THE PROBABILITY APPROACH TO NUMERICAL SOLUTION OF NONLINEAR PARABOLIC EQUATIONS

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Abstract. A number of new layer methods for solving semilinear parabolic equations and reaction-diffusion systems is derived by using probabilistic representations of their solutions. These methods exploit the ideas of weak sense numerical integration of stochastic differential equations. In spite of the probabilistic nature these methods are nevertheless deterministic. A convergence theorem is proved. Some numerical tests are presented.

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1. **Introduction**

Parabolic type quasilinear differential equations are of great interest both in theoretical and applied aspects. Their investigation is presented in many publications in which (see, e.g., [6], [9], [18], [19], [22] and references therein) a deterministic approach is applicable. A few authors only make use of a probabilistic approach (see [5], [8], [21] and references therein). A similar state takes place in numerical analysis as well.

The aim of this paper is to develop layer approximation methods for solving the Cauchy problem for semilinear parabolic equations

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i\partial x^j}+\sum_{i=1}^{d}b^i(t,x,u)\frac{\partial u}{\partial x^i}+g(t,x,u)=0,\ t_0\le t<T,\ x\in\mathbf{R}^d, \tag{1.1}$$

$$u(T,x)=\varphi(x). \tag{1.2}$$

The form of equation (1.1) is relevant to a probabilistic approach, i.e., the equation is considered under $t<T$, and "initial" conditions are prescribed at $t=T$. Assume a solution of (1.1) should be found at the moment $t_0<T$. Consider a time discretization $T=t_N>t_{N-1}>\cdots>t_0.$ The proposed here methods give an approximation $\bar{u}(t_k,x)$ of the solution $u(t_k,x),\ k=N,...,0$. Using the well known probabilistic representation of the solution to (1.1)-(1.2) (see [4], [5]), we get

$$u(t_k,x)=\mathbf{E}(u(t_{k+1},X_{t_k,x}(t_{k+1}))+\int_{t_k}^{t_{k+1}}g(s,X_{t_k,x}(s),u(s,X_{t_k,x}(s)))ds). \tag{1.3}$$

In (1.3) $X_{t_k,x}(s)$ is the solution of the Cauchy problem for the Ito system of stochastic differential equations

$$dX=b(s,X,u(s,X))ds+\sigma(s,X,u(s,X))dw(s),\ X(t_k)=x, \tag{1.4}$$

where $w(s)=(w^1(s),...,w^d(s))^\top$ is a standard Wiener process, $b(t,x,u)=(b^1(t,x,u),...,b^d(t,x,u))^\top$ is the column vector, and the matrix $\sigma=\sigma(t,x,u)$ is obtained from the equation $\sigma\sigma^\top=a=\{a^{ij}(t,x,u)\}$.

Further we exploit the ideas of weak sense numerical integration of stochastic differential equations (see [7], [11]) and obtain some approximate relations from (1.3)-(1.4). The relations allow to express $\bar{u}(t_k,x)$ recurrently in terms of $\bar{u}(t_{k+1},x),\ k=N-1,...,0$, i.e., to construct some layer methods which are discrete in the variable t only. Despite the probabilistic nature these methods turn out nevertheless to be deterministic. However the probabilistic approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic way.

Therefore it can be expected that the proposed methods allow to avoid the difficulties stemming from essentially changing coefficients and strong advection.

In Section 2, a comparison of difference and probabilistic methods in the case of linear parabolic equations is given. In Section 3, we derive a few methods, relying on the numerical integration of ordinary stochastic differential equations, for nonlinear parabolic equations. In Section 4, we give a proof of a convergence theorem for one of the proposed methods using deterministic type arguments. The recurrent realization of any of the proposed layer methods makes use of the function $\bar{u}(t_{k+1}, x)$, in general, at all points x. Because it is possible to find the next layer $\bar{u}(t_k, x)$ numerically for a finite number of knots only, we need a discretization in the variable x with some kind of interpolation at every step to turn an applied method into an algorithm. Such numerical algorithms are constructed in Section 5. All main ideas can be demonstrated at $d = 1$ though that we restrict ourselves to this case in Sections 3 - 5. The case $d \geq 2$ is shortly discussed in Section 6. In addition we show in Section 7 how the results obtained can be extended for reaction-diffusion systems. Numerical tests are presented in the last section.

This article is devoted to initial value problems. Boundary value problems for nonlinear parabolic equations will be considered in a separate work. The probability approach to boundary value linear problems is treated in [12], [13].

2. The probabilistic approach to linear parabolic equations

Consider the Cauchy problem for linear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x)\frac{\partial u}{\partial x^i} + c(t,x)u + g(t,x) = 0,$$

$$t_0 \leq \ t < T, \ x \in \mathbf{R}^d, \tag{2.1}$$

with the initial condition

$$u(T,x) = \varphi(x). \tag{2.2}$$

The matrix $a(t,x) = \{a^{ij}(t,x)\}$ is supposed to be symmetric and positive semidefinite.

Let $\sigma(t,x)$ be a matrix obtained from the equation

$$a(t,x) = \sigma(t,x)\sigma^{\top}(t,x) \ .$$

This equation is solvable with respect to σ (for instance, by a lower triangular matrix) at least for a positively definite a.

The solution to the problem (2.1)-(2.2) has various probabilistic representations:

$$u(t,x) = \mathbf{E}(\varphi(X_{t,x}(T))Y_{t,x,1}(T) + Z_{t,x,1,0}(T)) \ , \ t \leq T, \ x \in \mathbf{R}^d, \tag{2.3}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, is the solution of the Cauchy problem to the system of stochastic differential equations

$$dX = b(s,X)ds - \sigma(s,X)h(s,X)ds + \sigma(s,X)dw(s), \ X(t) = x, \tag{2.4}$$

$$dY = c(s,X)Yds + h^{\top}(s,X)Ydw(s), \ Y(t) = y, \tag{2.5}$$

$$dZ = g(s,X)Yds, \ Z(t) = z. \tag{2.6}$$

Here $w(s) = (w^1(s), ..., w^d(s))^{\top}$ is a d-dimensional standard Wiener process, $b(s,x)$ is the column-vectors of dimension d compounded from the coefficients $b^i(s,x)$, $h(s,x)$

is a column-vector of dimension d , Y and Z are scalars. The usual representation (see [4]) can be seen in (2.3)–(2.6) if $h = 0$, the others rest on Girsanov's theorem.

In what follows it is supposed that all the coefficients in (2.1) and in (2.4)-(2.6) and the solution of the problem (2.1)-(2.2) (which is supposed to exist and to be unique) are sufficiently smooth and satisfy needed conditions of growth under big $|x|$, so that these conditions are sufficient for applying the theory of weak methods (see, e.g., [11]).

Let us consider the time discretization (equidistant for definiteness)

$$T = t_N > t_{N-1} > \cdots > t_0 = t \ , \ \frac{T - t_0}{N} = h \ .$$

Remember that weak approximation of the system (2.4)–(2.6) consists in construction of the system of stochastic difference equations

$$X_0 = x, \ X_{m+1} = X_m + A(t_m, X_m, h; \xi_m) \tag{2.7}$$

$$Y_0 = 1, \ Y_{m+1} = Y_m + \alpha(t_m, X_m, h; \xi_m) Y_m \tag{2.8}$$

$$Z_0 = 0, \ Z_{m+1} = Z_m + \beta(t_m, X_m, h; \xi_m) Y_m, \ m = 0, 1, ..., N-1, \tag{2.9}$$

where X_m is a vector of dimension d, Y_m and Z_m are scalars, ξ_m is a random vector of a certain dimension, A is a vector function of dimension d, α and β are scalar functions, ξ_m is independent of $X_0, ..., X_m$ and $\xi_0, ..., \xi_{m-1}$.

Let the system (2.7)-(2.9) be a weak scheme of order p for the system (2.4)-(2.6). It means that (see [7], [11])

$$\bar{u}(t_0, x) = \bar{u}(t, x) := E(\varphi(X_N) Y_N + Z_N) = u(t, x) + R_N \ , \tag{2.10}$$

where

$$|R_N| \leq K(1 + |x|^\kappa) h^p,$$

and $K > 0$, $\kappa \geq 0$ are some constants.

The well known numerical methods, including the finite difference ones (see, e.g., [15], [16], [17], [20], [23]), can be applied successfully provided the dimension d of the space variable x is comparatively small ($d \leq 3$) while for larger dimensions these numerical procedures become unrealistic due to huge volume of computations. Fortunately in many cases, functionals only, or even individual values of a solution, have to be found. For such problems, a probabilistic approach has an essential advantage as long as the problem under consideration can be reduced to solving the corresponding system of ordinary stochastic differential equations.

The probabilistic representation (2.3)–(2.6) and its approximation (2.10), (2.7)-(2.9) give an example of such an approach which allows to find the individual values $u(t, x)$ of the solution to problem (2.1)-(2.2) even in the essentially multi-dimensional ($d > 3$) cases. In addition, the value $\bar{u}(t, x)$ is evaluated by applying the Monte-Carlo technique:

$$\bar{u}(t, x) \cong \frac{1}{L} \sum_{l=1}^{L} (\varphi(X_N^{(l)}) Y_N^{(l)} + Z_N^{(l)}),$$

where $(X_N^{(l)}, Y_N^{(l)}, Z_N^{(l)})$, $l = 1, ..., L$, are independent realizations of the process defined by the system (2.7)-(2.9).

But it should be noted that the probabilistic approach is useful not only in this respect. Here we apply it to constructing some layer methods. To show this let us consider the Cauchy problem

$$X_k = x, \ X_{m+1} = X_m + A(t_m, X_m, h; \xi_m) \tag{2.11}$$

$$Y_k = y,\ Y_{m+1} = Y_m + \alpha(t_m, X_m, h; \xi_m) Y_m \tag{2.12}$$

$$Z_k = z,\ Z_{m+1} = Z_m + \beta(t_m, X_m, h; \xi_m) Y_m, \tag{2.13}$$

$$m = k, k+1, ..., N-1;\ 0 \le k \le N-1,$$

which is connected with the system (2.7)-(2.9).

Denote the solution of the problem by $\bar{X}_{t_k,x}(t_m)$, $\bar{Y}_{t_k,x,y}(t_m)$, $\bar{Z}_{t_k,x,y,z}(t_m)$, $t_m \ge t_k$. Introduce the function (remember $T = t_N$)

$$\bar{u}(t_k, x, y, z) = E(\varphi(\bar{X}_{t_k,x}(T))\bar{Y}_{t_k,x,y}(T) + \bar{Z}_{t_k,x,y,z}(T)) \ .$$

Clearly, the function $\bar{u}(t_k, x, y, z)$ has the form

$$\bar{u}(t_k, x, y, z) = \bar{u}(t_k, x) y + z,$$

where

$$\bar{u}(t_k, x) = E(\varphi(\bar{X}_{t_k,x}(T))\bar{Y}_{t_k,x,1}(T) + \bar{Z}_{t_k,x,1,0}(T)) \ .$$

Let $t = t_0 \le t_k < t_m \le T$. Since

$$\bar{X}_{t_k,x}(T) = \bar{X}_{t_m,\bar{X}_{t_k,x}(t_m)}(T)$$

$$\bar{Y}_{t_k,x,1}(T) = \bar{Y}_{t_m,\bar{X}_{t_k,x}(t_m),\bar{Y}_{t_k,x,1}(t_m)}(T)$$

$$\bar{Z}_{t_k,x,1,0}(T) = \bar{Z}_{t_m,\bar{X}_{t_k,x}(t_m),\bar{Y}_{t_k,x,1}(t_m),\bar{Z}_{t_k,x,1,0}(t_m)}(T) \ ,$$

we have

$$\bar{u}(t_k, x) = EE(\varphi(\bar{X}_{t_m,\bar{X}_{t_k,x}(t_m)}(T))\bar{Y}_{t_m,\bar{X}_{t_k,x}(t_m),\bar{Y}_{t_k,x,1}(t_m)}(T)$$
$$+\bar{Z}_{t_m,\bar{X}_{t_k,x}(t_m),\bar{Y}_{t_k,x,1}(t_m),\bar{Z}_{t_k,x,1,0}(t_m)}(T) / \bar{X}_{t_k,x}(t_m), \bar{Y}_{t_k,x,1}(t_m), \bar{Z}_{t_k,x,1,0}(t_m))$$
$$= E(\bar{u}(t_m, \bar{X}_{t_k,x}(t_m))\bar{Y}_{t_k,x,1}(t_m) + \bar{Z}_{t_k,x,1,0}(t_m)) \ , \ \bar{u}(t_N, x) = \varphi(x) \ . \tag{2.14}$$

Using (2.14) sequentially with $m = k+1$:

$$\bar{u}(t_k, x) = E(\bar{u}(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1}))\bar{Y}_{t_k,x,1}(t_{k+1}) + \bar{Z}_{t_k,x,1,0}(t_{k+1})) \ , \ k = N-1, ..., 0, \tag{2.15}$$

one can recurrently find the approximate solution $\bar{u}(t_{N-1}, x)$, $\bar{u}(t_{N-2}, x), ..., \bar{u}(t_0, x)$ of the problem (2.1)-(2.2) beginning from

$$\bar{u}(t_N, x) = \varphi(x). \tag{2.16}$$

This method becomes a deterministic one indeed if we are able to calculate the mathematical expectations explicitly (see, for instance, formulas (2.20) or (2.23) below). For numerical realization of (2.15) it is sufficient to calculate the functions $\bar{u}(t_k, x)$ in some knots x_i with applying some kind of interpolation at every layer.

