and reduced solely to a few representative examples. A critical presentation of various approaches to the subject of sound waves in porous and granular materials shall be contained in the forthcoming review paper.

2. Balance laws

We consider the class of isothermal processes in the two-component porous material whose skeleton is elastic, and the fluid component is ideal. It means that the solid material is elastic in the case of the porosity equal to zero (lack of space for the fluid component) and the fluid is ideal, in the case of the porosity equal to one (lack of the skeleton). The set of fields in the Eulerian description, for which we formulate later the governing set of linear equations, is as follows

$$\forall \mathbf{x} \in \mathcal{B}_t,\ t \in \mathcal{T}: \quad (\mathbf{x}, t) \mapsto \left\{ \rho_t^F, \mathbf{v}^F, \mathbf{u}^S, n \right\} \in \mathcal{V}^8, \tag{2.1}$$

where $\mathcal{B}_t \subset \Re^3$ is the domain of the current configuration of the skeleton and $\mathcal{T}$ is the time interval in which we consider the thermodynamical processes.

The current mass density of the fluid is denoted by $\rho^F{}_t$, the velocity of the fluid is denoted by $\mathbf{v}^F$, the displacement of the skeleton is $\mathbf{u}^S$ and the porosity is n.

These fields are used in the linear theory. However, in the case of fully non-linear model, it is more convenient to use the consistent Lagrangian description which has been proposed in my paper [1995,1]. The set of fields is then defined in the following way

$$\forall \mathbf{X} \in \mathcal{B},\ t \in \mathcal{T}: \quad (\mathbf{X}, t) \mapsto \left\{ \rho^F, \mathbf{x}'^F, \chi^S(\mathbf{X}, t), n \right\} \in \mathcal{V}^8, \qquad \mathcal{B} \equiv \chi^{S-1}(\mathcal{B}_t, t), \quad (2.2)$$

where $\mathcal{B} \subset \Re^3$ denotes the reference configuration of the skeleton and

$$\begin{aligned} &\rho^F = \rho^F_t J^S, \quad \mathbf{x}'^F = \mathbf{v}^F\left(\chi^S(\mathbf{X}, t), t\right), \\ &\mathbf{x} = \chi^S(\mathbf{X}, t), \quad J^S = \det \mathbf{F}^S, \quad \mathbf{F}^S = \operatorname{Grad} \chi^S, \end{aligned} \quad (2.3)$$

i.e. the mass density ρ^F of the fluid refers to the unit volume of the skeleton in the reference configuration, the velocity field of the fluid is defined on the reference configuration of the skeleton, χ^S denotes the deformation function of the skeleton and $\mathbf{F}^S$ is the deformation gradient of the skeleton.

The field equations for the fields (2.2) follow from the balance equations of mass, momentum and porosity (see: K. WILMANSKI [1996, 1,2])

$$\begin{aligned} &\frac{\partial \rho^F}{\partial t} + \operatorname{Div}\left(\rho^F \mathbf{X}'^F\right) = 0, \quad \mathbf{X}'^F \equiv \mathbf{F}^{S-1}\left(\mathbf{x}'^F - \mathbf{x}'^S\right), \quad \mathbf{x}'^S \equiv \frac{\partial \chi^S}{\partial t}(\mathbf{X}, t), \\ &\rho^F \left(\frac{\partial \mathbf{x}'^F}{\partial t} + \left(\operatorname{Grad} \mathbf{x}'^F\right) \mathbf{X}'^F \right) = \operatorname{Div} \mathbf{P}^F + \hat{\mathbf{p}}^F + \rho^F \mathbf{b}^F, \\ &\rho^S \frac{\partial \mathbf{x}'^S}{\partial t} = \operatorname{Div} \mathbf{P}^S + \hat{\mathbf{p}}^S + \rho^S \mathbf{b}^S, \qquad \hat{\mathbf{p}}^S + \hat{\mathbf{p}}^F = 0, \\ &\frac{\partial n}{\partial t} + \operatorname{Div}\left(\Phi_0 \mathbf{X}'^F\right) = \hat{n}. \end{aligned} \quad (2.4)$$

In these equations, the tensors $\mathbf{P}^F$ and $\mathbf{P}^S$ denote the Piola-Kirchhoff partial stress tensors in the fluid and in the skeleton, respectively. Both of them refer to the same reference configuration $\mathcal{B}$. The vectors $\hat{\mathbf{p}}^F, \hat{\mathbf{p}}^S$ describe the exchange of momentum between the components and $\mathbf{b}^F$, $\mathbf{b}^S$ are the external mass forces. The scalar Φ_0 describes the flux of porosity and $\hat{n}$ is the intensity of the source of porosity. Both these quantities are described in details in my earlier papers (e.g. see: K. WILMANSKI [1996, 1,2,3]).

As already mentioned, we use further the Eulerian description of motion. It is easy to show that the above balance equations have, in this description, the following form

$$
\begin{aligned}
&\frac{\partial \rho_t^F}{\partial t} + \operatorname{div}\left(\rho_t^F \mathbf{v}^F\right) = 0,\\
&\rho_t^F\left(\frac{\partial \mathbf{v}^F}{\partial t} + \left(\operatorname{grad}\mathbf{v}^F\right)\mathbf{v}^F\right) = \operatorname{div}\mathbf{T}^F + J^{S-1}\hat{\mathbf{p}}^F + \rho_t^F \mathbf{b}^F,\\
&\rho_t^S\left(\frac{\partial \mathbf{v}^S}{\partial t} + \left(\operatorname{grad}\mathbf{v}^S\right)\mathbf{v}^S\right) = \operatorname{div}\mathbf{T}^S + J^{S-1}\hat{\mathbf{p}}^S + \rho_t^S \mathbf{b}^S,\\
&\frac{\partial\, nJ^{S-1}}{\partial t} + \operatorname{div}\left[nJ^{S-1}\mathbf{v}^S + \varphi\left(\mathbf{v}^F - \mathbf{v}^S\right)\right] = \hat{n}J^{S-1},
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
&\rho_t^S = \rho^S J^{S-1}, \quad \mathbf{v}^S = \mathbf{x}'^S\left(\chi^{S-1}(\mathbf{x}, t), t\right), \quad \varphi = \Phi_0 J^{S-1},\\
&\mathbf{T}^S = J^{S-1}\mathbf{P}^S\mathbf{F}^{ST}, \quad \mathbf{T}^F = J^{S-1}\mathbf{P}^F\mathbf{F}^{ST}.
\end{aligned}
\tag{2.6}
$$

The number of balance equations (2.5) is identical with the number of unknown fields (2.1). It remains to close them by means of constitutive relations. We shall do so in the next section.

