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# Topics in Transcendental Dynamics



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A thesis submitted for the degree of

*Doctor of Philosophy*

to

Department of Mathematics and Statistics,  
The Open University,  
Milton Keynes, United Kingdom

03 July 2013

## **Declaration**

This thesis is the result of my own work, except where explicit reference is made to the work of others, and has not been submitted for another qualification to this or any other university.

David Sixsmith

03 July 2013

## Abstract

We study the iteration of a transcendental entire function,  $f$ ; in particular, the fast escaping set,  $A(f)$ . This set consists of points that iterate to infinity as fast as possible, and plays a significant role in transcendental dynamics.

First we investigate functions for which  $A(f)$  has a structure called a spider's web. We construct several new classes of function with this property. We show that some of these classes have a degree of stability under changes in the function, and that new examples of functions with this property can be constructed by composition, by differentiation, and by integration of existing examples. We use a property of spiders' webs to give new results concerning functions with no unbounded Fatou components.

When  $A(f)$  is a spider's web, it contains a sequence of fundamental loops. We next explore the structure of these fundamental loops for functions with a multiply connected Fatou component, and show that there exist functions for which some fundamental loops are analytic curves and approximately circles, while others are geometrically highly distorted. We do this by introducing a real-valued function which measures the rate of escape of points in  $A(f)$ , and show that this function has a number of interesting properties.

Next we study functions with a simply connected Fatou component in  $A(f)$ . We give an example of a function with this property, which – in contrast to the only other known functions of this type – has no multiply connected Fatou components. To do this we also prove a new criterion for points to be in  $A(f)$ .

Finally, we investigate the much studied Eremenko-Lyubich class of transcendental entire functions with a bounded set of singular values. We give a new characterisation of this class, and a new result regarding direct singularities which are not logarithmic.

## Acknowledgements

It is not possible to thank everyone who has helped – in some way – with this work. So I highlight three significant groups of people, without whom this thesis could never have been completed.

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Most importantly, I would like to thank my family. Their support was constant, and they put up with a good deal. This thesis is dedicated to them.

## Publications

A significant part of the content of this thesis has led to publications, as follows:

- (1) Most of the content of Chapter 2 appeared in the *Mathematical Proceedings of the Cambridge Philosophical Society* [97].
- (2) Most of the content of Chapter 3 will appear in the *Journal of the London Mathematical Society* [99].
- (3) The content of Chapter 4 has appeared in the *Pure and Applied Mathematics Quarterly* [100].
- (4) The content of Chapter 5 will appear in the *Journal d'Analyse Mathématique* [98].

# Contents

<b>Contents</b>	<b>v</b>
<b>List of Figures</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Structure of this thesis . . . . .	1
1.2 Preliminary material . . . . .	1
1.3 History and background . . . . .	5
1.4 The Fatou and Julia sets . . . . .	10
1.5 Fatou components . . . . .	12
1.6 The escaping set and Eremenko's conjectures . . . . .	17
1.7 The fast escaping set . . . . .	19
1.8 Cantor bouquets . . . . .	20
1.9 Spiders' webs . . . . .	21
<b>2 Escaping set spiders' webs</b>	<b>25</b>
2.1 Introduction . . . . .	25
2.2 New results on regularity . . . . .	26
2.3 Using composition to give functions for which $A_R(f)$ is a spider's web . . . . .	37
2.4 Stability of $A_R(f)$ spiders' webs . . . . .	42
2.5 Functions of finite order for which $A_R(f)$ is a spider's web . . . . .	45
2.6 A function of infinite order with gaps for which $A_R(f)$ is a spider's web . . . . .	50

2.7	Transcendental entire functions with no unbounded Fatou components . . . . .	56
<b>3</b>	<b>On fundamental loops and the fast escaping set</b>	<b>59</b>
3.1	Introduction . . . . .	59
3.2	Background results . . . . .	63
3.3	The hyperbolic metric and hyperbolic distance . . . . .	67
3.4	A map on a nested sequence of domains . . . . .	68
3.5	The function $R_A$ defined in a multiply connected Fatou component	71
3.6	Proofs of Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3 . . . .	80
3.7	The function $R_A$ defined in $\mathbb{C}$ . . . . .	85
<b>4</b>	<b>Fast escaping Fatou components</b>	<b>95</b>
4.1	Introduction . . . . .	95
4.2	The definition of the function . . . . .	96
4.3	There are no multiply connected Fatou components . . . . .	99
4.4	There are simply connected Fatou components . . . . .	100
4.5	The simply connected Fatou components are fast escaping . . . .	103
4.6	Appendix: defining the sequences . . . . .	110
<b>5</b>	<b>On the Eremenko-Lyubich class</b>	<b>117</b>
5.1	Introduction . . . . .	117
5.2	Singularities of the inverse function . . . . .	120
5.3	Direct non-logarithmic transcendental singularities . . . . .	121
5.4	Proof of the main result . . . . .	125
<b>6</b>	<b>Questions for further research</b>	<b>130</b>
6.1	Introduction . . . . .	130
6.2	Some families of transcendental entire functions . . . . .	130
6.3	A partition of the fast escaping set . . . . .	132
6.4	The behaviour of $M^n(R_A(z))$ as $n \rightarrow \infty$ . . . . .	134
	<b>References</b>	<b>135</b>



# List of Figures

1.1	Multiply connected wandering Fatou components. . . . .	15
1.2	Simply connected wandering Fatou components. . . . .	16
1.3	Both simply and multiply connected wandering Fatou components.	17
2.1	A slice through parts of the boundaries of some of the domains used in the proof of Lemma 2.3.5. . . . .	40
2.2	An approximation to part of $A(g)$ , where $g(z) = \cos z + \cosh z$ . . .	50
3.1	Subsets of the boundary of a multiply connected Fatou component.	60
3.2	A fundamental loop contained in a multiply connected Fatou com- ponent. . . . .	62
3.3	Annuli within a multiply connected Fatou component. . . . .	64
3.4	The construction in the proof of Lemma 3.5.2. . . . .	73
3.5	The construction used in the proof of Lemma 3.6.1. . . . .	81
5.1	The construction used in the proof of Theorem 5.1.2. . . . .	124
5.2	The sets and functions used in part of the proof of Theorem 5.1.1.	128

# Chapter 1

## Introduction

### 1.1 Structure of this thesis

The structure of this thesis is as follows. In this introductory chapter we give some of the history of the study of complex dynamics, along with some parts of the classical theory of transcendental dynamics used in subsequent chapters. We give a number of necessary definitions, and various useful and well-known results.

All new results are given in the four subsequent chapters, during which we introduce additional background results and concepts when these are required. At the end of the thesis we give a brief chapter containing interesting questions and suggestions for further study. These questions arise from the earlier material.

### 1.2 Preliminary material

We first give some notation used throughout the thesis. We denote the complex plane by  $\mathbb{C}$ , and the Riemann sphere by  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . For a disc in the complex plane we define

$$B(\zeta, r) = \{z : |z - \zeta| < r\}, \quad \text{for } \zeta \in \mathbb{C}, r > 0. \quad (1.1)$$

In Section 5 only, we need to define a disc in the Riemann sphere. In the case

that the centre of the disc is finite we use (1.1), and otherwise we define

$$B(\infty, r) = \{ z : |z| > r \}, \quad \text{for } r > 0.$$

For an annulus we write

$$A(r_1, r_2) = \{ z : r_1 < |z| < r_2 \}, \quad \text{for } 0 < r_1 < r_2.$$

We use the following notation for three special domains in the complex plane. We denote the unit disc by

$$\mathbb{D} = B(0, 1),$$

the punctured unit disc by

$$\mathbb{D}^* = B(0, 1) \setminus \{0\},$$

and the left half-plane by

$$\mathbb{H} = \{ z : \operatorname{Re}(z) < 0 \}.$$

We denote the integers greater than zero by  $\mathbb{N}$ .

We often use phrases such as ‘for  $r \geq R_0$ ’. By this we mean that the condition is satisfied by *all* values of  $r$  greater than or equal to  $R_0$ .

If  $f$  is a transcendental entire function, then we denote by  $f^n$ ,  $n \in \mathbb{N}$ , the  $n$ th iterate of  $f$ . If  $D \subset \mathbb{C}$  and  $n \in \mathbb{N}$ , then we write  $f^{-n}(D)$  to denote the set

$$\{ z : f^n(z) \in D \}.$$

We call a point  $z$  *periodic* if  $f^n(z) = z$ , for some  $n \in \mathbb{N}$ . We say that  $z$  is a *repelling periodic point* if, in addition,  $|(f^n)'(z)| > 1$ .

We frequently need to use the *maximum modulus function*. This is defined by

$$M(r, f) = \max_{|z|=r} |f(z)|, \quad \text{for } r \geq 0. \quad (1.2)$$

We sometime write this as  $M(r)$ , provided that it is clear from the context which function  $f$  is being considered. We write  $M^n(r, f)$  to denote repeated iteration of  $M(r, f)$  with respect to the variable  $r$ .

We also use the *minimum modulus function* defined by

$$L(r, f) = \min_{|z|=r} |f(z)|, \quad \text{for } r \geq 0.$$

The *order*  $\rho(f)$  and *lower order*  $\lambda(f)$  of a transcendental entire function  $f$  are defined by

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} \quad \text{and} \quad \lambda(f) = \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}. \quad (1.3)$$

If  $f$  is a transcendental entire function, then  $\rho(f) = \rho(f')$ ; see, for example, [35, p.286].

We note from, for example, [62] that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$\rho(f) = \limsup_{n \rightarrow \infty} \frac{n \log n}{\log |a_n|^{-1}} \quad (1.4)$$

and

$$\lambda(f) = \max_{(n_p)} \liminf_{p \rightarrow \infty} \frac{n_p \log n_{p-1}}{\log |a_{n_p}|^{-1}}. \quad (1.5)$$

We note that, in (1.5), the maximum is taken over all increasing sequences of positive integers  $(n_p)$ . We note also that it is explicitly shown in [62] that the maximum is achieved.

Throughout the thesis we use the following well-known facts about the maximum modulus function  $M(r, f)$  of a transcendental entire function  $f$ .

**Lemma 1.2.1.** *Suppose that  $f$  is a transcendental entire function, and define  $\phi(t) = \log M(e^t, f)$ . Then the following hold.*

(a)  $M(r, f)$  is a continuous function of  $r$ .

(b)

$$\frac{\log M(r, f)}{\log r} \rightarrow \infty \text{ as } r \rightarrow \infty. \quad (1.6)$$

(c)

$$\text{if } k > 1 \text{ then } \frac{M(kr, f)}{M(r, f)} \rightarrow \infty \text{ as } r \rightarrow \infty. \quad (1.7)$$

(d)  $\phi(t)$  is a convex and increasing function of  $t$ .

(e)  $\phi(t)$  has a derivative at all but at most a countable set of points.

We also use the following [27, Theorem 2.2], generally with  $n = 1$ .

**Lemma 1.2.2.** *Let  $f$  be a transcendental entire function. Then there exists  $R_0 > 0$  such that, for  $0 < c' < 1 < c$  and all  $n \in \mathbb{N}$ ,*

$$M(r^c, f^n) \geq M(r, f^n)^c, \quad \text{for } r > R_0, \quad (1.8)$$

and

$$M(r^{c'}, f^n) \leq M(r, f^n)^{c'}, \quad \text{for } r > R_0^{1/c'}. \quad (1.9)$$

We also use a result similar to Lemma 1.2.2 which applies when  $f$  is a polynomial.

**Lemma 1.2.3.** *Let  $f$  be a non-constant polynomial, and suppose that  $0 < c' < c$ . Then there exists  $R_0 > 0$  such that,*

$$M(r^c, f) \geq M(r, f)^{c'}, \quad \text{for } r > R_0. \quad (1.10)$$

*Proof.* This follows because there exist  $a > 0$ ,  $n \in \mathbb{N}$  and  $R' > 0$  such that

$$\frac{1}{2}ar^n \leq M(r, f) \leq 2ar^n, \quad \text{for } r \geq R'.$$

□

We denote the inverse function of  $M$ , when this is defined, by  $M^{-1}$ . For simplicity we write  $M^{-n}$ , for  $n \in \mathbb{N}$ , to denote  $n$  repeated iterations of  $M^{-1}$ . Observe that  $M^{-1}(r)$  is defined for  $r \in [|f(0)|, \infty)$  and is strictly increasing. Moreover, by Lemma 1.2.1 (d) and [65, Theorem 7.2.2], we have that  $\log M^{-1}(e^s)$  is a concave and increasing function of  $s$ . Also, if  $R_0$  is the constant from Lemma 1.2.2, then it follows from (1.8) that

$$M^{-1}(r^c) \leq M^{-1}(r)^c, \quad \text{for } r > \max\{M(R_0), |f(0)|\}, \quad c > 1. \quad (1.11)$$

### 1.3 History and background

In this section we give a brief outline of the history of the study of complex dynamics, and some of the key developments which have motivated the research in this thesis. Further details are given in subsequent sections.

The origins of the detailed study of complex dynamics lie with Fatou [43–45] and Julia [61] in the early part of the last century. These early papers all consider the iteration of rational functions. In this thesis we are concerned only with the iteration of transcendental entire functions, which was considered by Fatou in 1926 [46]. An introduction to the theory of the iteration of transcendental entire and transcendental meromorphic functions can be found in [18], and also in [94].

In honour of the two mathematicians noted in the previous paragraph, the most fundamental objects of study in this field are named the Fatou and Julia sets. The *Fatou set*  $F(f)$  is defined as the set of points  $z \in \mathbb{C}$  such that  $(f^n)_{n \in \mathbb{N}}$  is a normal family in a neighbourhood of  $z$ . The *Julia set*  $J(f)$  is the complement in  $\mathbb{C}$  of  $F(f)$ .

After the work of mathematicians such as Fatou, Julia, Lattès (who, for example, gave a large family of rational functions for which the Julia set is the whole complex plane [68]) and Ritt (who also studied the iteration of rational maps [90]), the field of complex dynamics was relatively quiet in the 20th century for several decades. An important exception to this observation is that of Baker, who worked extensively in this area from the 1950s onwards. Baker proved a number of important results on the iteration of transcendental entire functions, and we often refer to these in this thesis. A summary of his life and work was

given in [80].

A renaissance in the study of complex dynamics occurred in the 1980s. One factor in this was, perhaps, the fact that improvements in computer technology enabled mathematicians to experiment with and to illustrate some of the complicated geometric objects involved in this study.

A second key factor was the introduction of new techniques from other areas of mathematics, which enabled long-standing problems to be solved. Arguably the most significant of these was Sullivan's use of quasiconformal mappings in [104] to show that it is not possible for a rational function to have a wandering Fatou component. In other words, for a rational function all components of the Fatou set are eventually periodic; see Section 1.5 for definitions of these concepts. This result led to significantly increased interest in the study of the dynamics of rational functions.

It is possible, however, for a transcendental entire function to have a wandering Fatou component. Baker gave an example of a transcendental entire function with this property in 1963 [5], although he did not show that the multiply connected Fatou component of the function in this paper is bounded and hence wandering until 1976 [7]. We give an example of a transcendental entire function with a wandering Fatou component with novel properties in Chapter 4.

A third set which has generated significant interest is the *escaping set*  $I(f)$ , which was first investigated for a general transcendental entire function by Eremenko [39]. It is defined by

$$I(f) = \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Eremenko proved that if  $f$  is a transcendental entire function then  $I(f) \cap J(f)$  is not empty,  $J(f) = \partial I(f)$ , and  $\overline{I(f)}$  has no bounded components. Eremenko also conjectured that  $I(f)$  has no bounded components. This important conjecture – which we refer to simply as *Eremenko's conjecture* – remains open, and attempts to resolve it have led to significant progress in the study of transcendental dynamics. We discuss the escaping set and Eremenko's conjecture in more detail in Section 1.6.

In general, one can often gain a better understanding of a set by studying

some of its subsets. Rippon and Stallard, first in [82] and then more fully in [86], took this approach by considering a subset of the escaping set known as the *fast escaping set*,  $A(f)$ . Much of the work in this thesis relates to  $A(f)$ . This set was introduced by Bergweiler and Hinkkanen [25], and can be defined by

$$A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\}; \quad (1.12)$$

see [86] for this form of the definition of  $A(f)$ . Here  $R > 0$  can be taken to be any value such that  $M(r) > r$  for  $r \geq R$ . For simplicity, we only write down this restriction on  $R$  in formal statements of results – elsewhere this should be assumed to be true.

As observed in [86], since  $|f(z)| < M(r)$ , for  $|z| < r$ , the conditions  $\ell \in \mathbb{N}$  and  $n \in \mathbb{N}$  in (1.12) could be replaced by the conditions  $\ell \geq 0$  and  $n \geq 0$ . We have chosen to write  $\ell \in \mathbb{N}$  and  $n \in \mathbb{N}$  in order to be consistent with the definition given in [86].

The arguments in [86] regarding the properties of  $A(f)$  were frequently based on properties of the set

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\},$$

where  $R > 0$  is such that  $M(r) > r$  for  $r \geq R$ . Many of the arguments use the fact that  $A_R(f)$  is closed, together with the set equality

$$A(f) = \bigcup_{\ell=0}^{\infty} f^{-\ell}(A_R(f)).$$

An important result is the following [86, Theorem 1.1] (see also [82, Theorem 1]).

**Theorem 1.3.1.** *Let  $f$  be a transcendental entire function, and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Then each component of  $A_R(f)$  is closed and unbounded, and hence each component of  $A(f)$  is unbounded.*

This result provides a partial result regarding Eremenko's conjecture; since  $A(f) \subset I(f)$ , and since it follows from [39] that  $A(f)$  is not empty, then  $I(f)$



certainly has at least one unbounded component. We discuss the fast escaping set in more detail in Section 1.7.

Many of the results in [86] relate to a structure known as spider's web, which is defined as follows. A set  $E$  is a *spider's web* if  $E$  is connected and there exists a sequence of bounded simply connected domains  $(G_n)_{n \in \mathbb{N}}$  such that

$$\partial G_n \subset E, G_n \subset G_{n+1}, \text{ for } n \in \mathbb{N}, \text{ and } \bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}. \quad (1.13)$$

It was shown in [86] that there are many transcendental entire functions  $f$  for which  $A_R(f)$  has this structure; for example, this is the case when  $f$  has a multiply connected Fatou component. As noted in [86] – and see also [83] – there are also many transcendental entire functions,  $f$ , such that  $f$  has no multiply connected Fatou components and  $A_R(f)$  is a spider's web. For example, if  $f$  has sufficiently small growth, then  $A_R(f)$  is a spider's web; see Theorem 1.9.2 (b), below, for a precise statement of this condition. Bergweiler and Eremenko showed [22] (see also [23]) that there are functions of arbitrarily small growth – and hence functions which satisfy this condition – for which the Fatou set is empty.

If  $A_R(f)$  is a spider's web then  $A(f)$  and  $I(f)$  are also spiders' webs [86, comments following Theorem 1.4],  $f$  has no unbounded Fatou components, see Theorem 1.9.1(d) below, and Eremenko's conjecture holds. We note that, trivially, the whole complex plane is a spider's web. However, if  $A_R(f)$  is a spider's web, then it cannot equal the whole complex plane since, for example, there exist periodic points, by Theorem 1.4.1(e). We give details of the intricate topological structure of an  $A_R(f)$  spider's web in Section 1.9.

In Chapter 2 we give a number of new classes of functions for which  $A_R(f)$  is a spider's web, including the simple example  $f(z) = \cos z + \cosh z$ .

To understand the structure of  $A_R(f)$  spiders' webs, Rippon and Stallard [86] introduced *fundamental holes* and *fundamental loops*. When  $A_R(f)$  is a spider's web, we define the fundamental hole  $H_R$  as the component of  $A_R(f)^c$  that contains the origin, and the fundamental loop  $L_R$  by  $L_R = \partial H_R$ . Since  $A_R(f)$  is closed, we have that  $L_R \subset A_R(f)$ . We use the following theorem, which is part of [86, Lemma 7.2].

**Theorem 1.3.2.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider's web. Then*

$$f^n(H_R) = H_{M^n(R)} \quad \text{and} \quad f^n(L_R) = L_{M^n(R)}, \quad \text{for } n \in \mathbb{N}.$$

Our notation here differs slightly from that in [86]. For  $R > 0$  fixed, Rippon and Stallard define sets

$$A_R^m(f) = \{z : |f^n(z)| \geq M^{n+m}(R), n \in \mathbb{N}\}, \quad \text{for } m \geq 0,$$

and define the sequence of fundamental holes to be the components of  $A_R^m(f)^c$  that contain the origin. Denoting this sequence by  $(H'_m)_{m \geq 0}$ , we observe that these notations are related by the equation

$$H_{M^m(R)} = H'_m, \quad \text{for } m \geq 0.$$

In Chapter 3 we investigate the structure of these fundamental loops for functions with a multiply connected Fatou component. We discuss spiders' webs in more detail in Section 1.9.

Suppose that  $f$  is a transcendental entire function and that  $f'(z) = 0$ . Then we say that  $z$  is a *critical point*, and  $w = f(z)$  is a *critical value* of  $f$ . We call a curve  $\Gamma : (0, 1) \rightarrow \mathbb{C}$  an *asymptotic curve* with *asymptotic value*  $a$  if, as  $t \rightarrow 1$ , we have both  $\Gamma(t) \rightarrow \infty$  and  $f(\Gamma(t)) \rightarrow a$ . The set formed by the union of the critical and finite asymptotic values of a transcendental entire function  $f$  coincides with the set of singularities of the inverse function, and is denoted by  $\text{sing}(f^{-1})$ .

If  $f$  is a transcendental entire function such that  $\text{sing}(f^{-1})$  is bounded, then we say that  $f$  belongs to the *Eremenko-Lyubich class*,  $\mathcal{B}$ . This class was introduced to complex dynamics in [41]. A particularly important result given in this paper is [41, Theorem 1] that  $F(f) \cap I(f) = \emptyset$ , for  $f \in \mathcal{B}$ .

Examples of functions in this class include functions in the *exponential family*

$$\{f : f(z) = \lambda \exp(z), \lambda \neq 0\},$$

and functions in the *cosine family*

$$\{f : f(z) = \cos(\alpha z + \beta), \alpha \neq 0\}.$$

Functions in the Eremenko-Lyubich class have a number of strong properties outside a bounded domain which includes  $\text{sing}(f^{-1})$ . As a result, this class has been widely studied. Papers which study the structure of the escaping set for functions in this class include the important paper [91] discussed in Section 1.6, and Rempe's paper [77], the results of which explain the observation that there are striking similarities between the Julia sets of many transcendental entire functions. Papers which concern the dimensions of the Julia set and the escaping set include Stallard's result [102] that the Julia sets of functions in this family have Hausdorff dimension strictly greater than one, and Barański, Karpińska, and Zdunik's paper [12] which generalised this result to show that the Julia sets of functions in this family have hyperbolic dimension strictly greater than one. Bishop [29] gave an example of a function in this class with a wandering Fatou component. Finally, papers studying the value distribution of functions in this class include [66], which concerns the Nevanlinna deficiency of functions in this class, [67], which gives results on the fixed points of functions in this class, and [73] which concerns the Nevanlinna deficiency of the derivatives of some functions in this class. We refer to the papers themselves for the definitions of some of these terms.

In Chapter 5 we give a new characterisation of this class of functions.

## 1.4 The Fatou and Julia sets

Fundamental properties of the Fatou and Julia sets of a transcendental entire function are given in the following theorem; see, for example, [18, Lemma 1, Lemma 2, Lemma 3, Theorem 3 and Theorem 4]. Here we say that a set  $S$  is *completely invariant* if  $z \in S$  implies  $f(z) \in S$ , and  $f(z) \in S$  implies  $z \in S$ . Recall also that a set is *perfect* if it is closed, non-empty and has no isolated points.

**Theorem 1.4.1.** *Suppose that  $f$  is a transcendental entire function. Then the following hold.*

- (a)  $F(f) = F(f^n)$  and  $J(f) = J(f^n)$ , for  $n \geq 2$ .
- (b)  $F(f)$  and  $J(f)$  are completely invariant.
- (c) Either  $J(f) = \mathbb{C}$  or  $\text{int}(J(f)) = \emptyset$ .
- (d)  $J(f)$  is perfect.
- (e)  $J(f)$  is the closure of the set of repelling periodic points of  $f$ .

The first example of a transcendental entire function for which  $J(f) = \mathbb{C}$  was given by Baker [6], who showed that, for a suitable value of  $k > 0$ , this is the case for  $f(z) = kze^z$ .

A useful sufficient condition for a point to lie in the Fatou set can be obtained from Montel's Theorem. There are a number of versions of this result, we use the simplest; see, for example, [93, p.54].

**Theorem 1.4.2** (Montel). *Suppose that  $\mathcal{V}$  is a family of analytic functions on a domain  $\Delta$ . Suppose also that there exist distinct points  $w_1, w_2 \in \mathbb{C}$  such that*

$$g(w) \notin \{w_1, w_2\}, \quad \text{for all } g \in \mathcal{V} \text{ and } w \in \Delta.$$

*Then  $\mathcal{V}$  is a normal family in  $\Delta$ .*

When  $f$  is a transcendental entire function and  $z \in \mathbb{C}$ , then we can often show that  $z \in F(f)$  by applying Theorem 1.4.2 with  $\mathcal{V} = \{f^n\}_{n \in \mathbb{N}}$ , and with  $\Delta$  a neighbourhood of  $z$ .

A complementary property of transcendental entire functions, which reflects the chaotic nature of the Julia set, is the following well-known result [83, Lemma 2.1].

**Theorem 1.4.3.** *Let  $f$  be a transcendental entire function, let  $K$  be a compact set with  $K \cap E(f) = \emptyset$  and let  $\Delta$  be a neighbourhood of  $z \in J(f)$ . Then there exists  $N \in \mathbb{N}$  such that  $f^n(\Delta) \supset K$ , for  $n \geq N$ .*

Here

$$E(f) = \{z : O^-(z) \text{ is finite}\} \tag{1.14}$$

and

$$O^-(z) = \{w : f^n(w) = z, \text{ for some } n \in \mathbb{N}\}.$$

It follows from the big Picard theorem that the set  $E(f)$  contains at most one point.

## 1.5 Fatou components

Since  $F(f)$  is open, it consists of at most countably many connected components, called *Fatou components*. Suppose that  $U = U_0$  is a Fatou component. We denote by  $U_n$  the Fatou component containing  $f^n(U)$ , for  $n \in \mathbb{N}$ . For  $n \in \mathbb{N}$ , we have that  $U_n \setminus f^n(U)$  can contain at most one point [56, Corollary 3]. If  $U$  is bounded, then  $f^n : U \rightarrow U_n$  is a proper map and so  $U_n = f^n(U)$  [56, Corollary 1]. Note that if  $U$  and  $V$  are domains and  $f : U \rightarrow V$  is continuous, then  $f$  is a *proper map* of  $U$  onto  $V$  if and only if the preimage of every relatively compact subset of  $V$  is a relatively compact subset of  $U$ . If  $f$  is a transcendental entire function and  $U$  and  $V$  are bounded domains, then it follows from the open mapping theorem that  $f$  is a proper map of  $U$  onto  $V$  if and only if  $f(\partial U) = \partial V$ .

Fatou components may be classified as follows. If  $U_p = U$ , for some least  $p \in \mathbb{N}$ , then we say that  $U$  is *periodic*, with period  $p$ . If  $U$  is not periodic, but  $U_q$  is periodic, for some  $q \in \mathbb{N}$ , then we say that  $U$  is *pre-periodic*. The remaining possibility is that  $U_n = U_m$  implies that  $n = m$ ; we call such a component *wandering*. These components are often called *wandering domains*. As noted earlier, Sullivan [104] showed that it is not possible for a rational function to have wandering Fatou components. It has further been shown that there are a number of classes of transcendental entire functions which do not have wandering Fatou components. For example, Eremenko and Lyubich [41], and also Goldberg and Keen [48], showed that this is the case for transcendental entire functions for which  $\text{sing}(f^{-1})$  is finite, and Stallard [101] showed that this is the case for entire

and meromorphic functions of the form

$$f(z) = z + R(z)e^{P(z)},$$

where  $P$  is a non-constant polynomial and  $R$  is a non-constant rational function. Note that the proofs of all these results use Sullivan's ideas.

There exists a well-known classification of periodic Fatou components; see, for example, [18, Theorem 6]. We do not need this classification here, as we are primarily interested in wandering Fatou components.

We distinguish between Fatou components which are simply connected and those which are multiply connected. The following important result regarding multiply connected Fatou components was proved by Baker [9, Theorem 3.1]. We often use this result without comment. Note that we say that a set  $U$  *surrounds* a set  $V$  if and only if  $V$  is contained in a bounded component of  $\mathbb{C} \setminus U$ . We also write  $\text{dist}(z, U) = \inf_{w \in U} |z - w|$ .

**Theorem 1.5.1.** *Suppose that  $f$  is a transcendental entire function and that  $U$  is a multiply connected Fatou component of  $f$ . Then each  $U_n$  is bounded and multiply connected,  $U_{n+1}$  surrounds  $U_n$  for large  $n$ , and  $\text{dist}(0, U_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .*

An immediate corollary of Theorem 1.5.1 is that all multiply connected components of a transcendental entire function are wandering. It also follows that if  $U$  is a multiply connected Fatou component, then  $U_n = f^n(U)$ , for  $n \in \mathbb{N}$ .

Zheng [107] showed that, if  $U$  is a multiply connected Fatou component, then for sufficiently large values of  $n$ ,  $U_n$  contains an annulus  $A(r_n, R_n)$ , such that  $R_n/r_n \rightarrow \infty$  as  $n \rightarrow \infty$ . This result was strengthened by Bergweiler, Rippon and Stallard in a recent paper [27] which gave a detailed study of the dynamics of a transcendental entire function in a multiply connected Fatou component. We use the following result, which is given in [27, comments following Theorem 1.2].

**Theorem 1.5.2.** *Suppose that  $f$  is a transcendental entire function with a multiply connected Fatou component  $U = U_0$ , and let  $z_0 \in U$  be fixed. Then there exists  $\alpha > 0$  such that, for large  $n$ , the maximum annulus centred at the origin, contained in  $U_n$  and containing  $f^n(z_0)$ , is of the form*

$$B_n = A(r_n^{a_n}, r_n^{b_n}), \text{ where } r_n = |f^n(z_0)|, \ 0 < a_n < 1 - \alpha < 1 + \alpha < b_n. \quad (1.15)$$

We use a number of other results from [27]; see Section 3.2.

The first example of a transcendental entire function with a multiply connected Fatou component was constructed by Baker in [5]. This function has the form

$$f(z) = cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right).$$

The sequence of positive real numbers  $(a_n)_{n \in \mathbb{N}}$  is chosen so that the annuli

$$A_n = A(a_n^2, a_{n+1}^{\frac{1}{2}})$$

have the property  $f(A_n) \subset A_{n+1}$ , for  $n \in \mathbb{N}$ . The existence of multiply connected Fatou components can be shown to follow from this fact; see Figure 1.1 for an illustration of the dynamics of this function. In this figure a multiply connected Fatou component, labeled  $U$ , is shown being mapped by iterates of  $f$  to multiply connected Fatou components which surround  $U$ . All these Fatou components are shown in grey. The figure is schematic, and it should not be interpreted, for example, that the boundaries of the Fatou components are smooth. For simplicity, Figure 1.1 suggests that these Fatou components are doubly connected. However, it was recently shown by Bergweiler and Zheng [28] that they are in fact infinitely connected.

Other papers giving functions with multiply connected wandering Fatou components include:

- Baker's paper [10], in which it is shown that multiply connected wandering Fatou components may occur for functions of any order;
- Hinkkanen's paper [57], in which it is shown that multiply connected wandering Fatou components may occur for functions of arbitrarily small growth;
- Bergweiler's paper [20], which is further discussed below (see text before Figure 1.3);
- Bergweiler, Rippon and Stallard's paper [27, Section 10], in which examples of transcendental entire functions with multiply connected wandering Fatou components of different geometries are given;

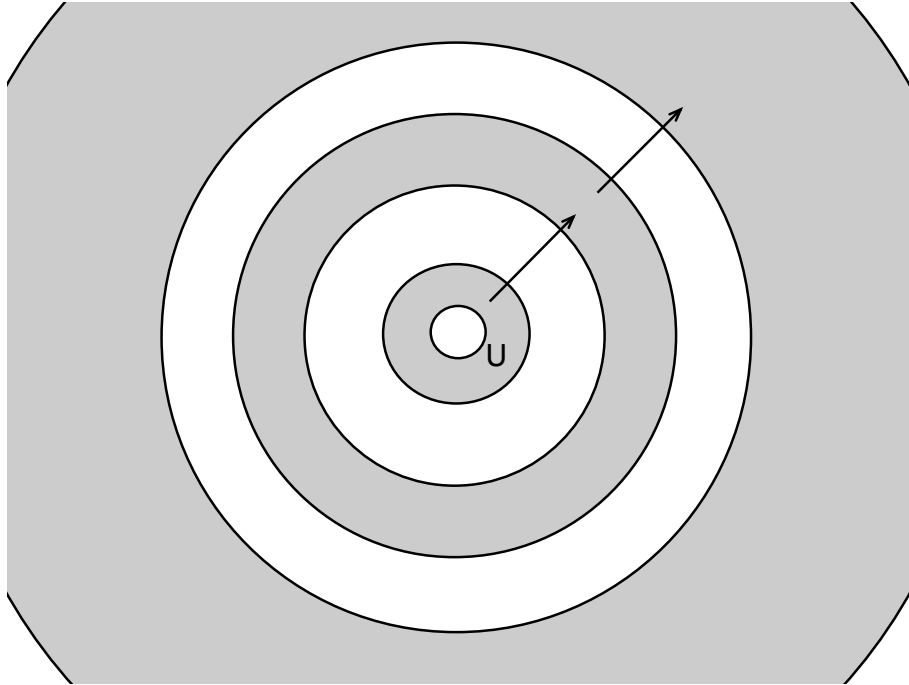


Figure 1.1: Multiply connected wandering Fatou components.

- and Kisaka and Shishikura's paper [63], which gave the first example of a transcendental entire function with a doubly connected wandering Fatou component.

A recent paper which gives a transcendental entire function with a multiply connected Fatou component is [30]. The transcendental entire function in [30] has a Julia set of Hausdorff dimension 1. This resolved a long-standing question whether, for a transcendental entire function  $f$ , it was always the case that the Hausdorff dimension of  $J(f)$  is strictly greater than 1.

An example of transcendental entire function with a simply connected wandering Fatou component is

$$f(z) = z - 1 + e^{-z} + 2\pi i,$$

which was introduced by Herman, quoted in [103] and described in detail by Baker [9].

It was shown in [9] that, for  $n \in \mathbb{Z}$ ,  $f$  has a simply connected wandering



Fatou component which contains a unique  $2n\pi i$  translation of the real line. On iteration of  $f$ , each of these wandering Fatou components is translated by  $2\pi i$ . See Figure 1.2 for an illustration of the dynamics of  $f$ . In this figure a simply connected Fatou component, labeled  $U$ , is shown being mapped by iterates of  $f$  to further simply connected Fatou components. Each iteration translates a simply connected Fatou component by  $2\pi i$ . All these Fatou components are shown in grey. The figure is schematic, and it should not be interpreted, for example, that the boundaries of the Fatou components are smooth.

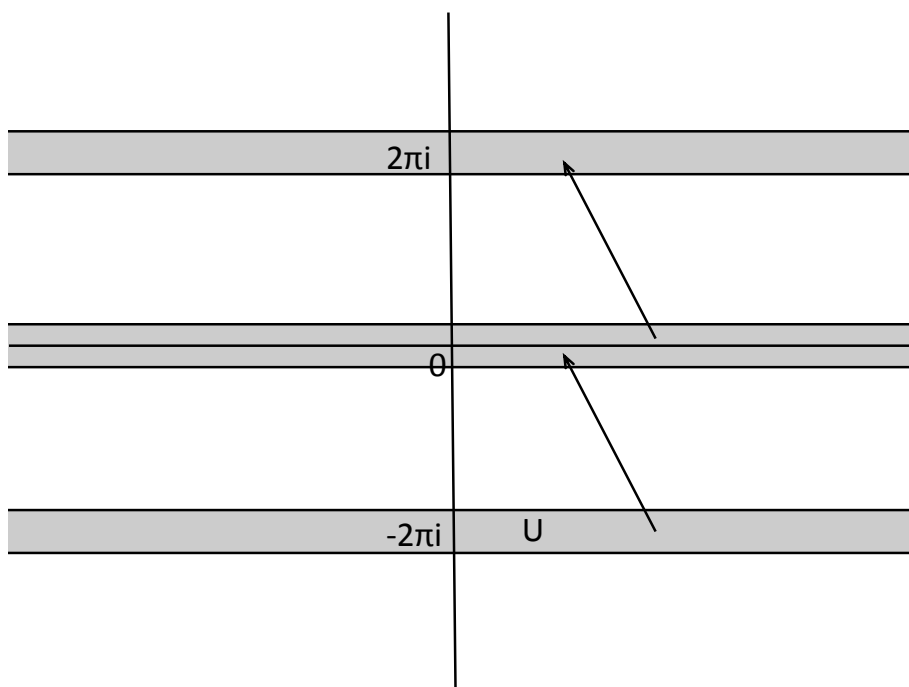


Figure 1.2: Simply connected wandering Fatou components.

Other functions with simply connected wandering Fatou components include:

- the function  $f(z) = 2 - \log 2 + 2z - \exp(z)$  discussed in [19];
- a function constructed in [40] using approximation theory, which has a wandering Fatou component which is not in the escaping set;
- and the function  $f(z) = z + \lambda \sin(2\pi z) + 1$ , for certain values of  $\lambda \in \mathbb{C}$ , discussed in [55, p.106].