It turns out that despite lack of probabilistic representations like (2.3)–(2.6) for solutions of nonlinear parabolic equations, such an approach as (2.15) can be adapted to nonlinear equations as well.

Further, it is more convenient to expound some additional ideas on simple examples. To this end let us consider the following one-dimensional ($d = 1$) problem

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} = 0,\ t < 0,\ -\infty < x < \infty,\ u(0, x) = \varphi(x). \tag{2.17}$$

Because $c = 0,\ g = 0$, we omit the equations for Y and Z. We have

$$dX = \sigma dw(s)\ ,\ X(t_0) = x,\ t_0 < 0. \tag{2.18}$$

Example 2.1. Consider the weak Euler scheme

$$X_{k+1} = X_k + \sigma\sqrt{h}\xi_k\ ,\ X_0 = x, \tag{2.19}$$

where $P(\xi_k = \pm 1) = \dfrac{1}{2}$.

If we set $m = k+1$ in (2.14), we obtain

$$\bar{u}(t_k, x) = E\bar{u}(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1}))$$

$$= \frac{1}{2}\bar{u}(t_{k+1}, x - \sigma\sqrt{h}) + \frac{1}{2}\bar{u}(t_{k+1}, x + \sigma\sqrt{h}),\ \bar{u}(t_N, x) = \varphi(x). \tag{2.20}$$

Here $t_N = 0,\ h = -t_0/N,\ t_k = -h(N-k) = t_{k+1} - h,\ k = N-1, ..., 0.$

The relation (2.20) is a linear difference equation. The equation (2.19) can be considered as a characteristic one for (2.20), and the formula

$$\bar{u}(t_k, x) = E\varphi(\bar{X}_{t_k,x}(t_N)) \tag{2.21}$$

gives the probabilistic representation of the solution to the equation (2.20).

It is well known that this solution is distinguished from the solution of the problem (2.17) by a quantity of the order $O(h)$.

It is easy to see that the layer determination of the values $\bar{u}(t_k, x_i)$ due to the formula (2.20) coincides with the simplest explicit difference method of solving (2.17) if we set $h_t = h,\ h_x = \sigma\sqrt{h}$, and consider the equidistant space discretization : $x_i = x_0 + i\sigma\sqrt{h},\ i = 0, \pm 1, \pm 2, \ldots$, x_0 is a point belonging to $\mathbf{R}^1$.

In Example 2.1, if we need the solution of (2.20) for all points (t_k, x_i), we can use (2.20) to find $\bar{u}(t_k, x_i)$ layerwise. But if we need it at a separate point (t_k, x), it is more convenient to use the formula (2.21). Of course, in the last case the Monte-Carlo error arises in addition.

Example 2.2. Now consider a more general scheme than (2.19):

$$X_{k+1} = X_k + \alpha\sqrt{h}\eta_k\ ,\ X_0 = x, \tag{2.22}$$

where the constant $\alpha \geq \sigma,\ P(\eta = \pm 1) = \dfrac{\sigma^2}{2\alpha^2},\ P(\eta = 0) = 1 - \dfrac{\sigma^2}{\alpha^2}.$

Instead of (2.20) we get

$$\bar{u}(t_k, x) = E\bar{u}(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1})) = (1 - \frac{\sigma^2}{\alpha^2})\bar{u}(t_{k+1}, x)$$

$$+\frac{\sigma^2}{2\alpha^2}\bar{u}(t_{k+1}, x - \alpha\sqrt{h}) + \frac{\sigma^2}{2\alpha^2}\bar{u}(t_{k+1}, x + \alpha\sqrt{h}),\ \bar{u}(t_N, x) = \varphi(x). \tag{2.23}$$

Again due to the theory of weak methods for stochastic differential equations the formula (2.21) with $\bar{X}$ from (2.22) gives the solution of the problem (2.17) to within $O(h)$. The formula (2.21) can be realized either by the Monte-Carlo method or layerwise in accord with (2.23). The layer realization (2.23) is deterministic and coincides (after a choice of the corresponding net) with the following difference method

$$\frac{\bar{u}(t_k, x_i) - \bar{u}(t_{k+1}, x_i)}{h_t} = \frac{\sigma^2}{2} \cdot \frac{\bar{u}(t_{k+1}, x_{i+1}) - 2\bar{u}(t_{k+1}, x_i) + \bar{u}(t_{k+1}, x_{i-1})}{h_x^2}\ ,$$

$$h_t = h,\ h_x = \alpha\sqrt{h}\ . \tag{2.24}$$

Due to the Lax-Richtmyer equivalence theorem, the method (2.24) (or, what is the same, the method (2.23)) converges with accuracy $O(h)$ if $\alpha \geq \sigma$. If $\alpha < \sigma$ the numerical approximation (2.24) is not stable from the point of view of the theory of difference methods, and the method (2.24) diverges. We underline that there does not exist any probabilistic scheme of the form (2.11)–(2.13), (2.15) corresponding to (2.24) under $\alpha < \sigma$, i.e., there is no such a bad probabilistic scheme. The convergence theorems for weak methods (in comparison with the theory of difference methods) do not include any conditions about stability of their approximations. The point is that X_{k+1} (and consequently the distribution of X_{k+1} which generalizes the step h_x) of a suitable weak scheme is in a reasonable way connected with the step h_t, with X_k, and with the coefficients of the problem. Thus, the methods having a probabilistic nature like (2.11)–(2.13), (2.15) are more adjusted (especially when the coefficients of the considered problem are nonconstant) because the suitable choice of h_x is achieved automatically.

Let us remark that the methods (2.20) and (2.23) do not need any interpolation because the layer $\bar{u}(t_k, x_i)$ makes use of the previous layer $\bar{u}(t_{k+1}, x)$ in the knots x_j only. But such a property of layer methods under consideration is rather exception than a rule.

In conclusion let us give another two examples.

Example 2.3. Consider the scheme (2.22) under $\alpha = \sigma\sqrt{3}$:

$$X_{k+1} = X_k + \sigma\sqrt{3h}\eta_k\ ,\ X_0 = x\ , \tag{2.25}$$

where $P(\eta = \pm 1) = \dfrac{1}{6}\ ,\ P(\eta = 0) = \dfrac{2}{3}\ .$

Because

$$E\eta = E\eta^3 = 0,\ E(\sqrt{3}\eta)^2 = 1,\ E(\sqrt{3}\eta)^4 = 3,$$

this scheme has the second order of accuracy.

From (2.25) we obtain the following difference method

$$\bar{u}(t_k, x_i) = \frac{1}{6}(\bar{u}(t_{k+1}, x_{i+1}) + \bar{u}(t_{k+1}, x_{i-1})) + \frac{2}{3}\bar{u}(t_{k+1}, x_i)\ , \tag{2.26}$$

where $x_{i+1} - x_i = \sigma\sqrt{3h}\ .$

Since the scheme (2.25) is of the second order, the method (2.26) is also of order 2, i.e., $|u(t_k, x_i) - \bar{u}(t_k, x_i)| = O(h^2)$. The method (2.26) is known as the difference method of excited accuracy.

Example 2.4. Consider one more scheme

$$X_{k+1} = X_k + \sigma\sqrt{h}\zeta_k\ ,\ X_0 = x\ , \tag{2.27}$$

where $P(\zeta = 0) = p,\ P(\zeta = \pm\alpha) = q,\ P(\zeta = \pm\beta) = r.$

If, for example, $\alpha = 1,\ \beta = \sqrt{6},\ p = \dfrac{1}{3}\ ,\ q = \dfrac{3}{10}\ ,\ r = \dfrac{1}{30}$, then

$$E\zeta = E\zeta^3 = E\zeta^5 = 0,\ E\zeta^2 = 1,\ E\zeta^4 = 3,\ E\zeta^6 = 15,$$

and the scheme is of order 3. The corresponding method

$$\bar{u}(t_k, x) = E\bar{u}(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1})) = E\bar{u}(t_{k+1}, x + \sigma\sqrt{h}\zeta_k)$$

$$= \frac{1}{30}\bar{u}(t_{k+1}, x - \sigma\sqrt{6h}) + \frac{3}{10}\bar{u}(t_{k+1}, x - \sigma\sqrt{h}) + \frac{1}{3}\bar{u}(t_{k+1}, x)$$

$$+\frac{3}{10}\bar{u}(t_{k+1}, x+\sigma\sqrt{h}) + \frac{1}{30}\bar{u}(t_{k+1}, x+\sigma\sqrt{6h}) \tag{2.28}$$

is of order 3 too. But an interpolation is necessary for numerical realization of (2.28) in some net of knots x_i, because of incommensurability of $\sigma\sqrt{h}$ and $(\sigma\sqrt{6h}-\sigma\sqrt{h})$.

Let us indicate in passing that, for example, the scheme

$$X_{k+1} = X_k + \sigma\sqrt{h}\nu_k \ , \ X_0 = x \ ,$$

where $P(\nu=0) = \dfrac{7}{18}$, $P(\nu=\pm 1) = \dfrac{1}{4}$, $P(\nu=\pm 2) = \dfrac{1}{20}$, $P(\nu=\pm 3) = \dfrac{1}{180}$, also induces a method of order 3. Evidently, this method has the form

$$\bar{u}(t_k, x_i) = \frac{1}{180}\bar{u}(t_{k+1}, x_{i-3}) + \frac{1}{20}\bar{u}(t_{k+1}, x_{i-2}) + \frac{1}{4}\bar{u}(t_{k+1}, x_{i-1})$$

$$+\frac{7}{18}\bar{u}(t_{k+1}, x_i) + \frac{1}{4}\bar{u}(t_{k+1}, x_{i+1}) + \frac{1}{20}\bar{u}(t_{k+1}, x_{i+2}) + \frac{1}{180}\bar{u}(t_{k+1}, x_{i+3}) \ , \tag{2.29}$$

where $x_{i+1} - x_i = \sigma\sqrt{h}$.

Remark 2.1. Consider the Cauchy problem for an autonomous linear parabolic equation in the usual form (with the positive direction of time θ)

$$\frac{\partial v}{\partial \theta} = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x)\frac{\partial v}{\partial x^i} + c(x)v + g(x), \ \theta > 0, \ x \in \mathbf{R}^d, \tag{2.30}$$

$$v(0, x) = \varphi(x). \tag{2.31}$$

Changing the variables $t = \theta, \ u(t,x) = v(-t,x)$, we get the Cauchy problem of the form (2.1)-(2.2) for the function $u(t,x)$ where $t < 0, \ x \in \mathbf{R}^d, \ T = 0, \ u(T,x) = \varphi(x)$. The system (2.4)-(2.6) in the considered case is autonomous as well (we suppose the function $h(s,x)$ in (2.4)-(2.6) to be independent of s). Therefore (see (2.3))

$$v(\theta, x) = u(-\theta, x) = \mathbf{E}(\varphi(X_{-\theta,x}(0))Y_{-\theta,x,1}(0) + Z_{-\theta,x,1,0}(0))$$

$$= \mathbf{E}(\varphi(X_{0,x}(\theta))Y_{0,x,1}(\theta) + Z_{0,x,1,0}(\theta)) \ , \ \theta > 0, \ x \in \mathbf{R}^d,$$

i.e., we can consider the positive direction of time for both the parabolic equation and its characteristic system of stochastic differential equations. Accordingly to this fact we can write the following more convenient procedure in place of (2.15), (2.16):

$$\bar{v}(0, x) = \varphi(x),$$

$$\bar{v}(\theta_{k+1}, x) = E(\bar{v}(\theta_k, \bar{X}_{0,x}(h))\bar{Y}_{0,x,1}(h) + \bar{Z}_{0,x,1,0}(h)) \ , \ k = 0, ..., N-1, \tag{2.32}$$

where $0 = \theta_0 < \theta_1 < ... < \theta_N = \theta$; $h = \theta/N$ (of course, we consider A, α, and β in the scheme (2.11)-(2.13) to be independent of t_m). At the same time we preferred to remain the general style of our exposition in Examples 2.1 - 2.4.

3. **Constructing some methods for semilinear parabolic equations**

For simplicity in writing we restrict ourselves to the case $d=1$ in this and the next two sections.