3. Governing equations

Apart from the assumptions that the porous medium has an elastic skeleton and that the fluid component is ideal which we have mentioned before, we assume in addition that the medium is uniform and isotropic and the processes deviate not too far from the thermodynamical equilibrium state. The last assumption should be understood in the following manner.

The set of constitutive variables, on which depend the constitutive functions of our problem, is as follows

$$
\mathcal{C} = \left\{\rho_t^F, \mathbf{B}^S, \Delta, \mathbf{w}; n_E\right\}, \quad \mathbf{B}^S = \mathbf{F}^S\mathbf{F}^{ST}, \quad \mathbf{w} = \mathbf{v}^F - \mathbf{v}^S, \quad \Delta = n - n_E, \tag{3.1}
$$

where the dependence on n_E is parametric. The argument n_E is the constant value of porosity which is approached by n as $\hat{n} \to 0$ and $\mathbf{w} \to 0$. These states, for which the source of porosity vanishes and the relative velocity $\mathbf{w}$ is also zero, are called the states of the thermodynamical equilibrium. It can be shown (see: K.WILMANSKI [1996, 2]) that the dissipation approaches its minimum (zero) value in these states. Consequently, the small deviation from the thermodynamical equilibrium means that we assume the constitutive relations to be linear functions of Δ and $\mathbf{w}$.

The exploitation of the second law of thermodynamics yields then the following form of constitutive relations

$$\mathbf{T}^S = \Im_0 \mathbf{1} + \Im_1 \mathbf{B}^S + \Im_{-1} \mathbf{B}^{S-1} + \frac{\Delta\varphi}{\tau \mathcal{M}} \mathbf{1}, \quad \varphi = \varphi(n_E),$$

$$\mathbf{T}^F = -p^F \mathbf{1} - \frac{\Delta\varphi}{\tau \mathcal{M}} \mathbf{1}, \quad p^F = p^F\left(\rho_t^F, n_E\right), \tag{3.2}$$

$$\hat{n} = -\frac{\Delta}{\tau}, \quad J^{S-1}\hat{\mathbf{p}}^S = -J^{S-1}\hat{\mathbf{p}}^F = \pi \mathbf{w}, \quad \tau > 0, \quad \pi > 0, \quad \mathcal{M} > 0,$$

where the material parameters $\Im_0$, $\Im_1$, $\Im_{-1}$, τ, π and $\mathcal{M}$ may be still the functions of invariants of the deformation tensor $\mathbf{B}^S$ and of the equilibrium porosity n_E.

By the choice of the constitutive variables (3.1), it is also convenient to write the balance equation for porosity $(2.5)_4$ in the following form

$$\frac{\partial \Delta}{\partial t} + \mathbf{v}^S \cdot \operatorname{grad} \Delta + J^S \varphi \operatorname{div}\left(\mathbf{v}^F - \mathbf{v}^S\right) = -\frac{\Delta}{\tau}. \tag{3.3}$$

In the case of small deformations of the skeleton the above set of equations simplifies considerably. The problem of propagation of monochromatic waves, which we consider further in this work, shall be limited solely to such cases of small deformations. Namely, if we make the assumption

$$\sup_{t \in \mathcal{T}} \sup_{x \in \mathcal{B}_t} \left|\mathbf{e}^S \cdot \mathbf{n} \otimes \mathbf{n}\right| << 1 \quad \text{for all } \mathbf{n}, \quad |\mathbf{n}| = 1, \qquad \mathbf{e}^S \equiv \tfrac{1}{2}\left(\mathbf{1} - \mathbf{B}^{S-1}\right), \tag{3.4}$$

$$\Rightarrow J^S \approx 1 + \operatorname{tr} \mathbf{e}^S,$$

then the first three terms in the constitutive relation for the partial stress tensor $\mathbf{T}^S$ reduce to the classical Hooke´s law and we obtain

$$\mathbf{T}^S = \lambda^S\left(\mathbf{e}^S \cdot \mathbf{1}\right)\mathbf{1} + 2\mu^S \mathbf{e}^S + \frac{\Delta\varphi}{\tau \mathcal{M}} \mathbf{1}, \qquad \mathbf{e}^S = \tfrac{1}{2}\left(\operatorname{grad} \mathbf{u}^S + \operatorname{grad}^T \mathbf{u}^S\right), \tag{3.5}$$

where the effective Lamé parameters λ^S and μ^S as well as the other material parameters φ, τ and $\mathcal{M}$ are now solely parametrically dependent on n_E.

Bearing in mind the above considerations, we can now choose the following fields

$$\left\{\rho_t^F, \mathbf{v}^F, \mathbf{v}^S, \mathbf{e}^S, \Delta\right\}, \tag{3.6}$$

where the single field of displacement $\mathbf{u}^S$ has been replaced by the fields $\mathbf{v}^S$ and $\mathbf{e}^S$ in order to have the field equations in the form of the set of the first order partial differential equations.

