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# Periodic solutions of autonomous systems under discretization


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**Abstract**

The existence of a sequence of periodic trajectories of a general one-step numerical scheme corresponding to a null sequence of constant time-steps is established under the assumption that the autonomous ordinary differential equation has an isolated periodic solution with non-zero topological index. The convergence of the linearly interpolated numerical curve to the original invariant curve with respect to the Hausdorff metric is also shown.

# 1 Introduction

We consider an autonomous dynamical system described by a nonlinear differential equation

$$\frac{dx}{dt} = f(x), \qquad x \in I\!R^N \tag{1}$$

with smooth right-hand side and suppose that this system has a periodic solution of minimal period $T_* > 0$ for which the corresponding invariant curve is denoted by $\Gamma$. Beyn [1], Doan [2] and Eirola [3] have established the existence of a nearby invariant curve for a $p$th order one step numerical scheme [6] with sufficiently small constant step size $h > 0$ applied to (1) under the assumption of hyperbolicity of the original periodic solution; see also van Veldhuizen [7]. Numerical evidence suggests that the discretized system itself has a nearby periodic solution for certain step sizes. The aim of this paper is to prove that this is true for a certain null sequence of constant step sizes. Our main tool is degree theory and we assume only that the original periodic solution is isolated and has nonzero topological degree, which includes the hyperbolic case.

We construct a polygonal curve $L$ approximating $\Gamma$ by linearly interpolating the successive iterates of a $p$th order one step scheme. Let $h > 0$ be a fixed step size and define the mapping $A(\cdot; h) : I\!R^N \to I\!R^N$ by

$$A(x; h) := x + h\, f_h(x),$$

where $f_h$ is the increment function of the $p$th order one step scheme [6] under consideration applied to the differential equation (1). We then construct a polygonal curve $L = L(h, x_0)$ with nodes at the iterated points of the numerical scheme, that is at

$$x_k = A^k(x_0; h), \qquad k = 0, 1, \ldots, \tag{2}$$

by linear interpolation. Such a polygonal curve represents a periodic solution or cycle of the discretized system (2) if for some integer $n$ the nodes $x_0$, $x_1$, ..., $x_{n-1}$ are all different and $x_n = x_0$. In this case we will call it a cyclic polygonal curve.

If the initial approximation $x_0$ lies in an appropriate $\varepsilon$-neighborhood of the cycle $\Gamma$, then for $k = 0, 1, \ldots, \lfloor T_*/h \rfloor$ the curve $L(h, x_0)$ lies in the $(c_1\varepsilon + c_2 h^p)$-neighborhood of the cycle $\Gamma$ where the constants $c_1$ and $c_2$ depend only on the right-hand side of the differential equation (1) and the corresponding increment function of the numerical scheme in a neighbourhood of $\Gamma$. We will show that there exists a null sequence of step sizes $\{h_n\}$ and a sequence of initial values $\{x_0^{(n)}\}$ such that the polygonal curves $L_n = L(h_n, x_0^{(n)})$ are cyclic with $L_n$ converging to $\Gamma$ in the Hausdorff metric as $n \to \infty$.

## 2 Main result

Suppose that the periodic curve $\Gamma$ of system (1) is isolated, fix a point $x_* \in \Gamma$, let $\Pi$ be a transversal hyperplane to the curve $\Gamma$ at $x_*$, and denote by $P$ the Poincaré mapping defined in a vicinity $V$ of the point $x_*$ on the hyperplane $\Pi$. The point $x_*$ is thus an isolated zero of the vector field

$$\varphi(x) = x - P(x), \qquad x \in V. \tag{3}$$

Let $\operatorname{ind}(x_*; \psi)$ be the topological index [5] of this zero. This index does not depend either on the choice of the specific point $x_*$ or on the choice of the hyperplane $\Pi$. It will be called the *topological index of the periodic curve* $\Gamma$, or for simplicity the *index of* $\Gamma$, and denoted by $\operatorname{ind}(\Gamma)$.

**Theorem 1.** *Suppose that the autonomous system* (1) *has an isolated cycle* $\Gamma$ *with minimal period* $T_*$ *such that* $\operatorname{ind}(\Gamma) \neq 0$. *Then for any pth order one step numerical scheme and each integer n sufficiently large there exists a cyclic polygonal curve* $L_n$ *with exactly n nodes corresponding to iterates of the numerical scheme with step size* $h_n > 0$ *such that* $h_n \to 0$ *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} H(L_n, \Gamma) = 0, \tag{4}$$

*where* $H(\cdot,\cdot)$ *is the Hausdorff metric between nonempty compact subsets of* $\mathbb{R}^N$.