Let us consider the Cauchy problem

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 u}{\partial x^2}+b(t,x,u)\frac{\partial u}{\partial x}+g(t,x,u)=0,\ t_0\leq\ t<T,\ x\in\mathbf{R}^1, \tag{3.1}$$

$$u(T,x)=\varphi(x)\ . \tag{3.2}$$

Let $u=u(t,x)$ be the solution of the problem (3.1)-(3.2), which is supposed to exist, to be unique, to be sufficiently smooth, and to satisfy needed conditions of boundedness. One can find many theoretical results on this topic in [6], [9], [18], [19], [22] (see also references therein). If we substitute $u=u(t,x)$ in the coefficients $\sigma^2,\ b,\ g$, we obtain a linear parabolic equation. We suppose that all the requirements mentioned above in connection with the equation (2.1) are fulfilled for the obtained linear equation as well. Let us note that in comparison with (2.1) this linear equation does not contain the linear term with u. It is so because of the general form of g in (3.1). Sometimes may occur that it is more preferable to represent $g(t,x,u)$ as $g(t,x,u)=c(t,x)u+g_0(t,x,u)$ (for instance, in the case of small $g_0(t,x,u)$) and to substitute $u=u(t,x)$ in the function g_0 only. Clearly, in that case we obtain another kind of linear equation and another kind of probabilistic representation. For definiteness, we shall consider the case without linear term of u and we take $c(s,x)\equiv 0$ and $h(s,x)\equiv 0$ in the equation (2.1) and in the system (2.4)-(2.6).

We have (see (2.3) under $Y\equiv 1$)

$$u(t,x)=\mathbf{E}(\varphi(X_{t,x}(T))+\int_t^T g(s,X_{t,x}(s),u(s,X_{t,x}(s)))ds)\ ,\ t\leq T,\ x\in\mathbf{R}^1, \tag{3.3}$$

where $X_{t,x}(s)$ is the solution of the Cauchy problem for the following equation

$$dX=b(s,X,u(s,X))ds+\sigma(s,X,u(s,X))dw(s),\ X(t)=x.$$

Consider the equidistant time discretization

$$T=t_N>t_{N-1}>\cdots>t_0=t\ ,\ \frac{T-t_0}{N}=h\ .$$

Due to (3.3) we have

$$u(t_k,x)=\mathbf{E}(u(t_{k+1},X_{t_k,x}(t_{k+1}))+\int_{t_k}^{t_{k+1}} g(s,X_{t_k,x}(s),u(s,X_{t_k,x}(s)))ds)$$

$$=\mathbf{E}(u(t_{k+1},X_{t_k,x}(t_{k+1}))+Z_{t_k,x,0}(t_{k+1}))\ , \tag{3.4}$$

where $X,\ Z$ satisfy the following system

$$dX=b(s,X,u(s,X))ds+\sigma(s,X,u(s,X))dw(s),\ X(t_k)=x, \tag{3.5}$$

$$dZ=g(s,X,u(s,X))ds,\ Z(t_k)=0. \tag{3.6}$$

Applying the explicit weak Euler scheme with the simplest simulation of noise to the system (3.5)-(3.6), we get

$$X_{t_k,x}(t_{k+1})\simeq\bar{X}_{t_k,x}(t_{k+1})=x+b(t_k,x,u(t_k,x))h+\sigma(t_k,x,u(t_k,x))\sqrt{h}\xi_k\ , \tag{3.7}$$

$$Z_{t_k,x,0}(t_{k+1}) \simeq \bar{Z}_{t_k,x,0}(t_{k+1}) = g(t_k, x, u(t_k, x))h \ , \tag{3.8}$$

where $\xi_{N-1}, \ \xi_{N-2}, ..., \ \xi_0$ are i.i.d. random variables which are distributed by the law: $P(\xi = \pm 1) = \dfrac{1}{2}$.

Using (3.4), we obtain

$$u(t_k, x) \simeq \mathbf{E}(u(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1})) + \bar{Z}_{t_k,x,0}(t_{k+1}))$$

$$= \frac{1}{2}u(t_{k+1}, \ x + b(t_k, x, u(t_k, x))h + \sigma(t_k, x, u(t_k, x))\sqrt{h})$$

$$+\frac{1}{2}u(t_{k+1}, \ x + b(t_k, x, u(t_k, x))h - \sigma(t_k, x, u(t_k, x))\sqrt{h}) + g(t_k, x, u(t_k, x))h. \tag{3.9}$$

Following (3.9) one can write for the approximations $\bar{u}(t_k, x)$:

$$\bar{u}(t_N, x) = \varphi(x), \ \bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, \ x + b(t_k, x, \bar{u}(t_k, x))h + \sigma(t_k, x, \bar{u}(t_k, x))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, \ x + b(t_k, x, \bar{u}(t_k, x))h - \sigma(t_k, x, \bar{u}(t_k, x))\sqrt{h})$$

$$+g(t_k, x, \bar{u}(t_k, x))h, \ k = N-1, ..., 1, 0. \tag{3.10}$$

The method (3.10) is an implicit layer method for solution of the Cauchy problem (3.1)-(3.2). This method is a deterministic one though the probabilistic approach is used for its constructing. Remember, it rests on the explicit Euler scheme.

Now let us use the following implicit scheme instead of (3.7)-(3.8):

$$\bar{X}_{t_k,x}(t_{k+1}) := \bar{X}_{k+1} = x + b(t_{k+1}, \bar{X}_{k+1}, u(t_{k+1}, \bar{X}_{k+1}))h$$

$$+\sigma(t_{k+1}, \bar{X}_k, u(t_{k+1}, \bar{X}_k))\sqrt{h}\xi_k \ , \tag{3.11}$$

$$\bar{Z}_{t_k,x,0}(t_{k+1}) := \bar{Z}_{k+1} = g(t_{k+1}, \bar{X}_{k+1}, u(t_{k+1}, \bar{X}_{k+1}))h \ , \tag{3.12}$$

where $\xi_{N-1}, \ \xi_{N-2}, ..., \ \xi_0$ are the same as in (3.7).

Let $\bar{X}_{k+1} = \bar{X}_{k+1}(\xi_k)$ be the solution of (3.11) (remember that the function $u(t_{k+1}, x)$ is considered to be known). The variable ξ_k gets two different values. Denote by $\bar{X}^1_{k+1}, \ \bar{X}^2_{k+1}$ the corresponding values of $\bar{X}_{k+1}$. Accept the analogous notation for two values of $\bar{Z}_{k+1}$. As a result we obtain the following method

$$\bar{u}(t_N, x) = \varphi(x), \ \bar{u}(t_k, x) = \frac{1}{2}(\bar{u}(t_{k+1}, \bar{X}^1_{k+1}) + \bar{Z}^1_{k+1}) + \frac{1}{2}(\bar{u}(t_{k+1}, \bar{X}^2_{k+1}) + \bar{Z}^2_{k+1}). \tag{3.13}$$

It is a deterministic one just as the method (3.10).

The formula (3.13) is explicit but to find $\bar{X}_{k+1}$ we have to use the implicit scheme (3.11). Therefore both the method (3.10) and the method (3.13) are implicit.

To search for $\bar{u}(t_k, x)$ from (3.10), one can apply the method of simple iteration. If we take $\bar{u}(t_{k+1}, x)$ as a null iteration, we get the following first iteration (we denote this iteration as $\bar{u}(t_k, x)$ again)

$$\bar{u}(t_N, x) = \varphi(x),$$

$$\bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, \ x + b(t_k, x, \bar{u}(t_{k+1}, x))h + \sigma(t_k, x, \bar{u}(t_{k+1}, x))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1},\ x+b(t_k,x,\bar{u}(t_{k+1},x))h-\sigma(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+g(t_k,x,\bar{u}(t_{k+1},x))h,\ k=N-1,...,1,0. \tag{3.14}$$

The formula (3.14) gives an explicit method for recurrent layer solving of the problem (3.1)-(3.2). Let us note that if we apply another approximate method for solving (3.10) (for example, taking the second iteration), we obtain some other explicit method which can be possessed of better properties than (3.14) (just as under numerical integration of ordinary differential equations).

Analogously applying the method of simple iteration to (3.11) with x as a null iteration and substituting the obtained first iteration in (3.12) and (3.13), we obtain the following explicit method which differs from (3.14) in a small way:

$$\bar{u}(t_N,x)=\varphi(x),$$

$$\bar{u}(t_k,x)=\frac{1}{2}\bar{u}(t_{k+1},\ x+b(t_{k+1},x,\bar{u}(t_{k+1},x))h+\sigma(t_{k+1},x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1},\ x+b(t_{k+1},x,\bar{u}(t_{k+1},x))h-\sigma(t_{k+1},x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+g(t_{k+1},x,\bar{u}(t_{k+1},x))h,\ k=N-1,...,1,0. \tag{3.15}$$

Consider the action of a higher order method of numerical integration of stochastic differential equations on an example of the equation (3.1) with the constant coefficient σ. Let us apply the second order (in the weak sense) Runge-Kutta scheme [11] to the system (3.5)-(3.6) with constant σ. We get (instead of (3.7)-(3.8))

$$X_{t_k,x}(t_{k+1})\simeq\bar{X}_{t_k,x}(t_{k+1})=x+\sigma\sqrt{h}\xi_k+\frac{1}{2}b(t_k,x,u(t_k,x))h$$

$$+\frac{1}{2}b(t_{k+1},x+b(t_k,x,u(t_k,x))h+\sigma\sqrt{h}\xi_k,u(t_{k+1},x+b(t_k,x,u(t_k,x))h+\sigma\sqrt{h}\xi_k))h, \tag{3.16}$$

$$Z_{t_k,x,0}(t_{k+1})\simeq\bar{Z}_{t_k,x,0}(t_{k+1})=\frac{1}{2}g(t_k,x,u(t_k,x))h$$

$$+\frac{1}{2}g(t_{k+1},x+b(t_k,x,u(t_k,x))h+\sigma\sqrt{h}\xi_k,u(t_{k+1},x+b(t_k,x,u(t_k,x))h+\sigma\sqrt{h}\xi_k))h, \tag{3.17}$$

where $\xi_{N-1},\ \xi_{N-2},...,\ \xi_0$ are i.i.d. random variables distributed by the law: $P(\xi=0)=\frac{2}{3},\ \ P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=\frac{1}{6}$.

Now instead of (3.10) we obtain the following implicit layer method

$$\bar{u}(t_N,x)=\varphi(x),\ \bar{u}(t_k,x)=\frac{2}{3}\bar{u}(t_{k+1},\ x+\frac{1}{2}\bar{b}h+\frac{1}{2}b(t_{k+1},x+\bar{b}h,\bar{u}(t_{k+1},x+\bar{b}h))h)$$

$$+\frac{1}{6}\bar{u}(t_{k+1},\ x+\sigma\sqrt{3h}+\frac{1}{2}\bar{b}h+\frac{1}{2}b(t_{k+1},x+\sigma\sqrt{3h}+\bar{b}h,\bar{u}(t_{k+1},x+\sigma\sqrt{3h}+\bar{b}h))h)$$

$$+\frac{1}{6}\bar{u}(t_{k+1},\ x-\sigma\sqrt{3h}+\frac{1}{2}\bar{b}h+\frac{1}{2}b(t_{k+1},x-\sigma\sqrt{3h}+\bar{b}h,\bar{u}(t_{k+1},x-\sigma\sqrt{3h}+\bar{b}h))h)$$

$$+\frac{1}{2}g(t_k,x,u(t_k,x))h+\frac{1}{3}g(t_{k+1},x+\bar{b}h,\bar{u}(t_{k+1},x+\bar{b}h))h$$

$$+\frac{1}{12}g(t_{k+1},x+\sigma\sqrt{3h}+\bar{b}h,\bar{u}(t_{k+1},x+\sigma\sqrt{3h}+\bar{b}h))h$$

$$+\frac{1}{12}g(t_{k+1},x-\sigma\sqrt{3h}+\bar{b}h,\bar{u}(t_{k+1},x-\sigma\sqrt{3h}+\bar{b}h))h, \qquad (3.18)$$

where $\bar{b}=b(t_k,x,\bar{u}(t_k,x))$.

This method has the one-step error of the third order. If we take $\bar{u}(t_{k+1},x)$ as a null iteration, we obtain the first iteration differing from the solution of (3.18) by a quantity of the order $O(h^2)$, and only beginning from the second iteration we attain the needed exactness. So, the implicit method (3.18) becomes explicit of the same order after two simple iterations.

Clearly, resting on the ideas led to the obtained methods, one can construct a lot of new methods using some other probabilistic representations or some other methods of numerical integration of stochastic differential equations. Of course, the development of suitable recommendations for applying any such a method requires both a theoretical studying and computational testing. Here we confine ourselves to problems of convergence of the method (3.14) and to construction of some numerical algorithms on its basis.

Remark 3.1. There are special methods of numerical integration in the weak sense for stochastic differential equations with small noise which are more effective than general ones [14]. They can be adapted for constructing new methods within the scope of our approach in the case of small diffusion σ. Nonlinear parabolic equations with small parameter at higher derivatives are of great significance both in mathematical physics and in numerical mathematics. Some special layer methods for such equations will be considered in a separate work.