We give an explicit construction of a new example of a transcendental entire function with this property in Chapter 4. Note that the function in [20], mentioned earlier, has both simply and multiply connected wandering Fatou components; see Figure 1.3 for an illustration of the dynamics of this function. In this figure a simply connected Fatou component, labeled  $V$  and shown in black, is shown being mapped by iterates of  $f$  to further simply connected Fatou components. A multiply connected Fatou component, labeled  $U$  and shown in grey, is shown being mapped by iterates of  $f$  to further multiply connected Fatou components. The figure is schematic, and it should not be interpreted, for example, that the boundaries of the Fatou components are smooth.

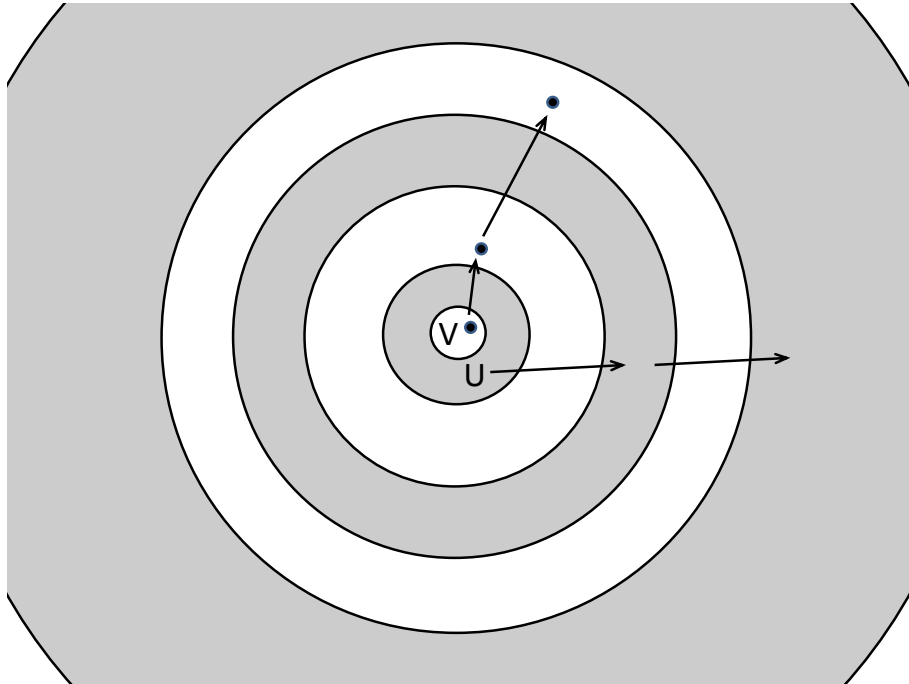


Figure 1.3: Both simply and multiply connected wandering Fatou components.

## 1.6 The escaping set and Eremenko's conjectures

In this section we briefly discuss some important properties of the escaping set,  $I(f)$ . Fundamental properties of the escaping set of a transcendental entire function are given in the following theorem; the first two parts of this theorem

follow immediately from the definition of  $I(f)$ , the remainder of the theorem is proved in the seminal paper of Eremenko [39].

**Theorem 1.6.1.** *Suppose that  $f$  is a transcendental entire function. Then the following hold.*

- (a)  $I(f) = I(f^n)$ , for  $n \geq 2$ .
- (b)  $I(f)$  is completely invariant.
- (c)  $J(f) \cap I(f) \neq \emptyset$ .
- (d)  $J(f) = \partial I(f)$ .
- (e)  $\overline{I(f)}$  has no bounded components.

If  $U$  is a Fatou component such that  $U \cap I(f) \neq \emptyset$ , then it follows by normality that  $U \subset I(f)$ . We call a Fatou component in  $I(f)$  *escaping*. Note, however, that it is not necessarily true that the boundary of an escaping Fatou component must lie in  $I(f)$ . For example, the function  $f(z) = z + 1 + e^{-z}$  has an escaping Fatou component the boundary of which contains periodic points; see [86, remark after Theorem 1.2].

We recall Eremenko's conjecture [39] that  $I(f)$  contains no bounded components. One noteworthy result regarding this conjecture is [85, Theorem 4.1(c)] that  $I(f) \cup \{\infty\}$  is connected. Hence, if there exists a transcendental entire function  $f$  such that  $I(f)$  has a bounded component, then  $I(f)$  must be very complicated topologically.

Fatou [46] noted that for many transcendental entire functions, such as

$$f_\lambda(z) = \lambda \sin z, \quad \text{for } \lambda \in \mathbb{R},$$

there is a set  $S$  of curves with the following property. If  $\Gamma \in S$ , then  $f_\lambda^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$ , for  $z \in \Gamma$ . He also asked if this was the case in general. Eremenko made this conjecture more precise in [39], by asking if it was the case that every point in  $I(f)$  can be joined to infinity by a curve in  $I(f)$ ; this is sometimes called the *strong form of Eremenko's conjecture*.

Arguably the most significant results regarding Eremenko's conjectures were given by Rottenfusser, Rückert, Rempe and Schleicher [91]. First [91, Theorem 1.1] they gave an example of a transcendental entire function,  $f \in \mathcal{B}$ , such that every path-connected component of  $J(f)$  is bounded. Together with the fact that  $F(f) \cap I(f) = \emptyset$ , for  $f \in \mathcal{B}$ , this result shows that the strong form of Eremenko's conjecture does not hold in general.

On the other hand, they also showed that the strong form of Eremenko's conjecture does hold for a large class of functions [91, Theorem 1.2].

**Theorem 1.6.2.** *Suppose that  $f \in \mathcal{B}$  is a function of finite order, or more generally a finite composition of such functions. Then every point  $z \in I(f)$  can be connected to infinity by a curve  $\gamma$  such that  $f^n(w) \rightarrow \infty$  uniformly for  $w \in \gamma$ .*

## 1.7 The fast escaping set

In this section we briefly discuss some important properties of the fast escaping set,  $A(f)$ . In fact,  $A(f)$  has several properties corresponding to the properties of  $I(f)$  given in Theorem 1.6.1.

**Theorem 1.7.1.** *Suppose that  $f$  is a transcendental entire function. Then the following hold.*

- (a)  $A(f) = A(f^n)$ , for  $n \geq 2$ .
- (b)  $A(f)$  is completely invariant.
- (c)  $J(f) \cap A(f) \neq \emptyset$ .
- (d)  $J(f) = \partial A(f)$ .
- (e)  $A(f)$  has no bounded components.

The first part of this theorem was shown in [82]. The second part was stated in [25] and proved in [82]. The relationships between  $A(f)$  and  $J(f)$  were proved in [25] and [82]. The fact that all the components of  $A(f)$  are unbounded was proved in [82] and implies that  $I(f)$  has at least one unbounded component, as noted earlier; see also [86] for a detailed account of all these properties of  $A(f)$ .

We call a Fatou component in  $A(f)$  *fast escaping*. The following results give three important properties of fast escaping Fatou components. The first property is part of [86, Theorem 4.4].

**Theorem 1.7.2.** *Suppose that  $f$  is a transcendental entire function and that  $U$  is a multiply connected Fatou component of  $f$ . Then  $\bar{U} \subset A(f)$ .*

The second is a version of [86, Theorem 1.2].

**Theorem 1.7.3.** *Suppose that  $f$  is a transcendental entire function and that  $R > 0$  is such that  $M(r) > r$  for  $r \geq R$ . If  $U$  is a simply connected Fatou component of  $f$  that meets  $A_R(f)$ , then  $\bar{U} \subset A_R(f)$ .*

Recall from the previous section that if  $U$  is Fatou component in  $I(f)$ , then it is not necessarily true that the boundary of  $U$  must lie in  $I(f)$ .

The third property, which was also proved in [25], is [86, Corollary 4.2].

**Theorem 1.7.4.** *Suppose that  $f$  is a transcendental entire function and that  $U$  is a Fatou component of  $f$  with  $U \cap A(f) \neq \emptyset$ . Then  $U$  is wandering.*

In the proof of Theorem 4.1.1 we construct a transcendental entire function with a simply connected fast escaping Fatou component, and no multiply connected Fatou components.

Suppose that  $0 < \epsilon < 1$ , and define  $\mu(r) = \epsilon M(r)$ , for  $r > 0$ . Rippon and Stallard proved the following alternative characterisation of  $A(f)$  [86, Theorem 2.7].

**Theorem 1.7.5.** *Suppose that  $f$  is a transcendental entire function and that  $R > 0$  is sufficiently large to ensure that  $\mu(r) > r$ , for  $r \geq R$ . Then*

$$A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu^n(R), \text{ for } n \in \mathbb{N}\}.$$

To prove Theorem 4.1.1 we require a stronger version of this result, in which  $\epsilon$  is taken to be a function of  $r$ . This is given in Theorem 4.1.2.

## 1.8 Cantor bouquets

Before discussing spiders' webs, we first briefly describe a contrasting structure, which has been known for much longer.

Devaney and Krych [36] studied the Julia set of many functions in the exponential family. They showed that the Julia set of one of these functions is a closed set consisting of an uncountable union of disjoint unbounded curves. Devaney and Tangerman [37] first used the name *Cantor bouquet* for this structure, and showed that there is a large class of functions, including many exponentials such as  $f(z) = \frac{1}{4}e^z$ , for which the Julia set is a Cantor bouquet. For a general study of Cantor bouquets, including a precise definition, we refer to [11].

All of these functions are in the class  $\mathcal{B}$ , for which it is known [41] that  $I(f)$  (and hence  $A(f)$ ) is a subset of  $J(f)$ . Schleicher and Zimmer [95] studied the whole exponential family, and showed that every point in the escaping set of any function in this family lies on an unbounded curve in the escaping set. Rottenfusser and Schleicher [92] showed that the same is true for functions in the cosine family. Clearly the strong form of Eremenko’s conjecture holds in both these cases.

As mentioned earlier, it was shown in [91, Theorem 1.2] that the strong form of Eremenko’s conjecture holds for a large class of functions in the class  $\mathcal{B}$ , which includes functions in the exponential and cosine families discussed above. For many of these functions the Julia set is a Cantor bouquet containing the escaping set. In other cases, however, the escaping set and the Julia set are both connected. For example, for  $f(z) = e^z$ , it was shown by Misiurewicz [72] that  $J(f) = \mathbb{C}$ , and, more recently, by Rempe [78] that  $I(f)$  is connected. For all of these functions, each point in  $A(f)$  lies on an unbounded curve in  $A(f)$ , and all other points in  $I(f)$  and  $J(f)$  are endpoints of these curves [79].

## 1.9 Spiders’ webs

As noted earlier, it was shown in [86] that  $A_R(f)$ ,  $A(f)$  and  $I(f)$  can have a structure known as a spider’s web, and that if  $A_R(f)$  is a spider’s web then so are  $A(f)$  and  $I(f)$  and, in many cases, so is  $J(f)$  – see Theorem 1.9.1(c). We observe that the spider’s web structure has several differences to the Cantor bouquet structure described in the previous section. In particular, the Cantor bouquet structure is closed and has uncountably many components with a single unbounded complementary component, whereas the spider’s web structure

is connected with infinitely many complementary components, each of which is bounded – see Theorem 1.9.1(a). Note that a spider’s web may contain a subset that is a Cantor bouquet, although no examples have yet been given of functions for which the Julia set or escaping set is known to be a spider’s web containing a Cantor bouquet.

Functions for which  $A_R(f)$  is a spider’s web have a number of strong dynamical properties. First, since  $I(f)$  is also a spider’s web,  $I(f)$  is connected. Since  $I(f)$  is unbounded, it follows at once that Eremenko’s conjecture holds in a particularly strong way whenever  $A_R(f)$  is a spider’s web.

As mentioned earlier, if  $A_R(f)$  is a spider’s web then it has an intricate topological structure, as shown by the following result.

**Theorem 1.9.1.** *Let  $f$  be a transcendental entire function, let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ , and let  $A_R(f)$  be a spider’s web. Then the following hold.*

- (a)  $A(f)^c$  has uncountably many components, each of which is compact.
- (b)  $A(f)^c$  has singleton periodic components which are dense in  $J(f)$ .
- (c) If  $f$  has no multiply connected Fatou components, then each of

$$A_R(f) \cap J(f), A(f) \cap J(f), I(f) \cap J(f) \text{ and } J(f)$$

*is a spider’s web.*

- (d) *The function  $f$  has no unbounded Fatou components.*

The first of these properties is a combination of [74, Theorem 1.2] and [86, Theorem 1.6(a)], the second is part of [74, Theorem 1.6]. The final two properties are [86, Theorem 1.5].

Theorem 1.9.1(d) provides a link between the study of  $A_R(f)$  spiders’ webs and another major open question in the field of transcendental dynamics. *Baker’s conjecture*, which arises from [8], is that if the order of a transcendental entire function  $f$  is less than  $\frac{1}{2}$ , then  $f$  has no unbounded Fatou components. A survey of progress on this question was given in [58]. It is known [106] that there are no

unbounded periodic Fatou components for functions of order less than  $\frac{1}{2}$ , and so it remains to show that such functions have no unbounded wandering domains. As described in [58], there are many papers showing that no such domains exist if the function also satisfies various regularity conditions. We strengthen some results on these classes of functions in Section 2.7.

The strongest results showing that functions of very small growth have no unbounded wandering domains were given – subsequent to the survey [58] – in [59] and [84]. It is still not known, however, whether the result holds even for all functions of order zero. It was observed in [84] that the techniques used to obtain all these partial results on Baker’s conjecture were in fact sufficient to imply the stronger result that  $A_R(f)$  is a spider’s web.

It has recently been shown, however, that there are functions for which Baker’s conjecture holds but  $A_R(f)$  is not a spider’s web. In [88] it was shown that if  $f$  is a transcendental entire function of order less than  $\frac{1}{2}$  and with all its zeros on the negative real axis, then all components of  $F(f)$  are bounded. Moreover,  $I(f)$  is a spider’s web and so Eremenko’s conjecture holds. On the other hand, it was shown in [87, Theorem 1.2] that there exist functions in this class for which  $A(f)$  is not a spider’s web. We note that the results in [88] do not require a regularity condition of the form discussed in the previous paragraph.

In view of these strong dynamical properties, it is desirable to determine functions for which  $A_R(f)$  is a spider’s web. In [86, Section 8] several classes of such functions were given. These were derived using the following [86, Theorem 1.9].

**Theorem 1.9.2.** *Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Then  $A_R(f)$  is a spider’s web if one of the following holds:*

- (a)  *$f$  has a multiply connected Fatou component;*
- (b)  *$f$  has very small growth; that is, there exist  $m \geq 2$  and  $r_0 > 0$  such that*

$$\log \log M(r, f) < \frac{\log r}{\log^m r}, \quad \text{for } r > r_0,$$

*where  $\log^m$  is the  $m$ th iterated logarithm;*



- (c)  $f$  has order less than  $\frac{1}{2}$  and regular growth;
- (d)  $f$  has finite order, Fabry gaps and regular growth;
- (e)  $f$  has a sufficiently strong version of the pits effect and has regular growth.

A definition of regular growth is given in Section 2.2, along with a number of new results regarding regular growth. We define Fabry gaps in Section 2.6. For a definition of the pits effect we refer to [86, Section 8].

A further class of transcendental entire functions for which  $A_R(f)$  is a spider's web was given by Mihaljević-Brandt and Peter [70].

In Chapter 2 we give several new classes of transcendental entire functions with this property.

As noted in Section 1.3, the structure of  $A_R(f)$  spiders' webs can be understood through fundamental holes and fundamental loops. In Chapter 3 we investigate the structure of these fundamental loops for functions with a multiply connected Fatou component. We show that there exist transcendental entire functions for which some fundamental loops are analytic curves and approximately circles, while others are geometrically highly distorted. We do this by introducing a real-valued function which measures the rate of escape of points in  $A(f)$ , and show that this function has a number of interesting properties.

In Section 6.2 we conjecture that there is a family of functions for which  $J(f) \cap A(f)$  is a spider's web of positive area, but not the whole complex plane.

# Chapter 2

## Entire functions for which the escaping set is a spider's web

### 2.1 Introduction

In this chapter we give several new classes of examples of transcendental entire functions such that  $A_R(f)$  is a spider's web. Recall that if  $A_R(f)$  is a spider's web, then so are  $I(f)$  and  $A(f)$ . We show that some of these classes have a degree of stability under changes in the function. We show that new examples of functions for which  $I(f)$  and  $A(f)$  are spiders' webs can be constructed by composition, by differentiation, and by integration of existing examples. Finally, we use a property of spiders' webs to give new results concerning functions with no unbounded Fatou components.

This chapter is structured as follows. First, in Section 2.2, we prove several new results concerning regular growth conditions, which we use in later sections. These results may also be of independent interest.

In Section 2.3, we demonstrate a technique for constructing new transcendental entire functions for which  $A_R(f)$  is a spider's web by taking finite compositions of functions that satisfy a minimum modulus condition and a regularity condition.

In Section 2.4, we show that in certain circumstances when  $A_R(f)$  is a spider's web, then so is  $A_R(P(f(Q(z))), z)$ , where  $P, Q$  are polynomials, and so also is  $A_R(f + h)$ , where the entire function  $h$  has smaller growth, in some sense, than  $f$ . These results allow us to construct a large class of functions for which  $A_R(f)$

is a spider's web. They also show that the property of having an  $A_R(f)$  spider's web can be stable under changes in  $f$ , unlike many other dynamical properties.

In Section 2.5, we establish a technique for constructing a large class of transcendental entire functions of finite order for which  $A_R(f)$  is a spider's web, by modifying the power series of a transcendental entire function of finite order. This technique is a generalisation of the method used to construct some of the examples in [86]. We show that this class of examples can be extended by differentiation or integration. By combining the results of Sections 2.3, 2.4 and 2.5, we give an unexpectedly simple function for which  $A_R(f)$  is a spider's web.

In Section 2.6, we present a technique for constructing new transcendental entire functions, of infinite order and with large gaps in their power series, for which  $A_R(f)$  is a spider's web.

Finally, in Section 2.7, we relate our results to previous work on classes of transcendental entire functions which have no unbounded Fatou components.

## 2.2 New results on regularity

In this section we set out conditions which ensure that  $A_R(f)$  is a spider's web. Many of these conditions require some form of regularity of growth. We prove several new results concerning forms of regularity of growth, which enable us to construct examples of functions with an  $A_R(f)$  spider's web later in the chapter.

A pair of conditions that are together necessary and sufficient for  $A_R(f)$  to be a spider's web were obtained in [86, Theorem 8.1]. Note that the sequence  $(G_n)_{n \geq 0}$  in the statement of this theorem is not the same as the sequence  $(G_n)_{n \in \mathbb{N}}$  in (1.13).

**Theorem 2.2.1.** *Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Then  $A_R(f)$  is a spider's web if and only if there exists a sequence  $(G_n)_{n \geq 0}$  of bounded simply connected domains such that, for  $n \geq 0$ ,*

$$G_n \supset B(0, M^n(R, f)) \tag{2.1}$$

and

$$G_{n+1} \text{ is contained in a bounded component of } \mathbb{C} \setminus f(\partial G_n). \quad (2.2)$$

This result is very general, and so, in order to construct examples, the following, more readily applicable, sufficient conditions for  $A_R(f)$  to be a spider's web were established in [86, Corollary 8.3].

**Lemma 2.2.2.** *Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Then  $A_R(f)$  is a spider's web if, for some real number  $m > 1$ ,*

(a) *there exists  $R_0 > 0$  such that, for  $r \geq R_0$ ,*

$$\text{there exists } \rho \in (r, r^m) \text{ with } L(\rho, f) \geq M(r, f), \text{ and} \quad (2.3)$$

(b)  *$f$  has regular growth in the sense that there exists a sequence  $(r_n)_{n \geq 0}$  with*

$$r_n > M^n(R, f) \text{ and } M(r_n, f) \geq r_{n+1}^m, \quad \text{for } n \geq 0. \quad (2.4)$$

We use the following condition, which is stronger than the regularity condition of Lemma 2.2.2(b), in order to construct a new class of functions with an  $A_R(f)$  spider's web. We define a transcendental entire function  $f$  to be  $\psi$ -regular if, for  $m > 1$ , there exist an increasing function  $\psi_m$  and  $R_m > 0$  such that, for  $r \geq R_m$ ,

$$\psi_m(r) \geq r \quad \text{and} \quad M(\psi_m(r), f) \geq (\psi_m(M(r, f)))^m. \quad (2.5)$$

For given  $m > 1$  we call  $\psi_m$  a *regularity function* for  $f$ .

This condition is slightly stronger than one used in [84, Theorem 5] in connection with transcendental entire functions with no unbounded Fatou components. That version did not require the regularity function to be increasing. However, all the regularity functions used in [84, 86] are, in fact, increasing.

We also use the following condition, which is stronger than  $\psi$ -regularity, in order to construct several classes of functions with an  $A_R(f)$  spider's web. Suppose that  $c > 0$ . We define a transcendental entire function  $f$  to be *log-regular*,

with constant  $c$ , if the function  $\phi(t) = \log M(e^t, f)$  satisfies

$$\frac{\phi'(t)}{\phi(t)} \geq \frac{1+c}{t}, \quad \text{for large } t. \quad (2.6)$$

By Lemma 1.2.1(e) there may be a countable set of points at which the derivative  $\phi'(t)$  fails to exist. At these points we understand  $\phi'(t)$  to be the right-hand derivative.

We say that  $f$  is *log-regular* if it is log-regular with constant  $c$ , for some  $c > 0$ . We observe also that in our choice of terminology we do not intend to suggest that a log-regular function is a  $\psi$ -regular function with  $\log$  as the regularity function.

The condition (2.6) was used by Anderson and Hinkkanen in [3, Theorem 2], also in connection with transcendental entire functions with no unbounded Fatou components. The name log-regular was suggested by Aimo Hinkkanen in a private communication. The condition was also used in [86, Section 8] in order to construct classes of functions with an  $A_R(f)$  spider's web.

We now state three new results concerning  $\psi$ -regularity and log-regularity. The first concerns the composition of  $\psi$ -regular functions.

**Theorem 2.2.3.** *Let  $f_1, f_2, \dots, f_k$  be transcendental entire functions. Suppose that, for  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is  $\psi$ -regular, each with regularity function  $\psi_m$  for  $m > 1$ . Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then, for any  $c > 1$ ,  $g$  is  $\psi$ -regular with regularity function  $c\psi_m$  for  $m > 1$ .*

In particular it follows that  $\psi$ -regularity is preserved under iteration.

**Corollary 2.2.4.** *If  $f$  is a  $\psi$ -regular transcendental entire function, then so is  $f^n$  for  $n \in \mathbb{N}$ .*

The second result relates to the composition of entire functions, one of which is log-regular.

**Theorem 2.2.5.** *Let  $f_1, f_2, \dots, f_k$  be non-constant entire functions such that, for some  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is a log-regular transcendental entire function. Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then  $g$  is log-regular.*

In particular it follows that log-regularity is preserved under iteration.

**Corollary 2.2.6.** *If  $f$  is a log-regular transcendental entire function, then so is  $f^n$  for  $n \in \mathbb{N}$ .*

Note that Theorem 2.2.3 requires all functions to be  $\psi$ -regular transcendental entire functions, whereas Theorem 2.2.5 requires just one to be a log-regular transcendental entire function and the others only to be entire.

The third result shows that if  $f$  is log-regular, then so is any transcendental entire function with similar growth.

**Theorem 2.2.7.** *Let  $f$  and  $g$  be transcendental entire functions. If  $f$  is log-regular and there exist  $a_1, a_2 \geq 1$  and  $R_0 > 0$  such that*

$$M(r^{a_1}, g) \geq M(r, f) \quad \text{and} \quad M(r^{a_2}, f) \geq M(r, g), \quad \text{for } r \geq R_0, \quad (2.7)$$

*then  $g$  is log-regular.*

We need three preparatory lemmas to prove these results. The first lemma is a version of [89, Corollary 4.3], and gives a necessary condition and a sufficient condition for  $f$  to be log-regular.

**Lemma 2.2.8.** *Let  $f$  be a transcendental entire function.*

(a) *If  $f$  is log-regular, with constant  $c$ , then there is an  $R_0 > 0$  such that, if  $k > 1$  and  $d = k^c$ , then*

$$M(r^k, f) \geq M(r, f)^{kd}, \quad \text{for } r \geq R_0. \quad (2.8)$$

(b) *If (2.8) holds for some  $d, k > 1$  and  $R_0 > 0$ , then  $f$  is log-regular.*

The second lemma comes from Wiman-Valiron theory, (see, for example, [51]), which was first used in connection with the escaping set by Eremenko [39]. We first need to introduce some terminology. Let  $g(z) = \sum_{n=0}^{\infty} a_n z^n$  be a transcendental entire function. Define

$$\mu(r) = \sup_n |a_n| r^n = |a_N| r^N, \quad \text{for } r > 0, \quad (2.9)$$

to be the *maximal term* of the power series, and call  $N = N(r)$  the *central index*; if (2.9) holds for several  $N$ , we take  $N(r)$  to be the largest of these. Note that

$N(r)$  is increasing and  $N(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . Wiman-Valiron theory uses  $\mu(r)$  to give results about the behaviour of  $g$  near points  $z(r)$ ,  $r > 0$ , that satisfy

$$|z(r)| = r \text{ and } |g(z(r))| = M(r, g). \quad (2.10)$$

A key result of Wiman-Valiron theory is the following.

**Lemma 2.2.9.** *Suppose that  $g$  is a transcendental entire function and  $\alpha > \frac{1}{2}$ . For  $r > 0$ , let  $z(r)$  be a point satisfying (2.10), and define*

$$D(r) = B\left(z(r), \frac{r}{(N(r))^\alpha}\right), \quad r > 0.$$

*Then there exists a measurable set  $E \subset (0, \infty)$  with*

$$\int_E 1/t \, dt < \infty \quad (2.11)$$

*such that, for  $r \notin E$  and  $z \in D(r)$ ,*

$$g(z) = \left(\frac{z}{z(r)}\right)^{N(r)} g(z(r))(1 + \epsilon), \quad (2.12)$$

*where  $\epsilon = \epsilon(r, z) \rightarrow 0$  uniformly with respect to  $z$  as  $r \rightarrow \infty$ ,  $r \notin E$ . In particular, if  $r$  is sufficiently large and  $r \notin E$ , then*

$$g(D(r)) \supset \{w : |w| = M(r, g)\}. \quad (2.13)$$

We use Lemma 2.2.9 to prove a result on the behaviour of the maximum modulus of the composite of two entire functions.

**Lemma 2.2.10.** *Suppose that  $f$  is a non-constant entire function and  $g$  is a transcendental entire function. Then, given  $\nu > 1$ , there exist  $R_0, R_1 > 0$  such that*

$$M(\nu r, f \circ g) \geq M(M(r, g), f) \geq M(r, f \circ g), \quad \text{for } r \geq R_0, \quad (2.14)$$

and

$$M(\nu r, g \circ f) \geq M(M(r, f), g) \geq M(r, g \circ f), \quad \text{for } r \geq R_1. \quad (2.15)$$

*Proof.* We first prove (2.14). Let  $\alpha > \frac{1}{2}$ , and let  $N(r)$ ,  $E$  and  $D(r)$  be related to  $g$  as in Lemma 2.2.9. It follows from (2.11), and the fact that  $N(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , that, for sufficiently large  $r$ , there exists  $r' \in (r, \frac{\nu+1}{2}r) \setminus E$ , with

$$D(r') \subset B(0, \nu r) \text{ and } g(D(r')) \supset \{w : |w| = M(r', g)\}. \quad (2.16)$$

Let  $w$  be such that  $|w| = M(r', g)$  and  $|f(w)| = M(M(r', g), f)$ . Then, by (2.16), there is a  $z \in D(r')$  with  $g(z) = w$ . Hence

$$|(f \circ g)(z)| = M(M(r', g), f) > M(M(r, g), f),$$

since  $r' > r$  and  $f$  is not constant. The first part of (2.14) now follows, by the first part of (2.16). The second part of (2.14) is immediate.

Equation (2.15) follows in the same way if  $f$  is transcendental. Otherwise, suppose that  $f$  is a polynomial. Then, for sufficiently large  $r$ , since  $f$  is not constant,

$$f(B(0, \nu r)) \supset \{w : |w| = M(r, f)\}. \quad (2.17)$$

Let  $w$  be such that  $|w| = M(r, f)$  and  $|g(w)| = M(M(r, f), g)$ . Then, by (2.17), there is a  $z \in B(0, \nu r)$  with  $f(z) = w$ . Hence

$$|(g \circ f)(z)| = M(M(r, f), g).$$

The first part of (2.15) follows. The second part of (2.15) is immediate.  $\square$

In passing, we note a related result discussed by Bergweiler and Hinkkanen [25, Lemma 1] that, if we also have  $g(0) = 0$ , then

$$M(6r, f \circ g) \geq M(M(r, g), f), \quad \text{for } r > 0.$$

Now we are ready to prove Theorems 2.2.3, 2.2.5 and 2.2.7.



*Proof of Theorem 2.2.3.* Suppose that  $m > 1$ . We note first a general result. Suppose that  $f$  is a  $\psi$ -regular transcendental entire function with regularity function  $\psi_m$ , and let  $\lambda > 1$ . Then, for sufficiently large  $r$ , by (1.7) and (2.5),

$$M(\lambda\psi_m(r), f) \geq \lambda^m M(\psi_m(r), f) \geq (\lambda\psi_m(M(r, f)))^m. \quad (2.18)$$

Hence  $\lambda\psi_m$  is also a regularity function for  $f$ .

We next claim that the following statement is true. Suppose that  $g_1$  and  $g_2$  are  $\psi$ -regular transcendental entire functions each with regularity function  $\psi_m$ , and that  $a > 1$ . Then  $g_1 \circ g_2$  is a  $\psi$ -regular transcendental entire function with regularity function  $a\psi_m$ .

We note that, for sufficiently large  $r$ ,

$$\begin{aligned} M(a\psi_m(r), g_1 \circ g_2) &\geq M(M(\psi_m(r), g_2), g_1) && \text{by Lemma 2.2.10} \\ &\geq M((\psi_m(M(r, g_2)))^m, g_1) && \text{by (2.5)} \\ &\geq M(\psi_m(M(r, g_2)), g_1)^m && \text{by (1.8)} \\ &\geq (\psi_m(M(M(r, g_2)), g_1))^{m^2} && \text{by (2.5)} \\ &\geq (\psi_m(M(r, g_1 \circ g_2)))^{m^2} && \text{since } \psi_m \text{ is increasing} \\ &\geq (a\psi_m(M(r, g_1 \circ g_2)))^m && \text{by (2.5)}. \end{aligned}$$

Hence  $g_1 \circ g_2$  is  $\psi$ -regular with regularity function  $a\psi_m$ , which completes the proof of our claim.

We now let  $a = c^{1/(k-1)} > 1$ . We apply the statement above with  $g_1 = f_1$  and  $g_2 = f_2$  to deduce that  $f_1 \circ f_2$  is a  $\psi$ -regular transcendental entire function with regularity function  $a\psi_m$ . Since, by (2.18), we have that  $a\psi_m$  is a regularity function for  $f_3$ , we may apply the statement above once again, with  $g_1 = f_1 \circ f_2$  and  $g_2 = f_3$  to deduce that  $f_1 \circ f_2 \circ f_3$  is a  $\psi$ -regular transcendental entire function with regularity function  $a^2\psi_m$ . We continue to apply the statement above, and after  $k - 1$  applications in total, we deduce that  $f_1 \circ f_2 \circ \cdots \circ f_k$  is a  $\psi$ -regular transcendental entire function with regularity function  $a^{k-1}\psi_m$ . This completes the proof, since  $a^{k-1} = c$ .  $\square$

*Proof of Theorem 2.2.5.* It is sufficient to prove the result for  $k = 2$ . Suppose

then that  $k = 2$ .

We consider first the case that  $f_2$  is a log-regular transcendental entire function. By Lemma 2.2.8(a) applied to  $f_2$ , there are  $k, d > 1$  and  $r_1 > 0$  such that

$$M(r^k, f_2) \geq M(r, f_2)^{kd}, \quad \text{for } r \geq r_1. \quad (2.19)$$

We consider the cases that  $f_1$  is a transcendental entire function and that  $f_1$  is a polynomial separately. Suppose that  $f_1$  is a transcendental entire function. Choose  $\nu$  such that  $1 < \nu < d$ , put  $k' = k\nu > 1$  and  $d' = \frac{d}{\nu} > 1$ . Then, for sufficiently large  $r$ ,

$$\begin{aligned} M(r^{k'}, f_1 \circ f_2) &\geq M(\nu r^k, f_1 \circ f_2) \\ &\geq M(M(r^k, f_2), f_1) && \text{by Lemma 2.2.10} \\ &\geq M(M(r, f_2)^{kd}, f_1) && \text{by (2.19)} \\ &\geq M(M(r, f_2), f_1)^{kd} && \text{by (1.8)} \\ &\geq M(r, f_1 \circ f_2)^{k'd'} && \text{by choice of } k', d'. \end{aligned}$$

Thus  $f_1 \circ f_2$  is log-regular by Lemma 2.2.8(b).

On the other hand, suppose that  $f_1$  is a polynomial. With the constants  $\nu, d, k$  and  $k'$  defined above, choose  $d''$  such that  $\nu < d'' < d$ , and set  $d^\# = \frac{kd''}{k'} > 1$ . Then, for sufficiently large  $r$ ,

$$\begin{aligned} M(r^{k'}, f_1 \circ f_2) &\geq M(M(r, f_2)^{kd}, f_1) && \text{as above} \\ &\geq M(M(r, f_2), f_1)^{kd''} && \text{by Lemma 1.2.3} \\ &\geq M(r, f_1 \circ f_2)^{k'd^\#} && \text{by choice of } d^\#. \end{aligned}$$

Once again,  $f_1 \circ f_2$  is log-regular by Lemma 2.2.8(b). This completes the proof in the case that  $f_2$  is a log-regular transcendental entire function.

The remaining case is that  $f_1$  is a log-regular transcendental entire function but  $f_2$  is not. By Lemma 2.2.8(a) applied to  $f_1$ , there are  $k, d > 1$  and  $r_1 > 0$  such that

$$M(r^k, f_1) \geq M(r, f_1)^{kd}, \quad \text{for } r \geq r_1. \quad (2.20)$$

We consider the cases that  $f_2$  is a transcendental entire function and that  $f_2$  is a polynomial separately. Suppose that  $f_2$  is a transcendental entire function. Choose  $\nu$  such that  $1 < \nu < d$ , put  $k' = k\nu > 1$  and  $d' = \frac{d}{\nu} > 1$ . Then, for sufficiently large  $r$ ,

$$\begin{aligned}
M(r^{k'}, f_1 \circ f_2) &\geq M(\nu r^k, f_1 \circ f_2) \\
&\geq M(M(r^k, f_2), f_1) && \text{by Lemma 2.2.10} \\
&\geq M(M(r, f_2)^k, f_1) && \text{by (1.8)} \\
&\geq M(M(r, f_2), f_1)^{kd} && \text{by (2.20)} \\
&\geq M(r, f_1 \circ f_2)^{k'd'} && \text{by choice of } k', d'.
\end{aligned}$$

Thus  $f_1 \circ f_2$  is log-regular by Lemma 2.2.8(b).

On the other hand, suppose that  $f_2$  is a polynomial. With the constants  $\nu, d$  and  $k$  defined above, choose  $k''$  and  $k^\#$  such that  $k < k'' < k^\# < kd$ , and set  $d^\# = \frac{kd}{k^\#} > 1$ . Then, for sufficiently large  $r$ ,

$$\begin{aligned}
M(r^{k^\#}, f_1 \circ f_2) &\geq M(\nu r^{k''}, f_1 \circ f_2) \\
&\geq M(M(r^{k''}, f_2), f_1) && \text{by Lemma 2.2.10} \\
&\geq M(M(r, f_2)^k, f_1) && \text{by Lemma 1.2.3} \\
&\geq M(M(r, f_2), f_1)^{kd} && \text{by (2.20)} \\
&\geq M(r, f_1 \circ f_2)^{k^\# d^\#} && \text{by choice of } k^\#, d^\#.
\end{aligned}$$

Once again,  $f_1 \circ f_2$  is log-regular by Lemma 2.2.8(b). This completes the proof of the lemma.  $\square$

*Proof of Theorem 2.2.7.* Suppose that  $f$  is log-regular with constant  $c$ , and  $a_1, a_2$  are as in (2.7). Choose  $k > 1$  sufficiently large that  $k^c > a_1 a_2$ . Set

$$d = k^c, \quad k' = a_1 a_2 k > 1 \quad \text{and} \quad d' = \frac{d}{a_1 a_2} > 1.$$

Then, for sufficiently large  $r$ ,

$$\begin{aligned}
M(r^{k'}, g) &= M(r^{a_1 a_2 k}, g) \\
&\geq M(r^{a_2 k}, f) && \text{by (2.7)} \\
&\geq M(r^{a_2}, f)^{kd} && \text{by (2.8)} \\
&\geq M(r, g)^{kd} && \text{by (2.7)} \\
&= M(r, g)^{k'd'} && \text{by choice of } k', d'.
\end{aligned}$$

Hence  $g$  is log-regular by Lemma 2.2.8(b).  $\square$

We now prove several useful corollaries of Theorem 2.2.7. The first relates to the derivatives and integrals of log-regular functions.

**Corollary 2.2.11.** *Let  $f$  be a transcendental entire function. Then  $f$  is log-regular if and only if  $f'$  is log-regular.*

*Proof.* Suppose that  $r > 0$ , and that  $z$  is such that  $|z| = r$  and  $M(r, f) = |f(z)|$ . Then

$$rM(r, f') \geq \left| \int_0^z f'(z) dz \right| = |M(r, f) - f(0)|.$$

Hence, by (1.6) and (1.8), for sufficiently large  $r$ ,

$$M(r^2, f') \geq rM(r, f') + |f(0)| \geq M(r, f). \quad (2.21)$$

Next, suppose that  $r > 0$ , and that  $z$  is such that  $|z| = r$  and  $M(r, f') = |f'(z)|$ . Then, by applying Cauchy's estimate on a circle centre  $z$  and radius  $r$ , we deduce that  $M(2r, f)/r \geq M(r, f')$ . We deduce that, for sufficiently large  $r$ ,

$$M(r^2, f) \geq M(2r, f)/r \geq M(r, f'). \quad (2.22)$$

The result follows by Theorem 2.2.7, with  $a_1 = a_2 = 2$ .  $\square$

The remaining corollaries of Theorem 2.2.7 are used later to give stability results about  $A_R(f)$  spiders' webs. While they could be combined, they are stated separately for clarity. The first concerns addition of a function to a log-regular function.