This set of field equations can be written in the following form

$$\frac{\partial \rho_t^F}{\partial t} + \operatorname{div}\left(\rho_t^F \mathbf{v}^F\right) = 0,$$

$$\rho_t^F\left(\frac{\partial \mathbf{v}^F}{\partial t} + \left(\operatorname{grad} \mathbf{v}^F\right)\mathbf{v}^F\right) = -\operatorname{grad}\left(p^F + \frac{\Delta \varphi}{\tau \mathcal{M}}\right) - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right) + \rho_t^F \mathbf{b}^F,$$

$$\rho^S \frac{\partial \mathbf{v}^S}{\partial t} = \operatorname{grad}\left(\lambda^S \operatorname{tr} \mathbf{e}^S + \frac{\Delta \varphi}{\tau \mathcal{M}}\right) + 2\mu^S \operatorname{div} \mathbf{e}^S + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right) + \rho^S \mathbf{b}^S, \qquad (3.7)$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = \tfrac{1}{2}\left(\operatorname{grad} \mathbf{v}^S + \operatorname{grad}^T \mathbf{v}^S\right),$$

$$\frac{\partial \Delta}{\partial t} + \mathbf{v}^S \cdot \operatorname{grad} \Delta + \varphi \operatorname{div}\left(\mathbf{v}^F - \mathbf{v}^S\right) = -\frac{\Delta}{\tau}.$$

We have left out the non-linear contributions of the fields, describing the skeleton in the same way as it is done in the classical linear theory of elasticity. The equation $(3.7)_4$ is the integrability condition which yields the existence of the displacement field $\mathbf{u}^S$.

Let us make in passing the following remark concerning the transport coefficient φ. If we multiply the equation $(2.5)_4$ by a constant ρ^{FR} and require

$$\rho_t^F = nJ^{S-1}\rho^{FR}, \qquad \hat{n} \equiv 0, \qquad (3.8)$$

then this equation becomes identical with the mass balance equation $(2.5)_1$ provided $\mathbf{\Phi}_0$=n. In such a case the field of porosity is controlled by the mass density of the fluid $\rho^F{}_t$ - the porosity n satisfies the constitutive relation $(3.8)_1$ rather than its own field equation. Consequently, the field equations reduce to the set with one equation less than before. The theory reminds now the theory of two-component miscible mixtures. Most likely the above simplifying assumption requires, apart from the small deformations of the skeleton, the small changes of the mass density of the fluid component as well. It is obvious that the assumptions (3.8) demand that there is no spontaneous relaxation of porosity. The first relation (3.8) with the constant „true mass density“ ρ^{FR} reminds these theories of porous materials which are based on the constraint of „incompressible“ true components. In the present case, there is, certainly, no constraint and no reaction force on this constraint. In addition, the assumption on the form of flux $\mathbf{\Phi}_0$ cannot be fulfilled exactly because it can be shown to be contradictory with the second law of thermodynamics. However, it may hold as an approximation in the case of small changes of porosity and small volume changes of the skeleton ($J^S \approx 1$). Bearing the relation $(2.6)_3$ in mind, we can write then approximately for such a case

$$\varphi \approx \gamma\, n_E, \quad \gamma = \text{const.} \qquad (3.9)$$

The constant γ is of the order of magnitude of the unity. In any case, it shall be assumed to be positive.

4. Propagation condition

We consider the propagation of the sound wave front through the porous material, described by the set of equations (3.7). Such a front is assumed to be the moving surface of the weak discontinuity. It means that the limits of fields and their first time derivatives are finite on both sides of this surface and that the fields are continuous, i.e.

$$
\begin{gathered}
[[\rho_t^F]] = 0, \quad [[\mathbf{v}^F]] = 0, \quad [[\mathbf{v}^S]] = 0, \quad [[\mathbf{e}^S]] = 0, \quad [[\Delta]] = 0, \\
r \equiv \left[\left[\frac{\partial \rho_t^F}{\partial t}\right]\right], \quad \mathbf{a}^F \equiv \left[\left[\frac{\partial \mathbf{v}^F}{\partial t}\right]\right], \quad \mathbf{a}^S \equiv \left[\left[\frac{\partial \mathbf{v}^S}{\partial t}\right]\right], \quad D \equiv \left[\left[\frac{\partial \Delta}{\partial t}\right]\right], \\
[[\cdots]] \equiv (\cdots)^+ - (\cdots)^-, \quad \max\left\{|r|, |\mathbf{a}^F|, |\mathbf{a}^S|, |D|\right\} < \infty.
\end{gathered}
\tag{4.1}
$$

Hence, the brackets [[...]] define the difference of limits on both sides of the surface. This is, certainly, the jump of the corresponding quantity across the singular surface.

The kinematical compatibility relations yield then the following jumps of the gradients

$$
\begin{gathered}
c[[\operatorname{grad} \rho_t^F]] = -r\,\mathbf{n}, \quad c[[\operatorname{grad} \mathbf{v}^F]] = -\mathbf{a}^F \otimes \mathbf{n}, \\
c[[\operatorname{grad} \mathbf{v}^S]] = -\mathbf{a}^S \otimes \mathbf{n}, \quad c[[\operatorname{grad} \Delta]] = -D\,\mathbf{n}, \\
c[[\operatorname{grad} \mathbf{e}^S]] = -\left[\left[\frac{\partial \mathbf{e}^S}{\partial t}\right]\right] \otimes \mathbf{n},
\end{gathered}
\tag{4.2}
$$

where c denotes the finite speed of propagation of the wave front and **n** is the outwart unit normal vector to this surface.

Simultaneously, the limits of the equation $(3.7)_4$ on both sides of the wave front give rise to the following relation

$$
c\left[\left[\frac{\partial \mathbf{e}^S}{\partial t}\right]\right] = -\frac{1}{2}\left(\mathbf{a}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a}^S\right) \quad \Rightarrow \quad \left\|\left[\left[\frac{\partial \mathbf{e}^S}{\partial t}\right]\right]\right\| < \infty. \tag{4.3}
$$

Consequently

$$
c^2[[\operatorname{grad} \mathbf{e}^S]] = \frac{1}{2}\left(\mathbf{a}^S \otimes \mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a}^S \otimes \mathbf{n}\right). \tag{4.4}
$$

Now it is easy to construct the similar jump conditions for the remaining equations (3.7). We obtain

$$
\begin{gathered}
r\left(c - \mathbf{v}^F \cdot \mathbf{n}\right) = \rho_t^F \mathbf{a}^F \cdot \mathbf{n}, \\
D\left(c - \mathbf{v}^F \cdot \mathbf{n}\right) = \varphi\left(\mathbf{a}^F \cdot \mathbf{n} - \mathbf{a}^S \cdot \mathbf{n}\right),
\end{gathered}
\tag{4.5$_1$}
$$

$$\rho^F_t\left(c - \mathbf{v}^F \cdot \mathbf{n}\right)\mathbf{a}^F = \frac{\partial p^F}{\partial \rho^F_t} r\mathbf{n} + \frac{\varphi}{\tau\mathcal{M}} D\mathbf{n},$$

$$\rho^S c^2 \mathbf{a}^S = \left(\lambda^S + \mu^S\right)\mathbf{a}^S \cdot \mathbf{n}\mathbf{n} + \mu^S \mathbf{a}^S - \frac{\varphi}{\tau\mathcal{M}} D\mathbf{n}, \tag{4.5$_2$}$$

where the body forces were assumed to be absent.