4. **Convergence theorem**

We continue to treat the problem (problem (3.1)-(3.2))

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 u}{\partial x^2}+b(t,x,u)\frac{\partial u}{\partial x}+g(t,x,u)=0,\ t_0\le\ t<T,\ x\in\mathbf{R}^1, \qquad (4.1)$$

$$u(T,x)=\varphi(x). \qquad (4.2)$$

We shall keep the following assumptions (remind that for simplicity in writing the case $d=1$ is taken).

(i) The coefficients $b(t,x,u),\ \sigma(t,x,u),\ g(t,x,u)$ are uniformly bounded:

$$|b|\le K,\ |\sigma|\le K,\ |g|\le K,\ t_0\le t\le T,\ x\in\mathbf{R}^1,\ u_\circ<u<u^\circ, \qquad (4.3)$$

where $-\infty\le\ u_\circ<u^\circ\le\infty$ are some constants.

(ii) The coefficients $b(t,x,u),\ \sigma(t,x,u),\ g(t,x,u)$ uniformly satisfy the Lipschitz condition with respect to x and u :

$$|b(t,x_2,u_2)-b(t,x_1,u_1)|+|\sigma(t,x_2,u_2)-\sigma(t,x_1,u_1)|+|g(t,x_2,u_2)-g(t,x_1,u_1)|$$

$$\le K(|x_2-x_1|+|u_2-u_1|),\ t_0\le t\le T,\ x_1,x_2\in\mathbf{R}^1,\ u_\circ<u_1,u_2<u^\circ. \qquad (4.4)$$

(iii) There exists the only bounded solution $u(t,x)$ of the problem (4.1)-(4.2) such that

$$u_{\circ} < u_{*} \leq u(t,x) \leq u^{*} < u^{\circ},\ t_0 \leq t \leq T,\ x \in \mathbf{R}^1, \tag{4.5}$$

and there exist the uniformly bounded derivatives:

$$|\frac{\partial^m u}{\partial t^i \partial x^l}| \leq K,\ i=0,\ l=1,2,3,4;\ i=1,\ l=0,1,2;\ i=2,\ l=0;\ t_0 \leq t \leq T,\ x \in \mathbf{R}^1. \tag{4.6}$$

Let us note that the various constants which depend only on the problem (4.1)-(4.2) and do not depend on $t,\ x$, and so on have been given by the same letter K (or C) without any index. In connection with this, instead of, e.g., $K+C,\ 2C,\ K^2$, etc., we write K (or C).

First of all let us evaluate the one-step error of the method (method (3.14))

$$\bar{u}(t_N,x) = \varphi(x),$$

$$\bar{u}(t_k,x) = \frac{1}{2}\bar{u}(t_{k+1},\ x + b(t_k,x,\bar{u}(t_{k+1},x))h + \sigma(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1},\ x + b(t_k,x,\bar{u}(t_{k+1},x))h - \sigma(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+g(t_k,x,\bar{u}(t_{k+1},x))h,\ k = N-1,...,1,0. \tag{4.7}$$

This error on the k-th layer (on the $(N-k)$-th step) is evidently equal to $v(t_k,x) - u(t_k,x)$, where

$$v(t_k,x) = \frac{1}{2}u(t_{k+1},\ x + b(t_k,x,u(t_{k+1},x))h + \sigma(t_k,x,u(t_{k+1},x))\sqrt{h})$$

$$+\frac{1}{2}u(t_{k+1},\ x + b(t_k,x,u(t_{k+1},x))h - \sigma(t_k,x,u(t_k,x))\sqrt{h}) + g(t_k,x,u(t_{k+1},x))h. \tag{4.8}$$

Lemma 4.1. *Under the assumptions $(i)-(iii)$ the one-step error of the method (4.7) has the second order of smallness with respect to h :*

$$|v(t_k,x) - u(t_k,x)| \leq Ch^2, \tag{4.9}$$

where C does not depend on $x,\ h,\ k$.

Proof. Expanding the functions $u(t_k+h,\ x+bh \pm \sigma\sqrt{h})$ at (t_k,x) in powers of h and $bh \pm \sigma\sqrt{h}$ and using the assumptions of boundedness (4.3) and (4.6), we get

$$v(t_k,x) = u(t_k,x) + \frac{\partial u}{\partial t}(t_k,x)h + \frac{\partial u}{\partial x}(t_k,x)bh + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t_k,x)\sigma^2 h + gh + O(h^2). \tag{4.10}$$

In (4.10) $b,\ \sigma^2,\ g$ have $t_k,\ x,\ u(t_{k+1},x)$ as their arguments, and

$$|O(h^2)| \leq Ch^2, \tag{4.11}$$

where C does not depend on $x,\ h,\ k$.

Now applying the Lipschitz condition (4.4) with respect to the variable u, it is not difficult to obtain

$$v(t_k,x) = u(t_k,x) + \frac{\partial u}{\partial t}(t_k,x)h + \frac{\partial u}{\partial x}(t_k,x)b(t_k,x,u(t_k,x))h$$

$$+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t_k,x)\sigma^2(t_k,x,u(t_k,x))h + g(t_k,x,u(t_k,x))h + O(h^2), \tag{4.12}$$

where $O(h^2)$ satisfies the relation (4.11) again.

Because $u(t,x)$ is a solution of the equation (4.1), the inequality (4.9) runs out from (4.12). Lemma 4.1 is proved.

Theorem 4.1. *Under the assumptions (i) − (iii) the method* (4.7) *has the first order, i.e.,*

$$|\bar{u}(t_k,x) - u(t_k,x)| \le Kh, \tag{4.13}$$

where K does not depend on x, h, k.

Proof. Denote the error of the method (4.7) on the k-th layer as $\varepsilon(t_k,x) := \bar{u}(t_k,x) - u(t_k,x)$. Thus, we have

$$\bar{u}(t_k,x) = u(t_k,x) + \varepsilon(t_k,x),\ \ \bar{u}(t_{k+1},x) = u(t_{k+1},x) + \varepsilon(t_{k+1},x). \tag{4.14}$$

By (4.7) and (4.14) we get

$$u(t_k,x) + \varepsilon(t_k,x) = \bar{u}(t_k,x) = \frac{1}{2}\bar{u}(t_{k+1},\ x+\bar{b}h+\bar{\sigma}\sqrt{h}) + \frac{1}{2}\bar{u}(t_{k+1},\ x+\bar{b}h-\bar{\sigma}\sqrt{h}) + \bar{g}h$$

$$= \frac{1}{2}u(t_{k+1},\ x+\bar{b}h+\bar{\sigma}\sqrt{h}) + \frac{1}{2}u(t_{k+1},\ x+\bar{b}h-\bar{\sigma}\sqrt{h}) + \bar{g}h$$

$$+\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h+\bar{\sigma}\sqrt{h}) + \frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h-\bar{\sigma}\sqrt{h}), \tag{4.15}$$

where $\bar{b}$, $\bar{\sigma}$, $\bar{g}$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ calculated at $t = t_k$, $x = x$, $u = \bar{u}(t_{k+1},x) = u(t_{k+1},x) + \varepsilon(t_{k+1},x)$. For example, $\bar{b} = b(t_k,x,u(t_{k+1},x) + \varepsilon(t_{k+1},x))$.

Here we have to suggest for a while that the value $u(t_{k+1},x) + \varepsilon(t_{k+1},x)$ remains in the interval $(u_\circ, u^\circ)$ (see the conditions (4.3) and (4.4)). Clearly, $\varepsilon(t_N,x) = 0$, and below we prove recurrently that $\varepsilon(t_k,x)$ is sufficiently small under a sufficiently small h. Thereupon thanks to (4.5) this suggestion will be justified.

We have

$$\bar{b} = b(t_k,x,u(t_{k+1},x) + \varepsilon(t_{k+1},x)) = b(t_k,x,u(t_{k+1},x)) + \Delta b = b + \Delta b,$$

where $b := b(t_k,x,u(t_{k+1},x))$ and Δb satisfies the inequality (thanks to (4.4))

$$|\Delta b| \le K|\varepsilon(t_{k+1},x)|. \tag{4.16}$$

Analogously

$$\bar{\sigma} = \sigma + \Delta\sigma,\ |\Delta\sigma| \le K|\varepsilon(t_{k+1},x)|,\ \bar{g} = g + \Delta g,\ |\Delta g| \le K|\varepsilon(t_{k+1},x)|. \tag{4.17}$$

From (4.16), (4.17) it is not difficult to obtain the following equalities

$$u(t_{k+1},\ x+\bar{b}h \pm \bar{\sigma}\sqrt{h}) = u(t_{k+1},\ x+bh \pm \sigma\sqrt{h})$$

$$+\frac{\partial u}{\partial x}(t_{k+1},\ x+bh)\cdot(\Delta b h \pm \Delta\sigma\sqrt{h}) + \Delta_\pm \cdot h\ , \tag{4.18}$$

where $\Delta_\pm$ satisfies the inequality of the type (4.16).

Substituting this in (4.15), we get

$$u(t_k,x)+\varepsilon(t_k,x)=\frac{1}{2}u(t_{k+1},\ x+bh+\sigma\sqrt{h})+\frac{1}{2}u(t_{k+1},\ x+bh-\sigma\sqrt{h})+gh$$

$$+\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h+\bar{\sigma}\sqrt{h})+\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h-\bar{\sigma}\sqrt{h})+r_k$$

$$=v(t_k,x)+\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h+\bar{\sigma}\sqrt{h})+\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h-\bar{\sigma}\sqrt{h})+r_k\ , \tag{4.19}$$

where

$$|r_k|\le K|\varepsilon(t_{k+1},x)|\cdot h\ . \tag{4.20}$$

Finally, using Lemma 4.1, we arrive at

$$\varepsilon(t_k,x)=\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h+\bar{\sigma}\sqrt{h})+\frac{1}{2}\varepsilon(t_{k+1},\ x+\bar{b}h-\bar{\sigma}\sqrt{h})+r_k+O(h^2). \tag{4.21}$$

Now introduce

$$\varepsilon_k:=\max_{-\infty<x<\infty}|\varepsilon(t_k,x)|\ . \tag{4.22}$$

From (4.20) and (4.21) we obtain (in addition remember that $\varepsilon(t_N,x)=0$)

$$\varepsilon_N=0,\ \varepsilon_k\le\varepsilon_{k+1}+K\varepsilon_{k+1}h+Ch^2,\ k=N-1,...,1,0. \tag{4.23}$$

From here

$$\varepsilon_k\le\frac{C}{K}(e^{K(T-t_0)}-1)\cdot h,\ k=N,...,0.$$

Theorem 4.1 is proved.

Remark 4.1. The result (4.13) for the method (4.7) can be justified under some other conditions as well. For instance, it is possible to allow a linear growth of the coefficients $b,\sigma,\ g$ under $|x|\to\infty$ instead of the condition (i), if at the same time to assume, that the derivatives of the solution $u(t,x)$ from (4.6) are not only bounded but some of them go to zero in a corresponding way under $|x|\to\infty$. Namely, if we assume that the expressions $|\frac{\partial^m u}{\partial t^i\partial x^l}|\cdot(1+|x|^l),\ i=0,\ l=1,2,3,4;\ i=1,\ l=1,2,$ are uniformly bounded. In addition it should be remarked that the conditions of Theorem 4.1 are not necessary and the action of the method (4.7) is much broader than it is determined by $(i)-(iii)$. At the same time, the conditions $(i)-(iii)$ are fairly suitable in many situations.

5. Numeric algorithms

A recursive procedure can be applied for implementation of the method (4.7). But under big $T-t_0$ and small h such a procedure requires too much computational expanses.

To avoid any recursive calculations and to have become a numerical algorithm, the method (4.7) (just as other layer methods) needs a discretization in the variable x. Consider the equidistant space discretization : $x_j=x_0+j\alpha h,\ j=0,\pm1,\pm2,...\ ,\ x_0$ is a point belonging to $\mathbf{R}^1$, $\alpha>0$ is a number, i.e., h_x is taken to be equal to $\alpha h=\alpha h_t$. Using, for example, the linear interpolation, we construct the following algorithm

$$\bar{u}(t_N,x)=\varphi(x),$$

$$\bar{u}(t_k,x_j)=\frac{1}{2}\bar{u}(t_{k+1},\ x_j+b(t_{k+1},x_j,\bar{u}(t_{k+1},x_j))h+\sigma(t_{k+1},x_j,\bar{u}(t_{k+1},x_j))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1},\ x_j+b(t_{k+1},x_j,\bar{u}(t_{k+1},x_j))h-\sigma(t_{k+1},x_j,\bar{u}(t_{k+1},x_j))\sqrt{h})$$

$$+g(t_k,x_j,\bar{u}(t_{k+1},x_j))h,\ j=0,\pm1,\pm2,..., \tag{5.1}$$

$$\bar{u}(t_k,x)=\frac{x_{j+1}-x}{\alpha h}\ \bar{u}(t_k,x_j)+\frac{x-x_j}{\alpha h}\ \bar{u}(t_k,x_{j+1}),\ x_j<x<x_{j+1},\ k=N-1,...,1,0. \tag{5.2}$$

Theorem 5.1. *Under the assumptions (i) – (iii) the algorithm (5.1)-(5.2) has the first order, i.e., the approximation $\bar{u}(t_k,x)$ from the formula (5.2) satisfies the relation*

$$|\bar{u}(t_k,x)-u(t_k,x)|\leq Kh, \tag{5.3}$$

where K does not depend on $x,\ h,\ k$.

