Immediate exchange of stabilities in singularly perturbed systems

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Abstract

We study the initial value problem for singularly perturbed systems of ordinary differential equations whose associated systems have two transversally intersecting families of equilibria (transcritical bifurcation) which exchange their stabilities. By means of the method of upper and lower solutions we derive a sufficient condition for the solution of the initial value problem to exhibit an immediate exchange of stabilities. Concerning its asymptotic behavior with respect to ε we prove that an immediate exchange of stabilities implies a change of the asymptotic behavior from $0(\varepsilon)$ to $0(\sqrt{\varepsilon})$ near the point of exchange of stabilities.

1 Introduction.

There are numerous processes in natural sciences and in techniques which can be modelled by singularly perturbed systems of ordinary differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,\varepsilon), \\ \varepsilon\,\frac{dy}{dt} &= g(x,y,t,\varepsilon) \end{aligned} \tag{1.1}$$

where $x \in R^n, y \in R^m$, and ε is a small positive parameter. A usual approach in studying such systems is based on the so-called quasi-steady state assumption which means that the fast variables y are in a quasi-steady state. Mathematically speaking, this assumption says that the behavior of the singularly perturbed system (1.1) in some region of the phase space can be approximated by the differential algebraic system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,0), \\ 0 &= g(x,y,t,0). \end{aligned} \tag{1.2}$$

This assumption can be justified by means of the theory of invariant manifolds for singularly perturbed systems [1, 15]. The crucial assumption in this theory is that the equation

$$g(x,y,t,0) = 0 \tag{1.3}$$

has an isolated solution $y = \varphi(x,t)$ on which all eigenvalues of the Jacobian $g_y(x,\varphi(x,t),t,0)$ are located in the left half plane, and bounded away from the imaginary axis for all (x,t) in the domain of interest.
In case that (1.3) has two solutions $\varphi_1(x,t)$ and $\varphi_2(x,t)$ which intersect and exchange their stability (expressed by the spectrum of g_y at theses solutions) there arises the following question: Will an exchange of stability of the solutions φ_1 and φ_2 imply that a solution of (1.1) which stays near φ_1 before the exchange of stability will immediately stay near φ_2 after the exchange of stability? This behavior is called

"immediate exchange of stabilities", and is commonly assumed to occur in natural and technical systems. In this paper we derive conditions which guarantee an immediate exchange of stabilitities in the codimension 1 case of transcritical bifurcation. Essentially the same problem has been considered by N.R. Lebovitz and R.J. Schaar [7], but they restricted themselves to singularly perturbed systems whose right hand side does not depend on ε. The singularly perturbed differential equation

$$\varepsilon \frac{dy}{dt} = y(y-t), \quad t \in (-1,2) \tag{1.4}$$

satisfies the conditions of the main result of N.R. Lebovitz and R.J. Schaar and exhibits an immediate exchange of stabilities. If we consider the singularly perturbed equation

$$\varepsilon \frac{dy}{dt} = y(y-t) + \varepsilon, \quad t \in (-1,2) \tag{1.5}$$

then the invariance of the straight line $y = t$ implies that this equation exhibits either a failure or delayed exchange of stabilities. Therefore, the result of N.R. Lebovitz and R.J. Schaar is not applicable to this equation.

In what follows we use the method of upper and lower solutions to derive conditions guaranteeing an immediate exchange of stabilities in the transcritical case for two-dimensional systems (1.1). The motivation to investigate this phenomenon comes from problems in biochemistry [11], a corresponding example is treated in Section 4.

2 Formulation of the problem.

We consider the two-dimensional singularly perturbed system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t,\varepsilon), \\ \varepsilon \frac{dy}{dt} &= g(x,y,t,\varepsilon) \end{aligned} \tag{2.1}$$

and study the initial value problem

$$x(t_0,\varepsilon) = x^0, \; y(t_0,\varepsilon) = y^0, \; t \in I_t := \{-\infty < t_0 < t < t_e < \infty\}. \tag{2.2}$$