The proof will be given in the section 4 following the proofs of several lemmas in the next section.

## 3 Several lemmas

In what follows we denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ a scalar product and the corresponding norm in $\mathbb{R}^N$, repectively, and denote by $p(t,x)$ the unique solution of the system (1) satisfying the initial $p(0,x) = x$.

Fix some sufficiently small $h > 0$ and consider the map

$$B(x; h) = p(h, x), \tag{5}$$

which is defined on every sufficiently small neighborhood $U(\Gamma)$ of the cycle $\Gamma$. Let us suppose that $\overline{U(\Gamma)}$ does not intersect any other cycle of system (1) and that all iterations $B^n(\cdot, h)$ for $n = 1, 2, \ldots, \lfloor 2T_*/h \rfloor$ are defined on $\overline{U(\Gamma)}$, which is possible by continuity considerations if the neighbourhood $U(\Gamma)$ is sufficiently small. From general results on the global discretization error of a $p$th order one step scheme (see, e.g. [6]) we have

**Lemma 1.** *There exists a constant $C = C(f; f_h; U(\Gamma); T_*)$ such that*

$$\sup_{x\in U(\Gamma)} \max_{1\le n\le \lfloor 2T_*/h\rfloor} |A^n(x;h) - B^n(x;h)| \le Ch^p. \tag{6}$$

Now choose some point $x_* \in \Gamma$, which we can suppose without loss of generality satisfies $\langle x_*, f(x_*)\rangle \neq 0$, and define

$$\ell(x) = \frac{T_*}{\langle x_*, f(x_*)\rangle}\langle x, f(x_*)\rangle.$$

Write $\Pi_0 = \{x \in I\!R^N : \ \ell(x) = 0\}$ and let $P_0$ be the orthogonal projector onto $\Pi_0$, with $P^0 = I - P_0$. Fix $r > 0$ sufficiently small so that the cylinder

$$T(r, x_*) = \left\{x \in I\!R^N : \ |P_0(x - x_*)| < r, \ |P^0(x - x_*)| < r\right\} \tag{7}$$

is a subset of $U(\Gamma)$. In view of the periodicity of the solution $p(t, x_*)$ in $\Gamma$, a continuous function $t(x)$ can thus be defined on on $\bar{T}(r, x_*)$ such that

$$t(x_*) = T_*, \quad p(t(x), x) \in \Pi, \qquad x \in \bar{T}(r, x_*) \tag{8}$$

where $\Pi = x_* + \Pi_0$. Now consider the vector field

$$\psi(x) = x - p(t(x), x) \tag{9}$$

on $\bar{T}(r, x_*)$. It is easy to see that the point $x_*$ is the unique zero of this vector field $\psi$ on $\bar{T}(r, x_*)$. Let $\operatorname{ind}(x_*; \psi)$ denote its the topological index.

**Lemma 2.** $\operatorname{ind}(x_*; \psi) = \operatorname{ind}(\Gamma)$.

**Proof.** The map $p(t(\cdot), \cdot)$ acts from $\bar{T}(r, x_*)$ to $\Pi$, so the topological index $\operatorname{ind}(x_*; \psi)$ of the point $x_*$ equals that of the restriction $p(t(\cdot), \cdot)|_\Pi$ of the map $p(t(\cdot), \cdot)$ on $\Pi$. But $p(t(\cdot), \cdot)|_\Pi = P$. Therefore

$$\operatorname{ind}(x_*; \psi) = \operatorname{ind}(x_*; \varphi) = \operatorname{ind}(\Gamma)$$

and Lemma 2 is proved. □

Now fix an integer $n \ge 2$ and define the interval

$$J_n = (T_*/(2n), 3T_*/(2n)). \tag{10}$$

Let $\Omega_n = \bar{T}(r, x_*) \times J_n \subset I\!R^{N+1} = I\!R^N \times I\!R$ and define on $\Omega_n$ the vector field

$$\Phi_n(u) = \left\{x - B^n(x;h), \ \ell(x) - T_*\right\}, \qquad u = \{x, h\} \in \bar{\Omega}_n.$$

Direct verification shows that the point $u_* = \{x_*, T_*/n\}$ is the unique zero of the vector field $\Phi_n(u)$ in $\bar{\Omega}_n$. Let $\operatorname{ind}(u_*; \Phi_n)$ denote the topological index of this zero.