Proof. Let us introduce the error of the algorithm (5.1)-(5.2) on the k-th layer

$$\varepsilon(t_k,x):=\bar{u}(t_k,x)-u(t_k,x)$$

and ε_k in accord with (4.22):

$$\varepsilon_k:=\max_{-\infty<x<\infty}|\varepsilon(t_k,x)|\ .$$

Of course, these new $\varepsilon(t_k,x)$ and ε_k differ from the old ones. Just as earlier we are able to obtain for the nodes x_j (cf. (4.21)):

$$\varepsilon(t_k,x_j)=\frac{1}{2}\varepsilon(t_{k+1},\ x_j+\bar{b}h+\bar{\sigma}\sqrt{h})+\frac{1}{2}\varepsilon(t_{k+1},\ x_j+\bar{b}h-\bar{\sigma}\sqrt{h})+r_k+O(h^2),$$

whence the following inequality runs out:

$$|\varepsilon(t_k,x_j)|\leq\varepsilon_{k+1}+K\varepsilon_{k+1}h+Ch^2. \tag{5.4}$$

We have

$$u(t_k,x)=\frac{x_{j+1}-x}{\alpha h}\ u(t_k,x_j)+\frac{x-x_j}{\alpha h}\ u(t_k,x_{j+1})+O(h^2),\ x_j<x<x_{j+1}, \tag{5.5}$$

where the interpolation error $O(h^2)$ satisfies the inequality of the form (4.11).

From (5.5) and (5.2) we get

$$\varepsilon(t_k,x)=\frac{x_{j+1}-x}{\alpha h}\ \varepsilon(t_k,x_j)+\frac{x-x_j}{\alpha h}\ \varepsilon(t_k,x_{j+1})+O(h^2),\ x_j<x<x_{j+1},$$

whence due to (5.4) for all x

$$|\varepsilon(t_k,x)|\leq\varepsilon_{k+1}+K\varepsilon_{k+1}h+Ch^2, \tag{5.6}$$

of course, with another constant C.

The inequality (5.6) implies (4.23). Theorem 5.1 is proved.

Remark 5.1. To reduce the amount of the nodes x_j, it is natural at first sight to take advantage of the cubic interpolation with step $h_x=\beta\sqrt{h}$ instead of the linear interpolation with step αh. Then in place of (5.2) we get

$$\bar{u}(t_k,x)=\sum_{i=0}^{3}\Phi_{j,i}(x)\ \bar{u}(t_k,x_{j+i}),\ x_j<x<x_{j+3},\ \Phi_{j,i}(x)=\prod_{k=0,k\neq i}^{3}\frac{x-x_{j+k}}{x_{j+i}-x_{j+k}}\ ,$$

in place of (5.5) we get

$$u(t_k,x)=\sum_{i=0}^{3}\Phi_{j,i}(x)\ u(t_k,x_{j+i})+O(h_x^4),\ x_j<x<x_{j+3},$$

and, consequently,

$$\varepsilon(t_k,x)=\sum_{i=0}^{3}\Phi_{j,i}(x)\varepsilon(t_k,x_{j+i})+O(h^2),\ x_j<x<x_{j+3}.$$

Though $\sum_{i=0}^{3}\Phi_{j,i}(x)=1$ for any x, the sum of the absolute values $\sum_{i=0}^{3}|\Phi_{j,i}(x)|$ can take values greater than one. And instead of the inequality (5.6), we can obtain the following one only:

$$|\varepsilon(t_k,x)|\le A\varepsilon_{k+1}+K\varepsilon_{k+1}h+Ch^2,$$

where the constant A is, unfortunately, more than one.

Therefore, our proof of Theorem 5.1 cannot be carried over for the case of the cubic interpolation.

Remark 5.2. Along with the linear interpolation (5.2) it is natural to use the spline approximation of the form

$$\bar{u}(t_k,x)=\sum_{i=-\infty}^{\infty}\bar{u}(t_k,x_i)B(\frac{x-ih}{h}),\ x_i<x<x_{i+1},\ k=N-1,...,1,0, \tag{5.7}$$

where $B(x)$ is the standard cubic B-spline

$$B(x)=\begin{cases}\frac{2}{3}-x^2+\frac{1}{2}|x|^3, & |x|\le 1,\\ \frac{1}{6}(2-|x|)^3, & 1\le|x|\le 2,\\ 0, & |x|\ge 2.\end{cases}$$

The spline (5.7) is twice continuously differentiable, and because $B(x)$ is locally supported, the series (5.7) has not more than four nonzero terms for any $x\in\mathbf{R}$.

It is known (see, e.g., [1]), that the spline $\Lambda(x)=\sum_{i=-\infty}^{\infty}f(x_i)B(\frac{x-ih}{h})$ possesses fairly good approximating and smoothing properties. In particular, if there exists the third derivative of $f(x)$ and it is bounded, then there exist constants C_1 and C_2 such that

$$|f(x)-\Lambda(x)|\le C_1h^2,\ |f'(x)-\Lambda'(x)|\le C_2h,\ x\in\mathbf{R}.$$

And since the sequence $B_i(x)=B(\frac{x-ih}{h})$ provides a nonnegative partition of unity:

$$\sum_{i=-\infty}^{\infty}B_i(x)=1,\ B_i(x)\ge 0,\ \text{all } i,$$

the proof of Theorem 5.1 can be carried over for the case of the approximation (5.7).

Remark 5.3. Consider the Cauchy problem for an autonomous semilinear parabolic equation with the positive direction of time t

$$\frac{\partial u}{\partial t}=\frac{1}{2}\sigma^2(x,u)\frac{\partial^2 u}{\partial x^2}+b(x,u)\frac{\partial u}{\partial x}+g(x,u),\ t>0,\ x\in\mathbf{R}^1, \tag{5.8}$$

$$u(0,x) = \varphi(x). \tag{5.9}$$

If we substitute a solution $u(t,x)$ to the problem (5.8)-(5.9) in the coefficients $\sigma,\ b,\ g,$ the equation (5.8) becomes nonautonomous and that is why the reasoning of Remark 2.1 cannot be carried over for the problem (5.8)-(5.9). Nevertheless, from (5.1)-(5.2) it is not difficult to obtain the following procedure with positive direction of time

$$\bar{u}(0,x) = \varphi(x),$$

$$\bar{u}(t_{k+1},x_j) = \frac{1}{2}\bar{u}(t_k,\ x_j + b(x_j,\bar{u}(t_k,x_j))h + \sigma(x_j,\bar{u}(t_k,x_j))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}(t_k,\ x_j + b(x_j,\bar{u}(t_k,x_j))h - \sigma(x_j,\bar{u}(t_k,x_j))\sqrt{h})$$

$$+g(x_j,\bar{u}(t_k,x_j))h,\ j = 0,\pm 1,\pm 2,...,\ t_k = kh,\ h = t/N, \tag{5.10}$$

$$\bar{u}(t_k,x) = \frac{x_{j+1}-x}{\alpha h}\ \bar{u}(t_k,x_j) + \frac{x-x_j}{\alpha h}\ \bar{u}(t_k,x_{j+1}),\ x_j < x < x_{j+1},\ k = 0,1,...,N-1. \tag{5.11}$$

Just the procedure (5.10)-(5.11) is used in Section 8 in numerical calculations.

6. **Many-dimensional case**

Consider the Cauchy problem for $d > 1$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x,u)\frac{\partial u}{\partial x^i} + g(t,x,u) = 0\ ,$$

$$t_0 \leq\ t < T,\ x \in \mathbf{R}^d, \tag{6.1}$$

$$u(T,x) = \varphi(x)\ . \tag{6.2}$$

Just as in Section 3 we can write the same relations (3.3)-(3.8) with the distinction that $x,\ X,$ and b are d-vectors, σ is a $d\times d$-matrix such that $\sigma\sigma^\top = a = \{a^{ij}\}$, and $\xi_{N-1},\ \xi_{N-2},...,\ \xi_0$ in (3.7) are i.i.d. vectors of dimension d with i.i.d. components $\xi_k^i,\ i = 1,...,d,$ and each component ξ^i is distributed by the law: $P(\xi = \pm 1) = \dfrac{1}{2}$.

Using (3.4) we obtain (here we restrict ourselves to the two-dimensional case in writing)

$$u(t_k,x) = u(t_k,x^1,x^2)$$

$$\simeq \mathbf{E}u(t_{k+1},\bar{X}^1_{t_k,x}(t_{k+1}),\bar{X}^2_{t_k,x}(t_{k+1})) + \mathbf{E}\bar{Z}_{t_k,x,0}(t_{k+1})$$

$$= \frac{1}{4}u(t_{k+1},\ x^1 + b^1h + \sigma^{11}\sqrt{h} + \sigma^{12}\sqrt{h},\ x^2 + b^2h + \sigma^{21}\sqrt{h} + \sigma^{22}\sqrt{h})$$

$$+\frac{1}{4}u(t_{k+1},\ x^1 + b^1h + \sigma^{11}\sqrt{h} - \sigma^{12}\sqrt{h},\ x^2 + b^2h + \sigma^{21}\sqrt{h} - \sigma^{22}\sqrt{h})$$

$$+\frac{1}{4}u(t_{k+1},\ x^1 + b^1h - \sigma^{11}\sqrt{h} + \sigma^{12}\sqrt{h},\ x^2 + b^2h - \sigma^{21}\sqrt{h} + \sigma^{22}\sqrt{h})$$

$$+\frac{1}{4}u(t_{k+1},\ x^1+b^1h-\sigma^{11}\sqrt{h}-\sigma^{12}\sqrt{h},\ x^2+b^2h-\sigma^{21}\sqrt{h}-\sigma^{22}\sqrt{h})\ +gh\ , \tag{6.3}$$

where $b^i = b^i(t_k, x^1, x^2, u(t_k, x^1, x^2)),\ \sigma^{ij} = \sigma^{ij}(t_k, x^1, x^2, u(t_k, x^1, x^2)),\ i,j = 1,2,\ g = g(t_k, x^1, x^2, u(t_k, x^1, x^2))$.

Now the analogous to (4.7) method has the following form:

$$\bar{u}(t_N, x^1, x^2) = \varphi(x^1, x^2),$$

$$\bar{u}(t_k, x^1, x^2) = \frac{1}{4}\bar{u}(t_{k+1},\ x^1+\bar{b}^1h+\bar{\sigma}^{11}\sqrt{h}+\bar{\sigma}^{12}\sqrt{h},\ x^2+\bar{b}^2h+\bar{\sigma}^{21}\sqrt{h}+\bar{\sigma}^{22}\sqrt{h})$$

$$+\frac{1}{4}\bar{u}(t_{k+1},\ x^1+\bar{b}^1h+\bar{\sigma}^{11}\sqrt{h}-\bar{\sigma}^{12}\sqrt{h},\ x^2+\bar{b}^2h+\bar{\sigma}^{21}\sqrt{h}-\bar{\sigma}^{22}\sqrt{h})$$

$$+\frac{1}{4}\bar{u}(t_{k+1},\ x^1+\bar{b}^1h-\bar{\sigma}^{11}\sqrt{h}+\bar{\sigma}^{12}\sqrt{h},\ x^2+\bar{b}^2h-\bar{\sigma}^{21}\sqrt{h}+\bar{\sigma}^{22}\sqrt{h})$$

$$+\frac{1}{4}\bar{u}(t_{k+1},\ x^1+\bar{b}^1h-\bar{\sigma}^{11}\sqrt{h}-\bar{\sigma}^{12}\sqrt{h},\ x^2+\bar{b}^2h-\bar{\sigma}^{21}\sqrt{h}-\bar{\sigma}^{22}\sqrt{h})+\bar{g}h\ , \tag{6.4}$$

where $\bar{b}^i = b^i(t_k, x^1, x^2, \bar{u}(t_{k+1}, x^1, x^2)),\ \bar{\sigma}^{ij} = \sigma^{ij}(t_k, x^1, x^2, \bar{u}(t_{k+1}, x^1, x^2)),\ i,j = 1,2,$ $\bar{g} = g(t_k, x^1, x^2, \bar{u}(t_{k+1}, x^1, x^2)),\ k = N-1, ..., 1, 0.$

This method is deterministic though the probabilistic approach is used for its constructing.