Our goal is to prove the existence of a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (2.1), (2.2) and to construct an asymptotic approximation of the type

$$\begin{aligned} x(t,\varepsilon) &= x_0(t) + \varepsilon(\Pi x(\tau,\varepsilon) + x_1(t)) + \cdots, \\ y(t,\varepsilon) &= \Pi y(\tau,\varepsilon) + y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots \end{aligned} \tag{2.3}$$

under conditions which do not fit into the standard theory of singularly perturbed differential equations [5, 12, 16, 17, 18, 19, 20]. The functions $\Pi x(\tau,\varepsilon)$ and $\Pi y(\tau,\varepsilon)$ are the initial layer corrections near $t = t_0$ where τ denotes the stretched variable $\tau = (t-t_0)/\varepsilon$.

If we set $\varepsilon = 0$ in (2.1) then we get the so-called degenerate system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, t, 0), \\ 0 &= g(x, y, t, 0). \end{aligned} \tag{2.4}$$

Let I_x and I_y be open bounded intervals, let $I_{\varepsilon_0} := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon_0 \ll 1\}$, $D := I_x \times I_y \times I_t \times I_{\varepsilon_0}$. Concerning the functions f and g we assume

(A_1). f and g map $\overline{D}$ into R, and are sufficiently smooth in D where all derivatives are continuous in $\overline{D}$.

It is obvious that the functions $x_0(t)$ and $y_0(t)$ in the asymptotic expansion (2.3) depend on the solution set of the equation

$$g(x, y, t, 0) = 0. \tag{2.5}$$

Concerning this solution set we assume

(A_2). The equation (2.5) has two different solutions $y = \varphi_1(x, t)$ and $y = \varphi_2(x, t)$ defined in $\bar{I}_x \times \bar{I}_t$, and with the same smoothness properties as g.

Different from Tichonov's theorem (see Theorem 5.1 in the Appendix), assumption (A_2) does not require φ_1 and φ_2 to be isolated. If we replace y by $\varphi_1(x, t)$ and $\varphi_2(x, t)$ resp. in the first equation of (1.1) then we impose on the reduced systems the following conditions.

(A_3). The initial value problem

$$\frac{dx}{dt} = f(x, \varphi_1(x, t), t, 0)\ , \ x(t_0) = x^0$$

has a unique solution $\tilde{x}^1(t, x^0)$ defined on I_t. There is a point t_c in I_t such that

(i) For $t_0 \le t < t_c$, $g_y(\tilde{x}^1(t, x^0), \psi_1(t), t, 0)$ is negative where $\psi_1(t)$ is defined by $\psi_1(t) := \varphi_1(\tilde{x}^1(t, x^0), t)$. That is, for $t \in [t_0, t_c)$, $\psi_1(t)$ is an asymptotically stable equilibrium point of the associated system

$$\frac{dy}{d\tau} = g(\tilde{x}^1(t, x^0), y, t, 0). \tag{2.6}$$

(ii) For $t \in (t_c, t_e]$, $g_y(\tilde{x}^1(t, x^0), \psi_1(t), t, 0)$ is positive.

(A_4). The initial value problem

$$\frac{dy}{d\tau} = g(x^0, y, t_0, 0),\ \ y(0) = y^0$$

has a unique solution $\tilde{y}(\tau, y^0)$ for $\tau \ge 0$ which tends to $\varphi_1(x^0, t_0)$ as $\tau \to \infty$.

Assumption (A_4) means that y^0 lies in the basin of attraction of the equilibrium point $\varphi_1(x^0, t_0)$ of (2.6).

From assumptions $(A_1)-(A_4)$ it follows that $\psi_1(t)$ is a differentiable one-parameter family of equilibria of the associated system (2.6) which exchanges its stability for $t = t_c$. The following assumption says that this exchange of stability is caused by the intersection with another family of equilibria related to the second root $\varphi_2(x,t)$ of (2.5).

(A_5). The initial value problem

$$\frac{dx}{dt} = f(x, \varphi_2(x,t), t, 0)\ , \ x(t_c) = x^1 := \tilde{x}^1(t_c, x^0)$$

has a unique solution $\tilde{x}^2(t, x^1)$ defined on I_t such that

(i) For $t_c < t \le t_e$, $g_y(\tilde{x}^2(t,x^1), \psi_2(t), t, 0)$ is negative where $\psi_2(t)$ is defined by $\psi_2(t) := \varphi_2(\tilde{x}^2(t,x^1), t)$.