**Lemma 3.** $\operatorname{ind}(u_*; \Phi_n) = \operatorname{ind}(\Gamma)$.

**Proof.** Consider the auxiliary vector field

$$\chi(u) = \{x - p(t(x), x),\ h - T_*/n\}, \qquad u = \{x, h\} \in \bar{\Omega}_n, \tag{11}$$

which is the Cartesian product of the field $\psi$ defined on $\bar{T}(r, x_*)$ and the one dimensional field $\theta_n(h) = h - T_*/n$ defined on $J_n$. The theorem on the rotation of the product of vector fields [4] gives

$$\text{ind}\,(u_*; \chi_n) = \text{ind}\,(x_*; \psi) \cdot \text{ind}\,(T_*/n; \theta_n). \tag{12}$$

Since $\text{ind}\,(T_*/n; \theta_n) = 1$, Lemma 2 implies the equality

$$\text{ind}\,(u_*; \chi_n) = \text{ind}\,(\Gamma). \tag{13}$$

Now consider the deformation

$$\Theta(u; \lambda) = \{x - p(\lambda nh - (1-\lambda)t(x), x)\ \lambda(\ell(x) - T_*) + (1-\lambda)(h - T_*/n)\},$$
$$u = \{x, h\} \in \bar{\Omega}_n,\ 0 \leq \lambda \leq 1.$$

We will show that the point $u_* = \{x_*; T_*/n\}$ is the unique zero of the field $\Theta(\cdot, \lambda)$ for every $\lambda$ in $\bar{\Omega}_n$.

If, otherwise, for some $u_0 = \{x_0, h_0\} \in \bar{\Omega}_n$ and $\lambda_0 \in [0, 1]$ the equality $\Theta(u_0; \lambda_0) = 0$ holds, then

$$x_0 = p(\lambda_0 nh_0 + (1-\lambda_0)t(x_0), x_0) \tag{14}$$

and

$$\lambda(\ell(x_0) - T_*) + (1-\lambda_0)(h_0 - T_*/n) = 0. \tag{15}$$

Equality (13) implies

$$x_0 \in \Gamma \tag{16}$$

and

$$\lambda_0 nh_0 + (1-\lambda_0)t(x_0) = T_*. \tag{17}$$

Suppose that $\ell(x_0) > T_*$. Then equality (15) guarantees that $nh_0 < T_*$, so by (17) we must have $t(x_0) > T_*$, but this is impossible since our construction implies that

$$(T_* - \ell(x))(T_* - t(x)) < 0$$

for $x \notin \Pi$. Hence

$$\ell(x_0) \leq T_*. \tag{18}$$

Analogously it is possible to prove that $\ell(x_0) \geq T_*$. Hence $\ell(x_0) = t(x_0) = T_*$, so

$$x_0 \in \Pi \tag{19}$$

and $h_0 = T_*/n$. Relations (16) and (19) imply that $x_0 = x_*$ and, consequently, $u_0 = u_*$.

Hence we have proved the uniqueness of the zero of the unique zero of the field $\Theta(\cdot; \lambda)$ for every $\lambda$, which means that the deformation $\Theta(\cdot, \lambda)$ is non-degenerate on the boundary $\partial\Omega_n$ of the cylinder $\Omega$. Therefore

$$\text{ind}\,(u_*; \Theta(\cdot, 0)) = \text{ind}\,(u_*; \Theta(\cdot, 1)). \tag{20}$$

On the other hand, the equalities $\Theta(\cdot; 0) = \chi_n$ and $\Theta(\cdot; 1) = \Phi_n$ hold. Thus (20) together with (12) complete the proof of Lemma 3. □

# 4 Proof of Theorem 1

As above, let $T(r, x_*)$ be the cylinder defined by (7), let $J_n$ be the interval (10) and let $\Omega_n = T(r, x_*) \times J_n$. Together with the fields $\Phi_n$, consider the sequence of the fields

$$\Psi_n(u) = \left\{x - A^n(x, h),\ \ell(x) - T_*\right\}, \qquad u = \{x, h\} \in \bar{\Omega}_n.$$