Consider the equidistant space discretization : $x^1_j = x^1_0 + j\alpha^1 h,\ x^2_l = x^2_0 + l\alpha^2 h,\ j,l = 0, \pm1, \pm2, \dots$, (x^1_0, x^2_0) is a point belonging to $\mathbf{R}^2$, $\alpha^1 > 0,\ \alpha^2 > 0$ are numbers, i.e., $h_{x^1},\ h_{x^2}$ are taken to be equal to $\alpha^1 h,\ \alpha^2 h$. Using the linear sequential interpolation, we construct the following algorithm based on the method (6.4):

$$\bar{u}(t_N, x^1, x^2) = \varphi(x^1, x^2),$$

$$\bar{u}(t_k, x^1_j, x^2_l) = \frac{1}{4}\bar{u}(t_{k+1},\ x^1_j+\bar{b}^1h+\bar{\sigma}^{11}\sqrt{h}+\bar{\sigma}^{12}\sqrt{h},\ x^2_l+\bar{b}^2h+\bar{\sigma}^{21}\sqrt{h}+\bar{\sigma}^{22}\sqrt{h})$$

$$+\frac{1}{4}\bar{u}(t_{k+1},\ x^1_j+\bar{b}^1h+\bar{\sigma}^{11}\sqrt{h}-\bar{\sigma}^{12}\sqrt{h},\ x^2_l+\bar{b}^2h+\bar{\sigma}^{21}\sqrt{h}-\bar{\sigma}^{22}\sqrt{h})$$

$$+\frac{1}{4}\bar{u}(t_{k+1},\ x^1_j+\bar{b}^1h-\bar{\sigma}^{11}\sqrt{h}+\bar{\sigma}^{12}\sqrt{h},\ x^2_l+\bar{b}^2h-\bar{\sigma}^{21}\sqrt{h}+\bar{\sigma}^{22}\sqrt{h})$$

$$+\frac{1}{4}\bar{u}(t_{k+1},\ x^1_j+\bar{b}^1h-\bar{\sigma}^{11}\sqrt{h}-\bar{\sigma}^{12}\sqrt{h},\ x^2_l+\bar{b}^2h-\bar{\sigma}^{21}\sqrt{h}-\bar{\sigma}^{22}\sqrt{h})+\bar{g}h\ , \tag{6.5}$$

where all the coefficients $\bar{b}$ and $\bar{\sigma}$ are calculated at $t_k, x^1_j, x^2_l, \bar{u}(t_{k+1}, x^1_j, x^2_l)$,

$$\bar{u}(t_k, x^1, x^2) = \frac{x^1_{j+1}-x^1}{\alpha^1 h}\cdot\frac{x^2_{l+1}-x^2}{\alpha^2 h}\bar{u}(t_k, x^1_j, x^2_l) + \frac{x^1_{j+1}-x^1}{\alpha^1 h}\cdot\frac{x^2-x^2_l}{\alpha^2 h}\bar{u}(t_k, x^1_j, x^2_{l+1})$$

$$+\frac{x^1-x^1_j}{\alpha^1 h}\cdot\frac{x^2_{l+1}-x^2}{\alpha^2 h}\bar{u}(t_k, x^1_{j+1}, x^2_l) + \frac{x^1-x^1_j}{\alpha^1 h}\cdot\frac{x^2-x^2_l}{\alpha^2 h}\bar{u}(t_k, x^1_{j+1}, x^2_{l+1})\ ,$$

$$x^1_j \le x^1 \le x^1_{j+1},\ x^2_l \le x^2 \le x^2_{l+1},\ (x^1, x^2) \ne (x^1_i, x^2_m),$$

$$i, m = 0, \pm1, \pm2, \dots\ ,\ k = N-1, , 1, 0\ . \tag{6.6}$$

Remark 6.1. The sequential linear interpolation in (6.6) is not linear with respect to both variables x^1 and x^2. The following triangular interpolation is linear one and just as the interpolation (6.6), to be applied to the solution $u(t,x^1,x^2)$, has an error of $O(h^2)$:

$$\bar{u}(t_k,x^1,x^2) = (1 - \frac{x^1 - x^1_j}{\alpha^1 h} - \frac{x^2 - x^2_l}{\alpha^2 h})\bar{u}(t_k,x^1_j,x^2_l)$$

$$+\frac{x^2 - x^2_l}{\alpha^2 h}\bar{u}(t_k,x^1_j,x^2_{l+1}) + \frac{x^1 - x^1_j}{\alpha^1 h}\bar{u}(t_k,x^1_{j+1},x^2_l). \quad (6.7)$$

This interpolation is not suitable for the all points (x^1,x^2) from the rectangle $\Pi_{j,l} = \{(x^1,x^2) : x^1_j \le x^1 \le x^1_{j+1},\ x^2_l \le x^2 \le x^2_{l+1}\}$ by the same reasons as it was mentioned in Remark 5.1. But for the points from the triangle with the corners (x^1_j,x^2_l), (x^1_j,x^2_{l+1}), (x^1_{j+1},x^2_l) such an interpolation is suitable because

$$\frac{x^1 - x^1_j}{\alpha^1 h} + \frac{x^2 - x^2_l}{\alpha^2 h} \le 1 \quad (6.8)$$

for those points.

For the other points of the rectangle $\Pi_{j,l}$

$$\frac{x^1_{j+1} - x^1}{\alpha^1 h} + \frac{x^2_{l+1} - x^2}{\alpha^2 h} < 1 \quad (6.9)$$

and we can use the following formula:

$$\bar{u}(t_k,x^1,x^2) = (1 - \frac{x^1_{j+1} - x^1}{\alpha^1 h} - \frac{x^2_{l+1} - x^2}{\alpha^2 h})\bar{u}(t_k,x^1_{j+1},x^2_{l+1})$$

$$+\frac{x^1_{j+1} - x^1}{\alpha^1 h}\bar{u}(t_k,x^1_j,x^2_{l+1}) + \frac{x^2_{l+1} - x^2}{\alpha^2 h}\bar{u}(t_k,x^1_{j+1},x^2_l). \quad (6.10)$$

Thus, the formulas (6.7) and (6.10) for (x^1,x^2) belonging to $\Pi_{j,l}$, satisfying (6.8) and (6.9) correspondingly, give another suitable rule of interpolation.

The theorems for the method (6.4) and for the algorithm (6.5) with both interpolations (6.6) and (6.7)-(6.10) are analogous to Theorems 4.1 and 5.1.

7. **Reaction-Diffusion systems**

The above constructed methods can also be applied to the Cauchy problem for systems of reaction-diffusion equations of the form (for simplicity we write them for the one-dimensional x):

$$\frac{\partial u_q}{\partial t} + L_q u_q + g_q(t,x,u) = 0,\ t_0 \le t < T,\ x \in \mathbf{R}^1,\ q = 1,\dots,n, \quad (7.1)$$

$$u_q(T,x) = \varphi_q(x), \quad (7.2)$$

where

$$u := (u_1,\dots,u_n)\ ,$$

$$L_q := \frac{1}{2}\sigma_q^2(t,x,u)\frac{\partial^2}{\partial x^2} + b_q(t,x,u)\frac{\partial}{\partial x}\ .$$

It is not difficult to derive the method which is analogous to (4.7):

$$\bar{u}_q(t_N,x) = \varphi_q(x),$$

$$\bar{u}_q(t_k,x)=\frac{1}{2}\bar{u}_q(t_{k+1},\ x+b_q(t_k,x,\bar{u}(t_{k+1},x))h+\sigma_q(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+\frac{1}{2}\bar{u}_q(t_{k+1},\ x+b_q(t_k,x,\bar{u}(t_{k+1},x))h-\sigma_q(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+g_q(t_k,x,\bar{u}(t_{k+1},x))h,\ k=N-1,...,1,0, \tag{7.3}$$

and then the corresponding algorithm (see (5.2)).

The system (7.1) is such that the linear system of parabolic equations, obtained after substituting $u=u(t,x)$ in the coefficients $\sigma_q,\ b_q,\ g_q$, splits and therefore every parabolic equation can be solved separately. In connection with this fact, one can consider n separate simple systems of the type (3.5)-(3.6). Such a way is impossible for reaction-diffusion systems containing equations with derivatives of different functions among $u_1,...,u_n$. Consider, for example, the system

$$\frac{\partial u_q}{\partial t}+\frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 u_q}{\partial x^2}+\sum_{j=1}^{n}\sigma(t,x,u)b_{jq}(t,x,u)\frac{\partial u_j}{\partial x}+g_q(t,x,u)=0 \tag{7.4}$$

with the conditions (7.2) (we pay attention that σ in (7.4) does not depend on q).

In this case one can use the following probabilistic representation (see [10]):

$$u_q(t_k,x)=E\sum_{l=1}^{n}u_l(t_{k+1},X_{t_k,x}(t_{k+1}))Y^l_{t_k,x,q}(t_{k+1})+EZ_{t_k,x,q,0}(t_{k+1}), \tag{7.5}$$

where $X_{t_k,x}(s),\ Y^l_{t_k,x,q}(s),\ Z_{t_k,x,q,0}(s)$ is the solution of the Cauchy problem to the system of stochastic differential equations

$$dX=\sigma(s,X,u(s,X))dw(s),\ X(t_k)=x,$$

$$dY^j=\sum_{l=1}^{n}b_{jl}(s,X,u(s,X))Y^l dw(s),\ Y^j(t_k)=\delta_{jq}=\begin{cases}0,\ j\neq q,\\ 1,\ j=q,\end{cases}$$

$$dZ=\sum_{l=1}^{n}g_l(s,X,u(s,X))Y^l ds,\ Z(t_k)=0. \tag{7.6}$$

Now it is not difficult to derive the method which is analogous to (4.7):

$$\bar{u}_q(t_N,x)=\varphi_q(x),$$

$$\bar{u}_q(t_k,x)=\frac{1}{2}\sum_{l=1}^{n}\bar{u}_l(t_{k+1},\ x+\sigma(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})\cdot(\delta_{lq}+b_{lq}(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+\frac{1}{2}\sum_{l=1}^{n}\bar{u}_l(t_{k+1},\ x-\sigma(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})\cdot(\delta_{lq}-b_{lq}(t_k,x,\bar{u}(t_{k+1},x))\sqrt{h})$$

$$+g_q(t_k,x,\bar{u}(t_{k+1},x))h,\ k=N-1,...,1,0, \tag{7.7}$$

and then the corresponding algorithm.

Convergence Theorems 4.1 and 5.1 can be carried over to these method and algorithm.

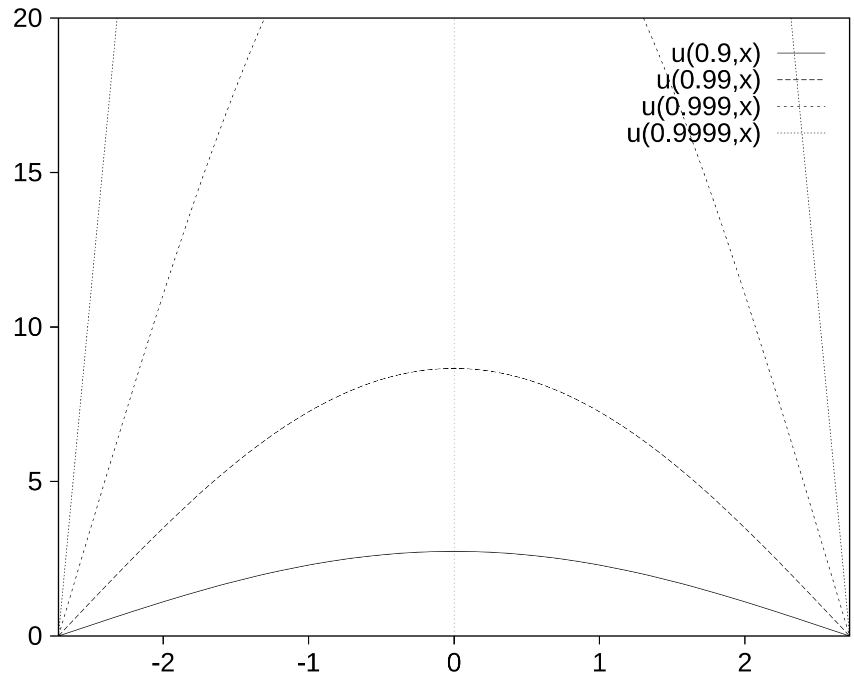

FIGURE 1. Solution (8.2), $\alpha = 2,\ T_0 = 1$

8. **Numerical examples**

Example 1. Consider the quasilinear equation with power law nonlinearities (see, e.g., [18])

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u^{\alpha}\frac{\partial u}{\partial x}) + u^{\alpha+1},\ t > 0,\ x \in \mathbf{R}, \tag{8.1}$$

where $\alpha > 0$ is a constant.