(ii) For $t \in [t_0, t_c)$, $g_y(\tilde{x}^2(t,x^1), \psi_2(t), t, 0)$ is positive.

Definition 2.1 *Under the assumptions* (A_3) *and* (A_5) *the vector function* $(\hat{x}(t), \hat{y}(t))$ *defined by*

$$\hat{x}(t) := \begin{cases} \tilde{x}^1(t,x^0) & t_0 \le t \le t_c, \\ \tilde{x}^2(t,x^1) & t_c \le t \le t_e \end{cases}\ , \ \hat{y}(t) := \begin{cases} \psi_1(t) & t_0 \le t \le t_c, \\ \psi_2(t) & t_c \le t \le t_e \end{cases} \tag{2.7}$$

is referred to as the composed stable solution of (2.1) *with respect to* $\psi_1(t), \psi_2(t)$.

From this definition we obtain

$$\begin{aligned} \frac{d\hat{x}}{dt} &= f(\hat{x}(t), \hat{y}(t), t, 0), \\ 0 &= g(\hat{x}(t), \hat{y}(t), t, 0). \end{aligned} \tag{2.8}$$

In what follows let ν be any fixed small positive number. It is obvious that under the hypotheses $(A_1)-(A_5)$, Theorem 5.1 in the Appendix describes the asymptotic behavior of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of the initial value problem (2.1), (2.2) on the interval $(t_0, t_c-\nu)$. A similar approach is valid for $t \in (t_c+\nu, t_e)$ provided $y(t_c+\nu, \varepsilon) = y^\nu$ lies in the basin of attraction of the equilibrium point $\varphi_2(\tilde{x}^2(t_c+\nu, x^1), t_c+\nu)$ of the associated system

$$\frac{dy}{d\tau} = g(\tilde{x}^2(t,x^1), y, t, 0). \tag{2.9}$$

The critical interval is the interval $I_\nu := [t_c-\nu, t_c+\nu]$. In order to prove the existence of a solution of (2.1), (2.2) defined on I_ν and to get an asymptotic approximation of it we will apply the method of ordered upper and lower solutions.

Definition 2.2 *The pairs of picewise continuously differentiable functions* $(\overline{x}(t,\varepsilon), \overline{y}(t,\varepsilon))$ *and* $(\underline{x}(t,\varepsilon), \underline{y}(t,\varepsilon))$ *are called ordered upper and lower solutions of* (2.1), (2.2) *respectively, provided they satisfy the following inequalities*

$$\underline{x}(t,\varepsilon) \le \overline{x}(t,\varepsilon),\ \ \underline{y}(t,\varepsilon) \le \overline{y}(t,\varepsilon),$$

$$\frac{d\underline{x}}{dt} - f(\underline{x},\underline{y},t,\varepsilon) \le 0 \le \frac{d\overline{x}}{dt} - f(\overline{x},\overline{y},t,\varepsilon), \tag{2.10}$$

$$\varepsilon\ \frac{d\underline{y}}{dt} - g(\underline{x},\underline{y},t,\varepsilon) \le 0 \le \varepsilon\ \frac{d\overline{y}}{dt} - g(\overline{x},\overline{y},t,\varepsilon)$$

for $\varepsilon \in I_{\varepsilon_0}$ *and for all* $t \in I_t$ *where* $(\overline{x}(t,\varepsilon),\overline{y}(t,\varepsilon))$ *and* $(\underline{x}(t,\varepsilon),\underline{y}(t,\varepsilon))$ *are differentiable, and*

$$\underline{x}(t_0,\varepsilon) \le x^0 \le \overline{x}(t_0,\varepsilon),\ \ \underline{y}(t_0,\varepsilon) \le y^0 \le \overline{y}(t_0,\varepsilon).$$