**Corollary 2.2.12.** *Let  $f$  be a log-regular transcendental entire function, and let  $h$  be an entire function. Suppose that there exist  $a \in (0, 1)$  and  $R_0 > 0$  such that*

$$aM(r, f) \geq M(r, h), \quad \text{for } r \geq R_0. \quad (2.23)$$

*Then  $g = f + h$  is log-regular.*

*Proof.* Suppose that  $r \geq R_0$ , and that  $z_f, z_g$  and  $z_h$  are points of modulus  $r$  such that  $M(r, f) = |f(z_f)|$ ,  $M(r, g) = |g(z_g)|$  and  $M(r, h) = |h(z_h)|$ . Then

$$\begin{aligned} (1 + a)M(r, f) &\geq M(r, f) + M(r, h) \\ &= |f(z_f)| + |h(z_h)| \\ &\geq |f(z_g)| + |h(z_g)| \\ &\geq |g(z_g)| \\ &= M(r, g). \end{aligned}$$

Moreover

$$\begin{aligned} (1 - a)M(r, f) &\leq M(r, f) - M(r, h) \\ &= |f(z_f)| - |h(z_h)| \\ &\leq |f(z_f)| - |h(z_f)| \\ &\leq |g(z_f)| \\ &\leq |g(z_g)| \\ &= M(r, g). \end{aligned}$$

The result now follows by (1.8), and Theorem 2.2.7 with  $a_1 = a_2 = 2$ .  $\square$

Note that, by (1.6), if  $h$  is a polynomial, then (2.23) is satisfied for any transcendental entire function  $f$  and any  $a \in (0, 1)$ .

The second corollary concerns a case where log-regularity is preserved under multiplication.

**Corollary 2.2.13.** *Let  $f$  be a log-regular transcendental entire function. Then  $g(z) = zf(z)$  is log-regular.*

*Proof.* By (1.8) and (1.6), for sufficiently large  $r$ ,  $M(r^2, f) \geq M(r, f)^2 \geq M(r, g)$ . Also, for sufficiently large  $r$ ,  $M(r, g) \geq M(r, f)$ . The result follows, by Theorem 2.2.7 with  $a_1 = 1$  and  $a_2 = 2$ .  $\square$

Our final corollary is quite general.

**Corollary 2.2.14.** *Let  $f$  be a log-regular transcendental entire function. Let  $P(w, z)$  be a polynomial of degree at least one in  $w$ , and let  $Q(z)$  be a polynomial of degree at least one. Then  $g(z) = P(f(Q(z)), z)$  is log-regular.*

*Proof.* Suppose that

$$P(f(Q(z)), z) = af(Q(z))^{N_1} z^{N_2} + h(z) = g_0(z) + h(z),$$

where  $N_1$  is the highest power of  $w$  in  $P(w, z)$ ,  $N_2$  is the highest power of  $z$  corresponding to  $f(Q(z))^{N_1}$ , and  $a \neq 0$ . By Theorem 2.2.5, the function  $z \mapsto af(Q(z))^{N_1}$  is log-regular. By Corollary 2.2.13, applied  $N_2$  times,  $g_0$  is log-regular. Since, by (1.6), we have

$$\frac{1}{2}M(r, g_0) \geq M(r, h), \quad \text{for large } r,$$

the result follows by Corollary 2.2.12.  $\square$

### 2.3 Using composition to give functions for which $A_R(f)$ is a spider's web

In this section we demonstrate that  $A_R(g)$  is a spider's web if  $g = f_1 \circ f_2 \circ \dots \circ f_k$ , and the entire functions  $f_j, j \in \{1, 2, \dots, k\}$ , satisfy certain conditions. We need a preparatory lemma before we can state the results. This lemma is a generalisation of Lemma 2.2.2, in which condition (a) is relaxed slightly although condition (b) is unchanged. Condition (a) was also used, independently, in [70].

**Lemma 2.3.1.** *Let  $f$  be a transcendental entire function and let  $R > 0$  be such that  $M(r, f) > r$  for  $r \geq R$ . Then  $A_R(f)$  is a spider's web if, for some real number  $m > 1$ ,*

(a) there exists  $R_0 > 0$  such that, for  $r \geq R_0$ , there is a simply connected domain  $G = G(r)$  with

$$B(0, r) \subset G \subset B(0, r^m) \text{ and } |f(z)| \geq M(r, f), \quad \text{for } z \in \partial G, \quad (2.24)$$

and

(b)  $f$  has regular growth in the sense that there exists a sequence  $(r_n)_{n \geq 0}$  with

$$r_n > M^n(R, f) \text{ and } M(r_n, f) \geq r_{n+1}^m, \quad \text{for } n \geq 0. \quad (2.25)$$

*Proof.* Let  $m$  and  $R_0$  be as in (a), and choose  $(r_n)_{n \geq 0}$  satisfying (2.25) with  $r_n > R_0$  for  $n \geq 0$ . For  $n \geq 0$ , let  $G_n = G(r_n)$ .

First, by (2.24) and (2.25),

$$G_n \supset B(0, r_n) \supset B(0, M^n(R, f)), \quad \text{for } n \geq 0, \quad (2.26)$$

and so  $(G_n)$  satisfies (2.1).

Second, by (2.24) and (2.25), if  $z \in \partial G_n$  then  $|f(z)| \geq M(r_n, f) \geq r_{n+1}^m$ . Thus  $f(G_n)$  contains  $B(0, r_{n+1}^m)$ , since  $f$  maps points of  $B(0, M^n(R, f))$  into  $B(0, M^{n+1}(R, f)) \subset B(0, r_{n+1}^m)$ . Now  $G_{n+1}$  is contained in  $B(0, r_{n+1}^m)$  and so is contained in a bounded component of  $\mathbb{C} \setminus f(\partial G_n)$ . Thus  $(G_n)$  satisfies (2.2). Hence, by Theorem 2.2.1,  $A_R(f)$  is a spider's web.  $\square$

We require one additional lemma in order to establish that certain classes of functions satisfy Lemma 2.3.1(b), for  $m > 1$ .

**Lemma 2.3.2.** *Let  $f$  be a transcendental entire function. If  $f$  is  $\psi$ -regular, or if  $f$  is log-regular, then  $f$  satisfies Lemma 2.3.1(b), for  $m > 1$ .*

*Proof.* Firstly, it was shown in [86, Section 8] that if  $f$  is  $\psi$ -regular, then it satisfies Lemma 2.3.1(b), for  $m > 1$ . Secondly, it was shown in [84, Section 7] that if  $f$  is log-regular with constant  $c$ , then  $f$  is  $\psi$ -regular with regularity function  $\psi_m(r) = r^{m^{1/c}}$ , for  $m > 1$ ; see also Lemma 2.2.8. These two observations complete the proof of the lemma.  $\square$

We note that if  $P$  is a non-constant polynomial, then  $P$  satisfies Lemma 2.3.1(a) for  $m > 1$ , taking  $G(r) = B(0, r^\alpha)$ , where  $\alpha \in (1, m)$ , and a suitable  $R_0$ .

We now state the main results of this section. The first relates to the composition of  $\psi$ -regular functions, and the second relates to the composition of entire functions, one of which is log-regular.

**Theorem 2.3.3.** *Let  $f_1, f_2, \dots, f_k$  be transcendental entire functions which satisfy the hypothesis of Lemma 2.3.1(a) for some  $m > 1$ . Suppose that, for  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is  $\psi$ -regular, with regularity function  $\psi_m$  for  $m > 1$ . Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .*

**Theorem 2.3.4.** *Let  $f_1, f_2, \dots, f_k$  be non-constant entire functions. Suppose that, for  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  satisfies the hypothesis of Lemma 2.3.1(a) for some  $m > 1$ . Suppose also that, for some  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is a log-regular transcendental entire function. Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .*

We need one further lemma before we can prove these results. This lemma also concerns the composition of entire functions.

**Lemma 2.3.5.** *Let  $f_1, f_2, \dots, f_k$  be non-constant entire functions. Suppose also that, for  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  satisfies the hypothesis of Lemma 2.3.1(a) with  $m = m_j > 1$ . Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then  $g$  satisfies the hypothesis of Lemma 2.3.1(a) with  $m = m_1 m_2 \dots m_k$ .*

*Proof.* It is sufficient to prove the result for  $k = 2$ . The result is immediate if  $f_1$  and  $f_2$  are both polynomials. Otherwise, let  $m_1$  and  $m_2$  be as given.

Consider first the case that  $f_2$  is a transcendental entire function; the reader may wish to refer to Figure 2.1 at this point, which shows a simplified version of a horizontal slice through some of the domains used in the proof. The boundaries of the discs constructed in the proof are shown as solid lines, and labeled below. The boundaries of the simply connected domains constructed in the proof are shown



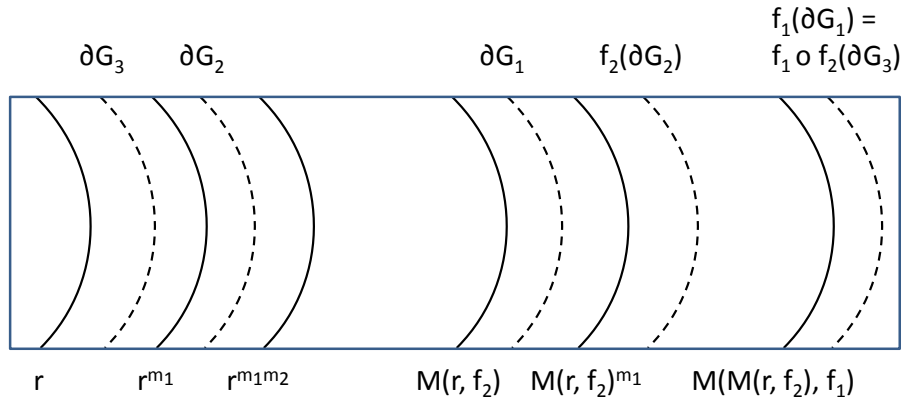


Figure 2.1: A slice through parts of the boundaries of some of the domains used in the proof of Lemma 2.3.5.

as dashed lines and labeled above. For sufficiently large  $r$ , let  $G_1 = G_1(r)$  be a simply connected domain such that

$$B(0, M(r, f_2)) \subset G_1 \subset B(0, M(r, f_2)^{m_1}), \quad (2.27)$$

and

$$|f_1(z)| \geq M(M(r, f_2), f_1), \quad \text{for } z \in \partial G_1. \quad (2.28)$$

For sufficiently large  $r$ , let  $G_2 = G_2(r)$  be a simply connected domain such that

$$B(0, r^{m_1}) \subset G_2 \subset B(0, r^{m_1 m_2}), \quad (2.29)$$

and

$$|f_2(z)| \geq M(r^{m_1}, f_2), \quad \text{for } z \in \partial G_2. \quad (2.30)$$

If  $f_2(z) \in \partial G_1$  then, by (2.27),  $|z| \geq r$ , and so there is a component  $G_3$  of  $f_2^{-1}(G_1)$  which contains  $B(0, r)$ . We observe that it follows from the open mapping theo-

rem that

$$f_2(z) \in \partial G_1, \quad \text{for } z \in \partial G_3. \quad (2.31)$$

If  $z \in \partial G_2$  then, by (2.30) and (1.8),

$$|f_2(z)| \geq M(r^{m_1}, f_2) \geq M(r, f_2)^{m_1}, \quad \text{for large } r. \quad (2.32)$$

If  $z \in \partial G_3$  then, by (2.27) and (2.31),  $|f_2(z)| \leq M(r, f_2)^{m_1}$ . Hence, by the maximum principle, if  $z \in G_3$  then  $|f_2(z)| < M(r, f_2)^{m_1}$ . Thus  $\partial G_2 \cap G_3 = \emptyset$ , by (2.32), and so  $B(0, r) \subset G_3 \subset B(0, r^{m_1 m_2})$ , by (2.29). Also, if  $z \in \partial G_3$  then, by (2.28) and (2.31),

$$|(f_1 \circ f_2)(z)| \geq M(M(r, f_2), f_1) \geq M(r, f_1 \circ f_2). \quad (2.33)$$

We note that  $G_3$  is simply connected; this follows from the maximum principle and the fact that  $G_1$  is simply connected. Hence  $f_1 \circ f_2$  satisfies Lemma 2.3.1(a), with  $m = m_1 m_2$ .

Secondly, we consider the case where  $f_2$  is a polynomial. Choose  $m'$  such that  $m' > m_1$ . For sufficiently large  $r$ , let  $G_1$  and  $G_3$  be the domains from the first part of the proof, and let  $G_2 = B(0, r^{m'})$ . Since  $f_2$  is a polynomial, for sufficiently large  $r$ ,

$$|f_2(z)| \geq M(r, f_2)^{m_1}, \quad \text{for } z \in \partial G_2.$$

As in the first part of the proof,  $\partial G_2 \cap G_3 = \emptyset$ , and so  $B(0, r) \subset G_3 \subset B(0, r^{m'})$ . Also, if  $z \in \partial G_3$  then  $|(f_1 \circ f_2)(z)| \geq M(r, f_1 \circ f_2)$ . Hence  $f_1 \circ f_2$  satisfies Lemma 2.3.1(a), with  $m = m' > m_1$ , in particular with  $m = m_1 m_2$ .  $\square$

In particular it follows from Lemma 2.3.5 that the property of satisfying Lemma 2.3.1(a) for some  $m > 1$  is preserved under iteration.

**Corollary 2.3.6.** *If  $f$  is a transcendental entire function that satisfies Lemma 2.3.1(a) for some  $m > 1$ , then so is  $f^n$  for  $n \in \mathbb{N}$ .*

We are now able to prove Theorems 2.3.3 and 2.3.4.

*Proof of Theorem 2.3.3.* By Lemma 2.3.5,  $g$  satisfies Lemma 2.3.1(a) for some  $m > 1$ . By Theorem 2.2.3,  $g$  is  $\psi$ -regular. Hence, by Lemma 2.3.2, it satisfies

Lemma 2.3.1(b) for  $m > 1$ . The result follows by Lemma 2.3.1.  $\square$

*Proof of Theorem 2.3.4.* As in the proof of Theorem 2.3.3,  $g$  satisfies Lemma 2.3.1(a) for some  $m > 1$ . By Theorem 2.2.5,  $g$  is log-regular. Hence, by Lemma 2.3.2, it satisfies Lemma 2.3.1(b) for  $m > 1$ . The result follows by Lemma 2.3.1.  $\square$

Rippon and Stallard [89, Example 6.1] gave an example of a  $\psi$ -regular functions which is not log-regular. This example has order zero, and so satisfies Lemma 2.3.1(a) for some  $m > 1$ ; see Lemma 2.5.2. This shows that there is a situation in which Theorem 2.3.3 can be applied, but not Theorem 2.3.4.

Finally, we note that the conditions of Theorem 2.3.4 are satisfied by many of the examples in [86, Section 8], and all the examples in this chapter (see Sections 2.5 and 2.6).

## 2.4 Stability of $A_R(f)$ spiders' webs

Many known dynamical properties of a transcendental entire function  $f$  are unstable under relatively small changes in  $f$ . For example, the functions

$$f_1(z) = e^{-z},$$

$$f_2(z) = f_1(z) + z + 2\pi i - 1,$$

and

$$f_3(z) = f_1(z) + z + 1,$$

all have very different Fatou sets (see, for example, [18, Section 4]). In this section we prove results which show that, in certain circumstances,  $A_R(f)$  spiders' webs can be very stable. The first result concerns composition with polynomials.

**Theorem 2.4.1.** *Suppose that  $f$  is a log-regular transcendental entire function which satisfies Lemma 2.3.1(a) for some  $m_0 > 1$ . Let  $P(w, z)$  be a polynomial of degree at least one in  $w$ , and let  $Q(z)$  be a polynomial of degree at least one.*

*Let  $g(z) = P(f(Q(z)), z)$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .*

*Proof.* By Corollary 2.2.14,  $g$  is log-regular. Thus, by Lemma 2.3.2, it satisfies Lemma 2.3.1(b) for  $m > 1$ . Hence we need only prove that  $g$  satisfies Lemma 2.3.1(a) for some  $m > 1$ .

As in the proof of Corollary 2.2.14, let

$$g(z) = af(Q(z))^{N_1} z^{N_2} + \dots,$$

where  $N_1$  is the highest power of  $w$  in  $P(w, z)$ , and  $N_2$  is the highest power of  $z$  corresponding to  $f(Q(z))^{N_1}$ . By Lemma 2.3.5,  $f \circ Q$  satisfies Lemma 2.3.1(a). Hence, there is an  $m_1 > 1$  such that, for sufficiently large  $r$ , there is a simply connected domain  $G = G(r)$  with  $B(0, r^{m_1}) \subset G \subset B(0, r^{m_1^2})$  and

$$|f(Q(z))| \geq M(r^{m_1}, f \circ Q), \quad \text{for } z \in \partial G. \quad (2.34)$$

Hence, when  $z \in \partial G$ , for sufficiently large  $r$ ,

$$\begin{aligned} |g(z)| &\geq \frac{1}{2}|a|M(r^{m_1}, f \circ Q)^{N_1} r^{N_2} && \text{by (2.34) and (1.6)} \\ &\geq 2|a|M(r, f \circ Q)^{N_1} r^{N_2} && \text{by (1.8)} \\ &\geq M(r, g) && \text{by (1.6)}. \end{aligned}$$

Thus  $g$  satisfies Lemma 2.3.1(a) with  $m = m_1^2$ , so the proof is complete.  $\square$

The second result concerns addition of an entire function to a transcendental entire function with an  $A_R(f)$  spider's web.

**Theorem 2.4.2.** *Suppose that  $f$  is a log-regular transcendental entire function which satisfies Lemma 2.3.1(a) for some  $m > 1$ , and that  $h$  is an entire function. Suppose also that there exist  $a \in (0, 1)$  and  $R_0 > 0$  such that*

$$aM(r, f) \geq M(r^m, h), \quad \text{for } r \geq R_0. \quad (2.35)$$

*Let  $g = f + h$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .*

*Proof.* First we note that, for sufficiently large  $r$ ,  $aM(r, f) \geq M(r, h)$ . Hence, by Corollary 2.2.12,  $g$  is log-regular and so, by Lemma 2.3.2, it satisfies Lemma 2.3.1(b)

for  $m > 1$ . Thus, by Lemma 2.3.1, it remains to prove that  $g$  satisfies Lemma 2.3.1(a) for some  $m > 1$ .

By hypothesis, for sufficiently large  $r$ , there is a simply connected domain  $G = G(r)$  with  $B(0, r^m) \subset G \subset B(0, r^{m^2})$  and

$$|f(z)| \geq M(r^m, f), \quad \text{for } z \in \partial G. \quad (2.36)$$

Thus, when  $z \in \partial G$ , for sufficiently large  $r$ ,

$$\begin{aligned} |g(z)| &\geq |f(z)| - |h(z)| \\ &\geq (1 - a)M(r^m, f) && \text{by (2.35), and (2.36)} \\ &\geq (1 + a)M(r, f) && \text{by (1.8)} \\ &\geq M(r, g). \end{aligned}$$

Hence  $g$  satisfies Lemma 2.3.1(a) with  $m$  replaced by  $m^2$ , so the proof is complete.  $\square$

**Remark 2.4.1.** Using the same method of proof it can be shown that in Theorem 2.4.2 the function  $h$  can also be of the form  $h(z) = f(z)/(z - c)$ , where  $f(c) = 0$ . First we note that, for large values of  $r$ , we have

$$\frac{1}{2}M(r, g) \leq M(r, f) \leq 2M(r, g).$$

The fact that  $g$  is log-regular then follows by Theorem 2.2.7. The fact that  $g$  satisfies Lemma 2.3.1(a) for some  $m > 1$  follows from the observation that, for large values of  $r$ , we have

$$\frac{r}{2}M(r, h) \leq M(r, f) \leq 2rM(r, h).$$

Finally, we note that the conditions on  $f$  in Theorems 2.4.1 and 2.4.2 are satisfied by functions in many of the classes given in [86, Section 8], in particular functions which satisfy Theorem 1.9.2 parts (c), (d) and (e); this follows from remarks in [86, Section 8]. These conditions are also satisfied for all the examples

in this chapter. It can be shown that these conditions are also satisfied by the functions in [70]. So we can produce new functions for which  $A_R(f)$  is a spider's web by taking these known examples and applying Theorems 2.4.1 and 2.4.2.

## 2.5 Functions of finite order for which $A_R(f)$ is a spider's web

In this section we develop a technique which enables us to take a transcendental entire function of finite order, modify its power series, and produce a class of transcendental entire functions of finite order for which  $A_R(f)$  is a spider's web. From the exponential function we obtain a class of such functions (Example 2.5.1) which contains the function

$$f(z) = \frac{1}{2}(\cos z^{\frac{1}{4}} + \cosh z^{\frac{1}{4}}) = \sum_{n=0}^{\infty} \frac{z^n}{(4n)!} \quad (2.37)$$

given in [86, Section 8], together with the related functions

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{pn}}{(qn)!}, \quad p, q \in \mathbb{N}, \quad p/q < \frac{1}{2}, \quad (2.38)$$

suggested by Halburd and also mentioned in [86, Section 8]. We obtain another class (Example 2.5.3) from the error function (see [1, p.297])

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1}. \quad (2.39)$$

Recall that the order  $\rho(f)$  and lower order  $\lambda(f)$  of a transcendental entire function  $f$  are defined in (1.3).

We use the following three lemmas, all discussed in [86, Corollary 8.3 and the following remarks]. The first is from [58, p.205], and gives a sufficient condition for a transcendental entire function to be log-regular.

**Lemma 2.5.1.** *If  $f$  is a transcendental entire function of finite order and positive lower order, then  $f$  is log-regular.*

The second is from, for example, [4, Satz 1].

**Lemma 2.5.2.** *If  $f$  is a transcendental entire function of order less than  $\frac{1}{2}$ , then  $f$  satisfies Lemma 2.3.1(a) for some  $m > 1$ .*

The third follows from Lemma 2.5.1, Lemma 2.5.2 and Lemma 2.3.1.

**Lemma 2.5.3.** *If  $f$  is a transcendental entire function of order less than  $\frac{1}{2}$  and positive lower order, then  $A_R(f)$  is a spider's web, where  $R > 0$  is such that  $M(r, f) > r$  for  $r \geq R$ .*

We use the following operator to produce classes of functions which satisfy the conditions of Lemma 2.5.3. For  $n, m \in \mathbb{N}$ , let  $T_{n,m}$  be defined by

$$T_{n,m}(f(z)) = \frac{1}{m} \sum_{k=1}^m f\left(e^{\frac{2\pi ik}{m}} z^{\frac{n}{m}}\right), \quad (2.40)$$

where  $f$  is an entire function, and we choose a consistent branch of the  $m$ th root for each term in the sum.

If  $f$  is a transcendental entire function, then the  $T_{n,m}$  operator extracts from the power series of  $f$  only those terms with exponents which are multiples of  $m$ , and these exponents are multiplied by  $n/m$  (see (2.41) below). For example, if  $f(z) = e^z$ , then

$$T_{2,3}(f(z)) = 1 + \frac{z^2}{3!} + \frac{z^4}{6!} + \cdots .$$

We note in passing that the  $T_{n,m}$  operator has some appealing properties; for example,  $T_{1,m} \circ T_{1,n} = T_{1,nm}$  and also  $T_{n,m}(f(z^m)) = f(z^n)$ .

The following result concerns a key property of this operator, namely its effect on the order of a function.

**Theorem 2.5.4.** *If  $f$  is a transcendental entire function of order  $\rho(f)$  and  $n, m \in \mathbb{N}$ , then  $T_{n,m}(f)$  is a well-defined entire function of order at most  $\frac{n}{m}\rho(f)$ .*

*Proof.* First, we consider the action of  $T_{n,m}$  on the power series  $f(z) = \sum_{l=0}^{\infty} a_l z^l$ . Since we have a consistent choice of the  $m$ th root, the sum of the complex roots of unity is zero, and with  $p = l/m$ , we obtain

$$T_{n,m}(f(z)) = \frac{1}{m} \sum_{k=1}^m \sum_{l=0}^{\infty} a_l e^{\frac{2\pi ikl}{m}} z^{\frac{ln}{m}} = \sum_{l=0}^{\infty} a_l z^{\frac{ln}{m}} \sum_{k=1}^m \frac{1}{m} e^{\frac{2\pi ikl}{m}} = \sum_{p=0}^{\infty} a_{pm} z^{pn}. \quad (2.41)$$

Hence the value of  $T_{n,m}(f)$  is independent of the choice of the  $m$ th root, and this power series has infinite radius of convergence.

We deduce from (1.4), with  $k = pm$ , that

$$\begin{aligned} \rho(T_{n,m}(f)) &= \limsup_{p \rightarrow \infty} \frac{pn \log pn}{\log |a_{pm}|^{-1}} \\ &\leq \limsup_{k \rightarrow \infty} \frac{(kn/m) \log(kn/m)}{\log |a_k|^{-1}} \\ &= \frac{n}{m} \limsup_{k \rightarrow \infty} \frac{k \log k}{\log |a_k|^{-1}} \\ &= \frac{n}{m} \rho(f), \end{aligned}$$

as required.  $\square$

We now seek to use this operator, together with Lemma 2.5.3, to generate transcendental entire functions for which  $A_R(f)$  is a spider's web. It is possible for the function  $T_{n,m}(f)$  to be simply a polynomial when  $f$  is a transcendental entire function. For example, if  $f(z) = z \exp(z^2)$  then  $T_{1,2}(f(z)) = 0$ , because the power series of  $f$  has only odd powers of  $z$  which are eliminated by the  $T_{1,2}$  operator.

Even if  $T_{n,m}(f)$  is transcendental,  $T_{n,m}(f)$  may not have positive lower order when  $f$  does. For example, if  $g$  is a transcendental entire function of order less than 1 and lower order zero, then  $f(z) = g(z^2) + z \exp(z^2)$  has both order and lower order 2, but  $T_{1,2}(f(z)) = T_{1,2}(g(z^2)) = g(z)$  has order less than 1 and lower order zero, reasoning as in the previous paragraph.

The following lemma gives two sufficient conditions for  $T_{n,m}(f)$  to have positive lower order.

**Lemma 2.5.5.** *Let  $f(z) = \sum_{p=0}^{\infty} a_p z^p$  be a transcendental entire function, and let  $n, m \in \mathbb{N}$ .*

(a) *If*

$$\liminf_{p \rightarrow \infty} \frac{p \log p}{\log |a_{pm}|^{-1}} > 0, \quad (2.42)$$

*then  $T_{n,m}(f)$  has positive lower order.*

(b) *If  $T_{n,m}(f)$  has positive lower order, and  $g(z) = \sum_{p=0}^{\infty} b_p z^p$  is a transcendental*



entire function with  $|b_p| \geq |a_p|$  for  $p$  sufficiently large, then  $T_{n,m}(g)$  has positive lower order.

*Proof.* For part (a) we note, by (2.41) and with  $n_p = np$  in (1.5), that

$$\lambda(T_{n,m}(f)) \geq \liminf_{p \rightarrow \infty} \frac{np \log(n(p-1))}{\log |a_{pm}|^{-1}} = n \liminf_{p \rightarrow \infty} \frac{p \log p}{\log |a_{pm}|^{-1}} > 0.$$

Part (b) follows immediately from (1.5).  $\square$

We now give some explicit examples of classes of functions for which  $A_R(g)$  is a spider's web. The first example includes (2.37) as a special case.

**Example 2.5.1.** Let  $g = T_{n,m}(f)$ , where  $f(z) = \exp(z)$  and where  $m > 2n$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .

*Proof.* The exponential function has order 1, and satisfies (2.42) for  $m > 1$ . Thus  $g$  has order less than  $\frac{1}{2}$  by Theorem 2.5.4. The result follows by Lemma 2.5.5(a) and Lemma 2.5.3.  $\square$

The second example illustrates the use of both parts of Lemma 2.5.5.

**Example 2.5.2.** Let  $g = T_{n,m}(f)$ , where  $f(z) = z \exp(z^2) + \exp(z)$ ,  $m > 4n$  and  $m$  is odd. Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .

*Proof.* The function  $z \mapsto z \exp(z^2)$  has order 2, and satisfies (2.42) when  $m$  is odd. Thus  $g$  has order less than  $\frac{1}{2}$  by Theorem 2.5.4, and the result follows by Lemma 2.5.5(b), with comparison function  $z \mapsto z \exp(z^2)$ , and Lemma 2.5.3.  $\square$

The technique of this section can be applied any transcendental entire function of finite order, provided its power series satisfies (2.42) for some  $m \in \mathbb{N}$ . We illustrate this with the error function.

**Example 2.5.3.** Let  $g = T_{n,m}(f)$ , where  $f(z) = \operatorname{erf}(z)$ ,  $m > 4n$  and  $m$  is odd. Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .

*Proof.* By (2.39) and (1.4),  $f$  has order 2, and satisfies (2.42) when  $m$  is odd. Thus  $g$  has order less than  $\frac{1}{2}$  by Theorem 2.5.4. The result follows by Lemma 2.5.5(a) and Lemma 2.5.3.  $\square$

Our final example combines earlier results to give an unexpectedly simple function with an  $A_R(g)$  spider's web.

**Example 2.5.4.** Let  $g(z) = \cos z + \cosh z$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .

*Proof.* This follows from Theorem 2.3.4 and the function  $f$  defined in (2.37), since  $g(z) = 2f(z^4)$ .  $\square$

Illustrating the  $A(g)$  spider's web for this function is difficult. We note that any point which iterates to the real line is certainly in  $A(g)$ ; this is because  $g$  achieves its maximum modulus on the positive real line, and all points on the real line map under  $g$  to the positive real line. In Figure 2.2 points in black represent those which iterate close to the real line, and so these points approximate a subset of  $A(g)$ . Indeed, in much of the figure  $A(g)$  appears to have non-empty interior, particularly near the origin. This is an artifact of the level of approximation required to obtain visible details elsewhere in the figure, and should not be interpreted as being the case. The scale of this figure has both real and imaginary parts between 0 and 5.

Our goal in this section has been to produce a class of log-regular transcendental entire functions of order less than  $\frac{1}{2}$ , which, by Lemmas 2.5.2 and 2.3.1, have an  $A_R(f)$  spider's web. Finally, we show that this class can be extended by differentiation or integration, thus giving a further method of constructing  $A_R(f)$  spiders' webs.

**Theorem 2.5.6.** *Let  $f$  be a log-regular transcendental entire function of order less than  $\frac{1}{2}$ , and let  $g$  be the derivative of  $f$  or an integral of  $f$ . Then  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .*

*Proof.* We observe that  $g$  has the same order as  $f$ , and is log-regular by Corollary 2.2.11. By Lemma 2.3.2 and Lemma 2.5.2, the hypotheses of Lemma 2.3.1 are satisfied. The result follows by Lemma 2.3.1.  $\square$

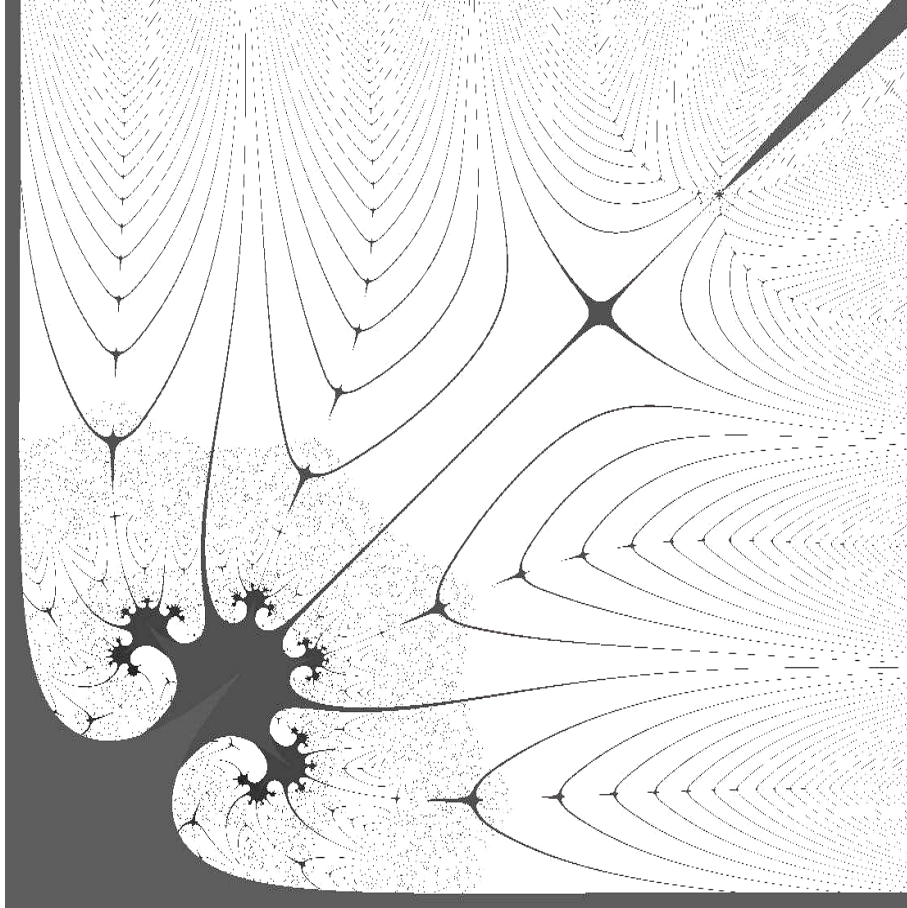


Figure 2.2: An approximation to part of  $A(g)$ , where  $g(z) = \cos z + \cosh z$ .

## 2.6 A function of infinite order with gaps for which $A_R(f)$ is a spider's web

We recall that a transcendental entire function  $f$  has Fabry gaps if

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

and  $n_k/k \rightarrow \infty$  as  $k \rightarrow \infty$ . By a result of Fuchs [47], an entire function  $f$  of finite order with Fabry gaps satisfies Lemma 2.3.1(a) for  $m > 1$ . This fact was used by Wang in [105, Theorem 1] to describe a class of entire functions with no unbounded Fatou components. Thus if  $f$  is also log-regular then, by

Lemma 2.3.2 and Lemma 2.3.1,  $A_R(f)$  is a spider's web. This fact was pointed out by Rippon and Stallard in Theorem 1.9.2(d). They gave an example of such a function [86, Example 1], shown to be log-regular by using Lemma 2.5.1.

It was also pointed out in [105] and in [86, Section 8] that, by a result of Hayman [50], Lemma 2.3.1(a) holds in the case of certain functions of infinite order with gaps:

**Lemma 2.6.1.** *Let  $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$  be a transcendental entire function where, for some  $\alpha > 2$ ,*

$$n_k > k \log k (\log \log k)^\alpha, \quad \text{for large } k. \quad (2.43)$$

*Then  $f$  satisfies Lemma 2.3.1(a) for  $m > 1$ .*

Wang [105, Theorem 2] used this result to show that if  $f$  satisfies (2.43) and has a property equivalent to log-regularity, then  $f$  has no unbounded Fatou components.

Suppose that  $g$  is a transcendental entire function of infinite order generated by omitting terms from the power series of another transcendental entire function,  $f$  say, and  $g$  satisfies (2.43). If  $g$  is also log-regular, then  $A_R(g)$  is a spider's web, by Lemma 2.3.2 and Lemma 2.3.1. If  $f$  has infinite order, then it does not seem straightforward to check that such a function  $g$  is log-regular. In this section we demonstrate a method for achieving this, and then give an explicit example of such a function.

We start with a general result.

**Theorem 2.6.2.** *Suppose that  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is a log-regular transcendental entire function and there exists  $N_0 \in \mathbb{N}$  such that*

$$0 < a_{n+1} \leq a_n, \quad \text{for } n \geq N_0. \quad (2.44)$$

*Suppose also that  $g$  is a transcendental entire function with*

$$g(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k}, \quad (2.45)$$

where, for some  $M > 1$  and  $\alpha > 2$ ,

$$1 < \frac{n_{k+1}}{n_k} < M, \quad \text{for large } k, \quad (2.46)$$

and

$$n_k > k \log k (\log \log k)^\alpha, \quad \text{for large } k. \quad (2.47)$$

Then  $g$  is log-regular and  $A_R(g)$  is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .

*Proof.* By Lemma 2.6.1,  $g$  satisfies Lemma 2.3.1(a) for  $m > 1$ . To complete the proof, we use Theorem 2.2.7 to show that  $g$  is log-regular.