Within the frame of the present linear model, it is also reasonable to assume that the mass density of the fluid $\rho^F{}_t$ deviates only a little from the initial value $\rho^F{}_0$ and that the normal particle velocities $\mathbf{v}^F \cdot \mathbf{n}$ and $\mathbf{v}^S \cdot \mathbf{n}$ are much smaller that the speed of propagation c, i.e.

$$|c| >> \max\left(\left|\mathbf{v}^F \cdot \mathbf{n}\right|, \left|\mathbf{v}^S \cdot \mathbf{n}\right|\right), \quad \left|\rho^F_t - \rho^F_0\right| << \rho^F_0. \tag{4.6}$$

The general non-linear case has been considered in my work [1995, 2].

Bearing the above assumptions in mind, we obtain

$$r = \frac{1}{c}\mathbf{a}^F \cdot \mathbf{n}, \quad D = \frac{1}{c}\varphi\left(\mathbf{a}^F \cdot \mathbf{n} - \mathbf{a}^S \cdot \mathbf{n}\right). \tag{4.7}$$

It means that the amplitudes of the mass density $\rho^F{}_t$ and of the changes of porosity Δ are determined by the normal components of the amplitudes of partial accelerations. The latter satisfy the following set of equations

$$\left(c^2\mathbf{1} - \left.\frac{\partial p^F}{\partial \rho^F_t}\right|_0 \mathbf{n}\otimes\mathbf{n} - \frac{\varphi^2}{\tau\mathcal{M}\rho^F_0}\mathbf{n}\otimes\mathbf{n}\right)\mathbf{a}^F + \left(\frac{\varphi^2}{\tau\mathcal{M}\rho^F_0}\mathbf{n}\otimes\mathbf{n}\right)\mathbf{a}^S = 0,$$

$$\left(\frac{\varphi^2}{\tau\mathcal{M}\rho^F_0}\mathbf{n}\otimes\mathbf{n}\right)\mathbf{a}^F + \left(c^2\mathbf{1} - \frac{\lambda^S + \mu^S}{\rho^S}\mathbf{n}\otimes\mathbf{n} - \frac{\mu^S}{\rho^S}\mathbf{1} - \frac{\varphi^2}{\tau\mathcal{M}\rho^S}\mathbf{n}\otimes\mathbf{n}\right)\mathbf{a}^S = 0, \tag{4.8}$$

where

$$\left.\frac{\partial p^F}{\partial \rho^F_t}\right|_0 \equiv \frac{\partial p^F}{\partial \rho^F_t}\left(\rho^F_t = \rho^F_0; n_E\right). \tag{4.9}$$

This is, certainly, the reduced eigenvalue problem for the matrix of coefficients of the set (3.7) with c^2 - the eigenvalues and ($\mathbf{a}^F$,$\mathbf{a}^S$) - the six-dimensional eigenvectors.

It is seen immediately from the first equation that the amplitude $\mathbf{a}^F$ has solely the normal component

$$\mathbf{a}^F = \mathbf{a}^F \cdot \mathbf{n}\mathbf{n}. \tag{4.10}$$

On the other hand, if we separate the normal and tangential parts in the second equation, we obtain

$$\frac{\varphi^2}{\tau \mathcal{M} \rho^S} \mathbf{a}^F \cdot \mathbf{n} + \left(c^2 - \frac{\lambda^S + 2\mu^S}{\rho^S} - \frac{\varphi^2}{\tau \mathcal{M} \rho^S} \right) \mathbf{a}^S \cdot \mathbf{n} = 0,$$
$$\left(c^2 - \frac{\mu^S}{\rho^S} \right) \mathbf{a}_\perp^S = 0, \quad \mathbf{a}_\perp^S \equiv \mathbf{a}^S - \mathbf{a}^S \cdot \mathbf{n}\, \mathbf{n}. \tag{4.11}$$

The latter relation describes the S-wave which, apart from the dependence on the porosity n_E, does not differ from the classical shear wave.

The equations $(4.8)_1$ and (4.11) yield now the following propagation condition

$$\left(c^2 - \left.\frac{\partial p^F}{\partial \rho_t^F}\right|_0 - \frac{\varphi^2}{\tau \mathcal{M} \rho_0^F} \right) \left(c^2 - \frac{\lambda^S + 2\mu^S}{\rho^S} - \frac{\varphi^2}{\tau \mathcal{M} \rho^S} \right) - \left(\frac{\varphi^2}{\tau \mathcal{M} \rho_0^F} \right)^2 \frac{\rho_0^F}{\rho^S} = 0. \tag{4.12}$$

Let us notice that the lack of stress coupling (e.g. for $\varphi \equiv 0$) would yield the following solutions of this equation

$$c^2 = U^{S2} \equiv \frac{\lambda^S + 2\mu^S}{\rho^S} \quad \text{or} \quad c^2 = U^{F2} \equiv \left.\frac{\partial p^F}{\partial_t^F}\right|_0 . \tag{4.13}$$

These are the classical relations for the speeds of propagation of longitudinal waves in the elastic solid and in the ideal fluid as if the second component was absent and the material properties were described by effective material constants rather than these for the porosity equal to one or zero, respectively.