Definition 2.3 *We call the vector function* (f,g) *quasimonotone nondecreasing in* $G_{(t,\varepsilon)} := \{(x,y) \in R^2 : \underline{x}(t,\varepsilon) \le x \le \overline{x}(t,\varepsilon), \underline{y}(t,\varepsilon) \le y \le \overline{y}(t,\varepsilon)\}$ *for* $(t,\varepsilon) \in I_t \times I_\varepsilon$ *iff* $f(\tilde{x},y,t,\varepsilon)$ *is nondecreasing in* y *for* $y \in [\underline{y}(t,\varepsilon),\overline{y}(t,\varepsilon)]$ *where* $\tilde{x}$ *is any point in* $[\underline{x}(t,\varepsilon),\overline{x}(t,\varepsilon)]$*, and* $g(x,\tilde{y},t,\varepsilon)$ *is nondecreasing in* x *for* $x \in [\underline{x}(t,\varepsilon),\overline{x}(t,\varepsilon)]$ *where* $\tilde{y}$ *is any point in* $[\underline{y}(t,\varepsilon),\overline{y}(t,\varepsilon)]$.

It is known [13] that if (f,g) is quasimonotone nondecreasing then under the assumption (A_1) the existence of an ordered lower and an upper solution of (2.1), (2.2) implies the existence of a unique solution $(x(t,\varepsilon),y(t,\varepsilon))$ of (2.1), (2.2) satisfying

$$\begin{array}{rcccl} \underline{x}(t,\varepsilon) & \le & x(t,\varepsilon) & \le & \overline{x}(t,\varepsilon), \\ \underline{y}(t,\varepsilon) & \le & y(t,\varepsilon) & \le & \overline{y}(t,\varepsilon). \end{array}$$

Remark 2.1 *The concept of quasimonotonicity has been introduced by E. Kamke* [6] *and M. Müller* [9] *and plays an important role in the theory of monotone systems* [3, 4].

The next condition implies that the composed stable solution is a lower solution of (2.1), (2.2) on $[t_0, t_c+\nu]$.

(A_6). For $t_0 \le t \le t_c+\nu, \varepsilon \in I_{\varepsilon_0}$ it holds

$$\begin{array}{rcl} \dfrac{d\hat{x}(t)}{dt} & \le & f(\hat{x}(t),\hat{y}(t),t,\varepsilon), \\ \varepsilon\ \dfrac{d\hat{y}(t)}{dt} & \le & g(\hat{x}(t),\hat{y}(t),t,\varepsilon). \end{array}$$

where ν is any small positive number independent of ε.

Furthermore we assume

(A_7). (f,g) is quasimonotone nondecreasing in (x,y) in $\hat{G}_{(t,\varepsilon)} := \{(x,y) \in R^2 : \hat{x}(t,\varepsilon) \le x \le \overline{x}(t,\varepsilon), \hat{y}(t,\varepsilon) \le y \le \overline{y}(t,\varepsilon)$ for $(t,\varepsilon) \in I_t \times I_\varepsilon$ where $(\overline{x}(t,\varepsilon),\overline{y}(t,\varepsilon))$ is an upper solution of (2.1), (2.2) which will be constructed in the sequel.

Finally we suppose

$(A_8.)$ There is a positive number r such that

$$-g_{yy}(\hat{x}(t_c),\hat{y}(t_c),t_c,0) \ge r.$$

3 Asymptotic behavior in case of immediate exchange of stability

The following theorem guarantees an immediate exchange of stabilities and characterizes the influence of an exchange of stability of the family $\psi_1(t)$ of equilibria of (2.6) on the asymptotic behavior of the solution of the initial value problem (2.1), (2.2): near the exchange point the usual $O(\varepsilon)$–behavior is replaced by an $O(\sqrt{\varepsilon})$–behavior. The proof of our main result is based on the application of the method of ordered lower and upper solutions.