Without loss of generality, by adding a polynomial, we can assume by Corollary 2.2.14 that  $N_0 = 0$  and (2.46) holds for  $k \geq 1$ . Because  $a_n > 0$  for  $n \geq 0$ ,

$$M(r, f) = f(r) > g(r) = M(r, g), \quad \text{for } r > 0.$$

Thus it remains to show that there exist  $a > 1$  and  $R_0 > 0$  such that

$$M(r^a, g) \geq M(r, f), \quad \text{for } r \geq R_0. \quad (2.48)$$

Choose  $a' > 1$  and  $K > 1$  sufficiently large such that

$$\frac{n_{k+1}}{n_k} < \frac{1}{2}(1 + a') < a', \quad \text{and} \quad K^{n_k} > n_{k+1} - n_k, \quad \text{for } k \geq 1. \quad (2.49)$$

Now let  $\mu = \frac{1}{2}(a' - 1) > 0$ , and define

$$M(r^{a'}, g) = \sum_{k=1}^{\infty} A_k, \quad A_k = a_{n_k} r^{a' n_k}, \quad (2.50)$$

$$M(r, f) = \sum_{n=0}^{a_{n_1}-1} a_n r^n + \sum_{k=1}^{\infty} B_k, \quad B_k = a_{n_k} r^{n_k} + \cdots + a_{n_{k+1}-1} r^{n_{k+1}-1}. \quad (2.51)$$

Because the  $a_n$  are decreasing,

$$B_k < (n_{k+1} - n_k) a_{n_k} r^{n_{k+1}}, \quad \text{for } r > 1 \text{ and } k \geq 1. \quad (2.52)$$

Thus, if  $r > \max\{1, K^{\frac{1}{\mu}}\}$ , then, by (2.50) and (2.49),

$$B_k < (n_{k+1} - n_k) r^{n_{k+1} - a' n_k} A_k < K^{n_k} r^{-n_k \mu} A_k < A_k, \quad \text{for } k \geq 1. \quad (2.53)$$

Thus, by (2.50) and (2.51),

$$M(r^{a'}, g) > M(r, f) - \sum_{n=0}^{a_{n_1} - 1} a_n r^n, \quad \text{for } r > \max\{1, K^{\frac{1}{\mu}}\}. \quad (2.54)$$

Finally, for any  $a > a'$  we can choose  $r$  sufficiently large such that

$$\begin{aligned} M(r^a, g) &\geq 2M(r^{a'}, g) && \text{by (1.7)} \\ &> 2M(r, f) - 2 \sum_{n=0}^{a_{n_1} - 1} a_n r^n && \text{by (2.54)} \\ &\geq M(r, f) && \text{by (1.6)}. \end{aligned}$$

This proves (2.48) as required.  $\square$

In the rest of this section we construct an explicit example of a transcendental entire function  $f$  of infinite order, defined by a gap series, for which  $A_R(f)$  is a spider's web. First we need a simple result about functions of infinite order.

**Lemma 2.6.3.** *Let  $f$  and  $g$  be transcendental entire functions, and suppose that  $f$  has infinite order. If there exist  $a, R_0 > 0$  such that*

$$M(r^a, g) \geq M(r, f), \quad \text{for } r \geq R_0,$$

*then  $g$  has infinite order.*

*Proof.* By (1.3),

$$\rho(g) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r^a, g)}{\log r^a} \geq \frac{1}{a} \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r} = \frac{1}{a} \rho(f),$$

and the result follows.  $\square$

The next lemma is needed in the construction of our example.

**Lemma 2.6.4.** *Let  $g(z) = \sum_{n=0}^{\infty} a_n z^n$  be a transcendental entire function, with  $a_n \in \mathbb{R}$  for  $n \geq 0$ ,  $a_1 \leq 1$ , and*

$$0 < (n+1)a_{n+1} \leq na_n, \quad \text{for } n \geq 1. \quad (2.55)$$

*Then  $f(z) = \exp(g(z))$  has power series  $f(z) = \sum_{n=0}^{\infty} b_n z^n$ , where*

$$0 < b_{n+1} \leq b_n, \quad \text{for } n \geq 1. \quad (2.56)$$

*Proof.* Clearly  $b_n > 0$  for  $n \geq 0$ . Since  $f'(z) = g'(z)f(z)$  we have

$$\sum_{n=0}^{\infty} (n+1)b_{n+1}z^n = \sum_{k=0}^{\infty} (k+1)a_{k+1}z^k \sum_{l=0}^{\infty} b_l z^l. \quad (2.57)$$

Equating powers of  $z$  gives

$$(n+1)b_{n+1} = \sum_{l=0}^n (n+1-l)a_{n+1-l}b_l, \quad \text{for } n \geq 0. \quad (2.58)$$

Hence, for  $n \geq 1$ ,

$$(n+1)b_{n+1} = \sum_{l=0}^{n-1} (n+1-l)a_{n+1-l}b_l + a_1 b_n \quad (2.59)$$

$$\leq \sum_{l=0}^{n-1} (n-l)a_{n-l}b_l + b_n, \quad \text{by (2.55) and as } a_1 \leq 1 \quad (2.60)$$

$$= nb_n + b_n, \quad \text{by (2.58),} \quad (2.61)$$

which proves that (2.56) holds.  $\square$

Finally, as promised, we give our explicit example.

**Theorem 2.6.5.** *Let*

$$f(z) = \exp(e^z - 1) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_{n^2} z^{n^2}.$$

*Then  $g$  is a log-regular transcendental entire function of infinite order, and  $A_R(g)$*

is a spider's web, where  $R > 0$  is such that  $M(r, g) > r$  for  $r \geq R$ .

*Proof.* We can see that  $f$  is log-regular because  $\phi(t) = \log M(e^t, f) = (e^t - 1)$  and

$$\frac{\phi'(t)}{\phi(t)} > e^t \geq \frac{2}{t}, \quad \text{for } t \geq 1.$$

Conditions (2.46) and (2.47) are satisfied, and the coefficients  $b_n$  are decreasing because the function  $z \mapsto e^z - 1$  satisfies the conditions of Lemma 2.6.4. Hence, by Theorem 2.6.2,  $g$  is log-regular and  $A_R(g)$  is a spider's web.

Finally,  $f$  has infinite order. We see from the proof of Theorem 2.6.2 that  $f$  and  $g$  satisfy (2.48). Hence, by Lemma 2.6.3,  $g$  has infinite order.  $\square$

Clearly this approach can be used with the function  $f$  of Theorem 2.6.5 to give a class of functions with  $A_R(f)$  spiders' webs, by suitably selecting terms from the power series of  $f$ . We can also use Lemma 2.6.4 to find other transcendental entire functions which can be manipulated in this way to give further classes of examples.

**Remark 2.6.1.** We note in passing that, in Theorem 2.6.5,  $b_n = B_n/n!$ , where  $(B_n)$  are the Bell numbers (see, for example, [17]). Thus, by (2.56), we have

$$B_{n+1} \leq (n+1)B_n, \quad \text{for } n \geq 1.$$

In fact the more precise estimates

$$2B_n < B_{n+1} < (n+1)B_n, \quad \text{for } n \geq 2,$$

hold (see [32, Corollary 8]). These can be deduced in a straightforward way from the identity

$$B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k, \quad \text{for } n \geq 0, \tag{2.62}$$

which follows from (2.58).



## 2.7 Transcendental entire functions with no unbounded Fatou components

As mentioned in the introduction, Baker [8] posed the question of whether the Fatou set of a transcendental entire function of sufficiently small growth can have any unbounded components. By Theorem 1.9.1(d), all the examples in this chapter have no unbounded Fatou components. In this section we give two results on functions with no unbounded Fatou components, which generalise existing results of this type.

Our first class of functions with no unbounded Fatou components consists of functions formed by composition of  $\psi$ -regular functions.

**Theorem 2.7.1.** *Let  $f_1, f_2, \dots, f_k$  be transcendental entire functions which satisfy Lemma 2.3.1(a) for some  $m > 1$ . Suppose that, for  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is  $\psi$ -regular, with regularity function  $\psi_m$  for  $m > 1$ . Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then every component of  $F(g)$  is bounded.*

*Proof.* By Theorem 2.3.3,  $A_R(g)$  is a spider's web, and the result follows by Theorem 1.9.1(d).  $\square$

To compare Theorem 2.7.1 to previous results, we need the following lemma, which is part of [84, Theorem 6]. This gives a sufficient condition for a transcendental entire function to be  $\psi$ -regular. We note that although the full statement of [84, Theorem 6] supposes order less than  $\frac{1}{2}$ , in order to establish a part of the result we do not use, finite order is sufficient for the proof of the part of the result we do use.

**Lemma 2.7.2.** *Let  $f$  be a transcendental entire function of finite order. Suppose that there exist  $n \in \mathbb{N}$  and  $q \in (0, 1)$  such that*

$$M(r, f) \geq \exp^{n+1}((\log^n r)^q), \quad \text{for large } r. \quad (2.63)$$

*Then  $f$  is  $\psi$ -regular with regularity function given, for  $m > 1$ , by*

$$\psi_m(r) = \exp^n((\log r)^p), \quad \text{where } pq > 1.$$

The next result now follows from Lemma 2.7.2 and Theorem 2.7.1.

**Corollary 2.7.3.** *Let  $f_1, f_2, \dots, f_k$  be transcendental entire functions of finite order which satisfy Lemma 2.3.1(a) for some  $m > 1$ . Suppose that there exist  $n \in \mathbb{N}$  and  $q \in (0, 1)$  such that, for  $j \in \{1, 2, \dots, k\}$ ,*

$$M(r, f_j) \geq \exp^{n+1}((\log^n r)^q), \quad \text{for large } r. \quad (2.64)$$

*Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then every component of  $F(g)$  is bounded.*

Rippon and Stallard, in [84, Theorem 6], showed that if  $f$  is a transcendental entire function of order less than  $\frac{1}{2}$ , which satisfies (2.63) for some  $n \in \mathbb{N}$  and  $q \in (0, 1)$ , then  $f$  has no unbounded Fatou components. By Lemma 2.5.2 this is included in Corollary 2.7.3, with  $k = 1$ .

Corollary 2.7.3, with  $n = 1$ , includes a result of Singh in [96, Theorem 1]. (We note that the statement of [96, Theorem 1] omits the requirement of finite order, but this was assumed in the proof of [96, Lemma 1].)

Our second class of functions with no unbounded Fatou components consists of functions formed by composition of entire functions, one of which is log-regular.

**Theorem 2.7.4.** *Let  $f_1, f_2, \dots, f_k$  be entire functions. Suppose that, for  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  satisfies Lemma 2.3.1(a) for some  $m > 1$ . Suppose also that, for some  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is a log-regular transcendental entire function. Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then every component of  $F(g)$  is bounded.*

*Proof.* By Theorem 2.3.4,  $A_R(g)$  is a spider's web, and the result follows by Theorem 1.9.1(d).  $\square$

As noted in Section 2.2, this result differs from Theorem 2.7.1 in that only one function in the composition needs to satisfy the regularity condition and be transcendental.

The final result follows from Theorem 2.7.4 and Lemma 2.5.2.

**Corollary 2.7.5.** *Let  $f_1, f_2, \dots, f_k$  be transcendental entire functions of order less than  $\frac{1}{2}$ . Suppose that, for some  $j \in \{1, 2, \dots, k\}$ ,  $f_j$  is log-regular. Let  $g = f_1 \circ f_2 \circ \dots \circ f_k$ . Then every component of  $F(g)$  is bounded.*

This corollary generalises a result of Anderson and Hinkkanen in [3, Theorem 2], which states that if a log-regular function has order less than  $\frac{1}{2}$ , then it has no unbounded Fatou components. Anderson and Hinkkanen's result is included in Corollary 2.7.5 with  $k = 1$ .

Cao and Wang [33] developed a similar result to Corollary 2.7.5, concerning composition of transcendental entire functions. They set  $g = f_1 \circ f_2 \circ \cdots \circ f_k$ , where  $f_1, f_2, \dots, f_k$  are transcendental entire functions of order less than  $\frac{1}{2}$ , at least one of which has positive lower order, and showed that  $g$  has no unbounded Fatou components. By Lemma 2.5.1, Cao and Wang's result is included in Corollary 2.7.5. We note that it is possible to construct a class of log-regular functions of lower order zero and any given finite order, in particular order less than  $\frac{1}{2}$ . This shows that there are situations in which Corollary 2.7.5 can be applied but not the result of [33].

# Chapter 3

## On fundamental loops and the fast escaping set

### 3.1 Introduction

As noted in Section 1.3, the structure of  $A_R(f)$  spiders' webs can be understood through fundamental holes and fundamental loops. Recall that, when  $A_R(f)$  is a spider's web, we define the fundamental hole  $H_R$  as the component of  $A_R(f)^c$  that contains the origin, and the fundamental loop  $L_R$  by  $L_R = \partial H_R$ .

By Theorem 1.9.2,  $A_R(f)$  is a spider's web whenever  $f$  is a transcendental entire function with a multiply connected Fatou component. In this chapter we give the first results on the properties of fundamental loops in this case. The first of these gives information on the location of some fundamental loops.

**Theorem 3.1.1.** *Suppose that  $f$  is a transcendental entire function. Then there exists  $R' = R'(f) > 0$  such that the following holds. If  $U$  is a multiply connected Fatou component of  $f$ , such that  $U$  surrounds the origin and  $\text{dist}(0, U) > R'$ , then there exist  $0 < R_1 < R_2$  such that*

(a)  $L_{R_1} = \partial_{\text{int}} U$ ;

(b)  $L_{R_2} = \partial_{\text{out}} U$ ;

(c) *if  $L_R$  is a fundamental loop such that  $L_R \cap U \neq \emptyset$ , then  $L_R \subset U$ . Moreover, this condition occurs if and only if  $R_1 < R < R_2$ .*

Here  $\partial_{\text{out}}U$  is defined as the boundary of the unbounded component of  $\mathbb{C}\setminus U$ , and  $\partial_{\text{int}}U$  is defined as the boundary of the component of  $\mathbb{C}\setminus\bar{U}$  that contains the origin. The related set  $\partial_{\text{inn}}U$  is defined in [27] as the boundary of the component of  $\mathbb{C}\setminus U$  that contains the origin.

These subsets of the boundary are illustrated in Figure 3.1. A multiply connected Fatou component  $U$  is shown in grey, with the origin at the centre of the diagram. Here  $\partial_{\text{out}}U$  is shown dashed and  $\partial_{\text{int}}U$  is shown dotted. The set  $\partial_{\text{inn}}U$  consists of  $\partial_{\text{int}}U$ , plus the boundary of the ‘pinch’ shown at the bottom of the inner boundary. Note that  $\partial_{\text{out}}U$  does include the boundary of the ‘pinch’ shown at the top of the outer boundary. It is not known if it is possible for a multiply connected Fatou component to have a ‘pinch’. Note also that, in this figure,  $U$  surrounds a component of the complement of  $U$  which does not contain the origin; the boundary of this set lies outside  $\partial_{\text{out}}U \cup \partial_{\text{inn}}U$ .

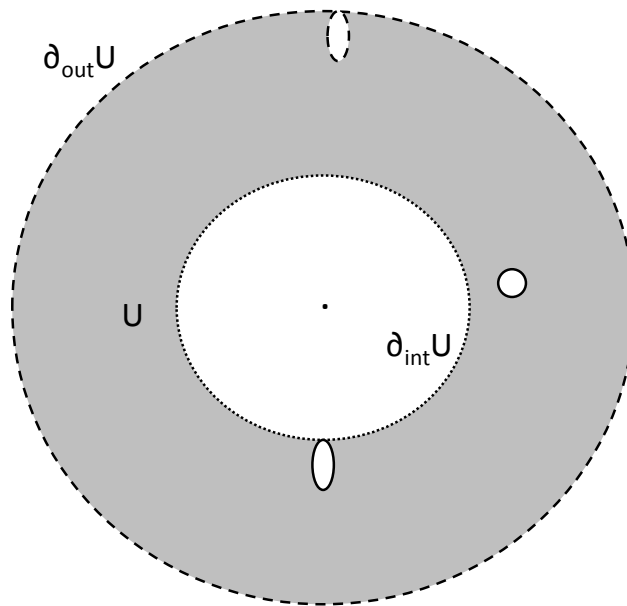


Figure 3.1: Subsets of the boundary of a multiply connected Fatou component.

Recall that, in general, if  $U$  is a Fatou component, we write  $U_n$ ,  $n \geq 0$ , for the Fatou component containing  $f^n(U)$ . Note that, by Theorem 1.5.1, if  $V$  is a multiply connected Fatou component then there is an  $N \in \mathbb{N}$  such that,

for  $n \geq N$ ,  $V_n$  is a multiply connected Fatou component which satisfies the hypotheses of Theorem 3.1.1.

Using Theorem 3.1.1 we prove the following result.

**Theorem 3.1.2.** *Suppose that  $f$  is a transcendental entire function and that  $L_R$  is a fundamental loop of  $f$ . Then either  $L_R \subset F(f)$  or  $L_R \subset J(f)$ .*

We observe that both alternatives in the conclusion of Theorem 3.1.2 are possible. This follows from Theorem 3.1.1.

A second consequence of Theorem 3.1.1 is that when a fundamental loop lies within a multiply connected Fatou component,  $U$ , it is often possible to say more about the nature of this set. In fact, there is a close relationship between some fundamental loops of  $f$  and some level sets of the function  $h$  that was introduced by Bergweiler, Rippon and Stallard in [27], and used to prove many geometric properties of multiply connected Fatou components. The function  $h$  is defined by

$$h(z) = \lim_{n \rightarrow \infty} \frac{\log |f^n(z)|}{\log |f^n(z_0)|}, \quad \text{for } z \in U, \text{ some } z_0 \in U. \quad (3.1)$$

It is shown in [27, Theorem 1.1] that this limit exists, and that the function  $h$  is non-constant, positive and harmonic. As observed in [27], the function  $h$  defined in (3.1) depends on the choice of  $z_0$ . However, if  $z_0$  is replaced by another point  $z'_0 \in U$ , then the resulting function is just  $h$  scaled by a positive factor equal to  $1/h(z'_0)$ .

Our result is as follows.

**Theorem 3.1.3.** *Suppose that  $f$  is a transcendental entire function. Then there exists  $R' = R'(f) > 0$  such that the following holds. If  $U$  is a multiply connected Fatou component of  $f$ , such that  $U$  surrounds the origin,  $\text{dist}(0, U) > R'$ , and  $h$  is as defined as in (3.1), then*

- (a) *if  $L_R \subset U$  is a fundamental loop, then  $h(z)$  is constant on  $L_R$  and so  $L_R$  is a piecewise analytic Jordan curve;*
- (b) *if  $\Gamma$  is a level set of  $h$ , then  $\Gamma$  has a component  $\gamma$  which surrounds the origin and there is a fundamental loop  $L_R$  such that  $L_R \subset \gamma$ .*

The situation of Theorem 3.1.3 is illustrated in Figure 3.2. A multiply connected Fatou component  $U$  is shown, with a fundamental loop  $L_R$  which is contained in  $U$ . The function  $h$  is constant on  $L_R$ .

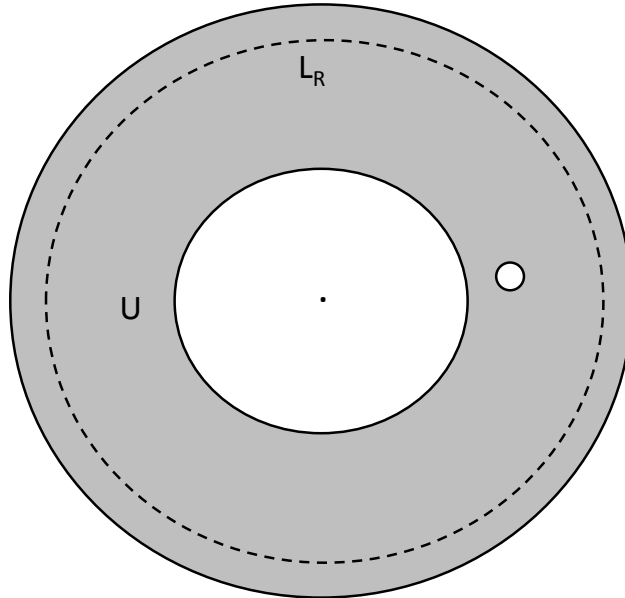


Figure 3.2: A fundamental loop contained in a multiply connected Fatou component.

It follows from these results that the fundamental loops of a transcendental entire function can have very varied geometrical properties. For example, consider the transcendental entire function  $f$  given in [27, Example 3]. The construction of this function is too intricate to give here, but it is shown in [27] that this function has a multiply connected Fatou component  $U$  with the property that

$$\lim_{n \rightarrow \infty} \frac{\max\{\log |z| : z \in \partial_{\text{out}} U_n\}}{\min\{\log |z| : z \in \partial_{\text{out}} U_n\}} = \infty.$$

By Theorem 3.1.1, there is a fundamental loop of  $f$  which coincides with  $\partial_{\text{out}} U_n$ , and so is far from circular for large values of  $n$ . However, there are also fundamental loops of  $f$  which lie inside  $U_n$ , for each  $n \in \mathbb{N}$ . By Theorem 3.1.3(a) these are analytic Jordan curves, and by [27, Theorem 7.1] can be approximately

circular.

A key tool in the proofs of these theorems is a function  $R_A$ , defined in (3.22) below, which for a point  $z$  is the largest  $R$  such that  $z \in A_R(f)$ . In general this function can only be defined in a subset of  $A(f)$ . In Section 3.7 we show that, subject to a certain normalisation, this definition can in fact be extended in a natural way to the whole complex plane. We show that, in this case, there is an alternative characterisation of  $A(f)$ . We also show that the function  $R_A$  has a number of interesting properties.

The structure of this chapter is as follows. First, in Section 3.2, we state a number of results required in the proof of our main theorems. With the exception of Lemma 3.2.4, these are all known results. In Section 3.3 we give some results regarding the hyperbolic metric, used only in this chapter. In Section 3.4 we prove a new result, which states that if a transcendental entire function has a certain property with respect to a nested sequence of bounded simply connected domains, then there is a fixed point which has a certain ‘attracting’ property. This may be of independent interest. In Section 3.5 we show that the function  $R_A$  can be defined in certain multiply connected Fatou components, and prove several preparatory lemmas. In Section 3.6 we prove Theorems 3.1.1, 3.1.2 and 3.1.3. Finally, in Section 3.7 we state and prove several results regarding the case when  $R_A$  can be defined in the whole complex plane.

## 3.2 Background results

We require a number of additional results from [27] concerning multiply connected Fatou components. We require part of [27, Theorem 1.5].

**Lemma 3.2.1.** *Suppose that  $f$  is a transcendental entire function with a multiply connected Fatou component  $U$ , and let  $z_0 \in U$ . For large  $n \in \mathbb{N}$ , let  $r_n$ ,  $a_n$  and  $b_n$  be as defined in (1.15), and let  $\underline{a}_n$  denote the smallest value such that*

$$\{z : |z| = r_n^{a_n}\} \cap \partial_{\text{inn}} U_n \neq \emptyset.$$



Then there exist  $a \in [0, 1)$  and  $b \in (1, \infty]$  such that, as  $n \rightarrow \infty$ ,

$$a_n \rightarrow a, \quad \underline{a}_n \rightarrow a, \quad \text{and} \quad b_n \rightarrow b. \quad (3.2)$$

We also need the following [27, Theorem 1.3] which shows that any compact subset of  $U$  eventually iterates into the maximal annulus  $B_n$ .

**Lemma 3.2.2.** *Let  $f$ ,  $U$ ,  $z_0$  be as in Lemma 3.2.1. For large  $n \in \mathbb{N}$ , let  $r_n$ ,  $a_n$ ,  $b_n$  and  $B_n$  be as in (1.15). Then, for each compact set  $C \subset U$ , there exists  $N \in \mathbb{N}$  such that*

$$f^n(C) \subset C_n \subset B_n, \quad \text{for } n \geq N, \quad (3.3)$$

where

$$C_n = A(r_n^{a_n+2\pi\delta_n}, r_n^{b_n(1-3\pi\delta_n)}), \quad \text{with } \delta_n = 1/\sqrt{\log r_n}. \quad (3.4)$$

Figure 3.3 gives a useful illustration of these sets. A multiply connected Fatou component  $U_n$  is shown. The annulus  $B_n$  has a dashed boundary, and the slightly smaller annulus  $C_n$  has a dotted boundary.

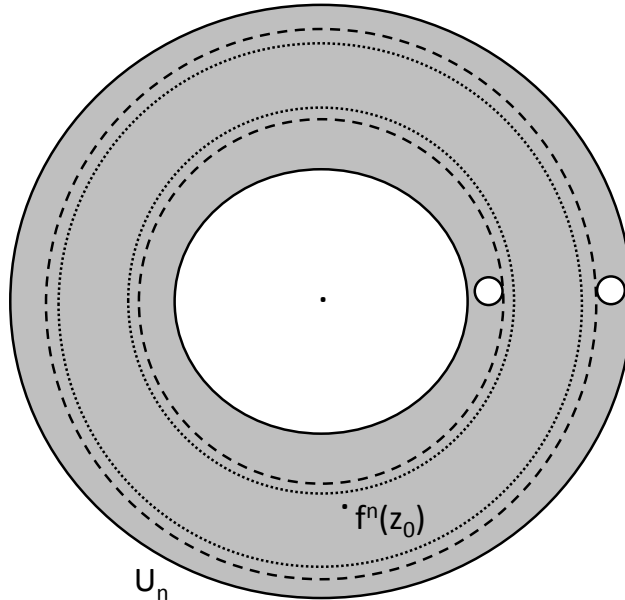


Figure 3.3: Annuli within a multiply connected Fatou component.

We also need the following, which shows that within  $C_n$  the modulus of  $f$  is very close to the maximum modulus, for large values of  $n$ . This is summarised from [27, Theorem 5.1(b)].

**Lemma 3.2.3.** *Let  $f$ ,  $U$  and  $z_0$  be as in Lemma 3.2.1. For large  $n \in \mathbb{N}$ , let  $r_n$ ,  $a_n$  and  $b_n$  be as in (1.15), and let  $\delta_n = 1/\sqrt{\log r_n}$ . Then, there exists  $N$  such that for  $n \geq N$ , and  $m \in \mathbb{N}$ ,*

$$\log |f^m(z)| \geq (1 - \delta_n) \log M(|z|, f^m), \quad \text{for } z \in A(r_n^{a_n+2\pi\delta_n}, r_n^{b_n-2\pi\delta_n}). \quad (3.5)$$

The following is a consequence of these lemmas.

**Lemma 3.2.4.** *Let  $f$  and  $U$  be as in Lemma 3.2.1, let  $z \in U$  and let  $0 < c < 1$ . Then there exists  $N \in \mathbb{N}$  such that*

$$|f^{n+m}(z)| \geq M^m(|f^n(z)|^c), \quad \text{for } n \geq N, m \in \mathbb{N}. \quad (3.6)$$

*Proof.* Fix  $z_0 \in U$ , and let  $\beta_n = 1 - 1/\sqrt{\log r_n}$ , where  $r_n = |f^n(z_0)|$ , for  $n \in \mathbb{N}$ . Choose  $R > 1$  sufficiently large that  $M^n(R) \rightarrow \infty$  as  $n \rightarrow \infty$  and also, by (1.6), that  $M(r) > r^2$ , for  $r \geq R$ .

Now, as mentioned in Section 1.7, we have that  $U \subset A(f)$ . Hence there exists  $\ell \in \mathbb{N}$  such that  $r_{n+\ell} \geq M^n(R)$ , for  $n \in \mathbb{N}$ . Since  $M^n(R) > R^{2^n}$ , for  $n \in \mathbb{N}$ , we have

$$\frac{1}{\sqrt{\log r_n}} < 2^{\frac{1}{2}(\ell-n)}, \quad \text{for large values of } n.$$

We note that

$$\log(1-x) > -2x, \quad \text{for } 0 < x < \frac{1}{2}. \quad (3.7)$$

It follows that

$$\log \beta_n > -2^{\frac{1}{2}(\ell-n+2)}, \quad \text{for large value of } n.$$

Hence, we can choose  $N \in \mathbb{N}$  sufficiently large that

$$\sum_{k=N}^{\infty} \log \beta_k > \log c,$$

or equivalently that

$$\prod_{k=N}^{\infty} \beta_k > c. \quad (3.8)$$

Now let  $z \in U$ . We can further assume that  $N$  is sufficiently large that

$$|f^n(z)|^c > R_0, \quad \text{for } n \geq N,$$

where  $R_0$  is the constant from Lemma 1.2.2. We note, by (3.2), that

$$b_n - 2\pi\delta_n > b_n(1 - 3\pi\delta_n), \quad \text{for large values of } n,$$

and so

$$C_n \subset A(r_n^{a_n+2\pi\delta_n}, r_n^{b_n-2\pi\delta_n}), \quad \text{for large values of } n,$$

where  $C_n$  is defined in (3.4). Hence, we can assume, by Lemma 3.2.2 and Lemma 3.2.3, that  $N$  is sufficiently large that

$$\log |f^{n+1}(z)| \geq \beta_n \log M(|f^n(z)|), \quad \text{for } n \geq N. \quad (3.9)$$

Hence, by (3.9) and (1.9),

$$|f^{n+1}(z)| \geq M(|f^n(z)|)^{\beta_n} \geq M(|f^n(z)|^{\beta_n}), \quad \text{for } n \geq N. \quad (3.10)$$

By repeated application of (3.10) and (1.9), and by (3.8), we have that

$$|f^{n+m}(z)| \geq M^m(|f^n(z)|^{\prod_{k=0}^{m-1} \beta_{n+k}}) \geq M^m(|f^n(z)|^c), \quad \text{for } n \geq N, m \in \mathbb{N},$$

as required.  $\square$

We also need the following [86, Theorem 2.3].

**Lemma 3.2.5.** *Let  $f$  be a transcendental entire function and let  $\eta > 1$ . There exists  $R'_0 = R'_0(f) > 0$  such that if  $r > R'_0$ , then there exists*

$$z' \in A(r, \eta r) \cap A(f)$$

with

$$|f^n(z')| > M^n(r, f), \quad \text{for } n \in \mathbb{N},$$

and hence

$$z' \in A_r(f) \quad \text{and} \quad M(\eta r, f^n) > M^n(r, f), \quad \text{for } n \in \mathbb{N}.$$

### 3.3 The hyperbolic metric and hyperbolic distance

In this chapter, we use the hyperbolic metric and hyperbolic distance; a detailed account of these topics can be found in, for example, [64], and we give here only the detail we require. We say that a domain  $V$  is *hyperbolic* if  $\partial V$  contains at least two points. For a hyperbolic domain  $V$ , we write  $[w, z]_V$  for the hyperbolic distance between  $w$  and  $z$  in  $V$ , and we let  $\rho_V$  denote the density of the hyperbolic metric in  $V$ . We use the following results, which are well-known.

**Lemma 3.3.1.** *Suppose that  $U$  is a hyperbolic domain and that  $U'$  is a domain contained in  $U$ . Then*

$$[w, z]_U \leq [w, z]_{U'}, \quad \text{for } w, z \in U'.$$

**Lemma 3.3.2.** *Suppose that  $U$  is a hyperbolic domain, and that  $f : U \rightarrow f(U)$  is analytic. Then*

$$[f(w), f(z)]_{f(U)} \leq [w, z]_U, \quad \text{for } w, z \in U,$$

with equality if and only if  $f$  is conformal.

We also use the following, which is well-known and follows in a straightforward way from [52, Theorem 9.14].

**Lemma 3.3.3.** *Suppose that  $U$  is an unbounded hyperbolic domain. Then there exist  $R > 2$  and  $C > 0$  such that*

$$\rho_U(z) \geq \frac{C}{|z| \log |z|}, \quad \text{for } |z| \geq R.$$

### 3.4 A map on a nested sequence of domains

In this section we prove a result about the existence and properties of a fixed point for certain transcendental entire functions. This may be of independent interest. The main result of this section is as follows.

**Theorem 3.4.1.** *Suppose that  $f$  is a transcendental entire function, and that  $(G_n)_{n \geq 0}$  is a sequence of bounded simply connected domains such that*

$$\overline{G_n} \subset G_{n+1} \quad \text{and} \quad f(\partial G_n) = \partial G_{n+1}, \quad \text{for } n = 0, 1, 2, \dots \quad (3.11)$$

Suppose also that

$$\bigcup_{n=0}^{\infty} G_n = \mathbb{C}. \quad (3.12)$$

Then there exists  $\alpha \in G_0$ , a fixed point of  $f$ , such that, if  $K \subset G_0$  is compact, then

$$[\alpha, f^n(z)]_{G_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } z \in K.$$

To prove Theorem 3.4.1 we require the following lemma, which is a new result.

**Lemma 3.4.2.** *Suppose that  $(B_n)_{n \geq 0}$  is a sequence of analytic functions from  $\mathbb{D}$  to  $\mathbb{D}$ . Suppose also that there exist  $\alpha \in \mathbb{D}$  and  $\lambda \in (0, 1)$  such that*

$$B_n(\alpha) = \alpha \quad \text{and} \quad |B'_n(\alpha)| \leq \lambda, \quad \text{for } n = 0, 1, 2, \dots \quad (3.13)$$

Then, if  $K'$  is a compact subset of  $\mathbb{D}$ ,

$$B_n \circ \dots \circ B_0(z) \rightarrow \alpha \quad \text{as } n \rightarrow \infty, \quad \text{uniformly for } z \in K'.$$

*Proof.* By conjugating with a Möbius map if necessary, we may assume that  $\alpha = 0$ , and so we may define functions  $C_n : \mathbb{D} \rightarrow \mathbb{C}$  by

$$C_n(z) = \frac{B_n(z)}{z}, \quad \text{for } n = 0, 1, 2, \dots, \quad z \in \mathbb{D}.$$

We note that

$$|C_n(0)| = |B'_n(0)| \leq \lambda < 1, \quad \text{for } n = 0, 1, 2, \dots \quad (3.14)$$

By Schwarz' lemma applied to  $B_n$ , we have that

$$|C_n(z)| < 1, \quad \text{for } n = 0, 1, 2, \dots, z \in \mathbb{D}. \quad (3.15)$$

It follows from (3.15), by Montel's theorem, that the family  $\{C_n\}_{n \in \{0, 1, \dots\}}$  is a normal family of maps from  $\mathbb{D}$  to  $\mathbb{D}$ .

Suppose that  $K' \subset \mathbb{D}$  is compact. By a normal family argument, and by (3.14), it follows that there exists  $\mu < 1$  such that

$$|C_n(z)| < \mu, \quad \text{for } n = 0, 1, 2, \dots, z \in K',$$

in which case

$$|B_n(z)| < \mu z, \quad \text{for } n = 0, 1, 2, \dots, z \in K'. \quad (3.16)$$

The assertion of the lemma follows at once from (3.16).  $\square$

We note that an alternative proof of this lemma, which uses a result of Bear-  
don and Carne [15, p.217], is given in [99].

We now prove Theorem 3.4.1.

*Proof of Theorem 3.4.1.* By (3.11), the triple  $(f, G_0, G_1)$  is a *polynomial-like map* in the sense of Douady and Hubbard [38]. (Explicitly, this means that  $f$  is an entire function, and  $G_0$  and  $G_1$  are domains such that  $f$  is a proper map of  $G_0$  onto  $G_1$ . See Section 1.5 for the definition of a proper map.) Since every polynomial-like map has a fixed point [13, Lemma 3] (see also [42, Lemma 3]), there exists a point  $\alpha \in G_0$  such that  $f(\alpha) = \alpha$ .

For  $n = 0, 1, 2, \dots$ , let  $\phi_n : \mathbb{D} \rightarrow G_n$  be a Riemann map such that  $\phi_n(0) = \alpha$ . By Lemma 3.3.2 we have that

$$[\alpha, f^n(z)]_{G_n} = [0, \phi_n^{-1} \circ f^n(z)]_{\mathbb{D}}, \quad \text{for } z \in G_0. \quad (3.17)$$

Hence it suffices to show that  $[0, \phi_n^{-1} \circ f^n(z)]_{\mathbb{D}} \rightarrow 0$  as  $n \rightarrow \infty$ , uniformly for  $z \in K$ .

Define functions  $B_n : \mathbb{D} \rightarrow \mathbb{D}$  by

$$B_n = \phi_{n+1}^{-1} \circ f \circ \phi_n, \quad \text{for } n = 0, 1, 2, \dots.$$

Then  $B_n$  is a proper map, by the definition of a proper map, and since  $\phi_n$  and  $\phi_{n+1}$  are conformal. Moreover,  $B_n(0) = 0$ . It follows that, for  $n = 0, 1, 2, \dots$ , we have that  $B_n$  is a finite Blaschke product

$$B_n(z) = c_n z^{q_n} \prod_{k=0}^{p_n} \left( \frac{z - a_{k,n}}{1 - \overline{a_{k,n}}z} \right)^{m_{k,n}}, \quad \text{for } z \in \mathbb{D}, \quad (3.18)$$

where  $p_n \in \{0, 1, 2, \dots\}$ ,  $q_n, m_{k,n} \in \mathbb{N}$ ,  $|c_n| = 1$ ,  $0 < |a_{k,n}| < 1$ , and  $a_{k,n} = a_{k',n}$  implies that  $k = k'$ , for  $n \in \{0, 1, 2, \dots\}$ , and  $k \in \{0, 1, \dots, p_n\}$ . (These facts are given in, for example, [54, p.35].)

Let  $q$  be the multiplicity of the fixed point of  $f$  at  $\alpha$ . Since  $B_n$  is conformally conjugate to  $f$ , we have that  $q$  is also the multiplicity of the fixed point of  $B_n$  at the origin; see, for example, [14, Lemma 2.6.1]. It follows that

$$q_n = q, \quad \text{for } n = 0, 1, 2, \dots. \quad (3.19)$$

We claim that there exists  $N \in \{0, 1, 2, \dots\}$ , such that  $p_n \neq 0$ , for  $n \geq N$ . This follows from (3.12) and (3.19), and from the fact that  $f$  is transcendental and so cannot be  $q$  to 1 in the whole complex plane. Without loss of generality, we may assume that  $N = 0$ .

We claim that there exists  $\lambda \in (0, 1)$  such that

$$|B'_n(0)| \leq \lambda, \quad \text{for } n = 0, 1, 2, \dots. \quad (3.20)$$

Suppose first that  $q \geq 2$ . Then  $B'_n(0) = 0$ , for  $n = 0, 1, 2, \dots$ .