In our case, the biquadratic equation (4.12) yields also two different speeds of propagation of longitudinal waves even though they are different from (4.13). Namely

$$c^2 = \frac{1}{2} \left\{ \left(U^{F2} + \frac{\varphi^2}{\tau \mathcal{M} \rho_0^F} \right) + \left(U^{S2} + \frac{\varphi^2}{\tau \mathcal{M} \rho^S} \right) \pm \right.$$
$$\left. \pm \sqrt{ \left[\left(U^{F2} + \frac{\varphi^2}{\tau \mathcal{M} \rho_0^F} \right) - \left(U^{S2} + \frac{\varphi^2}{\tau \mathcal{M} \rho^S} \right) \right]^2 + 4 \left(\frac{\varphi^2}{\tau \mathcal{M} \rho_0^F} \right) \frac{\rho_0^F}{\rho^S} } \right\}. \tag{4.14}$$

These two waves are called P1- and P2-waves in the porous materials.

It is easy to check that the speeds of propagation are real if

$$\frac{\lambda^S + 2\mu^S}{\rho^S} > 0, \quad \left.\frac{\partial p^F}{\partial \rho_t^F}\right|_0 > 0, \quad \frac{\varphi^2}{\tau \mathcal{M} \rho_0^F} > -\left(\frac{1}{U^{F2}} + \frac{\rho_0^F}{\rho^S} \frac{1}{U^{S2}} \right)^{-1} . \tag{4.15}$$

These conditions yield the <u>hyperbolicity</u> of the system of field equations (3.7). Let us notice that the inequalities

$$\tau > 0, \quad \mathcal{M} > 0, \tag{4.16}$$

required in my earlier works on thermodynamics of the model, are sufficient for $(4.15)_3$ to hold provided the first two inequalities (4.15) are satisfied.

5. Plane monochromatic waves

We proceed to consider the propagation of plane monochromatic waves in the porous material described by the linearized model presented in the third section. The waves are assumed to propagate in the direction of the x-axis and the amplitudes of the velocities and of the displacement of the skeleton are also assumed to have solely the x-direction. Consequently, we can expect to find only the longitudinal waves.

The set of unknown fields is now reduced in the following manner

$$(x,t) \mapsto \left\{\rho_t^F, v^F, v^S, e^S, \Delta\right\} \tag{5.1}$$

where v^S, v^F are the components of partial velocities in the direction of the x-axis and e^S denotes the small extension of the skeleton in the x-direction.

The field equations for these fields follow from (3.7) and have the form

$$\begin{aligned}
&\frac{\partial \rho_t^F}{\partial t} + \frac{\partial}{\partial x}\left(\rho_t^F v^F\right) = 0,\\
&\rho_t^F\left(\frac{\partial v^F}{\partial t} + \frac{\partial v^F}{\partial x} v^F\right) = -\frac{\partial}{\partial x}\left(p^F + \frac{n_E \gamma}{\tau \mathcal{M}}\Delta\right) - \pi\left(v^F - v^S\right), \quad p^F = p^F\left(\rho_t^F, n_E\right),\\
&\rho^S \frac{\partial \mathbf{v}^S}{\partial t} = \left(\lambda^S + 2\mu^S\right)\frac{\partial e^S}{\partial x} + \frac{n_E \gamma}{\tau \mathcal{M}}\frac{\partial \Delta}{\partial x} + \pi\left(v^F - v^S\right),\\
&\frac{\partial e^S}{\partial t} = \frac{\partial v^S}{\partial x},\\
&\frac{\partial \Delta}{\partial t} + v^S \frac{\partial \Delta}{\partial x} + n_E \gamma \frac{\partial\left(v^F - v^S\right)}{\partial x} = -\frac{\Delta}{\tau},
\end{aligned} \tag{5.2}$$

where the relation (3.9) has been used.

We seek the solution of the above set of equations in the form of the small disturbance of the uniform static initial state, i.e.

$$\begin{aligned}
&\rho_t^F = \rho_0^F + \varepsilon R^F \exp i\left(\omega t - k^* x\right), \quad \Delta = \varepsilon D \exp i\left(\omega t - k^* x\right),\\
&v^F = \varepsilon V^F \exp i\left(\omega t - k^* x\right), \quad v^S = \varepsilon V^S \exp i\left(\omega t - k^* x\right),\\
&e^S = \varepsilon E^S \exp i\left(\omega t - k^* x\right), \qquad 0 < \varepsilon << 1,
\end{aligned} \tag{5.3}$$

where ω is the (given) frequency of the disturbance, R^F, D, V^F, V^S, E^S are the constant amplitudes of the disturbance, $\rho^F{}_0$ is the constant initial value of the mass density of the fluid component and ε is the small parameter. The coefficient k* can be complex

$$k^* = k + i\alpha, \tag{5.4}$$

with the real part k describing the wave number and the imaginary part α being the attenuation coefficient.

Substitution of the relations (5.3) in the equations (5.2) yields the following set of algebraic relations for the amplitudes of the disturbance

$$\begin{aligned}
&R^F\omega - V^F\rho_0^F k^* = 0,\\
&D\left(i\omega + \tfrac{1}{\tau}\right) - i\left(V^F - V^S\right)n_E\gamma\, k^* = 0,\\
&E^S\omega + V^S k^* = 0,\\
&V^F\omega - R^F U^{F2}\frac{1}{\rho_0^F}k^* - D\frac{n_E\gamma}{\tau\mathcal{M}\rho_0^F}k^* - i\left(V^F - V^S\right)\frac{\pi}{\rho_0^F} = 0,\\
&V^S\omega + E^S U^{S2}k^* + D\frac{n_E\gamma}{\tau\mathcal{M}\rho^S}k^* + i\left(V^F - V^S\right)\frac{\pi}{\rho^S} = 0.
\end{aligned} \tag{5.5}$$