Theorem 3.1 *Assume hypotheses $(A_1) - (A_8)$ to be valid. Then to any given small positive $\nu > 0$ there exists a sufficiently small positive $\varepsilon^* = \varepsilon^*(\nu)$ such that for $0 < \varepsilon \leq \varepsilon^*(\nu)$ the initial value problem* (2.1),(2.2) *with $x^0 \geq \hat{x}(t_0)$, $y^0 \geq \hat{y}(t_0)$ has a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ satisfying*

$$\begin{aligned} \lim_{\varepsilon \to 0} x(t,\varepsilon) &= \hat{x}(t) \quad \text{for } t \in \bar{I}_t, \\ \lim_{\varepsilon \to 0} y(t,\varepsilon) &= \hat{y}(t) \quad \text{for } t_0 < t \leq t_e. \end{aligned}$$

Moreover we have

$$x(t,\varepsilon) = \begin{cases} \hat{x}(t) + O(\varepsilon) & \text{for } t \in \bar{I}_t \backslash I_\nu \\ \hat{x}(t) + O(\varepsilon^{\frac{1}{2}}) & \text{for } t \in I_\nu \end{cases}$$

$$y(t,\varepsilon) = \begin{cases} \hat{y}(t) + \Pi_0 y(\tau) + O(\varepsilon) & \text{for } t_0 \leq t \leq t_c - \nu \\ \hat{y}(t) + O(\varepsilon^{\frac{1}{2}}) & \text{for } t \in I_\nu \\ \hat{y}(t) + O(\varepsilon) & \text{for } t_0 + \nu \leq t \leq t_e \end{cases}$$

where $\Pi_0 y(\tau)$ is the zeroth order boundary layer function.

Proof. The proof proceeds in three steps. In the first step we consider the initial value problem (2.1), (2.2) on the interval $[t_0, t_c - \tilde{\nu}]$ where $\tilde{\nu}$ is any small positive number. It is obvious that under the hypotheses above, Theorem 5.3 in the Appendix can be applied. Thus, to given $\tilde{\nu}$, there is an $\varepsilon_1 = \varepsilon_1(\tilde{\nu}) \leq \varepsilon_0$ such that for $0 < \varepsilon \leq \varepsilon_1$ there exists a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (2.1), (2.2) on the interval $[t_0, t_c - \tilde{\nu}]$ with the asymptotic behavior as described in Theorem 3.1. Let

$$x_{\tilde{\nu}}^- := x(t_c - \tilde{\nu}, \varepsilon), \ y_{\tilde{\nu}}^- := y(t_c - \tilde{\nu}, \varepsilon). \tag{3.1}$$

Now we consider the initial value problem (2.1), (3.1) on the interval $I_{\tilde{\nu}}$. We prove the existence of a unique solution to this problem by applying the method of ordered lower and upper solutions. First we note that according to hypotheses (A_6) and (A_7), $(\hat{x}(t), \hat{y}(t))$ is a lower solution to (2.1), (3.1) on the interval $I_{\tilde{\nu}}$.

To construct an upper solution we use the functions

$$\overline{x}(t,\varepsilon) = \hat{x}(t) + \sqrt{\varepsilon}\exp(\lambda t), \quad \overline{y}(t,\varepsilon) = \hat{y}(t) + \gamma\sqrt{\varepsilon}\exp(\lambda t) \tag{3.2}$$

where the positive constants λ and γ will be choosen later in an appropriate way. It follows from Theorem 5.3 in the Appendix that to any given $\lambda, \gamma, \tilde{\nu}$ there is a sufficiently small positive $\varepsilon_2, \ \varepsilon_2 \leq \varepsilon_1$, such that for $0 < \varepsilon \leq \varepsilon_2$ the following inequalities hold

$$\hat{x}(t_c - \tilde{\nu}) \leq x_{\tilde{\nu}}^- \leq \overline{x}(t_c - \tilde{\nu}, \varepsilon), \quad \hat{y}(t_c - \tilde{\nu}) \leq y_{\tilde{\nu}}^- \leq \overline{y}(t_c - \tilde{\nu}, \varepsilon).$$

Now we prove that $\overline{x}(t,\varepsilon)$ and $\overline{y}(t,\varepsilon)$ satisfy the differential inequalities in (2.10).