Suppose, on the other hand, that  $q = 1$ . Then

$$|B'_n(0)| = \prod_{k=0}^{p_n} |a_{k,n}|^{m_{k,n}} < 1, \quad \text{for } n = 0, 1, 2, \dots. \quad (3.21)$$

For  $n = 0, 1, 2, \dots$ , and  $k = 0, 1, 2, \dots, p_n$ , set  $\alpha_{k,n} = \phi_n(a_{k,n})$ . Then  $f(\alpha_{k,n}) = \alpha$ ,

and so  $\phi_{n+1}^{-1}(\alpha_{k,n})$  is a zero of  $B_{n+1}$ . By renumbering the sequences if necessary, we can assume that

$$\alpha_{k,n} = \alpha_{k,n+1} = \phi_{n+1}(a_{k,n+1}), \quad \text{for } n = 0, 1, 2, \dots, k = 0, 1, 2, \dots, p_n.$$

Now, by (3.11),  $[\alpha_{k,n}, \alpha]_{G_n} \geq [\alpha_{k,n+1}, \alpha]_{G_{n+1}}$ . Hence, by Lemma 3.3.2, we have that  $|a_{k,n+1}| \leq |a_{k,n}|$ .

Moreover,  $m_{k,n}$  and  $m_{k,n+1}$  are both equal to the multiplicity of the zero  $\alpha_{k,n}$  of  $f(z) - \alpha$ , and so  $m_{k,n} = m_{k,n+1}$ . Thus, by (3.21),

$$|B'_{n+1}(0)| \leq |B'_n(0)|, \quad \text{for } n = 0, 1, 2, \dots.$$

This establishes (3.20).

Let  $K \subset G_0$  be compact. By Lemma 3.4.2, applied with  $K' = \phi_0^{-1}(K)$ , we obtain that

$$[0, B_{n-1} \circ \dots \circ B_1 \circ B_0 \circ \phi_0^{-1}(z)]_{\mathbb{D}} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{uniformly for } z \in K.$$

The result follows by (3.17), since

$$B_{n-1} \circ \dots \circ B_1 \circ B_0 \circ \phi_0^{-1} = \phi_n^{-1} \circ f^n.$$

□

**Remark 3.4.1.** In our application of Theorem 3.4.1, the hypothesis (3.12) holds. However this hypothesis is actually not necessary in Theorem 3.4.1 as it can be deduced from (3.11) by using a normal family argument; we omit the details.

### 3.5 The function $R_A$ defined in a multiply connected Fatou component

The main role of this section is to introduce the function  $R_A$ , which plays a key role in the proof of Theorem 3.1.1. Before stating and proving a sequence of lemmas, we outline how these results are used.



Suppose that  $U$  is a multiply connected Fatou component which surrounds the origin. We show that if  $U$  is sufficiently far from the origin, then we can define a real-valued function  $R_A$  which, for each  $z \in \bar{U}$ , is the largest value of  $R$  such that  $z \in A_R(f)$ ; see (3.22) below. It turns out that this function has a close relationship to fundamental loops. Indeed, where defined,  $R_A$  is strictly less than  $R$  in  $H_R$ , and is at least equal to  $R$  on  $L_R$ . We then prove that the function  $R_A$  has certain continuity properties, and shares level sets with the function  $h$  defined in (3.1). These facts allow us to show that;

- (a) on  $\partial_{\text{int}}U$ ,  $R_A$  is equal to its infimum in  $U$ ;
- (b) on  $\partial_{\text{out}}U$ ,  $R_A$  is at least equal to its supremum in  $U$ ;
- (c)  $R_A$  does not achieve a maximum or a minimum in  $U$ .

Because of the close relationship between the function  $R_A$  and the definition of fundamental loops, properties (a), (b) and (c) above can then be used to prove Theorem 3.1.1 parts (a), (b) and (c) respectively. Theorems 3.1.2 and 3.1.3 then follow quickly.

We start with a simple lemma.

**Lemma 3.5.1.** *Suppose that  $f$  is a transcendental entire function and that  $A_R(f)$  is a spider's web. Then  $f$  has a fixed point.*

*Proof.* Suppose that  $H_R$  is a fundamental hole of  $f$ . By Theorem 1.3.2 we have that the triple  $(f, H_R, f(H_R))$  is a polynomial-like map. The result follows because, as noted in the proof of Theorem 3.4.1, every polynomial-like map has a fixed point.  $\square$

The following lemma is central to our results.

**Lemma 3.5.2.** *Suppose that  $f$  is transcendental entire function. Then there exists  $R' = R'(f) > 0$  such that the following holds. Suppose that  $U$  is a multiply connected Fatou component of  $f$ , which surrounds the origin and satisfies  $\text{dist}(0, U) > R'$ . Define  $G_n$  as the complementary component of  $\bar{U}_n$  which contains the origin, for  $n = 0, 1, 2, \dots$ . Then*

- (a)  $\bar{G}_n \subset G_{n+1}$ , for  $n = 0, 1, 2, \dots$ ;

(b)  $f(\partial G_n) = \partial G_{n+1}$ , for  $n = 0, 1, 2, \dots$ ;

(c) for all  $z \in \bar{U}$  there exists  $R = R(z)$  such that  $z \in A_R(f)$ .

*Proof.* First we note by Theorem 1.9.2 that there exists  $R_1 > 0$  such that  $A_{R_1}(f)$  is a spider's web. Hence, by Lemma 3.5.1,  $f$  has a fixed point  $\alpha$ .

We now use properties of multiply connected Fatou components to establish a suitable value for  $R'$ . Let  $V$  be a multiply connected Fatou component of  $f$ . By Theorem 1.5.1 there is an  $N_1 \in \mathbb{N}$  such that  $f^{N_1}(V)$  surrounds both the origin and  $\alpha$ , and also  $f^{n+1}(V)$  surrounds  $f^n(V)$  for  $n \geq N_1$ .

Choose  $R_2 > 0$  such that  $M^n(R_2) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, by Theorem 1.7.2, there is an  $N_2 \in \mathbb{N}$  such that  $\overline{f^{N_2}(V)} \subset A_{R_2}(f)$ . Set  $N_3 = \max\{N_1, N_2\}$ , and let  $R' = \max\{|z| : z \in \overline{f^{N_3}(V)}\}$ .

Suppose that  $U$  is any multiply connected Fatou component such that  $U$  surrounds the origin and satisfies  $\text{dist}(0, U) > R'$ . It follows from our choice of  $R'$  that  $U$  surrounds  $f^{N_3}(V)$ ; see Figure 3.4. We now show that the results of the lemma follow from this fact.

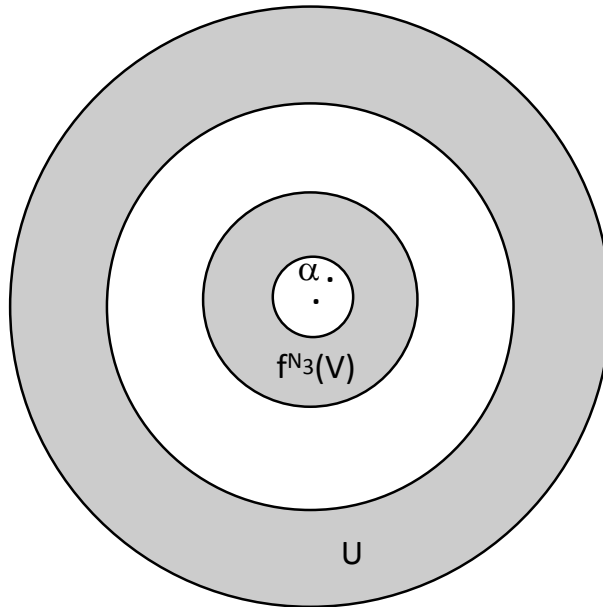


Figure 3.4: The construction in the proof of Lemma 3.5.2.

Since  $U$  surrounds  $\alpha$ , then  $U_1 = f(U)$  surrounds  $\alpha$  by the argument principle. Moreover,  $U_1$  cannot meet either  $f^{N_3+1}(V)$  or  $U$ , since  $\partial U_1 \subset J(f)$ . Hence, by the maximum principle,  $U_1$  surrounds both  $f^{N_3+1}(V)$  and  $U$ . Inductively,  $U_k$  surrounds both  $f^{N_3+k}(V)$  and  $U_{k-1}$ , for  $k \in \mathbb{N}$ . Parts (a) and (c) of the lemma follow from this fact and the choice of  $N_3$ .

Finally we establish part (b). Choose  $n = 0, 1, 2, \dots$ . Since  $f(G_n)$  is open and connected, and its boundary is in  $J(f)$ , it cannot meet the boundary of  $G_{n+1}$ . Now,  $\alpha \in G_n$ , and so  $f(G_n) \cap G_{n+1} \neq \emptyset$ . Hence  $\partial f(G_n)$  must lie in  $\overline{G_{n+1}}$ , and so

$$f(\partial G_n) \subset \overline{G_{n+1}}.$$

Moreover,  $f$  is a proper map on the Fatou component  $U_n$ , and so

$$f(\partial G_n) \subset f(\partial U_n) = \partial U_{n+1}.$$

Thus  $\partial f(G_n) = \partial G_{n+1}$ , as required.  $\square$

Suppose that  $U$  is a multiply connected Fatou component which surrounds the origin, and that  $\text{dist}(0, U) > R'$ , where  $R'$  is the constant from Lemma 3.5.2. Suppose that  $z \in \overline{U}$ , and let  $X_z = \{R \geq 0 : z \in A_R(f)\}$ . We see by Lemma 3.5.2(c) that  $X_z$  is not empty. Moreover, if  $R \notin X_z$ , then  $z \notin A_R(f)$  and so there is an  $n \in \mathbb{N}$  such that  $|f^n(z)| < M^n(R)$ . It follows by Lemma 1.2.1(a) that we may choose  $\epsilon$  sufficiently small that

$$|f^n(z)| < M^n(R - \epsilon),$$

and so  $z \notin A_{R-\epsilon}(f)$ . We deduce that  $X_z$  is a non-empty closed interval.

We may, therefore, introduce a new function  $R_A$  defined by

$$R_A(z) = \max\{R : z \in A_R(f)\}, \quad \text{for } z \in \overline{U}. \quad (3.22)$$

We note that it is possible here to define the function  $R_A$  in a larger set. However, for our current purposes a definition only in  $\overline{U}$  is sufficient. We explore a definition of the function  $R_A$  in the whole complex plane in the next section.

The function  $R_A$  has some strong continuity properties, and shares level sets

with the function  $h$ .

**Lemma 3.5.3.** *Suppose that  $f$  and  $U$  are as in Theorem 3.1.1, and that  $R_A$  is as in (3.22). Then  $R_A$  is upper semicontinuous in  $\bar{U}$  and continuous in  $U$ . Moreover, if  $h$  is as in (3.1), then there exists a continuous strictly increasing function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$R_A(z) = \phi(h(z)), \quad \text{for } z \in U. \quad (3.23)$$

*Proof.* We first prove that  $R_A$  is upper semicontinuous in  $\bar{U}$ . Suppose that  $z \in \bar{U}$  and that  $\epsilon > 0$ . By the definition of  $R_A$ , we have that  $z \notin A_{R_A(z)+\epsilon}(f)$ . Hence there is an  $N \in \mathbb{N}$  such that  $|f^N(z)| < M^N(R_A(z) + \epsilon)$ . By continuity, there exists a  $\delta > 0$  such that

$$|f^N(z')| < M^N(R_A(z) + \epsilon), \quad \text{for } z' \in B(z, \delta).$$

Hence  $R_A(z') < R_A(z) + \epsilon$ , for all  $z' \in \bar{U} \cap B(z, \delta)$ . This completes the proof that  $R_A$  is upper semicontinuous in  $\bar{U}$ .

To prove that  $R_A$  is continuous in  $U$  we need to prove that  $R_A$  is lower semicontinuous at  $z \in U$ . Suppose, to the contrary, that  $R_A$  is not lower semicontinuous at  $z$ . Then there exists  $\epsilon > 0$  such that the following holds. If  $\Delta \subset U$  is a neighbourhood of  $z$ , then there is a  $z' \in \Delta$  such that  $R_A(z) - \epsilon > R_A(z')$ , in which case  $z' \notin A_{R_A(z)-\epsilon}(f)$ . There exists, therefore, a sequence  $(z_k)_{k \in \mathbb{N}}$  of points of  $U$ , distinct from but tending to  $z$ , and a sequence  $(n_k)_{k \in \mathbb{N}}$  of integers such that

$$|f^{n_k}(z_k)| < M^{n_k}(R_A(z) - \epsilon), \quad \text{for } k \in \mathbb{N}.$$

Hence, for each  $k \in \mathbb{N}$ ,

$$|f^{n_k}(z_k)| < M^{n_k}(R_A(z) - \epsilon) < M^{n_k}(R_A(z)) \leq |f^{n_k}(z)|,$$

which implies that

$$\frac{\log M^{n_k}(R_A(z))}{\log M^{n_k}(R_A(z) - \epsilon)} < \frac{\log |f^{n_k}(z)|}{\log |f^{n_k}(z_k)|}, \quad \text{for } k \in \mathbb{N}. \quad (3.24)$$

We now establish a contradiction by showing that the right-hand side of (3.24) has an upper bound which tends to 1 as  $k \rightarrow \infty$ , but the left-hand side is greater than some  $c > 1$ , for sufficiently large values of  $k$ . Note that we can assume that  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

We may assume that  $\epsilon$  is sufficiently small that  $M^n(R_A(z) - \epsilon) \rightarrow \infty$  as  $n \rightarrow \infty$ . Hence we can choose  $N$  large enough that  $M^N(R_A(z) - \epsilon) > R_0$ , where  $R_0$  is the constant from Lemma 1.2.2. Set

$$r = M^N(R_A(z) - \epsilon) \quad \text{and} \quad c = \frac{\log M^N(R_A(z))}{\log r} > 1.$$

It follows by repeated application of (1.8) that we have

$$\log M^m(r^c) \geq c \log M^m(r), \quad \text{for } m \in \mathbb{N}.$$

Hence

$$\frac{\log M^{m+N}(R_A(z))}{\log M^{m+N}(R_A(z) - \epsilon)} = \frac{\log M^m(r^c)}{\log M^m(r)} \geq c > 1, \quad \text{for } m \in \mathbb{N}.$$

This establishes our claim regarding the left-hand side of (3.24).

To establish our claim regarding the right-hand side of (3.24) we use some techniques from [81], though we give the full details for completeness.

Choose any  $w_1, w_2 \in J(f)$  with  $w_1 \neq w_2$ , and put  $G = \mathbb{C} \setminus \{w_1, w_2\}$ . Note that, by Lemma 3.3.1 and Lemma 3.3.2,

$$[z, z_k]_U \geq [f^{n_k}(z), f^{n_k}(z_k)]_{f^{n_k}(U)} \geq [f^{n_k}(z), f^{n_k}(z_k)]_G = \int_{\Gamma_k} \rho_G(z) |dz|, \quad \text{for } k \in \mathbb{N},$$

where  $\Gamma_k$  is a hyperbolic geodesic in  $G$  joining  $f^{n_k}(z)$  to  $f^{n_k}(z_k)$ . Let  $R$  and  $C$  be the constants from Lemma 3.3.3 applied with  $U = G$ .

Choose  $K$  sufficiently large such that

$$|f^{n_k}(z)| > |f^{n_k}(z_k)| > 2R, \quad \text{for } k \geq K.$$

We then have, by Lemma 3.3.3,

$$[z, z_k]_U \geq \int_{\Gamma_k} \rho_G(z) |dz| \geq C \int_{|f^{n_k}(z_k)|}^{|f^{n_k}(z)|} \frac{dr}{r \log r} = C \log \left( \frac{\log |f^{n_k}(z)|}{\log |f^{n_k}(z_k)|} \right). \quad (3.25)$$

Hence

$$\frac{\log |f^{n_k}(z)|}{\log |f^{n_k}(z_k)|} \leq \exp([z, z_k]_U / C), \quad \text{for } k \in \mathbb{N}. \quad (3.26)$$

As  $k \rightarrow \infty$ ,  $z_k \rightarrow z$  and so  $[z, z_k]_U \rightarrow 0$ . Hence the right-hand side of (3.26) is indeed bounded above by a term tending to 1 as  $k \rightarrow \infty$ . This completes the proof that  $R_A$  is continuous in  $U$ .

Finally we need to prove that there exists a real function  $\phi$  which satisfies (3.23). Our method of proof is as follows. Suppose that  $w, z \in U$ . We claim that  $h(w) < h(z)$  if and only if  $R_A(w) < R_A(z)$ . This, combined with the fact that both  $h$  and  $R_A$  are continuous in  $U$ , proves that  $R_A(z) = \phi(h(z))$ , for  $z \in U$ , where  $\phi$  is continuous and strictly increasing.

Let  $w, z \in U$ . Suppose first that  $R_A(w) < R_A(z) = r$ , say. Then, there is an  $N \in \mathbb{N}$  such that

$$|f^n(z)| \geq M^n(r) > |f^n(w)|, \quad \text{for } n \geq N.$$

Assume also that  $N$  is sufficiently large that  $|f^n(w)| > R_0$ , for  $n \geq N$ , where  $R_0$  is the constant in Lemma 1.2.2. Set

$$c = \frac{\log M^N(r)}{\log |f^N(w)|} > 1.$$

Then, by (1.8),

$$M^{N+m}(r) = M^m(|f^N(w)|^c) \geq M^m(|f^N(w)|)^c \geq |f^{N+m}(w)|^c, \quad \text{for } m \in \mathbb{N}.$$

Hence

$$\frac{h(w)}{h(z)} = \lim_{m \rightarrow \infty} \frac{\log |f^{N+m}(w)|}{\log |f^{N+m}(z)|} \leq \lim_{m \rightarrow \infty} \frac{\log |f^{N+m}(w)|}{\log |M^{N+m}(r)|} \leq \frac{1}{c} < 1,$$

and so  $h(w) < h(z)$ . This completes the first part of the proof.

Suppose next that  $h(w) < h(z)$ . The proof is complete if we can show that  $R_A(w) < R_A(z)$ . Choose  $c$  such that  $h(w)/h(z) < c < 1$ . Choose  $N'$  sufficiently large such that

$$|f^n(z)|^c > |f^n(w)|, \quad \text{for } n \geq N'.$$

By Lemma 3.2.4, there exists  $N > N'$  such that

$$|f^{N+m}(z)| \geq M^m(|f^N(z)|^c), \quad \text{for } m \in \mathbb{N}.$$

Hence

$$R_A(f^N(z)) \geq |f^N(z)|^c > |f^N(w)| \geq M^N(R_A(w)). \quad (3.27)$$

Set  $R = M^{-N}(|f^N(z)|^c)$ , and note that  $R > R_A(w)$  by (3.27). Then

$$|f^{N+m}(z)| \geq M^m(|f^N(z)|^c) = M^{N+m}(R), \quad \text{for } m \in \mathbb{N}.$$

Hence  $R_A(z) \geq R > R_A(w)$  as required. This completes the proof of the lemma.  $\square$

**Remark 3.5.1.** In fact, with the conditions of Lemma 3.5.3, the stronger result holds that  $R_A$  is continuous in  $\bar{U} \setminus \partial_{\text{out}} U$ . This follows from Lemma 3.5.4 below, but is not pertinent to the proofs of the results of this chapter.

We use Lemma 3.5.3 to prove the following result regarding the values of the function  $R_A$  in  $\bar{U}$ . The reader may wish to refer to Figure 3.1 to recall the boundary sets involved in this lemma.

**Lemma 3.5.4.** *Suppose that  $f$  and  $U$  are as in Theorem 3.1.1 and that  $R_A$  is as in (3.22). Set*

$$R_1 = R_1(U) = \inf_{z \in U} R_A(z) \quad \text{and} \quad R_2 = R_2(U) = \sup_{z \in U} R_A(z). \quad (3.28)$$

*Then*

(a)  $R_A(z) = R_1$ , for  $z \in \partial U \setminus \partial_{\text{out}} U$ ;

(b)  $R_A(z) \geq R_2$ , for  $z \in \partial_{\text{out}}\overline{U}$ ;

(c)  $R_1 < R_A(z) < R_2$ , for  $z \in U$ .

*Proof.* First, suppose that  $z \in \partial_{\text{int}}U$ . Choose  $w \in U$ . By Lemma 3.2.2 applied with  $C = \{w\}$ , there exists  $N \in \mathbb{N}$  such that  $|f^n(w)| > |f^n(z)|$ , for  $n \geq N$ , in which case  $R_A(z) \leq R_A(w)$ . Hence  $R_A(z) \leq R_1$ . Equality follows by the upper semicontinuity of  $R_A$  at  $z \in \overline{U}$ .

We now show, more generally, that  $R_A(z) = R_1$  for  $z \in \partial U \setminus \partial_{\text{out}}U$ , by showing that  $R_A$  is constant on this set. Suppose also that there exist points  $z_1, z_2 \in \partial U \setminus \partial_{\text{out}}U$  with

$$R_A(z_1) = R > \rho > R_A(z_2), \quad \text{for some } \rho.$$

Choose  $N \in \mathbb{N}$  such that  $M^N(\rho) > R_0$ , where  $R_0$  is the constant from Lemma 1.2.2. Set  $c = \log M^N(R) / \log M^N(\rho) > 1$ . Then, for all sufficiently large  $n \in \mathbb{N}$ , we have by (1.8),

$$|f^n(z_2)|^c < (M^n(\rho))^c = (M^{n-N}(M^N(\rho)))^c \leq M^{n-N}((M^N(\rho))^c) = M^n(R) \leq |f^n(z_1)|. \quad (3.29)$$

We now claim that, for sufficiently large values of  $n$ ,

$$r_n^{\underline{a}_n} \leq |f^n(z)| \leq r_n^{a_n}, \quad \text{for } z \in \partial U \setminus \partial_{\text{out}}U, \quad (3.30)$$

where  $\underline{a}_n$  is as in Lemma 3.2.1. This fact is in part of the proof of [27, Theorem 1.6], but we give a brief justification for completeness. Suppose that  $K$  is a component of  $\partial U \setminus \partial_{\text{out}}U$  and  $\gamma$  is a Jordan curve in  $U$  that contains  $K$  in its interior  $\text{int}(\gamma)$ . For large  $n$  we have, by Lemma 3.2.2, that  $f^n(\gamma) \subset C_n \subset B_n$ , where  $C_n$  is the annulus defined in (3.4); see also Figure 3.3. Hence

$$f^n(\text{int}(\gamma)) \subset \{z : |z| < r_n^{b_n}\},$$

and (3.30) follows by the definitions of  $\underline{a}_n$  and  $a_n$ , and the fact that  $f^n(z) \notin B_n$ , for  $z \in \partial U \setminus \partial_{\text{out}}U$ . This completes the proof of our claim regarding equation (3.30).

Now, by Lemma 3.2.1, both  $\underline{a}_n$  and  $a_n$  tend to  $a$  as  $n \rightarrow \infty$ . Hence, for large



values of  $n \in \mathbb{N}$ , by (3.30),

$$|f^n(z_2)|^c \geq r_n^{ca_n} \geq r_n^{a_n} \geq |f^n(z_1)|, \quad (3.31)$$

which is a contradiction to (3.29). This completes the proof of part (a) of the lemma.

Next, suppose that  $z \in \partial_{\text{out}}U$ . Choose  $w \in U$ . By Lemma 3.2.2 applied with  $C = \{w\}$ , there exists  $N \in \mathbb{N}$  such that  $|f^n(z)| > |f^n(w)|$ , for  $n \geq N$ , in which case  $R_A(z) \geq R_A(w)$ . Thus  $R_A(z) \geq R_2$  and this completes the proof of part (b) of the lemma.

Finally, suppose that there exists  $z \in U$  such that  $R_A(z) = R_2$ , in which case  $R_A$  achieves a maximum in  $U$  at  $z$ . Then  $h$  also achieves a maximum in  $U$  at  $z$ , by Lemma 3.5.3. This is a contradiction, because  $h$  is harmonic in  $U$ . For a similar reason,  $R_A$  cannot equal  $R_1$  and so achieve a minimum in  $U$ . This completes the proof of the lemma.  $\square$

### 3.6 Proofs of Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3

In this section we prove Theorem 3.1.1, and then show that this can be used to prove Theorem 3.1.2 and Theorem 3.1.3. We begin by proving the following result. Recall that  $R_1 = R_1(U) = \inf_{z \in U} R_A(z)$ .

**Lemma 3.6.1.** *Suppose that  $f$  and  $U$  are as in Theorem 3.1.1, and let  $G_0$  be the complementary component of  $\overline{U}$  containing the origin. Then  $G_0 \subset A_{R_1}(f)^c$ .*

*Proof.* Suppose, to the contrary, that there exists  $z_0 \in G_0$  such that  $z_0 \in A_{R_1}(f)$ . Recall that  $U_n = f^n(U)$ ,  $G_n$  is the component of  $\mathbb{C} \setminus \overline{U_n}$  containing the origin, and  $\partial_{\text{int}}U_n = \partial G_n$ . Let  $r_n = \text{dist}(0, \partial_{\text{int}}U_n)$  and let  $z_n = f^n(z_0) \in G_n$ , for  $n \in \mathbb{N}$ . The reader may wish to refer to Figure 3.5 at this point.

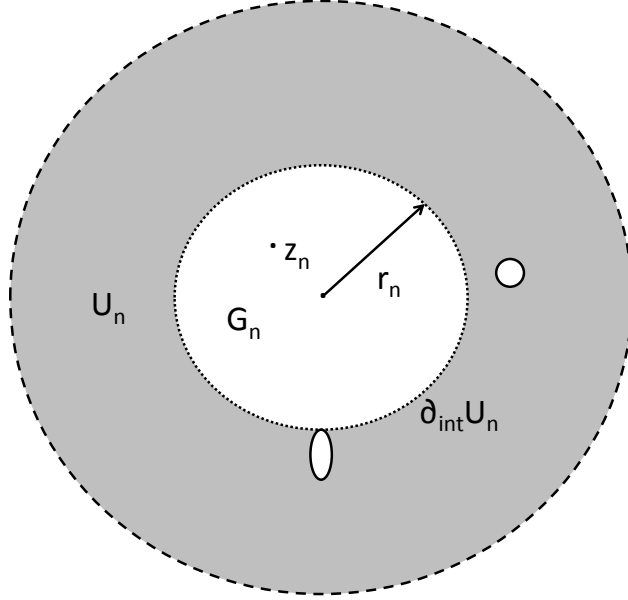


Figure 3.5: The construction used in the proof of Lemma 3.6.1.

In view of Lemma 3.5.2(a) and (b), we can apply Theorem 3.4.1, with  $G_n$  as above and with  $K = \{z_0\}$ . We obtain that  $f$  has a fixed point  $\alpha \in G_0$  such that

$$[\alpha, z_n]_{G_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We claim that there exists  $N \in \mathbb{N}$  such that  $|z_n| < r_n/2$  for  $n \geq N$ . Suppose, to the contrary, that  $|z_n| > r_n/2$  infinitely often. For these values of  $n$ , let  $\gamma_n$  be a curve in  $G_n$  joining  $\alpha$  and  $z_n$  such that

$$2[\alpha, z_n]_{G_n} \geq \int_{\gamma_n} \rho_{G_n}(w) |dw|.$$

Recall (for example, [34, Theorem 4.3]) that

$$\rho_{G_n}(w) \geq \frac{1}{2 \operatorname{dist}(w, \partial G_n)}, \quad \text{for } w \in G_n.$$

We can assume that  $n$  is sufficiently large that  $|\alpha| < r_n/4$ . Let

$$\gamma'_n = \gamma_n \cap B(0, r_n/2).$$

Note that  $\text{dist}(w, \partial G_n) \leq 2r_n$ , for  $w \in \gamma'_n$ . Moreover the length of  $\gamma'_n$  is certainly at least equal to  $r_n/4$ . Hence

$$2[\alpha, z_n]_{G_n} \geq \int_{\gamma'_n} \rho_{G_n}(w) |dw| \geq \frac{1}{4r_n} \int_{\gamma'_n} |dw| \geq \frac{1}{4r_n} \frac{r_n}{4} = \frac{1}{16},$$

which is a contradiction. Thus our claim is established.

We now set  $\eta = 3/2$  and  $\tau = r_N/2$ . By Lemma 3.2.5 and the above, there exists  $N \in \mathbb{N}$  such that the following conditions both hold. Firstly, there exists  $z' \in A(\tau, \eta\tau)$  such that  $z' \in A_\tau(f)$ . Secondly,  $|z_N| < \tau$ .

We have supposed that  $z_0 \in A_{R_1}(f)$ , and so this second condition implies that  $M^N(R_1) < \tau$ . Suppose that there exists  $w \in \partial_{\text{int}}U_N \cap A_\tau(f)$ . Since  $w = f^N(w')$ , for some  $w' \in \partial_{\text{int}}U$ , then

$$w' \in A_{M^{-N}(\tau)}(f).$$

This is impossible since  $M^{-N}(\tau) > R_1$ , but  $R_A(w') = R_1$ , by Lemma 3.5.4(a). Hence we have that  $\partial_{\text{int}}U_N \cap A_\tau(f) = \emptyset$ . This is a contradiction because  $\partial_{\text{int}}U_N$  surrounds  $z'$ , but  $A_\tau(f)$  has no bounded components by Theorem 1.3.1.  $\square$

We now prove Theorem 3.1.1.

*Proof of Theorem 3.1.1.* First we let  $R'$  be the constant from Lemma 3.5.2. Suppose that  $U$  is a multiply connected Fatou component of  $f$ , such that  $U$  surrounds the origin and  $\text{dist}(0, U) \geq R'$ . Let  $R_1 = R_1(U)$  and  $R_2 = R_2(U)$  be the constants from (3.28). Part (a) of the theorem, that  $\partial_{\text{int}}U$  is the fundamental loop  $L_{R_1}$ , follows because  $\partial_{\text{int}}U \subset A_{R_1}(f)$ , by Lemma 3.5.4(a), but the bounded component of  $\mathbb{C} \setminus \partial_{\text{int}}U$  is in  $A_{R_1}(f)^c$ , by Lemma 3.6.1.

Part (b) of the theorem, that  $\partial_{\text{out}}U$  is the fundamental loop  $L_{R_2}$ , follows immediately from Lemma 3.5.4(b) and (c).

Finally we prove part (c) of the theorem. Suppose that  $L_R$  is a fundamental loop, and that  $z \in L_R \cap U$ . Now  $z \in A_R(f)$ , and so  $R_A(z) \geq R$ . Moreover,  $R_A(w) < R$ , for  $w \in H_R \cap U$ . Hence, by the continuity of  $R_A$  in  $U$ ,  $R_A(z) = R$ . Thus, by Lemma 3.5.4(c),  $R_1 < R_A(z) = R < R_2$ .

Recall that  $R_A(w) = R_1$ , for  $w \in \partial U \setminus \partial_{\text{out}} U$ . It follows, by the upper semi-continuity of  $R_A$  in  $\bar{U}$ , that  $L_R \cap \partial U \setminus \partial_{\text{out}} U = \emptyset$ .

It remains to show that  $L_R \cap \partial_{\text{out}} U = \emptyset$ . Suppose, to the contrary, that  $L_R$  intersects  $\partial_{\text{out}} U$ . We recall from Theorem 1.3.2 that, in general, if  $L_\rho$  is a fundamental loop then  $f(L_\rho) = L_{M(\rho)}$ . By Lemma 3.2.2, applied to any closed subset of  $L_R \cap U$ , there exists  $N \in \mathbb{N}$  such that  $L_{M^n(R)} \cap C_n \neq \emptyset$ , where  $C_n$  is the annulus defined in (3.4), for  $n \geq N$ .

Next choose  $\eta > 1$ . We can assume that  $N$  is sufficiently large that, for  $n \geq N$  and  $z \in \bar{U}_n$ , we have that  $|z| > \max\{R_0, R'_0\}$ , where  $R_0$  is the constant from Lemma 1.2.2 and  $R'_0$  is the constant from Lemma 3.2.5. We can also assume that  $N$  is sufficiently large that the conclusions of Lemma 3.2.3 can be applied.

Define  $c_n = b_n - 2\pi\delta_n - \delta_n^2$ , for  $n \in \mathbb{N}$ . We can further assume that  $N$  is sufficiently large that we have both

$$b_N(1 - 3\pi\delta_N) < c_N < b_N - 2\pi\delta_N$$

and

$$\eta r_N^{b_N(1-3\pi\delta_N)} < r_N^{c_N(1-\delta_N)}.$$

The first inequality is easy to satisfy since, by Lemma 3.2.1,  $b_n \rightarrow b > 1$ , as  $n \rightarrow \infty$ . The second can be satisfied since

$$\eta r_n^{b_n(1-3\pi\delta_n)} = r_n^{b_n(1-3\pi\delta_n) + \log \eta \delta_n^2},$$

and

$$c_n(1 - \delta_n) = b_n(1 - (2\pi/b_n + 1)\delta_n) + \delta_n^2(2\pi - 1 + \delta_n),$$

and since  $\delta_n \rightarrow 0$  and  $b_n \rightarrow b > 1$  as  $n \rightarrow \infty$ .

Consider the fundamental loop  $L_{M^N(R)}$ . Since  $L_{M^N(R)} \cap C_N \neq \emptyset$ , there is a

point on  $L_{M^N(R)}$  of modulus less than  $r_N^{b_N(1-3\pi\delta_N)}$ . Hence

$$M^N(R) < r_N^{b_N(1-3\pi\delta_N)}. \quad (3.32)$$

Moreover, by assumption we have that  $L_{M^N(R)} \cap \partial_{\text{out}}U_N \neq \emptyset$ , and so there is a point on  $L_{M^N(R)}$  of modulus greater than  $r_N^{b_N}$ . Hence  $L_{M^N(R)}$  surrounds points in  $U_N$  which lie at all radii in  $(r_N^{b_N(1-3\pi\delta_N)}, r_N^{b_N})$ . In particular, there exists a point

$$z \in H_{M^N(R)} \cap U_N, \quad \text{such that } |z| = r_N^{c_N}. \quad (3.33)$$

Then, by Lemma 3.2.3, Lemma 1.2.2 and Lemma 3.2.5, we have that, for  $m \in \mathbb{N}$ ,

$$\begin{aligned} |f^m(z)| &\geq M(r_N^{c_N}, f^m)^{1-\delta_N} \\ &\geq M(r_N^{c_N(1-\delta_N)}, f^m) \geq M(\eta r_N^{b_N(1-3\pi\delta_N)}, f^m) \\ &\geq M^m(r_N^{b_N(1-3\pi\delta_N)}, f). \end{aligned}$$

Hence  $z \in A_\rho(f)$ , where  $\rho = r_N^{b_N(1-3\pi\delta_N)}$ . This is in contradiction to (3.32), since  $z \notin A_{M^N(R)}(f)$  by (3.33). This completes the first half of the proof of part (c).

Finally, suppose that  $R_1 < R < R_2$ . Then, by the continuity of  $R_A$  and the definitions of  $R_1$  and  $R_2$ , there exists  $z \in U$  such that  $R_A(z) = R$ . Hence the fundamental loop  $L_R$  must intersect  $U$ , and so  $L_R \subset U$ . This completes the proof.  $\square$

Next we prove Theorem 3.1.2, which states that if  $f$  is a transcendental entire function and that  $L_R$  is a fundamental loop of  $f$ , then either  $L_R \subset F(f)$  or  $L_R \subset J(f)$ .

*Proof of Theorem 3.1.2.* Suppose first that  $z \in L_R \cap U$ , where  $U$  is a simply connected Fatou component of  $f$ . Since  $L_R \subset A_R(f)$ , we have that  $\bar{U} \subset A_R(f)$ , by Theorem 1.7.3. This is a contradiction since  $L_R = \partial H_R$  and  $H_R \subset A_R(f)^c$ . Hence  $L_R$  cannot intersect any simply connected Fatou component of  $f$ .

Next suppose that  $z \in L_R \cap U$ , where  $U$  is a multiply connected Fatou component of  $f$ . Then there exists  $N \in \mathbb{N}$  such that  $\text{dist}(0, U_N) > R'$ , where  $R'$  is the constant from Theorem 3.1.1 and  $U_N = f^N(U)$ . Then  $f^N(L_R) = L_{M^N(R)}$  is

a fundamental loop which intersects  $U_N$  and so, by Theorem 3.1.1, is contained in  $U_N$ . The result follows.  $\square$

Finally we prove Theorem 3.1.3, which relates fundamental loops lying in  $U$  to level sets of  $h$ .

*Proof of Theorem 3.1.3.* First suppose that  $L_R \subset U$  is a fundamental loop. Then, because of the continuity of  $R_A$  in  $U$ , we have  $R_A(z) = R$ , for  $z \in L_R$ . Hence, by Lemma 3.5.3,  $h$  is also constant on  $L_R$ . This completes the first part of the proof.

Suppose next that  $\Gamma$  is a level set of  $h$ . By Lemma 3.5.3,  $\Gamma$  is also a level set of  $R_A$ , and so  $R_A(z) = R$ , say, for  $z \in \Gamma$ . Now  $R_1 < R < R_2$ , where  $R_1$  and  $R_2$  are as in (3.28), and so, by Theorem 3.1.1, there is a fundamental loop  $L_R \subset U$ . The result follows, since  $L_R \subset \Gamma$ , again by Lemma 3.5.3.  $\square$

### 3.7 The function $R_A$ defined in $\mathbb{C}$

The function  $R_A$  played a key role in proving Theorem 3.1.1. In general, however,  $R_A(z)$  cannot be defined for many values of  $z \in A(f)$ ; consider, for example,  $f(z) = e^z$  and  $z = \log 2\pi + i\pi/2$ . In this section we show that, with a certain normalisation of  $f$ , the definition of  $R_A(z)$  can be extended in a natural way to all  $z \in \mathbb{C}$ . The function  $R_A$  then has several interesting properties.