The equation (8.1) has the following automodelling solution (see Figure 1 as well)

$$u(t,x) = \begin{cases} (T_0 - t)^{-1/\alpha}(\frac{2(\alpha+1)}{\alpha(\alpha+2)}\cos^2\frac{\pi x}{L})^{1/\alpha}, & |x| < \frac{L}{2}, \\ 0, & |x| \geq \frac{L}{2},\ 0 < t < T_0, \end{cases} \tag{8.2}$$

where

$$L = \frac{2\pi}{\alpha}(\alpha+1)^{1/2}.$$

The temperature $u(t,x)$ grows infinitely under $t \to T_0$. At the same time the heat is localized in the interval $(-L/2,\ L/2)$. The function

$$v = \frac{1}{\alpha+1}u^{\alpha+1}$$

satisfies the equation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\cdot 2(\alpha+1)^{\alpha/(\alpha+1)}v^{\alpha/(\alpha+1)}\frac{\partial^2 v}{\partial x^2} + (\alpha+1)^{(2\alpha+1)/(\alpha+1)}\cdot v^{(2\alpha+1)/(\alpha+1)} \tag{8.3}$$

which has the form of (5.8).

TABLE 1. The absolute errors of algorithm (5.10)-(5.11) to the Cauchy problem (8.3)-(8.4) at $t = 0.5$

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$
$err_{\bar{v}}$	$0.9931 \cdot 10^{-1}$	$1.422 \cdot 10^{-2}$	$1.489 \cdot 10^{-3}$	$1.500 \cdot 10^{-4}$
$err_{\bar{u}}$	$0.9572 \cdot 10^{-1}$	$1.643 \cdot 10^{-2}$	$4.432 \cdot 10^{-3}$	$13.77 \cdot 10^{-4}$
$err_{\bar{u}}[-2,2]$	$0.7015 \cdot 10^{-1}$	$0.9552 \cdot 10^{-2}$	$1.215 \cdot 10^{-3}$	$1.003 \cdot 10^{-4}$

TABLE 2. The relative errors $\delta(t,h)$ of algorithm (5.10)-(5.11) to the Cauchy problem (8.3)-(8.4) and the explosion time

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$
$t = 0.9$	$3.644 \cdot 10^{-1}$	$1.024 \cdot 10^{-1}$	$1.313 \cdot 10^{-2}$	$1.353 \cdot 10^{-3}$
$t = 0.99$	- - - - - -	$5.298 \cdot 10^{-1}$	$1.815 \cdot 10^{-1}$	$2.585 \cdot 10^{-2}$
$t = 0.999$	- - - - - -	- - - - - -	$6.167 \cdot 10^{-1}$	$2.436 \cdot 10^{-1}$
$t = 0.9999$	- - - - - -	- - - - - -	- - - - - -	$6.704 \cdot 10^{-1}$
t^*	1.5	1.07	1.008	1.0001

We make use of algorithm (5.10)-(5.11) to find the solution of (8.3) under $\alpha = 2$ with the initial conditions

$$v(0,x) = \begin{cases} \frac{\sqrt{3}}{8} \cos^3 \frac{\pi x}{L}, & |x| < \frac{L}{2}, \\ 0, & |x| \geq \frac{L}{2} . \end{cases} \tag{8.4}$$

Table 1 presents the errors

$$err_{\bar{v}} = \max_{x_i} |\bar{v}(t,x_i) - v(t,x_i)|,$$

$$err_{\bar{u}} = \max_{x_i} |\bar{u}(t,x_i) - u(t,x_i)|, \ \bar{u}(t,x_i) = (3\bar{v}(t,x_i))^{1/3},$$

$$err_{\bar{u}}[-2,2] = \max_{|x_i|\leq 2} |\bar{u}(t,x_i) - u(t,x_i)|$$

for $t = 0.5$ depending on h ($h_t = h_x = h$).

The rather large values $err_{\bar{u}}$ are connected with the fact that under x_i, being close to the ends of the interval $(-L/2, L/2)$, the values $v(t,x_i)$ are very small and, consequently, for such x_i

$$|\bar{u}(t,x_i) - u(t,x_i)| = |(3\bar{v}(t,x_i))^{1/3} - (3v(t,x_i))^{1/3}| \simeq 3^{1/3} |\bar{v}(t,x_i) - v(t,x_i)|^{1/3},$$

i.e. $err_{\bar{u}} = O(h^{1/3})$.

But the difference $\bar{u}(t,x_i) - u(t,x_i)$ on a subinterval $(-a,a)$, $a < L/2$, behaves as $O(h)$ (see the row $err_{\bar{u}}[-2,2]$ in Table 1).

For times t which are close to the explosion time T_0, the errors $err_{\bar{v}}$ become fairly large (we pay attention that v in our example is proportional to cube of u). However if we are interested in finding the explosions time it is natural to consider another characteristic under $t \to T_0$. Table 2 presents the values

$$\delta(t,h) = \frac{err_{\bar{u}}}{u(t,0)}$$

and the time t^* at which the values of u become more than 10^4, i.e., this time evaluates the explosion time.

Example 2. Consider the one-dimensional Burger equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x},\ t > 0,\ x \in \mathbf{R}, \tag{8.5}$$

$$u(0,x) = \varphi(x). \tag{8.6}$$

Due to the Cole-Hopf transformation the solution to the problem (8.5)-(8.6) can be found explicitly:

$$u(t,x) = \frac{\int_{-\infty}^{\infty} K(t,x,y)\varphi(y)\exp(-\frac{1}{\sigma^2}\int_0^y \varphi(\xi)d\xi)dy}{\int_{-\infty}^{\infty} K(t,x,y)\exp(-\frac{1}{\sigma^2}\int_0^y \varphi(\xi)d\xi)dy}, \tag{8.7}$$

$$K(t,x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp(-\frac{(x-y)^2}{2\sigma^2 t}). \tag{8.8}$$

If

$$u(0,x) = \psi(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0, \end{cases} \tag{8.9}$$

then

$$u(t,x) = \psi(t,x) := 1 - \frac{\operatorname{erfc}(-\frac{x}{\sqrt{2\sigma^2 t}})}{\operatorname{erfc}(-\frac{x}{\sqrt{2\sigma^2 t}}) + \exp(\frac{t-2x}{2\sigma^2})\cdot(2-\operatorname{erfc}(\frac{t-x}{\sqrt{2\sigma^2 t}}))}, \tag{8.10}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp(-\xi^2)d\xi.$$

Under a sufficiently small σ the solution (8.9) is close to the travelling shock wave $\psi(x - \frac{1}{2}t)$ with speed $\frac{1}{2}$.

Tables 3 and 4 give numerical results obtained by using the algorithm (5.10)-(5.11) with $h_t = h_x = h$ to the Cauchy problem (8.5), (8.9). They present the errors of approximate solution $\bar{u}$ in the discrete Chebyshev norm (the top position) and in l^1-norm (the lower position):

$$err_{\bar{u}}^c = \max_{x_i} |\bar{u}(t,x_i) - u(t,x_i)|,$$

$$err_{\bar{u}}^l = \sum_i |\bar{u}(t,x_i) - u(t,x_i)| \cdot h.$$

These results illustrate the good properties of the algorithm (5.10)-(5.11). Besides they show more wide capabilities of the algorithm than it is ensured by Theorem 4.1 (we have in mind the discontinuity of the function $\psi(x)$). The big values of the errors (especially of $err_{\bar{u}}^c$) for small σ and t are easy explicable: the corresponding solution has the very large derivatives with respect to x in these cases. Clearly, the errors can be essentially decreased if we improve the exactness of interpolation, for instance, by means of choice a smaller h_x. In connection with this example, see the numerical experiments in [2], [3] as well.

Example 3. Consider the asymptotic behavior of some solutions to the problem (8.5)-(8.6). Figure 2 shows that the solution of the problem (8.5), (8.9) for large t is

TABLE 3. Dependence of the errors $err^c_{\bar u}$ and $err^l_{\bar u}$ in h and σ under fixed $t = 1$

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$
$\sigma = 0.05$	> 0.5 0.2516	> 0.5 0.1198	> 0.5 $0.2960 \cdot 10^{-1}$	0.3232 $0.3349 \cdot 10^{-2}$
$\sigma = 0.1$	> 0.5 0.1874	> 0.5 $0.7501 \cdot 10^{-1}$	0.2104 $0.8495 \cdot 10^{-2}$	$0.2198 \cdot 10^{-1}$ $0.8692 \cdot 10^{-3}$
$\sigma = 0.2$	0.4849 0.1057	0.1484 $0.2316 \cdot 10^{-1}$	$0.1582 \cdot 10^{-1}$ $0.2412 \cdot 10^{-2}$	$0.1625 \cdot 10^{-2}$ $0.2485 \cdot 10^{-3}$
$\sigma = 0.5$	$0.7295 \cdot 10^{-1}$ $0.5137 \cdot 10^{-1}$	$0.7704 \cdot 10^{-2}$ $0.5580 \cdot 10^{-2}$	$0.8448 \cdot 10^{-3}$ $0.6035 \cdot 10^{-3}$	$0.9010 \cdot 10^{-4}$ $0.6538 \cdot 10^{-4}$
$\sigma = 1$	$0.1033 \cdot 10^{-1}$ $0.2150 \cdot 10^{-1}$	$0.1151 \cdot 10^{-2}$ $0.2631 \cdot 10^{-2}$	$0.1351 \cdot 10^{-3}$ $0.2769 \cdot 10^{-3}$	$0.1506 \cdot 10^{-4}$ $0.3247 \cdot 10^{-4}$

TABLE 4. Dependence of the errors $err^c_{\bar u}$ and $err^l_{\bar u}$ in h and t under fixed $\sigma = 0.5$

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$
$t = 0.02$	— — — — — — — — — — — —	0.1348 $0.1923 \cdot 10^{-1}$	$0.4270 \cdot 10^{-1}$ $0.4703 \cdot 10^{-2}$
$t = 0.1$	0.1427 $0.2850 \cdot 10^{-1}$	$0.2216 \cdot 10^{-1}$ $0.6567 \cdot 10^{-2}$	$0.1099 \cdot 10^{-2}$ $0.3503 \cdot 10^{-3}$
$t = 0.5$	$0.6368 \cdot 10^{-1}$ $0.3198 \cdot 10^{-1}$	$0.5438 \cdot 10^{-2}$ $0.3463 \cdot 10^{-2}$	$0.6311 \cdot 10^{-3}$ $0.3824 \cdot 10^{-3}$
$t = 2.5$	0.1147 0.1018	$0.1296 \cdot 10^{-1}$ $0.1125 \cdot 10^{-1}$	$0.1362 \cdot 10^{-2}$ $0.1181 \cdot 10^{-2}$

close to a wave which preserves its shape and moves with speed 1/2. Figure 3 is related to the solution with the initial data

$$u(0,x) = \begin{cases} 1, & x < -10, \\ 0.75, & -10 < x < 0, \\ 0, & x > 0. \end{cases} \tag{8.11}$$

Comparing these two figures one can conclude that there exist the limit shape and the limit speed of the waves which are the same for the initial conditions (8.9) and (8.11).