From (3.2), (A_1), and (2.8) we get

$$\begin{aligned}
& \varepsilon \frac{d\overline{y}}{dt} - g(\overline{x}(t), \overline{y}(t), t, \varepsilon) \\
= \ & \varepsilon \frac{d\hat{y}}{dt} + \gamma \varepsilon^{3/2} \lambda \exp{(\lambda t)} - g(\hat{x}(t) + \sqrt{\varepsilon} \exp{(\lambda t)}, \hat{y}(t) + \sqrt{\varepsilon}\gamma \exp{(\lambda t)}, t, \varepsilon) \\
= \ & -(\hat{g}_x(t) + \gamma \hat{g}_y(t)) \sqrt{\varepsilon} \exp{(\lambda t)} \\
- \ & \left(\hat{g}_{xx}(t) + \gamma^2 \hat{g}_{yy}(t) + 2\gamma \hat{g}_{xy}(t) + \hat{g}_\varepsilon(t) + \frac{d\hat{y}}{dt} \right) \varepsilon \exp{(2\lambda t)} + o(\varepsilon)
\end{aligned}$$

where $\hat{g}$ denotes to replace x by $\hat{x}(t)$ and y by $\hat{y}(t)$ in g. From

$$\hat{g}_x(t) = -\hat{g}_y(t) \hat{\varphi}_x(t)$$

it follows

$$\hat{g}_x(t) + \gamma \hat{g}_y(t) \leq 0$$

for sufficiently large γ.

Thus, according to assumption (A_8), there are a positive number $\varepsilon_3, \ \varepsilon_3 \leq \varepsilon_2$, and a sufficiently large γ_0, such that for $0 < \varepsilon \leq \varepsilon_3, \ \gamma > \gamma_0$ and $t \in I_{\nu_2}$

$$\varepsilon \frac{d\overline{y}}{dt} - g(\overline{x}(t), \overline{y}(t), t, \varepsilon) \geq 0.$$

From (3.2), (2.7), (A_1), and (2.8) we get

$$\begin{aligned}
\frac{d\overline{x}}{dt} - f(\overline{x}(t), \overline{y}(t), t, \varepsilon) \ &= \ \frac{d\hat{x}}{dt} + \sqrt{\varepsilon} \lambda \exp(\lambda t) \\
&- \ f(\hat{x}(t) + \sqrt{\varepsilon} \exp(\lambda t), \hat{y}(t) + \gamma \sqrt{\varepsilon} \exp \lambda t, t, \varepsilon) \\
&= \ \sqrt{\varepsilon} \exp \lambda t \left(\lambda - \hat{f}_x(t) - \hat{f}_y(t) \gamma \right) + o(\sqrt{\varepsilon}).
\end{aligned}$$

Since $\hat{f}_x(t)$ and $\hat{f}_y(t)$ are continuous in t, to given ε_3, $\tilde{\nu}$, and γ_0 there is a sufficiently large λ_0 such that for $t \in I_{\tilde{\nu}}, \ \lambda \geq \lambda_0$, and $0 < \varepsilon \leq \varepsilon_3$

$$\lambda - \hat{f}_x(t) - \gamma \hat{f}_y(t) \geq \kappa > 0.$$

Hence, there is a sufficiently small positive number $\varepsilon_4, \ 0 < \varepsilon_4 \leq \varepsilon_3$, such that for $t \in I_{\tilde{\nu}}$ and $0 < \varepsilon \leq \varepsilon_4$

$$\frac{d\overline{x}}{dt} - f(\overline{x}, \overline{y}, t, \varepsilon) \geq 0.$$

Consequently, we have proved the existence of a lower and an upper solution of (2.1), (3.1) on $I_{\tilde{\nu}}$ which imply under our assumptions the existence of a unique solution of

(2.1), (3.1) on $I_{\tilde{\nu}}$ satisfying the estimate of Theorem 4.1.
Let

$$x_{\tilde{\nu}}^{+} := \overline{x}(t_c + \tilde{\nu}/2, \varepsilon), \; y_{\tilde{\nu}}^{+} := \overline{y}(t_c + \tilde{\nu}/2, \varepsilon). \tag{3.3}$$

In the last step we apply Theorem 5.3 to the initial value problem (2.1), (3.3) on the interval $[t_0 + \tilde{\nu}/2, t_e]$ for $0 < \varepsilon < \varepsilon_0(\tilde{\nu})$ where we assume that $\varepsilon_0(\tilde{\nu})$ is so small such that $x_{\tilde{\nu}}^{+}$ is in the domain of attraction of the equilibrium point $\varphi_2(\tilde{x}^2(t_c+\tilde{\nu}/\varepsilon, x^1), t_c+\tilde{\nu}/2)$ of the associated system (2.9) of the stable root φ_2 and that the corresponding boundary layer is contained in the interval $(t_c + \tilde{\nu}/2, t_c + \tilde{\nu})$ for $0 < \varepsilon \leq \varepsilon_0(\tilde{\nu})$. This completes the proof of the theorem.