First we adopt the normalisation  $f(0) = 0$ . We observe that, by Lemma 3.5.1, all transcendental entire functions for which  $A_R(f)$  is a spider's web have a fixed point and so, in this case, this normalisation is merely a change of coordinates. This suggests that this normalisation is not entirely unnatural when  $A_R(f)$  is a spider's web. Even when  $f$  does not have a fixed point the normalisation  $f(0) = 0$  is not as limiting as it might seem. If  $f(z)$  has no fixed point, then  $f(z)$  has the form  $f(z) = z + \exp(h(z))$ , for some entire function  $h$ . It then follows from Picard's Theorem that  $f^2$  has fixed points. We choose  $\alpha$ , a fixed point of  $f^2$ , and replace  $f$  by  $g$  where  $g(z) = f^2(z + \alpha) - \alpha$ . Then  $g(0) = 0$ , and the sets  $A(f)$  and  $A(g)$  differ only by a translation, since  $A(f^2) = A(f)$  by Theorem 1.7.1(a).

Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . Suppose that  $r$  is such that  $M^n(r, f) \rightarrow \infty$  as  $n \rightarrow \infty$ . Clearly  $M(r, f) > r$ , and so, by

Lemma 1.2.1(a), we may choose  $\epsilon > 0$  such that  $M(r - \epsilon, f) > r$ . It follows that  $M^n(r - \epsilon, f) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Consider the set

$$Y_f = \{r \geq 0 : M^n(r) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Clearly  $Y_f \neq \emptyset$ , and it follows from the previous paragraph that  $Y_f$  is closed. Thus  $Y_f$  has a maximum.

Hence, with the normalisation  $f(0) = 0$  we can define

$$R_f = \max\{r \geq 0 : M^n(r) \rightarrow \infty \text{ as } n \rightarrow \infty\}. \quad (3.34)$$

The following gives an alternative characterisation of  $A(f)$  as a continuous limit of the closed sets  $A_R(f)$ .

**Theorem 3.7.1.** *Suppose that  $f$  is a transcendental entire function, that  $f(0) = 0$ , and that  $R_f$  is as defined in (3.34). Then*

$$A(f) = \bigcup_{R > R_f} A_R(f). \quad (3.35)$$

*Proof.* If  $z \in \bigcup_{R > R_f} A_R(f)$ , then  $z \in A_R(f)$  for some  $R$  such that  $M^n(R) \rightarrow \infty$  as  $n \rightarrow \infty$ , and so  $z \in A(f)$  by definition.

Now, suppose that  $z \in A(f)$ . Then, by (1.12),  $f^\ell(z) \in A_R(f)$ , for some  $R > R_f$  and some  $\ell \in \mathbb{N}$ . Note next that, since  $f(0) = 0$ , we have that  $M^{-n}(r)$  is defined for all  $r \geq 0$  and  $n \in \mathbb{N}$ . Hence we can set  $R' = M^{-\ell}(R)$ , and we note that  $R' > R_f$ . Then  $z \in A_{R'}(f)$  and so  $z \in \bigcup_{R > R_f} A_R(f)$ , as required.  $\square$

For a transcendental entire function  $f$  with  $f(0) = 0$ , we extend the definition of  $R_A$  to the whole complex plane by setting

$$R_A(z) = \begin{cases} \max\{R : z \in A_R(f)\}, & \text{for } z \in A(f), \\ R_f, & \text{for } z \notin A(f). \end{cases} \quad (3.36)$$

The existence of the maximum, for  $z \in A(f)$ , follows from Lemma 1.2.1(a) and (3.35), in the same way as the existence of the maximum in (3.22).

We require the following result.

**Lemma 3.7.2.** *Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . Then*

$$M^n(R_A(z)) = R_A(f^n(z)), \quad \text{for } z \in A(f), \quad n \in \mathbb{N}. \quad (3.37)$$

*Proof.* Choose  $n \in \mathbb{N}$ . Suppose that  $z \in A(f)$ , and set  $R_A(z) = \rho_1$  and  $R_A(f^n(z)) = \rho_2$ . By the definition of  $R_A(z)$  we have that

$$|f^m(f^n(z))| = |f^{m+n}(z)| \geq M^{m+n}(\rho_1) = M^m(M^n(\rho_1)), \quad \text{for } m \in \mathbb{N},$$

and so  $\rho_2 \geq M^n(\rho_1)$ . It follows that  $M^{-n}(\rho_2)$  is defined and greater than or equal to  $\rho_1$ . By the definition of  $R_A(f^n(z))$  we have that

$$|f^m(z)| = |f^{m-n}(f^n(z))| \geq M^{m-n}(\rho_2) = M^m(M^{-n}(\rho_2)), \quad \text{for } m \geq n,$$

and so  $\rho_1 \geq M^{-n}(\rho_2)$ . The result follows.  $\square$

If  $f$  satisfies the normalization  $f(0) = 0$ , then a stronger version of Lemma 3.5.3 holds.

**Theorem 3.7.3.** *Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . Then*

- (a)  $R_A$  is upper semicontinuous in  $\mathbb{C}$ ;
- (b)  $R_A$  is nowhere continuous in  $A(f) \cap J(f)$ ;
- (c)  $R_A$  is constant in a simply connected Fatou component of  $f$ ;
- (d)  $R_A$  is continuous in  $A(f)^c \cup F(f)$ .

*Proof.* Part (a) follows in exactly the same way as the first part of the proof of Lemma 3.5.3, and so we omit the details.

Now we prove part (b). Observe that, in general, if  $w \in A(f)$ ,  $n > 1$  and  $f^n(w') = w$ , then

$$R_A(w') = M^{-n}(R_A(w)) < M^{-1}(R_A(w)) < R_A(w). \quad (3.38)$$



Suppose that  $z \in A(f) \cap J(f)$  and assume first that  $z \notin E(f)$ , where  $E(f)$  is as defined in (1.14). Let  $\Delta$  be a neighbourhood of  $z$ , sufficiently small that  $\overline{\Delta} \cap E(f) = \emptyset$ . Then, by Theorem 1.4.3, there is an  $n > 1$  such that  $f^n(\Delta) \supset \overline{\Delta}$ . Hence, since  $z$  cannot be periodic, there is a  $z' \in \Delta$  with  $z' \neq z$  and such that  $f^n(z') = z$ . Hence, by (3.38),

$$R_A(z') < M^{-1}(R_A(z)) < R_A(z). \quad (3.39)$$

This shows that  $R_A$  is not continuous at  $z$  in the case that  $z \notin E(f)$ , since  $\Delta$  was arbitrary.

In the case that  $z \in E(f)$ , we first observe that  $f(z) \notin E(f)$ . Let  $\Delta$  be a neighbourhood of  $z$ , sufficiently small that  $\overline{f(\Delta)} \cap E(f) = \emptyset$ . By the same argument as above, there is a  $z' \in \Delta$  such that

$$R_A(f(z')) < M^{-1}(R_A(f(z))) < R_A(f(z)). \quad (3.40)$$

Equation (3.39) now follows from (3.40) and (3.37). This completes the proof of part (b).

Next we prove part (c). Suppose that  $U$  is a simply connected Fatou component and that  $z_1, z_2 \in U$  are such that  $r = R_A(z_1) > R_A(z_2)$ . First we observe that  $z_1 \in A_r(f)$ . Hence  $\overline{U} \subset A_r(f)$ , by Theorem 1.7.3, and in particular  $z_2 \in A_r(f)$ . From this it follows that  $R_A(z_2) \geq r$ , which is a contradiction. This completes the proof of part (c).

Finally we prove part (d). The result when  $z \in A(f)^c$  is immediate from part (a), and the fact that  $R_A$  achieves its global minimum of  $R_f$  everywhere in  $A(f)^c$ . If  $z \in A(f) \cap F(f)$ , then we can assume that  $z$  is in a multiply connected Fatou component of  $f$ , since in a simply connected Fatou component  $R_A$  is constant, by part (c), and so continuous. The proof follows in exactly the same way as the second part of the proof of Lemma 3.5.3.  $\square$

In a multiply connected Fatou component, we can say more about the properties of the function  $R_A$ .

**Theorem 3.7.4.** *Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . Then the function  $v = -\log R_A$  is subharmonic in  $F(f)$ .*

**Remark 3.7.1.** It follows from Theorem 3.7.3(d) and Theorem 3.7.4 that  $1/R_A$  is in the class PL in each component of  $F(f)$ . Here (see [16]), a function  $u$  in a domain  $D$  is said to be in the class PL if  $u$  is continuous and non-negative in  $D$ , and  $\log u$  is subharmonic in the part of  $D$  where  $u > 0$ . This class is a generalisation of functions of the form  $|\phi|$ , where  $\phi$  is analytic in  $D$ . The weaker result that  $1/R_A$  is subharmonic in  $F(f)$  also follows from Theorem 3.7.4, since  $1/R_A(z) = \exp(v(z))$  and by [76, Corollary 2.6.4], which states that if  $u$  is subharmonic in a domain then so is  $\exp \circ u$ .

**Remark 3.7.2.** It seems natural to ask if  $v$  is harmonic in  $F(f)$ . This cannot be the case in general. For, by the last statement of Lemma 3.5.3, if  $v$  is harmonic in a multiply connected Fatou component  $U$  which satisfies the conditions of Theorem 3.1.1, then there is a continuous function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$v(z) = \psi(h(z)), \quad \text{for } z \in U. \quad (3.41)$$

If  $v$  is harmonic, then – since  $h$  is also harmonic – we can differentiate (3.41) to obtain that  $\psi''(h(z)) = 0$ , for  $z \in U$ . Hence  $v$  is a linear function of  $h$  in  $U$ . Now,  $v$  is finite in  $\bar{U}$ . In [27, Example 2 and Theorem 1.6] it is shown that there exist transcendental entire functions such that  $h$  is unbounded in  $U$ . For these functions the relationship between  $h$  and  $v$  cannot, therefore, be linear, and so  $v$  is not harmonic in  $U$ .

In order to prove Theorem 3.7.4 we need three further lemmas. The first concerns repeated iteration of the function  $M^{-1}$ .

**Lemma 3.7.5.** *Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . For each  $n \in \mathbb{N}$ , define the function  $v_n$  by*

$$v_n(z) = -\log M^{-n}(|f^n(z)|), \quad \text{for } z \in D_n = \{z : f^n(z) \neq 0\}. \quad (3.42)$$

*Then  $v_n$  is subharmonic in  $D_n$ .*

*Proof.* Since  $\psi(s) = \log M^{-1}(e^s)$  is a concave and increasing function of  $s$ , we have (see, for example, [65, Theorem 7.2.1]) that

$$\psi^n(s) = \log M^{-n}(e^s)$$

is also a concave function of  $s$ , for  $n \in \mathbb{N}$ . Now, for each  $n \in \mathbb{N}$ ,  $\log |f^n(z)|$  is a harmonic function of  $z$  in  $D_n$ , since  $f^n(z) \neq 0$  in  $D_n$ . The result follows since

$$v_n(z) = -\log M^{-n}(\exp(\log |f^n(z)|)) = -\psi^n(\log |f^n(z)|),$$

is a convex function of a harmonic function; see e.g. [76, p.47], which states that a convex function of a harmonic function is subharmonic.  $\square$

Note that, if  $f(0) = 0$ , then  $0 \notin A(f)$  and so  $v_n(z)$  is defined for all  $z \in A(f)$  and  $n \in \mathbb{N}$ .

The second lemma gives an alternative characterisation of the function  $R_A$  in  $A(f)$ . Here we say that a sequence of real numbers,  $(t_n)_{n \in \mathbb{N}}$ , is *non-increasing* if  $t_{n+1} \leq t_n$ , for  $n \in \mathbb{N}$ .

**Lemma 3.7.6.** *Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . Then, for each  $z \in A(f)$ ,  $(M^{-n}(|f^n(z)|))_{n \in \mathbb{N}}$  is a non-increasing sequence, with limit  $R_A(z)$ .*

*Proof.* Suppose that  $z \in A(f)$ . Since  $M(|f^n(z)|) \geq |f^{n+1}(z)|$  we have that

$$M^{-n}(|f^n(z)|) \geq M^{-(n+1)}(|f^{n+1}(z)|), \quad \text{for } n \in \mathbb{N}. \quad (3.43)$$

Hence the sequence  $(M^{-n}(|f^n(z)|))_{n \in \mathbb{N}}$  is non-increasing. In addition, since  $|f^n(z)| \geq M^n(R_A(z))$ , for  $n \in \mathbb{N}$ , we have that

$$M^{-n}(|f^n(z)|) \geq R_A(z), \quad \text{for } n \in \mathbb{N}.$$

So  $\lim_{n \rightarrow \infty} M^{-n}(|f^n(z)|)$  exists and is at least  $R_A(z)$ . It follows from (3.43) and the definition of  $A_R(f)$  that if this limit is  $R$ , then  $z \in A_R(f)$ . This completes the proof.  $\square$

We also need a result on subharmonic functions. Suppose that  $D$  is a domain, and  $u : D \rightarrow [-\infty, \infty)$  is a function which is locally bounded above in  $D$ . The *upper semicontinuous regularization* of  $u$  is the function  $u^* : D \rightarrow [-\infty, \infty)$  defined by

$$u^*(z) = \limsup_{w \rightarrow z} u(w).$$

It can be shown that  $u^*$  is the least upper semicontinuous function on  $D$  such that  $u^* \geq u$ ; see, for example, [76, p.62]. The result we require is the following [76, Theorem 3.4.2(a)].

**Lemma 3.7.7** (Brelot-Cartan Theorem). *Suppose that  $D$  is a domain, that  $\mathcal{V}$  is a family of subharmonic functions on  $D$  and that  $u = \sup_{v \in \mathcal{V}} v$  is locally bounded above on  $D$ . Then  $u^*$  is subharmonic on  $D$ .*

We now give the proof of Theorem 3.7.4, that the function  $v(z) = -\log R_A(z)$  is subharmonic, for  $z \in F(f)$ .

*Proof of Theorem 3.7.4.* Suppose that  $z \in A(f)^c \cap F(f)$ . The result follows because  $R_A$  is constant in a neighbourhood of  $z$ . On the other hand, suppose that we have  $z \in A(f) \cap F(f)$ , and let  $U$  be the Fatou component containing  $z$ . Since  $R_A$  is constant in any simply connected Fatou component, we can assume that  $U$  is multiply connected. Observe that, by Lemma 3.5.4, applied, if necessary, to  $U_N$  for some large  $N$ , there exists  $R_1 > 0$  such that  $R_A(z) \geq R_1$ , for  $z \in \bar{U}$ . Hence  $v$  is bounded above in  $\bar{U}$ .

Let  $v_n$  be as defined in Lemma 3.7.5. Then, by Lemma 3.7.6 and Lemma 3.7.5,  $v_n$  is a non-decreasing sequence of subharmonic functions, converging pointwise in  $\bar{U}$  to  $v$ . Hence,  $\sup_{n \in \mathbb{N}} v_n = v$ . By Lemma 3.7.7, applied with  $\mathcal{V} = \{v_n : n \in \mathbb{N}\}$ ,  $v^*$  is subharmonic in  $U$ . By Theorem 3.7.3 part (d),  $v$  is continuous in  $U$ , and so  $v^* = v$  there. This completes the proof.  $\square$

Another advantage of the normalisation  $f(0) = 0$  is that, if this condition is satisfied, then the conclusions of Theorems 3.1.1 and 3.1.3 hold for any multiply connected Fatou component which surrounds the origin, without the additional restriction of being a sufficient distance from the origin. This fact follows from the proof of Theorem 3.1.1 and from the following version of Lemma 3.5.2.

**Lemma 3.7.8.** *Suppose that  $f$  is a transcendental entire function and that  $f(0) = 0$ . Suppose that  $U$  is a multiply connected Fatou component of  $f$  which surrounds the origin, and define  $G_n$  as the complementary component of  $\bar{U}_n$  which contains the origin, for  $n = 0, 1, 2, \dots$ . Then*

(a)  $\bar{G}_n \subset G_{n+1}$ , for  $n = 0, 1, 2, \dots$ ;

(b)  $f(\partial G_n) = \partial G_{n+1}$ , for  $n = 0, 1, 2, \dots$ ;

(c) for all  $z \in \bar{U}$  there exists  $R = R(z)$  such that  $z \in A_R(f)$ .

*Proof.* Parts (a) and (b) follow as in the proof of Lemma 3.5.2, since the origin is a fixed point of  $f$ . Part (c) follows from Theorem 3.7.1.  $\square$

Given a transcendental entire function  $f$  and  $N \in \mathbb{N}$ , it is not hard to show that there is a point  $z \in A(f)$  such that  $|f^{n+1}(z)|$  is small compared to  $|f^n(z)|$ , for  $n \leq N$ . Hence  $R_A(z)$  can be much smaller than  $|z|$ . It does seem reasonable, however, to expect that  $M^n(R_A(z))$  should be comparable to  $|f^n(z)|$ , for large values of  $n \in \mathbb{N}$ . We use results from [27] to prove the following.

**Theorem 3.7.9.** *Suppose that  $f$  is a transcendental entire function, that  $f(0) = 0$ , and that  $z$  is in a multiply connected Fatou component of  $f$ . Then*

$$\lim_{n \rightarrow \infty} \frac{\log |f^n(z)|}{\log M^n(R_A(z))} = 1. \quad (3.44)$$

*Proof.* Let  $U$  be the Fatou component containing  $z$ . It follows from (3.37) that we need to prove that

$$\lim_{n \rightarrow \infty} \frac{\log |f^n(z)|}{\log R_A(f^n(z))} = 1. \quad (3.45)$$

By definition  $|f^n(z)| \geq R_A(f^n(z))$ , for  $n \in \mathbb{N}$ . Suppose that, contrary to (3.45), there exists  $0 < c < 1$  and a sequence of natural numbers  $(n_k)_{k \in \mathbb{N}}$  such that  $|f^{n_k}(z)|^c > R_A(f^{n_k}(z))$ , for  $k \in \mathbb{N}$ , and  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Then, by the definition of  $R_A$ , for each  $k \in \mathbb{N}$  there exists  $m_k \in \mathbb{N}$  such that

$$|f^{n_k+m}(z)| < M^m(|f^{n_k}(z)|^c), \quad \text{for } m \geq m_k. \quad (3.46)$$

Since  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ , we see that (3.46) is contrary to Lemma 3.2.4. This completes the proof of Theorem 3.7.9.  $\square$

**Remark 3.7.3.** If  $f$  is a transcendental entire function and  $f(0) \neq 0$ , then the conclusions of Theorem 3.7.4 and Theorem 3.7.9 still hold for a multiply connected Fatou component  $U$  which satisfies the conditions of Theorem 3.1.1. This is readily seen from a review of the proofs of these results.

The relationship between  $h$  and  $R_A$  may be closely related to the growth of  $M(r)$  for values of  $r$  close to  $R_f$ , and so it seems unlikely that, in general, there is a simple form for the function  $\phi$  defined in (3.23). It is possible, however, to obtain bounds on the relationship between  $R_A$  and  $h$ . In the following result we consider only the case when  $h(z) > 1$ . The case when  $h(z) < 1$  is similar, and it follows from Lemma 3.5.3 that  $R_A(z_0) = R_A(z)$  when  $h(z) = 1$ .

**Lemma 3.7.10.** *Suppose that  $f$  is a transcendental entire function with  $f(0) = 0$ , that  $U$  is a multiply connected Fatou component of  $f$  with  $z, z_0 \in U$ , and that  $h$  is defined as in (3.1). Suppose also that  $h(z) > 1$  and that  $R_A(z_0) > R_0$ , where  $R_0$  is the constant from Lemma 1.2.2. Then*

$$\log R_A(z_0) < \log R_A(z) \leq h(z) \log R_A(z_0). \quad (3.47)$$

*Proof.* Since  $h(z) > 1$  we have that  $|f^n(z_0)| < |f^n(z)|$ , for large values of  $n \in \mathbb{N}$ . Hence  $\log R_A(z_0) \leq \log R_A(z)$ , and equality is impossible by the last statement of Lemma 3.5.3.

To prove the right-hand inequality, we proceed as follows. For each  $n \in \mathbb{N}$ , set

$$\alpha_n = \frac{\log |f^n(z)|}{\log |f^n(z_0)|},$$

and so  $\alpha_n \rightarrow h(z)$  as  $n \rightarrow \infty$ . Since  $h(z) > 1$ , there is an  $N \in \mathbb{N}$  such that  $\alpha_n > 1$ , for  $n \geq N$ . Hence, for  $n \geq N$ , by repeated application of (1.11),

$$\log M^{-n}(|f^n(z)|) = \log M^{-n}(|f^n(z_0)|^{\alpha_n}) \leq \alpha_n \log M^{-n}(|f^n(z_0)|). \quad (3.48)$$

Observe that the smallest term to which we apply (1.11) has  $r$  replaced by

$$M^{-(n-1)}(|f^n(z_0)|) = M(M^{-n}(|f^n(z_0)|)) \geq M(R_A(z_0)).$$

This explains the condition  $R_A(z_0) > R_0$  in the statement of the lemma. The result follows by letting  $n \rightarrow \infty$  in (3.48), and by Lemma 3.7.6.  $\square$

The following simple result shows that, in a sense, if we replace  $z$  and  $z_0$  in (3.47) with  $f^n(z)$  and  $f^n(z_0)$ , and take the limit as  $n \rightarrow \infty$ , then the central term tends to its upper bound.

**Lemma 3.7.11.** *Suppose that  $f$  is a transcendental entire function with  $f(0) = 0$ , that  $U$  is a multiply connected Fatou component of  $f$  with  $z, z_0 \in U$ , and that  $h$  is defined as in (3.1). Then*

$$\log R_A(f^n(z)) \sim h(z) \log R_A(f^n(z_0)), \text{ as } n \rightarrow \infty. \quad (3.49)$$

*Proof.* By Theorem 3.7.9 we have

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{\log |f^n(z_0)|}{\log R_A(f^n(z_0))} \lim_{n \rightarrow \infty} \frac{\log R_A(f^n(z))}{\log |f^n(z)|} \\ &= \lim_{n \rightarrow \infty} \frac{\log |f^n(z_0)|}{\log |f^n(z)|} \lim_{n \rightarrow \infty} \frac{\log R_A(f^n(z))}{\log R_A(f^n(z_0))} \\ &= \frac{1}{h(z)} \lim_{n \rightarrow \infty} \frac{\log R_A(f^n(z))}{\log R_A(f^n(z_0))}, \end{aligned}$$

and the result follows. Note that the existence of these limits follows from Theorem 3.7.9 and the existence of the function  $h$  defined in (3.1).  $\square$

# Chapter 4

## Simply connected fast escaping Fatou components

### 4.1 Introduction

In this chapter we give an example of a transcendental entire function with a simply connected fast escaping Fatou component, but with no multiply connected Fatou components. We also give a new criterion for points to be in the fast escaping set.

As noted earlier, the first example of a simply connected fast escaping Fatou component was given by Bergweiler [20], using a quasi-conformal surgery technique from [63]. This function also has multiply connected Fatou components. In fact, in [20], the properties of the multiply connected Fatou components are used to show that the simply connected Fatou components are fast escaping. A similar example was given in [75, Example 6.3].

This prompts the question of whether a transcendental entire function can have simply connected fast escaping Fatou components without having multiply connected Fatou components. We answer this in the affirmative, using a direct construction and Theorem 1.5.2 to prove the following.

**Theorem 4.1.1.** *There is a transcendental entire function with a simply connected fast escaping Fatou component, and no multiply connected Fatou components.*



To prove Theorem 4.1.1 we require a new sufficient condition for points to be in  $A(f)$ , which may be of independent interest.

**Theorem 4.1.2.** *Suppose that  $f$  is a transcendental entire function. Suppose also that there exists  $R_0 > 0$  and a nonincreasing function  $\epsilon : [R_0, \infty) \rightarrow (0, 1)$  such that*

$$\epsilon(M^n(r)) \geq \epsilon(r)^{n+1}, \quad \text{for } r \geq R_0 \text{ and } n \in \mathbb{N}. \quad (4.1)$$

Define  $\eta(r) = \epsilon(r)M(r)$ , for  $r \geq R_0$ . Then there exists  $R_1 \geq R_0$  such that

$$A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \eta^n(R'), \text{ for } n \in \mathbb{N}\},$$

for  $R' \geq R_1$ .

Note that this is a generalisation of Theorem 1.7.5, which is obtained from Theorem 4.1.2 when  $\epsilon$  is constant.

## 4.2 The definition of the function

In this section we define a transcendental entire function,  $f$ , which has all the properties defined in Theorem 4.1.1. Since  $f$  is very complicated, we first outline informally the construction of  $f$ , starting with simpler functions which only have some of these properties. We then give the full construction. A detailed proof of Theorem 4.1.1 is given in subsequent sections.

Consider first a transcendental entire function defined by a product;

$$g(z) = z \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right)^2, \quad 0 < a_1 < a_2 < \dots$$

The sequence  $(a_n)_{n \in \mathbb{N}}$  can be chosen so that the following holds: we can define another sequence,  $(b_n)_{n \in \mathbb{N}}$ , such that  $b_n$  is approximately equal to  $a_n$ ,  $-b_n$  is close to a critical point of  $g$ , and  $g(-b_n)$  is close to  $-b_{n+1}$ . It can then be shown that a small disc centred at  $-b_n$  is mapped by  $g$  into a small disc centred at  $-b_{n+1}$ . By Montel's theorem, these discs must be in the Fatou set of  $g$ . Moreover, these discs cannot be in multiply connected Fatou components of  $g$  since, by [27, Theorem 1.2], any open set contained in a multiply connected Fatou component of  $g$  must,

after a finite number of iterations of  $g$ , cover an annulus surrounding the origin. Finally, it can be shown, by comparing  $|g(-b_n)|$  to  $M(b_n, g) = g(b_n)$ , that these discs are contained in fast escaping Fatou components of  $g$ .

However,  $g$  does not have all the properties asserted in Theorem 4.1.1. In particular, by considering the behaviour of  $g$  in large annuli which omit the zeros of  $g$ , it can be shown that  $g$  has multiply connected Fatou components. Thus  $g$  has very similar properties to the example in [20].

We note that no zero of  $g$  can be in a multiply connected Fatou component, since 0 is a fixed point. In order to prevent the existence of multiply connected Fatou components, we add further zeros to the function, along the negative real axis. This requires some care. The addition of too many zeros – for example, spaced linearly along the negative real axis – leads to a breakdown of other parts of the construction. The addition of a zero with modulus insufficiently distant from  $a_n$  leads to a similar breakdown.

We use Theorem 1.5.2 to show that only a relatively small number of additional zeros are required. In particular, suppose that  $h$  is a transcendental entire function with  $h(0) = 0$  and with zeros of modulus  $0 < r_0 < r_1 < r_2 < \dots$ . Then, by Theorem 1.5.2,  $h$  has no multiply connected Fatou components if  $\lim_{k \rightarrow \infty} \log r_{k+1} / \log r_k$  exists and is equal to 1.

To use this result, we need to understand the behaviour of  $\log a_{n+1} / \log a_n$ , for large  $n$ . From the recursive definition that we use to ensure that the sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  have the required properties, (see (4.35)), we find that, for large  $n$ ,  $\log a_{n+1} / \log a_n$  is close to  $n^3$ . See (4.36) for a precise statement of how the term  $n^3$  arises here.

This suggests the following. Define  $\mu_n = n^{3/n}$ , for  $n \in \mathbb{N}$ . To simplify some displays we set  $\mu_{n,m} = \mu_n^m$ , and observe that  $\mu_{n,0} = 1$  and  $\mu_{n,n} = n^3$ , for  $n \in \mathbb{N}$ . We now define a more complicated transcendental entire function

$$h(z) = z \prod_{k=1}^{\infty} \prod_{l=0}^{k-1} \left( 1 + \frac{z}{a_k^{\mu_{k,l}}} \right)^2, \quad 0 < a_1 < a_2 < \dots$$

The sequence  $(a_n)_{n \in \mathbb{N}}$  in this definition is not the same as in the definition of  $g$ , but serves the same purpose, and is chosen similarly. This function has zeros of

modulus

$$\cdots, a_n, a_n^{\mu_n}, a_n^{\mu_{n,2}}, \cdots, a_n^{\mu_{n,n-1}}, a_{n+1}, a_{n+1}^{\mu_{n+1}} \cdots$$

Since it is readily seen that  $\mu_n \rightarrow 1$  as  $n \rightarrow \infty$ , this function does not have multiply connected Fatou components. However, two further adjustments are required. Firstly, the zero of modulus  $a_n^{\mu_{n,n-1}}$  is sufficiently close to the zero of modulus  $a_{n+1}$  that the original construction breaks down. We resolve this by omitting this zero. Secondly, the value of  $\log a_{n+1}/\log a_n$  is not close enough to  $n^3$ , for large  $n$ , to ensure that  $\lim_{k \rightarrow \infty} \log a_{k+1}/\log a_k^{\mu_{k,k-2}} = 1$ . We resolve this by adding one additional zero, which serves no other purpose in the construction. This zero is defined using two additional sequences,  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$ , which we choose to keep  $\log a_{n+1}/\log a_n$  sufficiently close to  $n^3$ .

Now we are able to indicate the form of the function  $f$  in Theorem 4.1.1. Let  $f$  be the transcendental entire function;

$$f(z) = z \prod_{k=3}^{\infty} \left\{ \left( 1 + \frac{z}{a_k^{\beta_k}} \right)^{2\alpha_k} \prod_{l=0}^{k-2} \left( 1 + \frac{z}{a_k^{\mu_{k,l}}} \right)^2 \right\}, \quad (4.2)$$

where  $0 < a_3 < a_4 < \cdots$ ,  $\alpha_n \in \{0, 1, 2, \dots\}$ ,  $\beta_n \in \mathbb{R}$ , for  $n \in \mathbb{N}$ . Again, the sequence  $(a_n)_{n \in \mathbb{N}}$  in this definition is not the same as that in the definition of  $g$  or  $h$ , but serves the same purpose, and is chosen similarly. The related sequence  $(b_n)_{n \in \mathbb{N}}$ , discussed after the definition of  $g$ , is defined for  $f$  by (4.34). The sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  are the two sequences mentioned at the end of the previous paragraph. In Section 4.6, at the end of this chapter, we give the definition of the various sequences in (4.2), and we prove a number of estimates on the modulus of the zeros of  $f$ .

The structure of the proof of Theorem 4.1.1 is as follows. In Section 4.3, we show that there are no multiply connected Fatou components of  $f$ . In Section 4.4 we show that there are intervals on the negative real axis each contained in some Fatou component of  $f$ . Finally, in Section 4.5 we prove Theorem 4.1.2 and then use this to show that these Fatou components of  $f$  are fast escaping. It is clear that Theorem 4.1.1 follows from these results.

**Remark 4.2.1.** Rippon and Stallard asked [86, Question 1] if there can be unbounded fast escaping Fatou components of a transcendental entire function. It can be shown that the Fatou components of the function  $f$  are all bounded. Indeed, it is straightforward to prove that the number of zeros of  $f$  in the disc  $\{z : |z| < r\}$  is  $O(\log r)$ , and hence that  $\log M(r, f) = O((\log r)^2)$ . It follows, by Theorem 1.9.2(b) and Theorem 1.9.1(d), that the set  $A(f)$  is a spider's web and that  $f$  has no unbounded Fatou components.

### 4.3 There are no multiply connected Fatou components

In this section we prove the following result.

**Lemma 4.3.1.** *The transcendental entire function  $f$  does not have multiply connected Fatou components.*

*Proof.* Observe that, for large  $n$ , in the closed annulus  $\overline{A(a_n, a_{n+1})}$  there are zeros of  $f$  on the negative real axis of modulus  $a_n, a_n^{\mu_n}, a_n^{\mu_{n,2}}, \dots, a_n^{\mu_{n,n-2}}$  and  $a_{n+1}$ . Note also that 0 is a fixed point of  $f$ , and so all zeros of  $f$  lie in  $A(f)^c$ . Now, by (4.36),

$$a_{n+1} \leq a_n^{n^3+2/n} < (a_n^{\mu_{n,n-2}})^{\mu_{n,2}+2/n}.$$

Hence, for large  $n$ , there is at least one zero of  $f$  in any annulus  $A(r, r^{\mu_{n,2}+2/n})$ , for  $a_n \leq r \leq a_{n+1}$ . Note that  $\mu_{n,2} + 2/n \rightarrow 1$  as  $n \rightarrow \infty$ . It follows that, given  $d > 1$  there exists  $R > 0$  such that

$$A(r, r^d) \cap A(f)^c \neq \emptyset, \quad \text{for } r \geq R. \quad (4.3)$$

Now, by Theorem 1.5.2, if  $f$  has a multiply connected Fatou component, then there is a  $d > 1$ , and a sequence  $(r_i)_{i \in \mathbb{N}}$ , tending to infinity, such that the annuli  $A(r_i, r_i^d)$  are contained in multiply connected Fatou components of  $f$ . This is a contradiction, by (4.3) and Theorem 1.7.2. Hence there can be no multiply connected Fatou components of  $f$ .  $\square$

## 4.4 There are simply connected Fatou components

Next we show that  $f$  has simply connected Fatou components.

**Lemma 4.4.1.** *Define  $B_n = \{z : |z + b_n| < \delta_n b_n\}$ , where  $\delta_n = n^{-9}$ . Then, for large  $n$ , we have  $f(B_n) \subset B_{n+1}$ , and  $B_n$  is contained in a simply connected Fatou component of  $f$ .*

*Proof.* Suppose that  $z \in B_n$ , in which case  $z = -b_n + wb_n$  where  $|w| < \delta_n$ . We assume throughout this section that  $n$  is sufficiently large for various estimates to hold.

We consider the quotient of  $f(z)$  and  $-b_{n+1}$ , gathering together the terms in the product for  $f(z)$  which also occur in the product for  $b_{n+1}$ , and the terms in the product for  $f(z)$  which do not occur in the product for  $b_{n+1}$ . By (4.2), (4.34) and (4.35),

$$\frac{f(z)}{-b_{n+1}} = I_1 I_2, \quad (4.4)$$

where

$$I_1 = (1 - w) \left(1 + w \frac{b_n}{a_n - b_n}\right)^2 \prod_{k=3}^{n-1} \left\{ \left(1 + w \frac{b_n}{a_k^{\beta_k} - b_n}\right)^{2\alpha_k} \prod_{l=0}^{k-2} \left(1 + w \frac{b_n}{a_k^{\mu_{k,l}} - b_n}\right)^2 \right\}, \quad (4.5)$$

and

$$I_2 = \prod_{l=1}^{n-2} \left(1 + \frac{z}{a_n^{\mu_{n,l}}}\right)^2 \prod_{k=n}^{\infty} \left(1 + \frac{z}{a_k^{\beta_k}}\right)^{2\alpha_k} \prod_{k=n+1}^{\infty} \prod_{l=0}^{k-2} \left(1 + \frac{z}{a_k^{\mu_{k,l}}}\right)^2.$$

To consider the terms  $I_1$  and  $I_2$  we make use of the facts that, as  $z \rightarrow 0$ , we have

$$\log(1 + z) = z + O(|z|^2), \quad (4.6)$$

and

$$\log(1 + z) = O(z). \quad (4.7)$$

Here we take the branch of the logarithm with argument close to zero.

First we consider  $I_1$ . It follows from (4.5) and (4.6) that

$$\log I_1 = \eta + O(\iota),$$

where

$$\eta = -w \left( 1 + 2 \frac{b_n}{b_n - a_n} + 2 \sum_{k=3}^{n-1} \left\{ \alpha_k \frac{b_n}{b_n - a_k^{\beta_k}} + \sum_{l=0}^{k-2} \frac{b_n}{b_n - a_k^{\mu_{k,l}}} \right\} \right),$$

and

$$\iota = |w|^2 \left( 1 + 2 \left( \frac{b_n}{b_n - a_n} \right)^2 + 2 \sum_{k=3}^{n-1} \left\{ \alpha_k \left( \frac{b_n}{b_n - a_k^{\beta_k}} \right)^2 + \sum_{l=0}^{k-2} \left( \frac{b_n}{b_n - a_k^{\mu_{k,l}}} \right)^2 \right\} \right).$$

Now, for large  $n$ ,

$$\begin{aligned} |\eta| &= |w| \left| 1 + 2 \frac{b_n}{b_n - a_n} + 2 \sum_{k=3}^{n-1} \left\{ \alpha_k \frac{b_n}{b_n - a_k^{\beta_k}} + \sum_{l=0}^{k-2} \frac{b_n}{b_n - a_k^{\mu_{k,l}}} \right\} \right| \\ &= |w| \left| 1 - T_n + 2 \sum_{k=3}^{n-1} \left\{ \alpha_k \left( 1 + \frac{a_k^{\beta_k}}{b_n - a_k^{\beta_k}} \right) + \sum_{l=0}^{k-2} \left( 1 + \frac{a_k^{\mu_{k,l}}}{b_n - a_k^{\mu_{k,l}}} \right) \right\} \right| \quad \text{by (4.34)} \\ &= 2|w| \left( \sum_{k=3}^{n-1} \left\{ \alpha_k \frac{a_k^{\beta_k}}{b_n - a_k^{\beta_k}} + \sum_{l=0}^{k-2} \frac{a_k^{\mu_{k,l}}}{b_n - a_k^{\mu_{k,l}}} \right\} \right) \quad \text{by (4.29)} \\ &\leq 2n^2 |w| \left( \frac{\alpha_{n-1} a_{n-1}^{\beta_{n-1}}}{b_n - a_{n-1}^{\beta_{n-1}}} + \frac{a_{n-1}^{\mu_{n-1, n-3}}}{b_n - a_{n-1}^{\mu_{n-1, n-3}}} \right) \\ &\leq 4n^2 |w| \left( \frac{\alpha_{n-1} a_{n-1}^{\beta_{n-1}}}{a_n - a_{n-1}^{\beta_{n-1}}} + \frac{a_{n-1}^{\mu_{n-1, n-3}}}{a_n - a_{n-1}^{\mu_{n-1, n-3}}} \right) \quad \text{by (4.34)} \\ &\leq 8n^2 |w| \left( \exp(-e^{(n-1)/2}) + \exp(-e^{(n-1)/2}) \right) \quad \text{by (4.39)} \\ &\leq |w| \exp(-e^{n/4}). \end{aligned}$$

Note that the cancellation in the second line occurs because, due to the choice of  $b_n$  and  $T_n$ ,  $-b_n$  is very close to a critical point of  $f$ .