The above homogeneous set of equations yields non-trivial solutions if the determinant vanishes. This condition leads to the following dispersion relation

$$\left(i\frac{\pi\omega}{\rho_0^F}\frac{\rho_0^F}{\rho^S} - \omega^2 + U^{S2}k^{*2} + \frac{n_E^2\gamma^2 i\omega\tau}{(1+i\omega\tau)\tau\mathcal{M}\rho_0^F}\frac{\rho_0^F}{\rho^S}k^{*2}\right)\times\left(i\frac{\pi\omega}{\rho_0^F} - \omega^2 + U^{F2}k^{*2} + \right.$$
$$\left. + \frac{n_E^2\gamma^2 i\omega\tau}{(1+i\omega\tau)\tau\mathcal{M}\rho_0^F}k^{*2}\right) + \left(\frac{\pi\omega}{\rho_0^F} + \frac{n_E^2\gamma^2\omega\tau}{(1+i\omega\tau)\tau\mathcal{M}\rho_0^F}k^{*2}\right)^2\frac{\rho_0^F}{\rho^S} = 0. \tag{5.6}$$

Obviously, the lack of coupling between the components would give the biquadratic dispersion relation for k* which predicts two classical longitudinal waves (compare: section 4)

$$\gamma \equiv 0 \vee \pi \equiv 0 \quad \Rightarrow \quad \left(\omega^2 - U^{S2}k^{*2}\right)\left(\omega^2 - U^{F2}k^{*2}\right) = 0. \tag{5.7}$$

Consequently, one class of monochromatic waves would propagate with the phase velocity U^S independently of the frequency ω and the other class would propagate with the phase velocity U^F independently of the frequency ω. Both waves would not be attenuated (*Im* $k^* \equiv 0$).

In general, the dispersion relation (4.6) remains the biquadratic equation for k* but its solutions are complex and they depend on the frequency ω in the non-linear manner. Hence, the phase velocities are not identical for different frequencies and we have the attenuation of the monochromatic waves defined by the imaginary part α of k*. We proceed to investigate these solutions in relation to the material parameters of the model.

6. Parameter analysis of the dispersion relation

Let us introduce the following dimensionless quantities

$$\bar{k}^* \equiv U^S \tau_0 k^*, \qquad \bar{\omega} \equiv \omega\tau_0,$$
$$G \equiv \frac{n_E^2 \gamma^2}{\tau \mathscr{M} \rho_0^F U^{S2}}, \quad P \equiv \frac{\pi \tau_0}{\rho_0^F}, \quad \eta \equiv \frac{\tau}{\tau_0}, \tag{6.1}$$

where τ_0 is an arbitrary reference relaxation time. Then the dispersion relation (5.6) has the following form

$$\left(\bar{\omega}^2 - \bar{k}^{*2} - G \frac{\rho_0^F}{\rho^S} \frac{\bar{\omega}^2 \eta^2 + i \bar{\omega} \eta}{\bar{\omega}^2 \eta^2 + 1} \bar{k}^{*2} - P \frac{\rho_0^F}{\rho^S} i \bar{\omega} \right) \times$$
$$\times \left(\bar{\omega}^2 - \frac{U^{F2}}{U^{S2}} \bar{k}^{*2} - G \frac{\bar{\omega}^2 \eta^2 + i \bar{\omega} \eta}{\bar{\omega}^2 \eta^2 + 1} \bar{k}^{*2} - P\, i \bar{\omega} \right) - \tag{6.2}$$
$$- \left(P\, i \bar{\omega} + G \frac{\bar{\omega}^2 \eta^2 + i \bar{\omega} \eta}{\bar{\omega}^2 \eta^2 + 1} \bar{k}^{*2} \right)^2 \frac{\rho_0^F}{\rho^S} = 0.$$

Consequently the solutions of this equation depend only on the five combinations of material parameters

$$\bar{k}^* = \bar{k}^* \left(\bar{\omega}; G, P, \eta, \frac{U^F}{U^S}, \frac{\rho_0^F}{\rho^S} \right). \tag{6.3}$$

This is, certainly, the artefact following from the one-dimensional character of plane monochromatic waves. In spite of this property it is worthwhile to investigate this dependence, in particular for the parameters G, η and P. They contain the new material constants τ, $\tau \mathscr{M}$, γ of the present model. We shall investigate the phase velocity V_{ph} and the attenuation coefficient α defined by the relations

$$V_{ph} = \frac{\omega}{Re\, k^*} = \frac{\bar{\omega}}{Re\, \bar{k}^*} U^S, \quad \alpha = Im\, k^* = \left(U^S \tau_0 \right)^{-1} Im\, \bar{k}^*. \tag{6.4}$$

In the previous papers, (K.WILMANSKI [1995,2], [1996,2,4], [1997]) the following data have been attributed to the material constants

$$U^S = 3.1 \times 10^3\ \mathrm{m/s}, \quad U^F = 0.9 \times 10^3\ \mathrm{m/s},$$
$$\rho^S = 2.4 \times 10^3\ \mathrm{kg/m^3}, \quad \rho_0^F = o.23 \times 10^3\ \mathrm{kg/m^3},$$
$$n_E = 0.23, \quad \tau \mathscr{M} \rho_0^F = 7.347 \times 10^{-8}\ \mathrm{s^2/m^2}, \tag{6.5}$$
$$\pi = 2.602 \times 10^9\ \mathrm{kg/m^3 s}, \quad \tau = 3.7 \times 10^{-6}\ \mathrm{s}.$$

They correspond roughly to the data of measurements on the Massillon sandstone quoted by T.BOURBIE, O.COUSSY, B.ZINSZNER [1987]. However it should be born in mind that the measurements were not done in the conditions corresponding exactly to the problem we consider in this work. Therefore, the numerical results which we present below, have a rather qualitative character.

If we choose $\tau_0=10^{-6}$s then the parameters appearing in (6.3) have for this case the following values

$$G = 0.03555\gamma^2, \quad P = 11.31304, \quad \frac{U^F}{U^S} = 0.29032, \quad \frac{\rho_0^F}{\rho^S} = 0.09583, \quad \eta = 3.7. \qquad (6.6)$$

As we have already argued, the parameter γ has approximately the value 1.0. However, we shall vary it in much broader limits using it as the control parameter for the combination of material constants G. Consequently it should be understood as reflecting the changes of γ as well as the changes of $\mathcal{M}$. In order to see its influence on the results, we show in Figures 1-2 the attenuation coefficient α for different values of γ. Quite clearly, the changes of attenuation of P1-waves are rather small for the values of γ between 0 and 5.0, i.e. for those values of γ and $\mathcal{M}$ which do not differ considerably from one and from the data quoted in (6.5), respectively. On the other hand the attenuation of P2-waves is dependent on γ much stronger and, in this respect, the P2-waves can be used to determine the constant G for a particular multicomponent system.