Recently the following two-parameter solution of Burger's equation (8.5) is found in [2]:

$$u^{a,b}(t,x) = b - a \tanh \frac{a(x-bt)}{\sigma^2}, \tag{8.12}$$

where a and b are constants and (we remind)

$$\tanh x = \operatorname{sign}(x) \frac{1 - e^{-2|x|}}{1 + e^{-2|x|}}.$$

Clearly, $u^{a,b}(t,x) = u^{|a|,b}(t,x)$ (therefore one can consider the case $a \geq 0$ only) and

$$b - a \leq u^{a,b}(t,x) \leq b + a, \; a > 0.$$

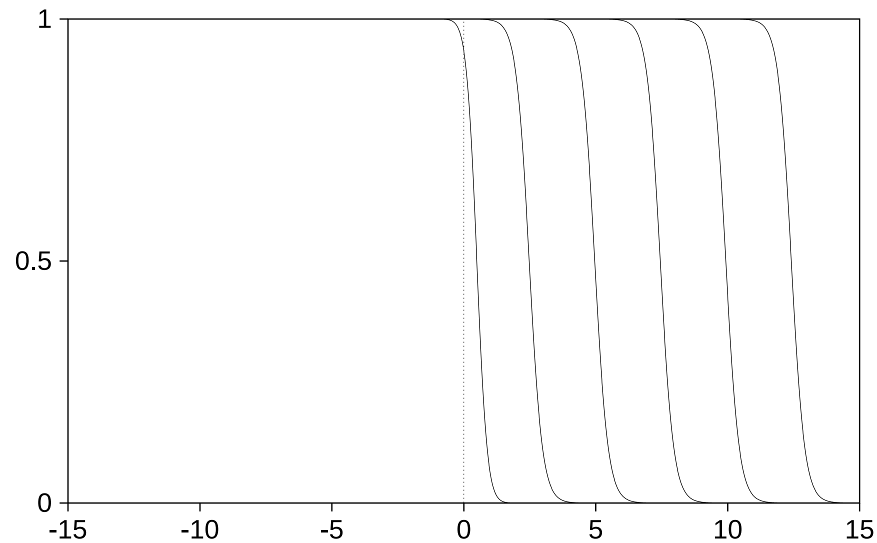

FIGURE 2. The solution to the problem (8.5), (8.9) at the moments $t = 1,\ t = 5,\ t = 10,\ t = 15,\ t = 20,\ t = 25;\ \sigma = 0.5$

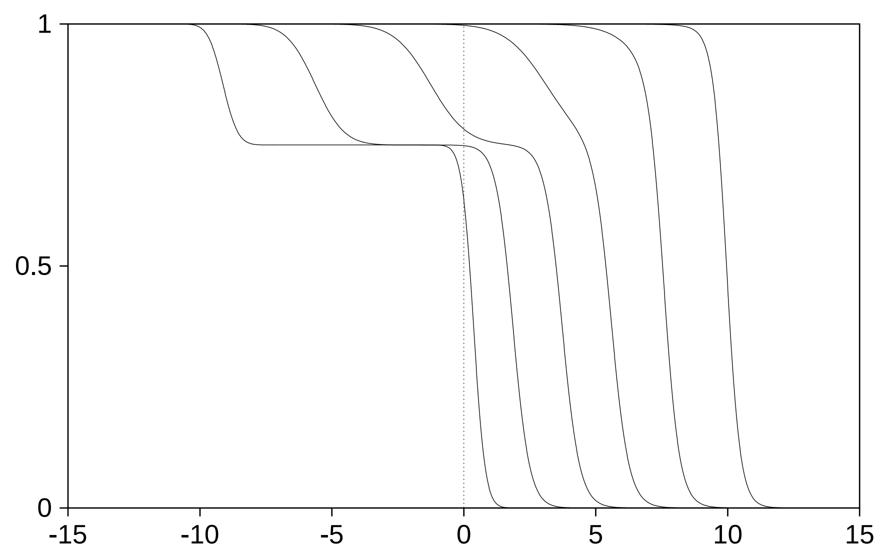

FIGURE 3. The solution to the problem (8.5), (8.11) at the moments $t = 1,\ t = 5,\ t = 10,\ t = 15,\ t = 20,\ t = 25;\ \sigma = 0.5$

Let us also note that the function

$$u_0(x) = -\tanh \frac{x}{\sigma^2}$$

is a stationary solution of Burger's equation, i.e.,

$$\frac{1}{2}\sigma^2 \frac{d^2 u_0}{dx^2} - u_0 \frac{du_0}{dx} = 0, \tag{8.13}$$

and if some function $u_0(x)$ is a solution of the steady-state Burger equation (8.13) then the function

$$u(t, x) = b + a u_0(a(x - bt))$$

is a solution of the general equation (8.5).

The solution $u^{a,b}(t,x)$ is a traveling wave which runs with speed b, i.e., it runs from the left to the right if $b > 0$, it runs conversely if $b < 0$, and it is immovable if $b = 0$. The shape of this wave is determined by the function $u^{a,b}(0,x)$.

We say that the shape of $u(t,x)$ converges to $f(x)$ as t becomes infinite if there exists the function $m(t)$ such that

$$\lim_{t\to\infty} \sup_x |u(t, x + m(t)) - f(x)| = 0.$$

It is not difficult to prove that the shape of $\psi(t,x)$ (see (8.10)) converges to $u^{0.5,0.5}(0,x) = \frac{1}{2} - \frac{1}{2}\tanh\frac{x}{2\sigma^2}$. To show it you should to set $m(t) = \frac{t}{2}$ (let us remind that $\operatorname{erfc}(-x) + \operatorname{erfc} x = 2$ and therefore $\psi(t, \frac{t}{2}) = \frac{1}{2}$).

Theorem 8.1. *Let*

$$u(0,x) = \begin{cases} c, & x < l_0, \\ \lambda(x), & l_0 \le x \le l_0 + l, \\ d, & x > l_0 + l, \end{cases} \tag{8.14}$$

where c, d, l_0, l are some constants: $c > d$, $l \ge 0$; $\lambda(x)$ is a measurable function and $d \le \lambda(x) \le c$.

Then the shape of $u(t,x)$ converges to $u^{a,b}(0,x)$ with

$$a = \frac{c-d}{2} > 0, \ b = \frac{c+d}{2} .$$

More exactly:

$$\lim_{t\to\infty} \sup_x |u(t, x + l_0 + bt + \alpha) - u^{a,b}(0,x)| = 0, \tag{8.15}$$

where

$$\alpha = \frac{S - d\cdot l}{c-d}, \ S = \int_0^l \lambda(\xi) d\xi.$$

Thus, the limit shape of a solution of Burger's equation with initial data of the form (8.14) *depends on c and d only and for large t it is close to the traveling symmetric wave of the shape $u^{a,b}(0,x)$ with speed b and with center*

$$m(t) = l_0 + bt + \alpha \, . \tag{8.16}$$

Proof. Because $u(t, x + l_0)$ is also a solution of Burger's equation, it is sufficient to prove the theorem for the case $l_0 = 0$. Let us make use of the formula (8.7). We obtain

$$u(t,x) = \frac{c \cdot I_1(t,x) + d\cdot I_2(t,x) + I_4(t,x) - d\cdot I_5(t,x)}{I_1(t,x) + I_2(t,x) + I_3(t,x) - I_5(t,x)} ,$$

where

$$I_1(t,x) = \int_{-\infty}^0 K(t,x,y) \exp(-\frac{c}{\sigma^2} y) dy = \exp(\frac{c^2 t - 2cx}{2\sigma^2}) \cdot (1 - \frac{1}{2}\operatorname{erfc}(\frac{ct - x}{\sqrt{2\sigma^2 t}})),$$

$$I_2(t,x) = \int_0^\infty K(t,x,y) \exp(-\frac{S + d\cdot(y - l)}{\sigma^2}) dy$$

$$= \exp(\frac{d^2\cdot t - 2d\cdot x}{2\sigma^2} - \frac{S - d\cdot l}{\sigma^2}) \cdot \frac{1}{2}\operatorname{erfc}(\frac{d\cdot t - x}{\sqrt{2\sigma^2 t}}),$$

$$I_3(t,x)=\int_0^l K(t,x,y)\exp(-\frac{1}{\sigma^2}\int_0^y \lambda(z)dz)dy,$$

$$I_4(t,x)=\int_0^l K(t,x,y)\lambda(y)\exp(-\frac{1}{\sigma^2}\int_0^y \lambda(z)dz)dy,$$

$$I_5(t,x)=\int_0^l K(t,x,y)\exp(-\frac{S+d\cdot(y-l)}{\sigma^2})dy.$$

The direct calculations give

$$I_1(t,x+m(t))=\exp(-\frac{cd\cdot t}{2\sigma^2}-\frac{c\alpha}{\sigma^2})\cdot\exp(-\frac{cx}{\sigma^2})\cdot(1-\frac{1}{2}\operatorname{erfc}(\frac{(c-d)t-2x-2\alpha}{2\sqrt{2\sigma^2 t}})),$$

$$I_2(t,x+m(t))=\exp(-\frac{cd\cdot t}{2\sigma^2}-\frac{c\alpha}{\sigma^2})\cdot\exp(-\frac{d\cdot x}{\sigma^2})\cdot\frac{1}{2}\operatorname{erfc}(\frac{(d-c)t-2x-2\alpha}{2\sqrt{2\sigma^2 t}})).$$

We have (see (8.8) and (8.16))

$$K(t,x+m(t),y)=\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp(-\frac{(c+d)^2\cdot t}{8\sigma^2})\cdot$$

$$\cdot\exp(-\frac{(c+d)\cdot x}{2\sigma^2})\cdot\exp(-\frac{(x-y+\alpha)^2}{2\sigma^2 t}-\frac{(\alpha-y)(c+d)}{2\sigma^2})\ .$$

Because

$$-\frac{(c+d)^2\cdot t}{8\sigma^2}\le-\frac{cd\cdot t}{2\sigma^2}\ ,$$

it is not difficult to obtain the following representation for the integrals $I_3(t,x+m(t)),\ I_4(t,x+m(t)),\ I_5(t,x+m(t))$:

$$I_j(t,x+m(t))=\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp(-\frac{cd\cdot t}{2\sigma^2}-\frac{c\alpha}{\sigma^2})\cdot\exp(-\frac{(c+d)\cdot x}{2\sigma^2})\cdot J_j(t,x),\ j=3,4,5,$$

where the functions $J_j(t,x)$ are bounded: there exists a constant $C>0$ such that

$$|J_j(t,x)|\le C,\ j=3,4,5,\ t\ge 0,\ x\in\mathbf{R}. \tag{8.17}$$

Now the function $u(t,x+m(t))$ with $m(t)$ from (8.16) (remind $l_0=0$) can be represented in the form

$$u(t,x+m(t))=$$

$$=\frac{c\cdot\exp(-\frac{cx}{\sigma^2})\cdot p(t,x)+d\cdot\exp(-\frac{d\cdot x}{\sigma^2})\cdot q(t,x)+\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp(-\frac{(c+d)\cdot x}{2\sigma^2})\cdot(J_4-d\cdot J_5)}{\exp(-\frac{cx}{\sigma^2})\cdot p(t,x)+\exp(-\frac{d\cdot x}{\sigma^2})\cdot q(t,x)+\frac{1}{\sqrt{2\pi\sigma^2 t}}\exp(-\frac{(c+d)\cdot x}{2\sigma^2})\cdot(J_4-J_5)}, \tag{8.18}$$

where

$$p(t,x):=1-\frac{1}{2}\operatorname{erfc}(\frac{(c-d)t-2x-2\alpha}{2\sqrt{2\sigma^2 t}}),\ q(t,x):=\frac{1}{2}\operatorname{erfc}(\frac{(d-c)t-2x-2\alpha}{2\sqrt{2\sigma^2 t}}).$$

We have

$$\lim_{t\to\infty}p(t,x)=1,\ \lim_{t\to\infty}q(t,x)=1. \tag{8.19}$$

Due to (8.17) and (8.19) we get from (8.18)

$$\lim_{t\to\infty} u(t, x + m(t)) = \frac{c \cdot \exp(-\frac{cx}{\sigma^2}) + d \cdot \exp(-\frac{d \cdot x}{\sigma^2})}{\exp(-\frac{cx}{\sigma^2}) + \exp(-\frac{d \cdot x}{\sigma^2})} \equiv b - a \tanh \frac{ax}{\sigma^2} = u^{a,b}(0, x).$$

Thus, the pointwise convergence in (8.15) is proved. It is not too difficult to justify the uniform convergence as well. Theorem 8.1 is proved.

TABLE 5. Dependence of the errors $err_{\bar{u}}$ and $sherr_{\bar{u}}$ in h and t under fixed $\sigma = 0.5$

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$
$t = 10$	0.3098 $0.1866 \cdot 10^{-1}$	$0.3751 \cdot 10^{-1}$ $0.7637 \cdot 10^{-3}$	$0.3829 \cdot 10^{-2}$ $0.2068 \cdot 10^{-3}$
$t = 20$	0.5324 $0.1906 \cdot 10^{-1}$	$0.7048 \cdot 10^{-1}$ $0.9203 \cdot 10^{-3}$	$0.7153 \cdot 10^{-2}$ $0.1264 \cdot 10^{-3}$
$t = 30$	0.6970 $0.1900 \cdot 10^{-1}$	0.1033 $0.9212 \cdot 10^{-3}$	$0.1048 \cdot 10^{-1}$ $0.1274 \cdot 10^{-3}$

Table 5 gives numerical results obtained by using the algorithm (5.10)-(5.11) with $h_t = h_x = h$ to the Cauchy problem (8.5), (8.9) under $\sigma = 0.5$. The table presents the usual errors $err_{\bar{u}} = err^c_{\bar{u}}$ of approximate solution $\bar{u}$ and the distances $sherr_{\bar{u}}$ of $\bar{u}$ (shape errors) from the shape determined by $u^{0.5,0.5}(0, x) = \frac{1}{2} - \frac{1}{2} \tanh \dfrac{x}{2\sigma^2}$. These distances are calculated by the formula

$$sherr_{\bar{u}} = \max_{x_i} |\bar{u}(t, x_i + \bar{m}(t)) - u^{0.5,0.5}(0, x_i)|,$$

where $\bar{m}(t)$ is a root of the equation $\bar{u}(t, x) = \dfrac{1}{2}$.

We see that the shape error $sherr_{\bar{u}}$ is stabilized as t becomes infinite and it tends to zero if h tends to zero. This proves that the solution of the procedure (5.10)-(5.11) has a limit shape which is close to the limit shape of the solution of the problem (8.5), (8.9) under small h.

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