We next consider  $\iota$ . We observe that  $b_n/(a_n - b_n) = T_n/2 < n^3$ . Hence, for

large  $n$ ,

$$\begin{aligned}
\iota &\leq |w|^2 \left( 1 + 2n^6 + 6n^4 \left( \frac{b_n}{b_n - a_{n-1}^{\mu_{n-1, n-3}}} \right)^2 \right) && \text{by (4.25)} \\
&\leq |w|^2 \left( 1 + 2n^6 + 12n^4 \left( \frac{1}{1 - a_{n-1}^{\mu_{n-1, n-3}}/a_n} \right)^2 \right) && \text{by (4.34)} \\
&\leq |w|^2 \left( 1 + 2n^6 + 12n^4 \left( 1 + 4 \frac{a_{n-1}^{\mu_{n-1, n-3}}}{a_n} \right) \right) \\
&\leq |w|^2 (1 + 2n^6 + 24n^4) && \text{by (4.39)} \\
&\leq |w|^2 n^7 \\
&< n^{-11}.
\end{aligned}$$

It follows that

$$|I_1 - 1| < n^{-10}, \quad \text{for large } n. \quad (4.8)$$

Now we consider  $I_2$ . For large  $n$  we have

$$\begin{aligned}
\log I_2 &= 2 \left( \sum_{l=1}^{n-2} \log \left( 1 + \frac{z}{a_n^{\mu_{n,l}}} \right) + \sum_{k=n}^{\infty} \alpha_k \log \left( 1 + \frac{z}{a_k^{\beta_k}} \right) + \sum_{k=n+1}^{\infty} \sum_{l=0}^{k-2} \log \left( 1 + \frac{z}{a_k^{\mu_{k,l}}} \right) \right) \\
&= O \left( \sum_{l=1}^{n-2} \frac{|z|}{a_n^{\mu_{n,l}}} + \sum_{k=n}^{\infty} \alpha_k \frac{|z|}{a_k^{\beta_k}} + \sum_{k=n+1}^{\infty} \sum_{l=0}^{k-2} \frac{|z|}{a_k^{\mu_{k,l}}} \right) && \text{by (4.7)} \\
&= O \left( \sum_{l=1}^{n-2} \frac{a_n}{a_n^{\mu_{n,l}}} + \sum_{k=n}^{\infty} \frac{\alpha_k a_n}{a_k^{\beta_k}} + \sum_{k=n+1}^{\infty} \sum_{l=0}^{k-2} \frac{a_n}{a_k^{\mu_{k,l}}} \right) && \text{by (4.34)} \\
&= O \left( n \exp(-e^{n/2}) + \sum_{k=n+1}^{\infty} \frac{\alpha_k a_n}{a_k^{\beta_k}} + \sum_{k=n+1}^{\infty} \frac{k a_n}{a_k} \right) && \text{by (4.38)} \\
&= O \left( n \exp(-e^{n/2}) + \sum_{k=1}^{\infty} \left( \frac{\alpha_{n+k} a_n}{a_n^{n^{3k}/2^k}} + \frac{(n+k) a_n}{a_n^{n^{3k}/2^k}} \right) \right) && \text{by (4.36)} \\
&= O \left( n \exp(-e^{n/2}) + \sum_{k=1}^{\infty} \left( a_n^{-n^{3k}/2^{k+1}} (\alpha_{n+k} + n + k) \right) \right) \\
&= O \left( n \exp(-e^{n/2}) \right) && \text{by (4.37)}.
\end{aligned}$$

Thus, for sufficiently large  $n$ ,

$$|I_2 - 1| \leq \exp(-e^{n/4}).$$

This, together with (4.8), establishes the first part of the lemma. It follows from Montel's theorem that, for large  $n$ ,  $B_n$  is contained in a Fatou component, which must be simply connected by Lemma 4.3.1.  $\square$

**Remark 4.4.1.** Let  $V_n$  be the Fatou component containing  $B_n$ . These Fatou components are distinct. For, suppose that  $V_m = V_n$  with  $m \neq n$ . Because all the coefficients of  $z$  in (4.2) are real, the Fatou set  $F(f)$  must be invariant under reflection in the real axis. Hence, all points on the negative real axis between  $B_n$  and  $B_m$  must be in  $V_m$ , as otherwise  $V_m$  would be multiply connected. This is a contradiction since these points include the zeros of  $f$ .

## 4.5 The simply connected Fatou components are fast escaping

In this section we first prove Theorem 4.1.2, and then we use this result to prove the following.

**Lemma 4.5.1.** *Let  $V_n$ ,  $n \in \mathbb{N}$ , be the simply connected Fatou components defined at the end of Section 4.4. Then  $V_n \subset A(f)$ , for large  $n$ .*

*Proof of Theorem 4.1.2.* Fix  $r_0 \geq R_0$  such that  $M(r) > r$ , for  $r \geq r_0$ . Whenever  $r \geq r_0$  there is a unique  $n \in \mathbb{N}$  such that  $M^{n-1}(r_0) \leq r < M^n(r_0)$ . Hence, since  $\epsilon$  is nonincreasing, by (4.1) and (1.6)

$$\epsilon(r)r \geq \epsilon(M^n(r_0))M^{n-1}(r_0) \geq \epsilon(r_0)^{n+1}M^{n-1}(r_0) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Hence

$$\epsilon(r)r \rightarrow \infty \text{ as } r \rightarrow \infty. \quad (4.9)$$

By (1.6) and (4.9) we see that, given  $k > 0$ , we can ensure that

$$\frac{\log M(\epsilon(r)R)}{\log(\epsilon(r)R)} > k, \quad \text{for large } r, R \geq r. \quad (4.10)$$



A little algebra shows that this is equivalent to

$$M(\epsilon(r)R)^{\frac{-\log \epsilon(r)}{\log(\epsilon(r)R)}} > \epsilon(r)^{-k}, \quad \text{for large } r, R \geq r. \quad (4.11)$$

In (1.8) we replace  $c$  with  $\log R / \log(\epsilon(r)R)$ , and replace  $r$  with  $\epsilon(r)R$ . We obtain, using (4.11) with  $k = 3$ , that there exists  $R_1 \geq R_0$  such that

$$M(R) \geq \epsilon(r)^{-3}M(\epsilon(r)R), \quad \text{for } R \geq r \geq R_1. \quad (4.12)$$

We can assume that  $R_1$  is sufficiently large that  $M(r) > r$ , for  $r \geq R_1/\epsilon(r)$ , and also, by (4.9), that  $\eta(r) > r$ , for  $r \geq R_1$ .

We claim next that we have

$$\eta^k(r) \geq \epsilon(r)^{-k-1}M^k(\epsilon(r)r) > M^k(\epsilon(r)r), \quad \text{for } r \geq R_1, k \in \mathbb{N}. \quad (4.13)$$

This can be seen by induction. When  $k = 1$  we have, by (4.12),

$$\eta(r) = \epsilon(r)M(r) \geq \epsilon(r)^{-2}M(\epsilon(r)r), \quad \text{for } r \geq R_1.$$

Hence, by induction, for  $r \geq R_1$ ,

$$\begin{aligned} \eta^{k+1}(r) &= \epsilon(\eta^k(r))M(\eta^k(r)) \\ &\geq \epsilon(M^k(r))M(\eta^k(r)) && \text{as } \epsilon \text{ is nonincreasing} \\ &\geq \epsilon(r)^{k+1}M(\eta^k(r)) && \text{by (4.1)} \\ &\geq \epsilon(r)^{k+1}M(\epsilon(r)^{-k-1}M^k(\epsilon(r)r)) && \text{by (4.13)} \\ &\geq \epsilon(r)^{k+1}\epsilon(r)^{-3k-3}M^{k+1}(\epsilon(r)r) && \text{by repeated use of (4.12)} \\ &\geq \epsilon(r)^{-(k+1)-1}M^{k+1}(\epsilon(r)r) && \text{as required.} \end{aligned}$$

(Note that in the penultimate step above we have also made use of the fact that  $\epsilon(r)r \leq M(\epsilon(r)r)$ , for  $r \geq R_1$ .)

It follows from (4.13) that, for  $r \geq R_1$ ,  $\eta^n(r) \rightarrow \infty$  as  $n \rightarrow \infty$ . Define

$$A'(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \eta^n(R') \text{ for } n \in \mathbb{N}\},$$

for  $R' \geq R_1$ . We complete the proof by showing that  $A'(f) = A(f)$ .

First, suppose that  $z \in A(f)$ , in which case for some  $\ell \in \mathbb{N}$  we have

$$|f^{n+\ell}(z)| \geq M^n(R), \quad \text{for } n \in \mathbb{N},$$

and some  $R$  with  $M(r) > r$ , for  $r \geq R$ . Choose  $K \in \mathbb{N}$  such that  $M^K(R) = R' \geq R_1$ .

Then

$$|f^{n+\ell+K}(z)| \geq M^{n+K}(R) = M^n(R') \geq \eta^n(R'), \quad \text{for } n \in \mathbb{N}.$$

Hence  $z \in A'(f)$ .

Conversely, suppose that  $z \in A'(f)$ , in which case for some  $\ell \in \mathbb{N}$  and  $R' \geq R_1$  we have

$$|f^{n+\ell}(z)| \geq \eta^n(R'), \quad \text{for } n \in \mathbb{N}.$$

Choose  $K \in \mathbb{N}$  so that  $M^K(\epsilon(R')R') = R \geq R_1$ . Then, by (4.13).

$$|f^{n+\ell+K}(z)| \geq \eta^{n+K}(R') \geq M^{n+K}(\epsilon(R')R') \geq M^n(R), \quad \text{for } n \in \mathbb{N}.$$

Hence  $z \in A(f)$ . □

Finally, we give the

*Proof of Lemma 4.5.1.* For some large  $R_0$  define, for  $r > R_0$ ,

$$\epsilon(r) = \frac{1}{16n^6}, \quad \text{for } a_{n-1}(1 + \delta_{n-1}) < r \leq a_n(1 + \delta_n), \quad (4.14)$$

where  $\delta_n = n^{-9}$  as in Lemma 4.4.1. Define also  $\eta(r) = \epsilon(r)M(r)$ , for  $r \geq R_0$ .

Suppose that  $x' \in B_n \cap \mathbb{R} \subset V_n$ , for some  $n$ , where  $B_n$  is as defined in the statement of Lemma 4.4.1. We can assume that  $n$  is chosen sufficiently large for the various estimates in this section to hold. We claim that  $x' \in A(f)$ , and so, by Theorem 1.7.3,  $V_n \subset A(f)$ .

Our approach to proving this claim is as follows. Set  $x = -x'$ , recalling that

$x > 0$ . We first show that

$$|f(x')| \geq \frac{1}{144n^6} M(x). \quad (4.15)$$

It follows from this, and since  $f(x') \in B_{n+1} \cap \mathbb{R}$ , that

$$|f^m(x')| \geq \eta^m(x), \quad \text{for } m \in \mathbb{N}. \quad (4.16)$$

Second, we show that  $\epsilon$  satisfies (4.1). Thus, by (4.16) and Theorem 4.1.2,  $x' \in A(f)$ , as required.

First we need to establish (4.15). We consider the quotient of  $|f(x')|$  and  $M(x) = f(x)$ , gathering together the terms in the products for these quantities prior to the  $n$ th term, the  $n$ th term, and the remaining terms. We obtain

$$\frac{|f(x')|}{M(x)} = J_1 J_2 J_3, \quad (4.17)$$

where

$$J_1 = \prod_{k=3}^{n-1} \left\{ \left( \frac{a_k^{\beta_k} - x}{a_k^{\beta_k} + x} \right)^{2\alpha_k} \prod_{l=0}^{k-2} \left( \frac{a_k^{\mu_{k,l}} - x}{a_k^{\mu_{k,l}} + x} \right)^2 \right\},$$

$$J_2 = \left( \frac{a_n - x}{a_n + x} \right)^2,$$

and

$$J_3 = \prod_{k=n+1}^{\infty} \left\{ \left( \frac{a_k^{\beta_k} - x}{a_k^{\beta_k} + x} \right)^{2\alpha_k} \prod_{l=0}^{k-2} \left( \frac{a_k^{\mu_{k,l}} - x}{a_k^{\mu_{k,l}} + x} \right)^2 \right\} \left( \frac{a_n^{\beta_n} - x}{a_n^{\beta_n} + x} \right)^{2\alpha_n} \prod_{l=1}^{n-2} \left( \frac{a_n^{\mu_{n,l}} - x}{a_n^{\mu_{n,l}} + x} \right)^2.$$

We consider these three terms separately. We note that

$$\log \frac{1-x}{1+x} > -4x, \quad \text{for } 0 < x < \frac{1}{2}. \quad (4.18)$$

Hence, for large  $n$ ,

$$\begin{aligned}
\log J_1 &= 2 \left( \sum_{k=3}^{n-1} \alpha_k \log \left( \frac{1 - a_k^{\beta_k}/x}{1 + a_k^{\beta_k}/x} \right) + \sum_{k=3}^{n-1} \sum_{l=0}^{k-2} \log \left( \frac{1 - a_k^{\mu_{k,l}}/x}{1 + a_k^{\mu_{k,l}}/x} \right) \right) \\
&> -8 \left( \sum_{k=3}^{n-1} \frac{\alpha_k a_k^{\beta_k}}{x} + \sum_{k=3}^{n-1} \sum_{l=0}^{k-2} \frac{a_k^{\mu_{k,l}}}{x} \right) && \text{by (4.18)} \\
&> -16 \left( \sum_{k=3}^{n-1} \frac{\alpha_k a_k^{\beta_k}}{a_n} + \sum_{k=3}^{n-1} \sum_{l=0}^{k-2} \frac{a_k^{\mu_{k,l}}}{a_n} \right) && \text{by (4.34)} \\
&> -16n^2 \left( \frac{\alpha_{n-1} a_{n-1}^{\beta_{n-1}}}{a_n} + \frac{a_{n-1}^{\mu_{n-1, n-3}}}{a_n} \right) \\
&\geq -16n^2 \exp(-e^{(n-1)/2}) && \text{by (4.39)}.
\end{aligned}$$

It follows that by choosing  $n$  large we may assume that  $J_1 \geq \frac{1}{2}$ .

Secondly, recalling that  $x = b_n + \omega b_n$ , with  $|\omega| < \delta_n = n^{-9}$ , we have for large  $n$ , by (4.26),

$$J_2 = \left( \frac{a_n - b_n - \omega b_n}{a_n + b_n + \omega b_n} \right)^2 = \left( \frac{2 - \omega T_n}{2T_n + 2 + \omega T_n} \right)^2 \geq \left( \frac{1}{3T_n} \right)^2 \geq \frac{1}{36n^6}.$$

Thirdly we have, for large  $n$ ,

$$\begin{aligned}
\frac{1}{2} \log J_3 &= \sum_{k=n+1}^{\infty} \left( \alpha_k \log \left( \frac{1-x/a_k^{\beta_k}}{1+x/a_k^{\beta_k}} \right) + \sum_{l=0}^{k-2} \log \left( \frac{1-x/a_k^{\mu_{k,l}}}{1+x/a_k^{\mu_{k,l}}} \right) \right) \\
&+ \alpha_n \log \left( \frac{1-x/a_n^{\beta_n}}{1+x/a_n^{\beta_n}} \right) + \sum_{l=1}^{n-2} \log \left( \frac{1-x/a_n^{\mu_{n,l}}}{1+x/a_n^{\mu_{n,l}}} \right) \\
&> -4 \left( \sum_{k=n+1}^{\infty} \left( \frac{\alpha_k x}{a_k^{\beta_k}} + \frac{kx}{a_k} \right) + \frac{\alpha_n x}{a_n^{\beta_n}} + n \frac{x}{a_n^{\mu_n}} \right) && \text{by (4.18)} \\
&> -8 \left( \sum_{k=1}^{\infty} \left( \frac{\alpha_{n+k} a_n}{a_{n+k}^{\beta_{n+k}}} + \frac{(n+k)a_n}{a_{n+k}} \right) + \frac{\alpha_n a_n}{a_n^{\beta_n}} + n \frac{a_n}{a_n^{\mu_n}} \right) && \text{by (4.34)} \\
&> -8 \left( \sum_{k=1}^{\infty} \frac{a_n}{a_{n+k-1}} \left( \frac{\alpha_{n+k} a_{n+k-1}}{a_{n+k}^{\beta_{n+k}}} + \frac{(n+k)a_{n+k-1}}{a_{n+k}} \right) + 2n \exp(-e^{n/2}) \right) && \text{by (4.38)} \\
&> -8 \left( \sum_{k=1}^{\infty} \frac{a_n}{a_{n+k-1}} (n+k+1) \exp(-e^{(n+k)/2}) + 2n \exp(-e^{n/2}) \right) && \text{by (4.39)} \\
&> -8 \exp(-e^{n/2}) \left( \sum_{k=1}^{\infty} (n+k+1) \frac{a_n}{a_{n+k-1}} + 2n \right) \\
&> -8 \exp(-e^{n/2}) \left( \sum_{k=2}^{\infty} (n+k+1) a_{n+k-1}^{\frac{2(k-1)}{3(k-1)}-1} + 4n \right) && \text{by (4.36)} \\
&> -8 \exp(-e^{n/2}) \left( \sum_{k=2}^{\infty} (n+k+1) a_{n+k-1}^{-\frac{1}{2}} + 4n \right) \\
&> -8 \exp(-e^{n/2}) \left( \sum_{k=2}^{\infty} (n+k+1) \exp(-e^{n+k-1}/2) + 4n \right) && \text{by (4.37)} \\
&> -\exp(-e^{n/4}).
\end{aligned}$$

Thus, for large  $n$ , we have  $J_3 \geq \frac{1}{2}$ . This establishes our first claim.

To complete the proof of the lemma, we need to show that  $\epsilon$  satisfies (4.1). We claim that, for large  $n$ ,

$$M(a_n(1 + \delta_n)) \leq a_{n+2}.$$

To do this, we consider the quotient of  $f(2b_n)$  and  $a_{n+1}$ , gathering together the terms in the products for these quantities prior to the  $n$ th term, the  $n$ th term, and the remaining terms (which occur only in the product for  $f(2b_n)$ ). We obtain

$$\frac{f(2b_n)}{a_{n+1}} = K_1 K_2 K_3, \quad (4.19)$$

where

$$K_1 = \prod_{k=3}^{n-1} \left\{ \left( \frac{2b_n + a_k^{\beta_k}}{b_n - a_k^{\beta_k}} \right)^{2\alpha_k} \prod_{l=0}^{k-2} \left( \frac{2b_n + a_k^{\mu_{k,l}}}{b_n - a_k^{\mu_{k,l}}} \right)^2 \right\},$$

$$K_2 = \frac{2T_{n+1}}{T_{n+1} + 2} \left( \frac{a_n}{a_n - b_n} \right)^2 \left( 1 + \frac{2b_n}{a_n} \right)^2,$$

$$K_3 = \prod_{k=n+1}^{\infty} \left\{ \left( 1 + \frac{2b_n}{a_k^{\beta_k}} \right)^{2\alpha_k} \prod_{l=0}^{k-2} \left( 1 + \frac{2b_n}{a_k^{\mu_{k,l}}} \right)^2 \right\} \left( 1 + \frac{b_n}{a_n^{\beta_n}} \right)^{2\alpha_n} \prod_{l=1}^{n-2} \left( 1 + \frac{2b_n}{a_n^{\mu_{n,l}}} \right)^2.$$

It follows by (4.34) and (4.39) that each term being squared in the product in  $K_1$  is less than 4. Hence, by (4.25) we have that  $K_1 < 2^{12n^4}$ .

It follows by (4.26) and (4.34) that  $K_2 < 50n^6$ .

A calculation almost identical to that for  $I_2$  shows that  $K_3 < 2$ ; we omit the details. It follows that  $f(2b_n)/a_{n+1} < 2^{24n^4}$ . Hence, by (4.34), (4.36) and (4.37),

$$M(a_n(1 + \delta_n)) \leq M(2b_n) = f(2b_n) < 2^{24n^4} a_{n+1} \leq a_{n+1}^2 \leq a_{n+2}, \quad (4.20)$$

as required.

Suppose then that  $r$  is such that  $a_{n-1}(1 + \delta_{n-1}) < r \leq a_n(1 + \delta_n)$ . Since  $\epsilon$  is nonincreasing, we deduce that, for  $k \in \mathbb{N}$ ,

$$\begin{aligned} \epsilon(M^k(r)) &\geq \epsilon(M^k(a_n(1 + \delta_n))), \\ &\geq \epsilon(a_{n+2k}), && \text{by (4.20)} \\ &= \frac{1}{144(n+2k)^6} \geq \frac{1}{(144n^6)^{k+1}} = \epsilon(r)^{k+1}. \end{aligned}$$

Thus  $\epsilon$  satisfies (4.1). This completes the proof of Lemma 4.5.1 and hence the proof of Theorem 4.1.1.  $\square$

## 4.6 Appendix: defining the sequences

In this section we first define the sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$ , and then define the sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$ .

Recall from Section 4.2 that  $\mu_n = n^{3/n}$  and  $\mu_{n,m} = \mu_n^m$ ; we also define, for  $n \geq 3$ ,

$$\sigma_n = \sum_{l=1}^{n-2} \mu_{n,l} = \mu_n \frac{\mu_{n,n-2} - 1}{\mu_n - 1}. \quad (4.21)$$

We define  $(\alpha_n)_{n \in \mathbb{N}}$  to be a sequence of integers and  $(\beta_n)_{n \in \mathbb{N}}$  to be a sequence of real numbers. Assume that  $N_0$  is even and chosen sufficiently large for subsequent estimates to hold. Set

$$2\alpha_n = \begin{cases} 0, & \text{for } n < N_0, \\ N_0^3 + 2N_0^2 + 6N_0 + 2, & \text{for } n = N_0, \\ 3n^2 + n + 6, & \text{for } n > N_0, \text{ } n \text{ even,} \\ 3n^2 + n + 4, & \text{for } n > N_0, \text{ } n \text{ odd.} \end{cases} \quad (4.22)$$

Note that  $\alpha_n$  is an integer, for  $n \in \mathbb{N}$ . Set

$$\beta_n = \begin{cases} 0, & \text{for } n < N_0, \\ \frac{1}{\alpha_n}(n^4 - \sigma_n), & \text{for } n \geq N_0, \text{ } n \text{ even,} \\ \frac{1}{\alpha_n}\left(\frac{n^3(2n-1)}{2} - \sigma_n\right), & \text{for } n \geq N_0, \text{ } n \text{ odd.} \end{cases}$$

We observe that these choices imply that

$$\tau_n = \frac{2}{n^3}(\alpha_n \beta_n + \sigma_n) \quad (4.23)$$

satisfies

$$\tau_n = \begin{cases} 2n, & \text{for } n \geq N_0, \text{ } n \text{ even,} \\ 2n - 1, & \text{for } n \geq N_0, \text{ } n \text{ odd,} \end{cases} \quad (4.24)$$

and

$$2\alpha_n = 3n^2 + n + 3 + \tau_n - \tau_{n-1}, \quad \text{for } n > N_0. \quad (4.25)$$

We also define a sequence of integers  $(T_n)_{n \in \mathbb{N}}$  by

$$T_n = \begin{cases} n^3 + 2n - 3, & \text{for } n \text{ even,} \\ n^3 + 2n - 2, & \text{for } n \text{ odd.} \end{cases} \quad (4.26)$$

Next we prove a result which gives various relationships between these sequences.

**Lemma 4.6.1.** *The following all hold for the choice of sequences above:*

$$\alpha_n \sim \frac{3}{2}n^2, \quad \beta_n \sim \frac{2}{3}n^2, \quad \text{as } n \rightarrow \infty, \quad (4.27)$$

$$2 \sum_{k=3}^n \alpha_k = n^3 + 2n^2 + 4n + 2 + \tau_n, \quad \text{for } n \geq N_0, \quad (4.28)$$

$$1 + 2 \sum_{k=3}^{n-1} \alpha_k + \sum_{k=3}^{n-1} \sum_{l=0}^{k-2} 2 = n^3 + \tau_{n-1} = T_n, \quad \text{for } n > N_0, \quad (4.29)$$

and

$$\mu_{n,2} < \beta_n < \mu_{n,n-3}, \quad \text{for large } n. \quad (4.30)$$

*Proof.* The first half of (4.27) is immediate from (4.22). Now, by (4.22), (4.23) and (4.24),

$$\beta_n \sim \frac{2n}{3} \left( n - \frac{\sigma_n}{n^3} \right), \quad \text{as } n \rightarrow \infty. \quad (4.31)$$

We have that

$$\frac{x}{2} \leq \log(1+x) \leq x, \quad \text{for } 0 < x < \frac{1}{2}. \quad (4.32)$$

Putting  $x = \mu_n - 1$ , we obtain

$$\frac{3}{n} \log n \leq \mu_n - 1 \leq \frac{6}{n} \log n, \quad \text{for large } n. \quad (4.33)$$

Hence, by (4.21),

$$\frac{\sigma_n}{n^3} = \frac{\mu_n}{\mu_{n,n}} \frac{\mu_{n,n-2} - 1}{\mu_n - 1} \sim \frac{1}{\mu_n(\mu_n - 1)} = O\left(\frac{n}{\log n}\right) \quad \text{as } n \rightarrow \infty,$$



and the second half of (4.27) follows by (4.31).

We can see that (4.28) holds by induction. For, it is immediately satisfied when  $n = N_0$ . When  $n = m > N_0$  we have, by (4.25) and (4.28) with  $n = m - 1$ ,

$$2 \sum_{k=3}^m \alpha_k = 2\alpha_m + 2 \sum_{k=3}^{m-1} \alpha_k = m^3 + 2m^2 + 4m + 2 + \tau_m.$$

Finally (4.29) follows from (4.24), (4.26), and (4.28), and (4.30) follows from (4.27).  $\square$

Next we define the sequence  $(a_n)_{n \in \mathbb{N}}$  recursively, and for each  $n \in \mathbb{N}$  put

$$b_n = a_n - \frac{2}{T_n + 2} a_n = \frac{T_n}{T_n + 2} a_n. \quad (4.34)$$

Choose  $a_3$  and  $N_1$  large, and set  $a_{n+1} = a_n^{n^3}$ , for  $3 \leq n < N_1$ . We assume that  $a_3$  and  $N_1$  are chosen sufficiently large for various estimates in the sequel to hold. For  $n \geq N_1$ , we define

$$a_{n+1} = \frac{(T_{n+1} + 2)}{T_{n+1}} b_n \left(1 - \frac{b_n}{a_n}\right)^{2n-1} \prod_{k=3}^{n-1} \left\{ \left(1 - \frac{b_n}{a_k^{\beta_k}}\right)^{2\alpha_k} \prod_{l=0}^{k-2} \left(1 - \frac{b_n}{a_k^{\mu_{k,l}}}\right)^2 \right\}. \quad (4.35)$$

Finally in this section we prove a set of inequalities which concern the growth of the sequence  $(a_n)$ , and the ratios of these numbers to the modulus of the other zeros of  $f$ . Note that (4.27) and (4.37) imply that the product in (4.2) is locally uniformly convergent in  $\mathbb{C}$ ; see, for example, [2, Theorem 6 p.192].

**Lemma 4.6.2.** *The following inequalities hold for the sequence  $(a_n)$  defined above. For  $n \geq 3$ ,*

$$a_n^{n^3-2/n} \leq a_{n+1} \leq a_n^{n^3+2/n}, \quad (4.36)$$

$$a_n > \exp(e^n), \quad (4.37)$$

and, for large  $n$ ,

$$\frac{a_n}{a_n^{\mu_n}} \leq \exp(-e^{n/2}), \quad \frac{\alpha_n a_n}{a_n^{\beta_n}} \leq \exp(-e^{n/2}), \quad (4.38)$$

$$\frac{a_n^{\mu_{n,n-2}}}{a_{n+1}} \leq \exp(-e^{n/2}), \quad \frac{\alpha_n a_n^{\beta_n}}{a_{n+1}} \leq \exp(-e^{n/2}). \quad (4.39)$$

*Proof.* First, assume that (4.36) holds for  $3 \leq n \leq m$ . Equation (4.37) follows for  $3 \leq n \leq m$  by a simple induction. Hence, for sufficiently large  $m$ , by (4.27), (4.33), (4.36) and (4.37):

$$\begin{aligned} \frac{a_m}{a_m^{\mu_m}} &= a_m^{1-\mu_m} \leq \exp(e^m(1-\mu_m)) \leq \exp(-e^{m/2}); \\ \frac{\alpha_m a_m}{a_m^{\beta_m}} &\leq 3m^2 a_m^{1-m^2/2} \leq \exp(-e^{m/2}); \\ \frac{a_m^{\mu_{m,m-2}}}{a_{m+1}} &\leq (a_m^{\mu_{m,m-2}})^{1-\mu_{m,2}+\frac{2}{m}} \leq \exp(e^m(1-\mu_{m,2}+\frac{2}{m})) \leq \exp(-e^{m/2}); \\ \frac{\alpha_m a_m^{\beta_m}}{a_{m+1}} &\leq 3m^2 a_m^{m^2-m^3+2/m} \leq \exp(-e^{m/2}). \end{aligned}$$

It remains to prove (4.36). We can assume, by taking  $N_1$  sufficiently large, that (4.36) holds for  $3 \leq n \leq m-1$ , for some large  $m$ . We can assume also that  $m$  is sufficiently large that (4.29), (4.30) and various other estimates used in the following hold. We need to prove that (4.36) holds for  $n = m$ . Now, by (4.35),

$$\begin{aligned} a_{m+1} &= \frac{(T_{m+1}+2)}{T_{m+1}} b_m^{\kappa_m} \left(1 - \frac{b_m}{a_m}\right)^2 \frac{L_1}{\prod_{k=3}^{m-1} \left\{ a_k^{2\alpha_k \beta_k} \prod_{l=0}^{k-2} a_k^{2\mu_{k,l}} \right\}} \\ &= k_m \frac{a_m^{\kappa_m}}{a_{m-1}^{2\alpha_{m-1} \beta_{m-1}} \prod_{l=1}^{m-3} a_{m-1}^{2\mu_{m-1,l}}} \frac{L_1}{L_2} \\ &= k_m \frac{a_m^{\kappa_m}}{a_{m-1}^{(m-1)^3 \tau_{m-1}}} \frac{L_1}{L_2}, \end{aligned} \quad (4.40)$$

where, by (4.34),

$$k_m = \frac{(T_{m+1}+2)}{T_{m+1}} \left( \frac{T_m}{T_m+2} \right)^{\kappa_m} \left( \frac{2}{T_m+2} \right)^2,$$

$$L_1 = \prod_{k=3}^{m-1} \left\{ \left( 1 - \frac{a_k^{\beta_k}}{b_m} \right)^{2\alpha_k} \prod_{l=0}^{k-2} \left( 1 - \frac{a_k^{\mu_{k,l}}}{b_m} \right)^2 \right\},$$

by (4.23) again,

$$\begin{aligned} L_2 &= a_{m-1}^2 \prod_{k=3}^{m-2} \left\{ a_k^{2\alpha_k \beta_k} \prod_{l=0}^{k-2} a_k^{2\mu_{k,l}} \right\} \\ &= a_{m-1}^2 \prod_{k=3}^{m-2} a_k^{2(1+\alpha_k \beta_k + \sum_{l=1}^{k-2} \mu_{k,l})} && \text{by gathering powers} \\ &= a_{m-1}^2 \prod_{k=3}^{m-2} a_k^{2+k^3 \tau_k} && \text{by (4.21) and (4.23),} \end{aligned}$$

and, by (4.29),

$$\kappa_m = 1 + 2 \sum_{k=3}^{m-1} \alpha_k + \sum_{k=3}^{m-1} \sum_{l=0}^{k-2} 2 = m^3 + \tau_{m-1} = T_m.$$

Note that the calculation of  $L_1$  in the first step follows by writing terms of (4.35) such as

$$\left( 1 - \frac{b_m}{a_k} \right)^2$$

in the form

$$\left( \frac{b_m}{a_k} \right)^2 \left( 1 - \frac{a_k}{b_m} \right)^2.$$

We now estimate the terms in this equality. Firstly, by (4.26), and noting that  $(T_m/(T_m + 2))^{T_m} > 1/e^2$ , we obtain

$$\frac{1}{8} m^{-6} < k_m < 8m^{-6}.$$

Secondly, by (4.36), with  $n = m - 1$ ,

$$a_{m-1}^{(m-1)^3} \leq a_m^{(1 - \frac{2}{(m-1)^4})^{-1}} \leq a_m^{1 + \frac{4}{(m-1)^4}},$$

and so, by (4.24),

$$\frac{a_m^{\kappa_m}}{a_{m-1}^{(m-1)^3\tau_{m-1}}} \geq a_m^{\kappa_m - \tau_{m-1} - \frac{4\tau_{m-1}}{(m-1)^4}} = a_m^{m^3 - \frac{4\tau_{m-1}}{(m-1)^4}} \geq a_m^{m^3 - \frac{1}{m}}.$$

Similarly, by (4.24) and (4.36),

$$\frac{a_m^{\kappa_m}}{a_{m-1}^{(m-1)^3\tau_{m-1}}} \leq a_m^{m^3 + \frac{1}{m}}.$$

Thirdly, we consider  $L_1$ . For large  $m$  we have

$$\begin{aligned} \log L_1 &= 2 \sum_{k=3}^{m-1} \alpha_k \log \left( 1 - \frac{a_k^{\beta_k}}{b_m} \right) + 2 \sum_{k=3}^{m-1} \sum_{l=0}^{k-2} \log \left( 1 - \frac{a_k^{\mu_{k,l}}}{b_m} \right) \\ &\geq 2m\alpha_{m-1} \log \left( 1 - \frac{a_{m-1}^{\beta_{m-1}}}{b_m} \right) + m^2 \log \left( 1 - \frac{a_{m-1}^{\mu_{m-1,m-3}}}{b_m} \right) \\ &> -4m \frac{\alpha_{m-1} a_{m-1}^{\beta_{m-1}}}{b_m} - 2m^2 \frac{a_{m-1}^{\mu_{m-1,m-3}}}{b_m} && \text{by (3.7)} \\ &> -8m \frac{\alpha_{m-1} a_{m-1}^{\beta_{m-1}}}{a_m} - 4m^2 \frac{a_{m-1}^{\mu_{m-1,m-3}}}{a_m} && \text{by (4.34)} \\ &\geq -8m^2 \exp \left( -e^{\frac{m-1}{2}} \right) && \text{by (4.39)}. \end{aligned}$$

It follows that, for large  $m$ , we have  $\frac{1}{2} < L_1 < 1$ .

Finally, we consider  $L_2$ . For large  $m$  we have

$$\begin{aligned} L_2 &= a_{m-1}^2 a_{m-2}^{2+(m-2)^3\tau_{m-2}} \prod_{k=3}^{m-3} a_k^{2+k^3\tau_k} \\ &< a_{m-1}^2 a_{m-2}^{2+(m-2)^3\tau_{m-2}} a_{m-3}^{(2+(m-3)^3\tau_{m-3})(m-5)} \\ &< a_{m-1}^2 a_{m-2}^{2m^4} a_{m-3}^{2m^5} && \text{by (4.24)} \\ &< a_{m-1}^2 a_{m-1}^{8m} a_{m-1}^{32/m} && \text{by (4.36)} \\ &< a_{m-1}^{16m} \\ &< a_m^{64/m^2} && \text{by (4.36)}. \end{aligned}$$

Hence  $1 < L_2 < a_m^{64/m^2}$ .

Thus, by (4.40), for sufficiently large  $m$ ,

$$a_{m+1} \geq \frac{1}{16} m^{-6} a_m^{m^3 - \frac{1}{m} - \frac{64}{m^2}} \geq a_m^{m^3 - \frac{2}{m}},$$

and similarly,

$$a_{m+1} \leq 8m^{-6} a_m^{m^3 + \frac{1}{m}} \leq a_m^{m^3 + \frac{2}{m}}.$$

This completes the proof of Lemma 4.6.2. □

# Chapter 5

## A new characterisation of the Eremenko-Lyubich class

### 5.1 Introduction

Recall from Section 1.3 that the Eremenko-Lyubich class of transcendental entire functions consists of those with a bounded set of singular values. In this chapter we give a new characterisation of this class of functions. We also give a new result regarding direct singularities which are not logarithmic.

Note that this chapter is of a slightly different character to the rest of the thesis; although the Eremenko-Lyubich class of functions has been much studied in complex dynamics, here our domain of study is complex analysis.

As elsewhere in this thesis, we assume that  $f$  is a transcendental entire function. An important property of functions in the Eremenko-Lyubich class is that they are expanding, in the following sense. Define

$$D_R = \{z : |f(z)| > R\}, \quad \text{for } R > 0.$$

If  $f \in \mathcal{B}$ , then it follows easily from [41, Lemma 1] that there is a constant  $R_0 > 0$  such that

$$\left| z \frac{f'(z)}{f(z)} \right| \geq \frac{1}{4\pi} (\log |f(z)| - \log R_0), \quad \text{for } z \in D_{R_0}. \quad (5.1)$$

This property has many applications in complex dynamics and value distribution

theory; for example, it was used in [41] to show that functions in the Eremenko-Lyubich class cannot have escaping Fatou components.

Define also

$$\eta_f = \lim_{R \rightarrow \infty} \inf_{z \in D_R} \left| z \frac{f'(z)}{f(z)} \right|. \quad (5.2)$$

It follows from (5.1) that if  $f$  is a transcendental entire function in the Eremenko-Lyubich class, then  $\eta_f = \infty$ . The first main result of this chapter is the following, which shows that this property has a strong converse.