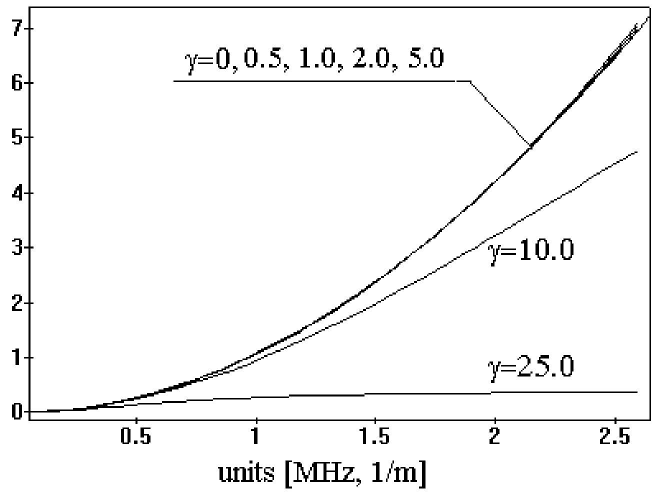

Figure 1: *Attenuation of* P1-*waves vs. frequency for different values of* γ

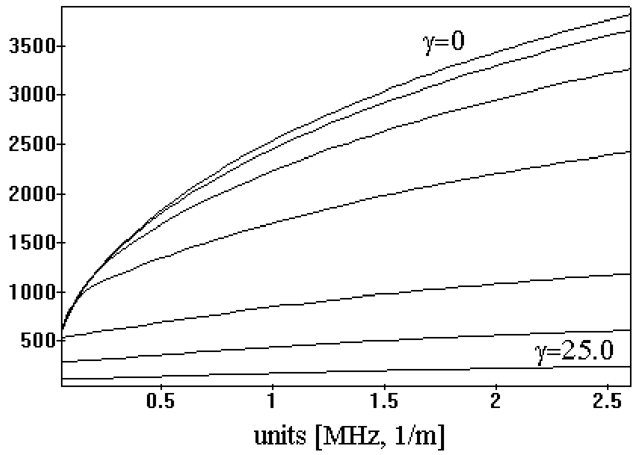

Figure 2: *Attenuation of* P2-*waves for the same values of* γ *as in Fig.1*

The similar conclusion can be drown from the diagram for the phase velocities shown in Figures 3 and 4. The phase velocity of P1-waves is not influenced by the changes of γ and, consequently by the changes of G at all. Simultaneously, the influence of G on the phase velocity of P2-waves is quite dramatic. The relative small values of γ yield rather small changes of this velocity (Figure 3) but the large values of γ provide unreasonable results shown in Figure 4. We can roughly say that the values G>0.889 (i.e. $\gamma=5 \Rightarrow G\sim0.03555\times5^2$) are, for other parameters given by (6.5), not acceptable any more.

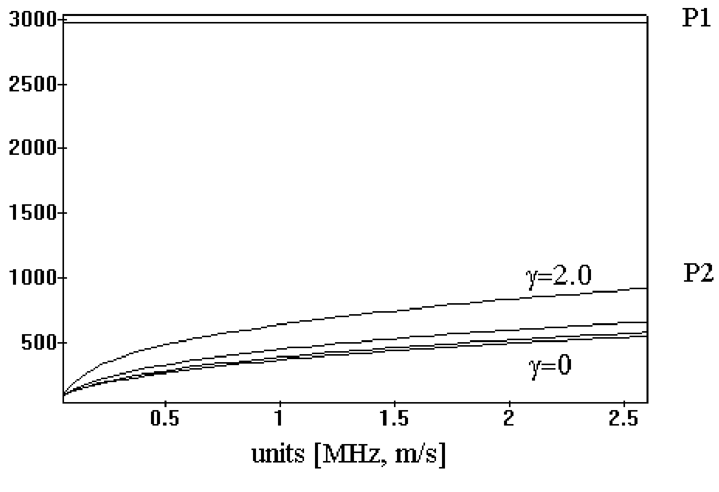

Figure 3: *Phase velocities vs. frequency for* γ=0, 0.5, 1.0, 2.0

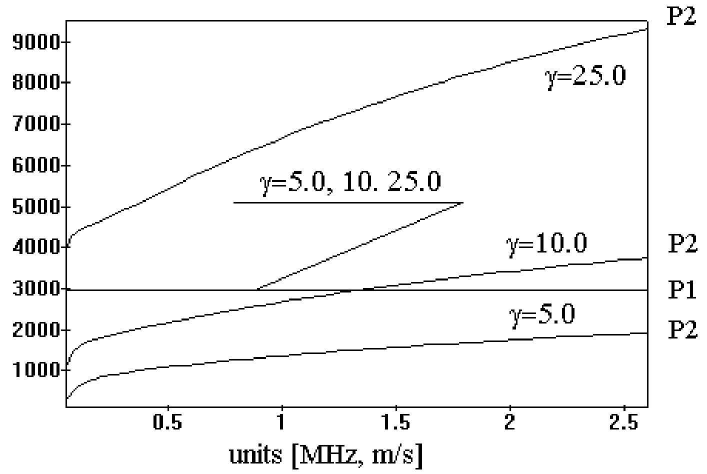

Figure 4: *Phase velocities vs. frequency for* γ=5.0, 10.0, 20.0

In Figure 5, we show in details the phase velocity of the P1-wave for γ=2.0. This example has been chosen to show that the phase velocity of P1-waves may possess a maximum as the function of frequencies. This property seems to be observed in some experiments on granular materials (J. KUBIK, *private communication*).