We will prove that for $n$ large enough the fields $\Psi_n$ and $\Phi_n$ are homotopical on $\partial\Omega_n$. For the proof we need to formulate some estimates. The boundary $\partial T(r, x_*)$ of the cylinder $T(r, x_*)$ can obviously be decomposed as $\partial T(r, x_*) = M_0 \bigcup M_1$, where

$$M_0 = \{x \in I\!R^N : \ |P_0(x - x_*)| \leq r,\ |P^0(x - x_*)| = r\},$$

$$M_1 = \{x \in I\!R^N : \ |P_0(x - x_*)| = r,\ |P^0(x - x_*)| \leq r\}.$$

If $x \in M_0$, then

$$|\ell(x) - T_*| = \frac{rT_*|f(x_*)|^2}{\langle x_*, f(x_*)\rangle}. \tag{21}$$

If $x \in M_1$ then, since the cycle $\Gamma$ is isolated, there exists an $\alpha(r) > 0$ such that for every $h \in J_n$ the inequality

$$|x - B^n(x; h)| \geq \alpha(r) \tag{22}$$

holds. Relations (21) and (22) imply that

$$|\Phi_n(u)| \geq \beta(r), \qquad \{u \in x, h\},\ x \in \partial T(r, x_*),\ h \in J_n, \tag{23}$$

where

$$\beta(r) = \min\left\{\frac{rT_*|f(x_*)|^2}{\langle x_*, f(x_*)\rangle},\ \alpha(r)\right\}.$$

Now the boundary $\partial\Omega_n$ can be decomposed as $\partial\Omega_n = \mathcal{N}_n^0 \bigcup \mathcal{N}_n^1$, where

$$\mathcal{N}_n^0 = \left\{u = \{x, h\} \in I\!R^{N+1} : \ x \in \bar{T}(r, x_*),\ h \in \partial J_n\right\},$$

$$\mathcal{N}_n^1 = \left\{u = \{x, h\} \in I\!R^{N+1} : \ x \in \partial T(r, x_*),\ h \in \bar{J}_n\right\}.$$

If $u \in \mathcal{N}_n^0$, then for $r > 0$ small enough, the minimality of the period $T_*$ of the cycle $\Gamma$ guarantees that

$$|\Phi_n(u)| \geq \gamma(r).$$

If $u \in \mathcal{N}_n^1$, then the field $\Phi_n$ satisfies (23), so

$$|\Phi_n(u)| \geq \delta(r), \qquad u \in \partial\Omega_n, \tag{24}$$

where $\delta(r) = \min\{\beta(r), \gamma(r)\}$.

Let us now estimate the norm $|\Phi_n(u) - \Psi_n(u)|$ on $\partial\Omega_n$. According to (6)

$$|\Phi_n(u) - \Psi_n(u)| = |B^n(x; h) - A^n(x, h)| \leq Ch^p \leq 3CT_*/(2n). \tag{25}$$

Estimates (24) and(25) and the Rouché theorem [4] imply that for $n \geq 3CT_*/2\delta(r)$ the fields $\Phi_n$ and $\Psi_n$ are homotopical, which means that the rotations $\gamma(\Phi_n; \partial\Omega_n)$ and $\gamma(\Psi_n; \partial\Omega_n)$ of these fields on $\partial\Omega_n$ coincide. Thus Lemma 3 guarantees that

$$\gamma(\Psi_n; \partial\Omega_n) = \gamma(\Phi_n; \partial\Omega_n) = \text{ind}\,(u_*; \Phi_n) = \text{ind}\,(\Gamma) \neq 0.$$

Hence every field $\Psi_n$ for sufficiently large $n$ has at least one zero $u_n = \{x_n, h_n\} \in \Omega_n$. By the definition of the mapping $B(\cdot, \cdot)$, the point $x_n$ defines a closed, that is cyclic, polygonal curve $L_n = L(h_n, x_n)$ with nodes $x_n, B(x_n, h_n), \ldots, B^{n-1}(x_n, h_n)$. Since $u_n \in \bar{\Omega}_n$ we have $|h_n - T_*/n| \leq T_*/(2n)$ and $h_n \to 0$ as $n \to \infty$. The limit (4) follows immediately from the estimate (6). The theorem is proved. □

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