**Theorem 5.1.1.** *Suppose that  $f$  is a transcendental entire function. Then, either  $\eta_f = \infty$  and  $f \in \mathcal{B}$ , or  $\eta_f = 0$  and  $f \notin \mathcal{B}$ .*

It is clear that if  $f$  has an unbounded set of critical values, then  $\eta_f = 0$ . Thus the proof of Theorem 5.1.1 requires detailed analysis of the behaviour of functions with an unbounded set of asymptotic values. Since every asymptotic value of  $f$  gives rise to a transcendental singularity of  $f^{-1}$ , we need a number of results on singularities of the inverse function. In particular we require the following result on the density of transcendental singularities of a certain type, which may be of independent interest. Definitions of terms used in the statement of this theorem are given in Section 1.3 and Section 5.2.

**Theorem 5.1.2.** *Suppose that  $f$  is a transcendental entire function, with a direct non-logarithmic singularity with projection  $a \in \widehat{\mathbb{C}}$ . Then at least one of the following holds:*

- (i)  *$a$  is the limit of critical values of  $f$ ;*
- (ii) *every neighbourhood of this singularity contains a neighbourhood of another transcendental singularity of  $f^{-1}$  that is either indirect or logarithmic, and whose projection is different from  $a$ .*

We observe that Theorem 5.1.2 is complementary to the following result of Bergweiler and Eremenko [24, Theorem 5], which has almost the same hypothesis although in this result the projection of the transcendental singularity must be finite.

**Theorem 5.1.3.** *Suppose that  $f$  is a transcendental entire function, with a direct non-logarithmic singularity with projection  $a \in \mathbb{C}$ . Then every neighbourhood of this singularity is also a neighbourhood of other direct singularities of  $f^{-1}$  with projection  $a$ .*

Taken together, these results show that if  $a \in \mathbb{C}$  is the projection of a direct non-logarithmic singularity and is not the limit of critical values, then there is an infinite number of singularities both over  $a$  and over points arbitrarily close to  $a$ .

We mention two examples of transcendental entire functions with direct non-logarithmic singularities which illustrate some of the possibilities described above.

**Example 5.1.1.** Heins [53, p.435] gave the example  $f_1(z) = e^z \sin(e^z)$ , which has precisely one direct non-logarithmic singularity over  $\infty$ . Since the set of critical values of  $f_1$  is unbounded, case (i) of Theorem 5.1.2 holds for this function. This example also shows that Theorem 5.1.3 cannot be strengthened to  $a \in \widehat{\mathbb{C}}$ .

**Example 5.1.2.** Herring [56] gave the example  $f_2(z) = \int_0^z \exp(-e^t) dt$ . This function has no critical points. It follows from results in [56] that  $f_2$  has a direct non-logarithmic singularity over  $\infty$ , every neighbourhood of which contains a left half-plane. It also follows that within each set

$$A_k = \{z : \operatorname{Re}(z) > 0, |\operatorname{Im}(z) - 2k\pi| \leq \pi/2\}, \quad \text{for } k \in \mathbb{Z},$$

there is a neighbourhood of a logarithmic singularity with projection

$$\alpha_k = \alpha + 2k\pi i, \quad \text{where } \alpha \in \mathbb{C} \text{ is constant.}$$

Moreover, each neighbourhood of the direct non-logarithmic singularity over  $\infty$  contains neighbourhoods of these logarithmic singularities. Hence case (ii) of Theorem 5.1.2 holds for  $f_2$ .

The structure of this chapter is as follows. In Section 5.2 we give details of Iversen's classification of singularities. We then prove Theorem 5.1.2 in Section 5.3. Finally, in Section 5.4, we use Theorem 5.1.2 to prove Theorem 5.1.1.



## 5.2 Singularities of the inverse function

We recall Iversen's classification of singularities; see, for example, [21], [24], and [60]. Note that the definitions of this section coincide with those in Section 1.3, but we require slightly more detail to classify transcendental singularities.

Suppose that  $f$  is a transcendental entire function, and suppose that  $a \in \widehat{\mathbb{C}}$ . For each  $r > 0$ , we can choose a component  $U(r)$  of  $f^{-1}(B(a, r))$  so that  $r_1 < r_2$  implies that  $U(r_1) \subset U(r_2)$ . Then we have two possibilities:

- (a)  $\bigcap_{r>0} U(r)$  consists of a single point  $w$ , say, or
- (b)  $\bigcap_{r>0} U(r) = \emptyset$ .

In the first case, if  $f'(w) = 0$ , then  $w$  is a *critical point* of  $f$ ,  $a$  is a *critical value* of  $f$ , and we say that the singularity is *algebraic*. A simple example is when  $f(z) = \exp(z^2)$ . This function has a critical point at the origin, with critical value equal to 1.

In the second case we say that the choice  $r \mapsto U(r)$  defines a *transcendental singularity* of  $f^{-1}$ , and we say that  $a$  is the *projection* of the transcendental singularity or equivalently that the transcendental singularity is *over*  $a$ . Any of the sets  $U(r)$  is called a *neighbourhood* of the transcendental singularity. Note that, by a compactness argument, we have that  $\text{dist}(U(r), 0) \rightarrow \infty$  as  $r \rightarrow 0$ . A simple example is when  $f(z) = \exp(z)$ . This function has a transcendental singularity over the origin.

We say that a transcendental singularity over a point  $a$  is *direct* if there exists  $r > 0$  such that  $f(z) \neq a$ , for  $z \in U(r)$ . Otherwise we call the transcendental singularity *indirect*. A simple example of a direct singularity occurs when  $f(z) = \exp(z)$ , which has a direct singularity over the origin. An example of an indirect singularity occurs when

$$f(z) = \frac{\sin z}{z},$$

which can be seen to have two indirect singularities over the origin.

In order to give a more detailed classification of direct singularities, we need an additional definition. Suppose that  $V$  and  $W$  are domains, and that  $\phi : V \rightarrow W$  is an analytic function such that the following holds. For each  $w \in W$  there exists an open neighbourhood of  $w$ ,  $\Delta \subset W$ , such that each connected component of  $\phi^{-1}(\Delta)$  is mapped conformally by  $\phi$  onto  $\Delta$ . Then we say that  $\phi : V \rightarrow W$  is a *covering map*.

We call a direct transcendental singularity over a point  $a$  *logarithmic* if, for some  $r > 0$ , the restriction  $f : U(r) \rightarrow B(a, r) \setminus \{a\}$  is a covering map, and  $U(r)$  is simply connected. If a transcendental singularity is direct but not logarithmic, we use the term *direct non-logarithmic*.

Recall from Section 1.3 that a curve  $\Gamma : (0, 1) \rightarrow \mathbb{C}$  is an asymptotic curve with asymptotic value  $a$  if, as  $t \rightarrow 1$ , we have both  $\Gamma(t) \rightarrow \infty$  and  $f(\Gamma(t)) \rightarrow a$ . Given a transcendental singularity over a point  $a$  it is possible to construct an asymptotic curve with asymptotic value  $a$ , and *vice versa*; see [21, p.356] for details.

### 5.3 Direct non-logarithmic transcendental singularities

In this section we prove Theorem 5.1.2. We need the following theorem of Heins [53, Theorem 4'].

**Theorem 5.3.1.** *Suppose that  $f$  is a transcendental entire function,  $D \subset \mathbb{C}$  is a domain, and  $W$  is a component of  $f^{-1}(D)$ . Then either  $f_W$ , the restriction of  $f$  to  $W$ , has finite constant valence in  $D$ , or else there is at most one point of  $D$  at which the valence of  $f_W$  is finite.*

Here the *valence* of a point  $a \in D$  is the number of solutions of  $f(z) = a$ , for  $z \in W$ . It follows from Theorem 5.3.1 that there cannot be two distinct points  $a, a' \in D$  such that  $f(z) \in \{a, a'\}$  has no solutions, for  $z \in W$ .

We also need the following result, and two corollaries of it. This seems to be well-known, and follows from results such as [71, Example 4.2]. See also [108, Theorem 6.1.1] for a detailed proof.

**Theorem 5.3.2.** *Suppose that  $W \subset \mathbb{C}$  is a domain, and that  $g : W \rightarrow \mathbb{D}^*$  is a covering map. Then exactly one of the following holds:*

- (i) there exists a conformal map  $\phi : W \rightarrow \mathbb{H}$  such that  $g = \exp \circ \phi$ ;
- (ii) there exists a conformal map  $\phi : W \rightarrow \mathbb{D}^*$  such that  $g = (\phi)^m$ , for some  $m \in \mathbb{N}$ .

The first corollary is similar to [108, Theorem 6.2.2], and differs from that result in that it specifies the location of the neighbourhoods of the singularities, which is necessary for the proof of Theorem 5.1.2. We give a proof for completeness.

**Corollary 5.3.3.** *Suppose that  $f$  is a transcendental entire function with a transcendental singularity which is not logarithmic, over a point  $a \in \widehat{\mathbb{C}}$ . Then at least one of the following holds:*

- (i)  $a$  is the limit of critical values of  $f$ ;
- (ii) every neighbourhood of this singularity contains a neighbourhood of another transcendental singularity of  $f^{-1}$  whose projection is different from  $a$ .

*Proof.* Suppose that, contrary to the conclusion of the corollary, we can choose a sufficiently small  $r > 0$  such that there are no critical points of  $f$  in

$$W = U(r) \setminus \{z : f(z) = a\},$$

and all transcendental singularities of  $f^{-1}$  with a neighbourhood contained in  $U(r)$  have projection  $a$ . It follows that the restriction of  $f$ ,  $f_W : W \rightarrow B(a, r) \setminus \{a\}$  is a covering map.

Let  $h$  be a conformal map from  $B(a, r) \setminus \{a\}$  to  $\mathbb{D}^*$ . We apply Theorem 5.3.2 with  $g = h \circ f_W$ . If case (i) of the theorem holds, then  $W$  is simply connected, and the singularity is logarithmic, which is a contradiction. If case (ii) of the theorem holds, then the conformal mapping  $\phi$  has a punctured disc in the Riemann sphere as its domain, and at the puncture  $\phi$  has a removable singularity. Hence, since  $f_W = h^{-1} \circ (\phi)^m$ , the singularity is algebraic; this is also a contradiction.  $\square$

The second corollary of Theorem 5.3.2 is similarly straightforward, and we omit the proof.

**Corollary 5.3.4.** *Suppose that  $f$  is a transcendental entire function with a logarithmic singularity over a point  $a \in \widehat{\mathbb{C}}$ . Then there exist a neighbourhood of the singularity,  $W = U(r)$ , and conformal maps  $h : B(a, r) \setminus \{a\} \rightarrow \mathbb{D}^*$  and  $\phi : W \rightarrow \mathbb{H}$  such that  $h \circ f = \exp \circ \phi$ .*

We now prove Theorem 5.1.2.

*Proof of Theorem 5.1.2.* Suppose that  $f$  has a direct non-logarithmic singularity over a point  $a \in \widehat{\mathbb{C}}$ , and that  $a$  is not the limit of critical values of  $f$ . The existence of transcendental singularities, over points other than  $a$ , in any neighbourhood of this direct non-logarithmic singularity follows from Corollary 5.3.3; we need to show that in any neighbourhood of this singularity there are singularities over points other than  $a$ , which are either logarithmic or indirect.

The structure of the proof is as follows. We assume the contrary, and construct a sequence of direct non-logarithmic singularities the projections of which have a limit. We show that this limit is itself the projection of a direct non-logarithmic singularity, and use the comment after Theorem 5.3.1 to obtain a contradiction. Figure 5.1 illustrates the points and sets constructed, and is intended to be consulted alongside the text. On the right-hand side of the figure we see some of the asymptotic values constructed in the proof, and discs surrounding these. In particular  $D$  is the disc with centre  $a'$ , shown with dashed boundary. Contained in  $D$  is a disc with centre  $a_N$ , shown with dotted boundary. On the left-hand side of the figure, using corresponding styles of lines for the boundaries, are the components of the preimages of these sets used in the proof; for example  $f(W) = D$ . We also show the asymptotic curve  $\Gamma$  constructed in the proof.

Let  $r_0 > 0$  be such that there are no critical values of  $f$  in  $B(a, r_0)$  and also, since the transcendental singularity is assumed to be direct, such that  $f(z) \neq a$ , for  $z$  in the neighbourhood  $U(r_0)$ . Suppose also that  $r_0$  is sufficiently small that all transcendental singularities, over points other than  $a$  and with a neighbourhood contained in  $U(r_0)$ , are direct non-logarithmic.

Let  $(R_n)$  be any increasing sequence of positive real numbers such that  $R_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Recalling the definition of a neighbourhood of a transcendental singularity from Section 5.2, we construct a sequence of neighbourhoods of direct non-logarithmic singularities  $W_n(r_n)$ , with projection  $a_n$  say, such that, for  $n \geq 0$ ,

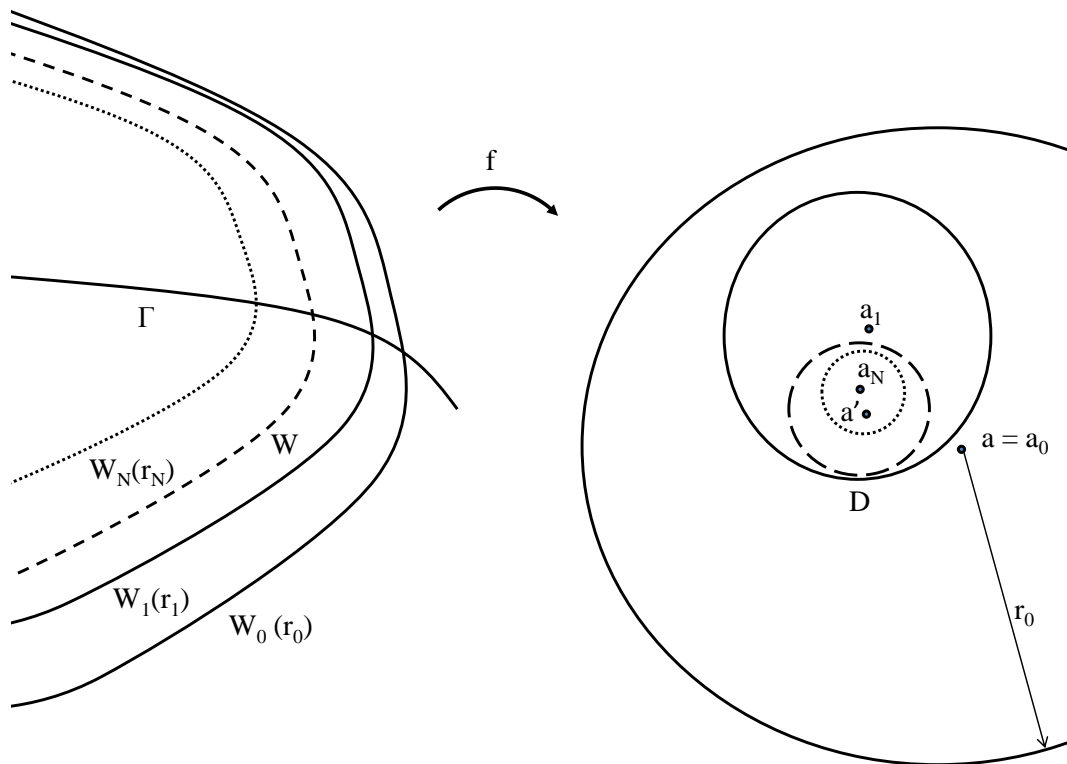


Figure 5.1: The construction used in the proof of Theorem 5.1.2.

- $W_{n+1}(r_{n+1}) \subset W_n(r_n) \subset U(r_0)$ ;
- $W_{n+1}(r_{n+1}) \cap B(0, R_{n+1}) = \emptyset$ ;
- $\overline{B(a_{n+1}, r_{n+1})} \subset B(a_n, r_n)$ ;
- $a_{n+1} \neq a$ ;
- the equation  $f(z) = a_n$  has no solutions for  $z \in W_n(r_n)$ ;
- $r_n \rightarrow 0$  as  $n \rightarrow \infty$ .

We set  $W_0(r_0) = U(r_0)$ ,  $a_0 = a$ , note that  $r_0$  and  $R_0$  are already defined, and then construct this sequence inductively. By assumption,  $W_n(r_n)$  is a neighbourhood of a direct non-logarithmic singularity. Hence we can use Corollary 5.3.3

to choose a transcendental singularity with projection  $a_{n+1}$  say and with neighbourhoods  $W_{n+1}(r)$ ,  $r > 0$ , such that

$$W_{n+1}(r'_{n+1}) \subset W_n(r_n), \quad \text{for some } r'_{n+1} > 0,$$

and also such that  $0 < |a_{n+1} - a_n| < r_n/2$ .

Next, recalling from Section 5.2 that  $\text{dist}(W_{n+1}(r), 0) \rightarrow \infty$  as  $r \rightarrow 0$ , we choose  $r''_{n+1} > 0$  such that  $W_{n+1}(r''_{n+1}) \cap B(0, R_{n+1}) = \emptyset$ . By assumption  $W_{n+1}(r'_{n+1})$  is a neighbourhood of a direct singularity with projection  $a_{n+1}$ . Hence, there exists  $r_{n+1}$  with

$$0 < r_{n+1} < \min\{r'_{n+1}, r''_{n+1}, |a_{n+1} - a_n|/4\}$$

such that  $f(z) = a_{n+1}$  has no solutions for  $z \in W_{n+1}(r_{n+1})$ . Finally, both  $\overline{B(a_{n+1}, r_{n+1})} \subset B(a_n, r_n)$  and  $a_{n+1} \neq a$ , by the choice of  $a_{n+1}$  and  $r_{n+1}$ . This completes the construction.

Let  $a' = \lim_{n \rightarrow \infty} a_n$ , which exists by our choice of  $r_n$ . Note that, by construction,  $a' \neq a$ . Let  $\Gamma$  be a curve produced inductively by joining a point in  $W_n(r_n)$  to a point in  $W_{n+1}(r_{n+1})$  using a curve lying in  $W_n(r_n)$ . By construction,  $\Gamma$  is an asymptotic curve with asymptotic value  $a'$ . Hence  $a'$  is the projection of a transcendental singularity of  $f^{-1}$  which, by assumption, is direct non-logarithmic.

Choose  $r > 0$  sufficiently small that  $D = B(a', r) \subset B(a, r_0)$ , and such that  $f(z) = a'$  has no solutions in  $W$ , where  $W$  is the component of  $f^{-1}(D)$  which has unbounded intersection with  $\Gamma$ .

Now, by construction,  $a' \in B(a_n, r_n)$ , for  $n \in \mathbb{N}$ , and so there is an  $N > 0$  such that  $a' \in B(a_N, r_N) \subset D$  and also  $a' \neq a_N$ . Note that  $W_N(r_N) \subset W$ , since the intersection of  $\Gamma$  and  $W_N(r_N)$  is unbounded. Then  $a'$  and  $a_N$  are two distinct points in  $B(a_N, r_N)$  such that  $f(z) \in \{a', a_N\}$  has no solutions in  $W_N(r_N)$ , which is contrary to Theorem 5.3.1.  $\square$

## 5.4 Proof of the main result

In this section we prove Theorem 5.1.1. We need the following, [34, Theorem I.2.2].

**Theorem 5.4.1.** *Suppose that  $W \subset \widehat{\mathbb{C}}$  is simply connected and  $\partial W$  has more than one point. Let  $\psi$  be a conformal map from  $W$  to  $\mathbb{D}$ , and let  $\Gamma$  be a Jordan arc in  $W$  except for one endpoint  $z_0 \in \partial W$ . Then the curve  $\psi(\Gamma)$  terminates in a point  $s_0 \in \partial\mathbb{D}$ , and  $\psi^{-1}(s) \rightarrow z_0$  as  $s \rightarrow s_0$  inside any Stolz angle at  $s_0$ .*

Here a *Stolz angle* at  $s_0 \in \partial\mathbb{D}$  is a set of the form;

$$\{s \in \mathbb{D} : |\arg(1 - \overline{s_0}s)| < \alpha, |s - s_0| < d\}, \quad \text{for } 0 < \alpha < \pi/2, d < 2 \cos \alpha.$$

We also need the following result, which is a version of [21, Theorem 1] that includes some assertions that appear only in the proof of that result; see also, [108, Theorem 6.2.3].

**Theorem 5.4.2.** *Suppose that  $f$  is a transcendental entire function with an indirect singularity with projection  $a \in \widehat{\mathbb{C}}$ . Suppose that  $a$  is not the limit of critical values of  $f$ . Then there exists a sequence of asymptotic values  $(a_n)$ , which converge to  $a$ , a sequence of disjoint unbounded simply connected domains  $(U_n)$  such that  $D_n = f(U_n)$  is a disc with  $a_n \in \partial D_n$ , and a sequence of asymptotic curves  $(\Gamma_n)$  such that  $\Gamma_n \subset U_n$ ,  $f(\Gamma_n)$  is a radius of  $D_n$  ending at  $a_n$ , and  $f$  is univalent in  $U_n$ .*

Finally, we need the following lemma.

**Lemma 5.4.3.** *Let  $f$  be a transcendental entire function. Suppose that for every  $R > 0$  there exist  $r > 0$ ,  $a_0 \in \mathbb{C}$  with  $|a_0| > R$ , an asymptotic curve  $\Gamma'$  with asymptotic value  $a_0$ ,  $W$  a simply connected neighbourhood of  $\Gamma'$ , and an analytic map  $\phi$ , univalent on  $W$ , such that  $\phi(\Gamma')$  is an interval  $(-\infty, x_0)$ , and*

$$f(z) = re^{\phi(z)} + a_0, \quad \text{for } z \in W. \quad (5.3)$$

Then  $\eta_f = 0$ .

*Proof.* Suppose that  $\eta_f \neq 0$ . Then there exist  $\epsilon, R > 0$  such that

$$\left| z \frac{f'(z)}{f(z)} \right| > \epsilon, \quad \text{for } |f(z)| > R. \quad (5.4)$$

Choose  $a_0$  such that  $|a_0| > 2R$ , let  $h = \phi^{-1}$  and put  $t = \phi(z)$ . Then, as  $z \rightarrow \infty$  along  $\Gamma'$ , by (5.3) and (5.4),

$$\epsilon < \left| z \frac{\phi'(z) r e^{\phi(z)}}{r e^{\phi(z)} + a_0} \right| = \left| h(t) \frac{\phi'(h(t)) r e^t}{r e^t + a_0} \right| = \left| \frac{h(t) r e^t}{h'(t)(r e^t + a_0)} \right| \sim \left| \frac{h(t) r e^t}{h'(t) a_0} \right|. \quad (5.5)$$

Hence, for sufficiently large negative values of  $t$ ,

$$\left| \frac{h'(t)}{h(t)} \right| < \frac{2r e^t}{\epsilon |a_0|}. \quad (5.6)$$

Without loss of generality, by choosing an unbounded subset of  $\Gamma'$  and relabeling, if necessary, we can assume that (5.6) applies for all  $t \in (-\infty, x_0)$  and that  $0 \notin W$ . Since  $W$  is simply connected, we can define a branch of the logarithm,  $L$ , in  $W$ . Then, by (5.6),

$$\left| \frac{d}{dt} L(h(t)) \right| < \frac{2r e^t}{\epsilon |a_0|}. \quad (5.7)$$

We set  $\zeta = L(h(t))$  and integrate (5.7), to obtain

$$\frac{2r}{\epsilon |a_0|} \int_{-\infty}^{x_0} e^t dt > \int_{-\infty}^{x_0} \left| \frac{d}{dt} L(h(t)) \right| dt \geq \left| \int_{-\infty}^{x_0} \frac{d}{dt} L(h(t)) dt \right| = \left| \int_{L(\Gamma')} d\zeta \right|. \quad (5.8)$$

Now,  $L(\Gamma')$  is an unbounded curve, and so the right-hand side of (5.8) is infinite. However, the left-hand side of (5.8) is finite. This contradiction completes the proof.  $\square$

We now prove Theorem 5.1.1.

*Proof of Theorem 5.1.1.* As mentioned in the introduction, it is clear that if  $f \in \mathcal{B}$  then  $\eta_f = \infty$ . Suppose, then, that  $\eta_f \neq \infty$ . It is immediate from (5.2) that the set of critical values of  $f$  is bounded. To complete the proof, we show that  $f$  cannot have an unbounded set of finite asymptotic values, and so  $f \in \mathcal{B}$ , and hence  $\eta_f = \infty$ . To achieve this we show first that  $f$  cannot have an unbounded set of projections of logarithmic singularities. We then show that  $f$  cannot have an unbounded set of projections of indirect singularities. Finally, we show that



$f$  cannot have an unbounded set of projections of direct non-logarithmic singularities.

Our first claim then is that  $f$  cannot have an unbounded set of projections of logarithmic singularities. Figure 5.2 illustrates some of the sets and functions used in the next part of the proof.

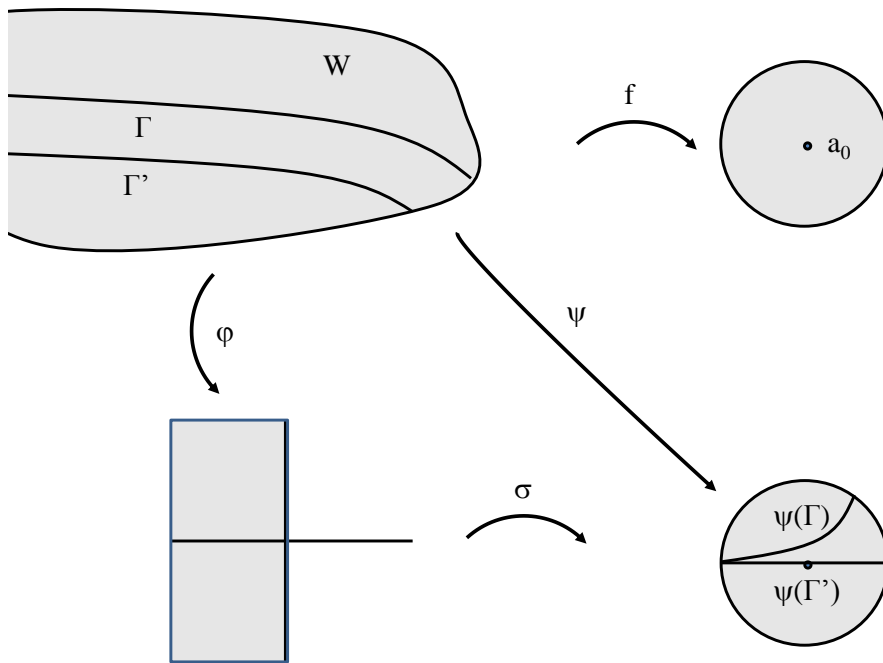


Figure 5.2: The sets and functions used in part of the proof of Theorem 5.1.1.

Suppose that, for every  $R > 0$ ,  $f$  has a logarithmic singularity with projection  $a_0 \in \mathbb{C}$ , such that  $|a_0| > R$ . Noting that  $a_0$  is finite, we apply Corollary 5.3.4 to obtain a simply connected neighbourhood,  $W = U(r)$ , of the singularity, and a conformal map  $\phi : W \rightarrow \mathbb{H}$  such that (5.3) holds for some  $r > 0$ . Let  $\Gamma$  be an asymptotic curve in  $W$  associated with the logarithmic singularity.

Put  $t = \phi(z)$  and let

$$s = \sigma(t) = \frac{1-t}{1+t}.$$

Then  $\psi = \sigma \circ \phi$  is a conformal mapping of  $W$  to  $\mathbb{D}$ . Moreover  $\psi(\Gamma)$  is a curve in

$\mathbb{D}$  tending to  $-1$ .

We now construct another curve to  $\infty$  in  $W$  which satisfies the hypotheses of Lemma 5.4.3. Let  $\Gamma' = \phi^{-1}((-\infty, 0))$ . Then  $\gamma = \psi(\Gamma')$  is a curve in  $\mathbb{D}$  tending to  $-1$  within a Stolz angle. By Theorem 5.4.1,  $\psi^{-1}(s) \rightarrow \infty$  as  $s \rightarrow -1$  along  $\gamma$ , and so  $\Gamma'$  is an asymptotic curve. Moreover,  $re^t + a_0 \rightarrow a_0$  as  $t \rightarrow -\infty$  along  $\phi(\Gamma')$ , and so  $\Gamma'$  has asymptotic value  $a_0$ . A contradiction follows by Lemma 5.4.3, since we have assumed that  $|a_0| > R$ . This establishes our initial claim.

We next show that  $f$  cannot have an unbounded set of projections of indirect singularities. Suppose that, for every  $R > 0$ ,  $f$  has an indirect singularity with projection  $a \in \mathbb{C}$ , such that  $|a| > 2R$ . By Theorem 5.4.2,  $f$  has an asymptotic value  $a_0$ , with  $|a_0| > R$ , an asymptotic curve  $\Gamma'$  associated with  $a_0$ , and an unbounded simply connected domain  $W$  containing  $\Gamma'$  such that  $f$  is univalent in  $W$ . Moreover,  $f(W)$  is a disc,  $D$ , with  $a_0 \in \partial D$ , and  $f(\Gamma')$  is a radius in  $D$  ending at  $a_0$ .

Without loss of generality, by composing with a rotation if necessary, assume that the centre of  $D$  is at  $a_0 + e^{x_0}$ , for some  $x_0 \in \mathbb{R}$ . Define a branch of the logarithm,  $L_1$ , such that  $\psi(w) = L_1(w - a_0)$  is a univalent map on  $D$ . Let  $\phi$  be the univalent map  $\phi = \psi \circ f$ . Note that  $\phi(\Gamma') = (-\infty, x_0)$ , and (5.3) holds with  $r = 1$ . A contradiction follows by Lemma 5.4.3, since we have assumed that  $|a_0| > R$ . This establishes our second claim.

Finally we show that the projections of direct non-logarithmic singularities are bounded. This follows immediately from the fact that the projections of other types of transcendental singularities are bounded and from Theorem 5.1.2. This completes the proof.  $\square$

**Remark 5.4.1.** It seems possible to generalise the result of Theorem 5.1.1 to transcendental meromorphic functions with direct tracts (see, for example, [26] for more background on this concept). We have not done this here, for reasons of simplicity. However, the proof seems to work similarly, although a number of results used in this chapter need to be generalised. In addition, we need to replace Theorem 5.3.1 with [31, Corollary 1], and [41, Lemma 1] with [26, Lemma 6.3].

# Chapter 6

## Questions for further research

### 6.1 Introduction

In this final chapter we briefly consider three areas of further study, which arise from or are closely related to the work in this thesis.

### 6.2 Some families of transcendental entire functions

For  $n \in \mathbb{N}$ , let  $\omega_n = e^{2\pi i/n}$  be an  $n$ th root of unity. Consider the collection of families of transcendental entire functions defined by

$$\mathcal{E} = \bigcup_{n=1}^{\infty} E_n,$$

where

$$E_n = \left\{ f : f(z) = \sum_{k=1}^n a_k \exp(\omega_n^k z), \text{ where } a_k \neq 0 \text{ for } k = 1, 2, \dots, n \right\}, \quad \text{for } n \in \mathbb{N}.$$

The collection  $\mathcal{E}$  forms a natural generalisation of the families  $E_1$  and  $E_2$ , the dynamics of which have been studied extensively. The family  $E_1$  is the well-known exponential family, defined in Section 1.3. The family  $E_2$  consists of functions of the form

$$f(z) = \alpha e^z + \beta e^{-z}, \quad \alpha \neq 0, \quad \beta \neq 0,$$

and so, up to a conjugacy, is the same family as the cosine family, also defined in Section 1.3. It can be shown that

$$\mathcal{E} \cap \mathcal{B} = E_1 \cup E_2.$$

The size of the Julia sets of functions in  $E_1$  and  $E_2$  was considered by McMullen [69]. In particular he proved the following results. Here  $\dim_H V$  denotes the Hausdorff dimension of a set  $V$ .

**Theorem 6.2.1.** *If  $f \in E_1$ , then  $\dim_H J(f) = 2$ .*

**Theorem 6.2.2.** *If  $f \in E_2$ , then  $J(f)$  has positive area.*

In fact, it can be seen from the constructions in [69] that these results hold, more strongly, with  $J(f)$  replaced by  $J(f) \cap A(f)$ . The following question is suggested by an analysis of McMullen's proof of Theorem 6.2.2.

*Question 1.* Is it true that if  $f \in E_n$ , for  $n \geq 2$ , then  $J(f) \cap A(f)$  has positive area?

We recall from Section 1.8 that Schleicher and Zimmer [95] showed that if  $f \in E_1$ , then  $J(f) \cap I(f)$  is contained in a Cantor bouquet. Rottenfusser and Schleicher [92] showed that the same is true when  $f \in E_2$ . In Example 2.5.4, however, it was shown that for the function

$$g(z) = \cos z + \cosh z,$$

we have that  $A_R(g)$  is a spider's web. We observe that  $g \in E_4$ . This suggests the following question.

*Question 2.* Is it true that if  $f \in E_n$ , for  $n \geq 3$ , then  $A_R(f)$  is a spider's web?

We note that it can be shown that all functions in  $\mathcal{E}$  are log-regular. Hence, by a remark in [89], functions in  $\mathcal{E}$  do not have multiply connected Fatou components. Suppose that  $f \in E_n$ , for  $n \geq 3$ . It follows that if Question 2 is answered in the affirmative, then, by Theorem 1.9.1(c), both  $J(f)$  and  $J(f) \cap I(f)$  are spiders' webs. Moreover, if Question 1 is also answered in the affirmative, then  $J(f)$  is a spider's web with positive area.

In a forthcoming paper, we hope to give positive answers to both Question 1 and Question 2, and so give the first result concerning the size of a spider’s web Julia set.

Schleicher and Zimmer [95] showed that if  $f \in E_1$  then the Julia set of  $f$  contains *dynamic rays*. Roughly speaking, a dynamic ray is an unbounded curve of points which escape to infinity in a precisely defined manner. Rottenfusser and Schleicher [92] showed that if  $f \in E_2$  then the Julia set of  $f$  contains a set with similar properties. If  $f \in E_n$ , for  $n \geq 2$ , then in very large parts of the plane  $f$  behaves similarly to a function in  $E_1$  of large modulus. This suggests the following question.

*Question 3.* Suppose that  $f \in \mathcal{E}$ . Is it possible to define unbounded curves, contained in  $J(f) \cap I(f)$ , with some of the properties of the dynamic rays discussed in [92] and [95]?

If Question 2 and Question 3 were both answered in the affirmative, then this would show that it is possible for a spider’s web Julia set to have a subset with some of the dynamical properties of a Cantor bouquet.

The following question seems to be a natural consequence of Question 3.

*Question 4.* Suppose that  $f \in \mathcal{E}$ . Is it the case that the strong form of Eremenko’s conjecture holds for  $f$ ?

### 6.3 A partition of the fast escaping set

The definition of the fast escaping set leads to a natural partition of  $A(f)$  into two completely invariant components. Firstly we define

$$A'(f) = \{z \in A(f) : \text{there exists } N \in \mathbb{N} \text{ s.t. } |f^{n+1}(z)| = M(|f^n(z)|), \text{ for } n \geq N\}.$$

The set  $A'(f)$  consists of points which, after at most a finite number of iterations, always achieve the maximum possible growth. These points can perhaps truly be described as escaping ‘as fast as possible’.

It is possible to construct functions for which  $A'(f)$  is not empty. For example, suppose that  $f_\lambda(z) = \lambda e^z$ , for  $\lambda > 0$ . It is easy to see that there exists

$\beta = \beta(\lambda) \geq 0$  such that

$$A'(f_\lambda) = \{z : f_\lambda^n(z) \geq \beta, \text{ for some } n \in \mathbb{N}\}, \quad (6.1)$$

and so  $A'(f_\lambda)$  consists of an unbounded interval of the real line together with all the preimages of this interval. Indeed, any transcendental entire function which has only positive non-zero real coefficients in its power series expansion satisfies (6.1), for some  $\beta \geq 0$ .

*Question 5.* Is it the case that  $A'(f)$  consists of, at most, a countable union of analytic curves?

It is also possible to construct functions for which  $A'(f) = \emptyset$ . For example, if  $f(z) = ie^z$ , then  $f$  achieves its maximum modulus only on the positive real axis, but the image of any point on the real axis is imaginary. It follows that  $A'(f) = \emptyset$ .

Finally, it may be possible to show that  $A'(f)$  can have unexpected properties. Hardy [49] introduced the function

$$f(z) = \exp(e^{z^2} + \sin z).$$

*Question 6.* Is it the case, as seems likely, that  $A'(f)$  is totally disconnected, in which case  $A'(f)$  – in contrast to  $A(f)$  – may contain bounded components?

Finally, we define

$$A''(f) = A(f) \setminus A'(f).$$

The set  $A''(f)$  can be described as the set of points which escape quickly, but not quite as fast as possible.

*Question 7.* Is it the case that if  $f$  is a transcendental entire function, then  $A''(f) \neq \emptyset$ ?

*Question 8.* Is it possible to use the properties of  $A'(f)$  and  $A''(f)$  to obtain further information on the structure or properties of  $A(f)$  and  $I(f)$ ?

## 6.4 The behaviour of $M^n(R_A(z))$ as $n \rightarrow \infty$

In Theorem 3.7.9 we showed that if  $f$  is a transcendental entire function with  $f(0) = 0$ , and  $z$  is in a multiply connected Fatou component of  $f$ , then

$$\lim_{n \rightarrow \infty} \frac{\log |f^n(z)|}{\log M^n(R_A(z))} = 1.$$

The only known examples of functions with simply connected fast escaping Fatou components are given in [20], [75] and Chapter 4. It can be shown that, for the example in Chapter 4, if  $z$  is in one of the simply connected fast escaping Fatou components, then we have the stronger result that

$$\lim_{n \rightarrow \infty} \frac{|f^n(z)|}{M^n(R_A(z))} = 1. \tag{6.2}$$

The example in [20] is not given in an explicit form, so it seems harder to check if (6.2) applies in this case. However, the following question does appear natural.

*Question 9.* Does a result similar to (6.2) hold, in general, for simply connected fast escaping Fatou components?

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