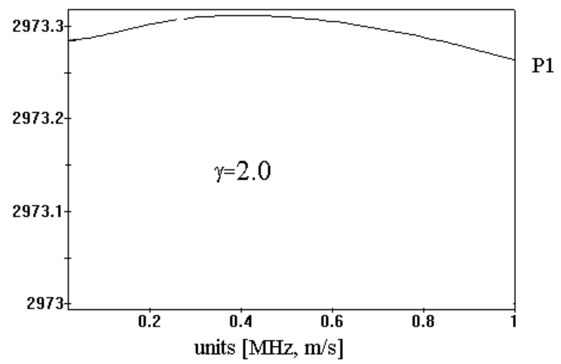

Figure 5: *Phase velocity vs. frequency for* P1-*wave for* γ=2.0

The attenuation, for this value of γ, is shown for P2-waves in Figure 6, and for P1-waves in Figure 7, respectively.

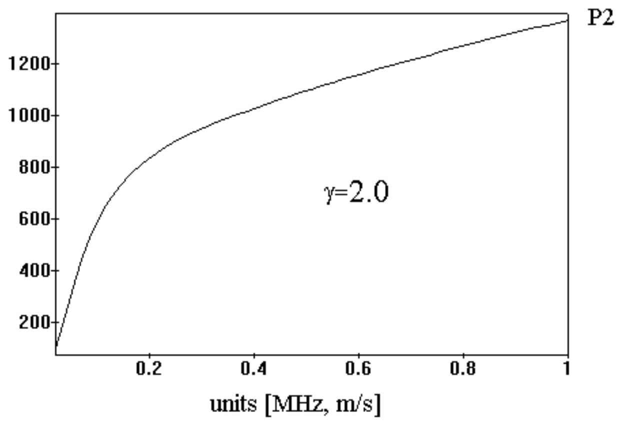

Figure 6: *Attenuation of* P2-*waves for* γ=2.0

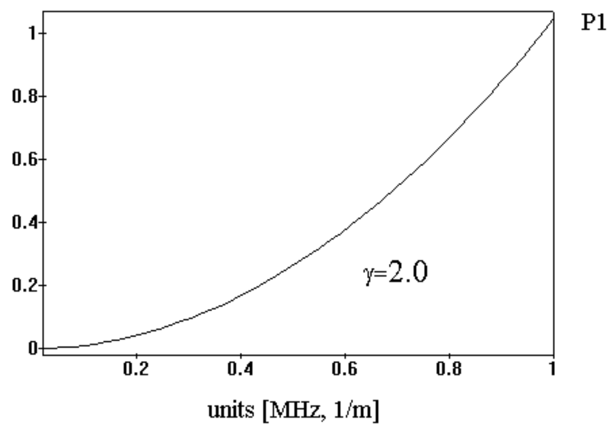

Figure 7: *Attenuation of* P1-*waves for* γ=2.0

We proceed to the presentation of the sensitivity of the model on the changes of the relaxation time τ. The attenuation of P1-waves is very weakly influenced by the change of the relaxation time. In the case of P2-waves a bigger sensitivity of the attenuation appears in the case of short relaxation times than for long times. This is seen in the Figure 8. For the above quoted data the changes of the relaxation time in the range above 10^{-2} sec. yield almost no changes of the attenuation any more. Consequently for the materials with the shorter relaxation times the measurements of the attenuation of P2-waves can be used to find the relaxation time of porosity for a given porous material.

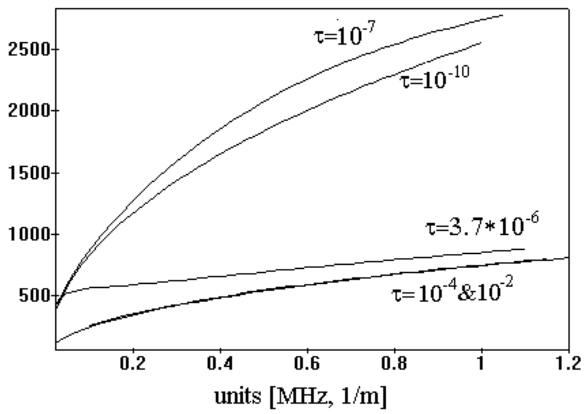

Figure 8: *Attenuation for* P2-*waves*, γ=5
and the relaxation times τ=: 10^{-10}, 10^{-7}, 3.7×10^{-6}, 10^{-4}, 10^{-2} *sec.*

We proceed to demonstrate one more property of the model connected with the sensitivity to the changes of the permeability π which is represented by the parameter P. In Figure 9, we show the attenuation for the P2-waves in the case of lack of permeability (i.e. π=0 which corresponds, for instance, to the case of empty or almost empty pores - weak interactions of components due to the relative motion). Comparison with Figure 6 shows that the attenuation of P2-waves is caused primarily by the relative motion of components. However the influence of the pore relaxation yields still a considerable and measurable effect which can be used to determine the relaxation time.

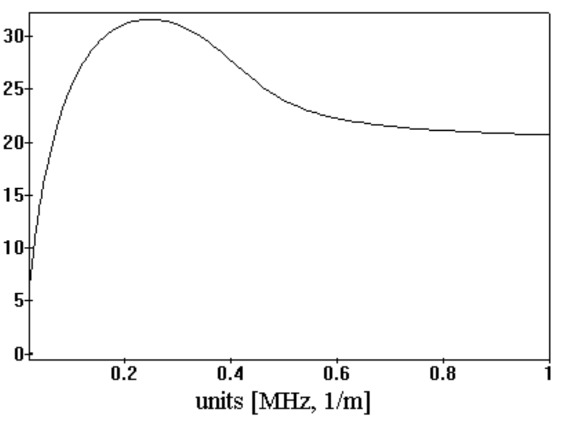

Figure 9: *Attenuation of* P2-*waves for* π=0, τ=3.7×10^{-6} *sec. and* γ=2

The above numerical example shows clearly that the P2-waves supply the most important data concerning the properties of microstructure of porous materials. These features of acoustic waves have been recognised by experimentalists to the very small extent as yet.

It should also be pointed out that the method of diagnosis by means of the sound waves, as indicated in this work, can play a particularly important role in the case of the static background of large deformations. This is, for instance, the case in the ultrasound devices used for the medical purposes. Such problems shall be considered in the forthcoming paper.

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