Now we consider the initial value problem (2.1), (2.2) with $x^0 < \hat{x}(t_0), \; y^0 < \hat{y}(t_0)$. To this end we replace hypotheses (A_6) and (A_7) as follows:

$(\tilde{A}_6)$. For $\varepsilon \in I_{\varepsilon_0}$, $t \in I_\nu$, where ν is any given small positive number, we have

$$\begin{aligned} \frac{d\tilde{x}^2}{dt} &\leq f(\tilde{x}^2(t, x^1), \psi_2(t), t, \varepsilon), \\ \varepsilon\,\frac{d\psi_2}{dt} &\leq g(\tilde{x}^2(t, x^1), \psi_2(t), t, \varepsilon). \end{aligned}$$

Let $I_\nu^- := \{t \in R : t_c - \nu \leq t \leq t_c\}$. It is easy to see that assumption (A_6) differs from assumption $(\tilde{A}_6)$ only on the interval I_ν^-.

$(\tilde{A}_7)$. For $\varepsilon \in I_{\varepsilon_0}$ and $t \in I_\nu$ the function (f, g) is quasimonotone nondecreasing in $\tilde{G}_{(t,\varepsilon)} := \{(x, y) \in R^2 : \tilde{x}^2(t, x^1) \leq x \leq \overline{x}(t, \varepsilon), \psi_2(t) \leq y \leq \overline{y}(t, \varepsilon)\}$ where $(\overline{x}(t, \varepsilon), \overline{y}(t, \varepsilon))$ is the upper solution of (2.1), (2.2).

Then the following theorem is valid.

Theorem 3.2 *Assume the hypotheses* $(A_1)-(A_5), (\tilde{A}_6), (\tilde{A}_7), (A_8)$ *to be valid. Then to any small* $\nu > 0$ *there exists a sufficiently small* $\varepsilon^* = \varepsilon^*(\nu)$ *such that for* $0 < \varepsilon \leq \varepsilon^*(\nu)$ *the initial value problem* (2.1), (2.2) *with* $x^0 < \hat{x}(0)$, $y^0 < \hat{y}(0)$ *has a unique solution* $(x(t, \varepsilon), \; y(t, \varepsilon))$ *satisfying*

$$x(t, \varepsilon) \;=\; \hat{x}(t) + 0(\varepsilon) \quad \textit{for} \quad t \in \overline{I_t} \backslash I_\nu^-$$

$$y(t, \varepsilon) \;=\; \begin{cases} \hat{y}(t) + \Pi_0 y(\tau) + O(\varepsilon) & \textit{for} \quad t_0 < t \leq t_c - \nu \\ \hat{y}(t) + O(\varepsilon^{\frac{1}{2}}) & \textit{for} \quad t_c \leq t < t_c + \nu \\ \hat{y}(t) + O(\varepsilon) & \textit{for} \quad t_c + \nu \leq t \leq t_e. \end{cases}$$

For $t \in I_\nu^-$ *we have*

$$\begin{aligned} \tilde{x}^2(t, x^1) &\leq x(t, \varepsilon) \leq \hat{x}(t) + \sqrt{\varepsilon}\exp(\lambda t), \\ \varphi_2(\tilde{x}^2(t, x^1), t) &\leq y(t, \varepsilon) \leq \hat{y}(t) + \gamma\sqrt{\varepsilon}\exp(\lambda t). \end{aligned}$$

Proof. The proof of this theorem proceeds essentially in the same line as the proof of Theorem 3.1. The upper solution on I_ν is exactly the same, the lower solution differs from that one in Theorem 3.1 on the interval I_ν^- and implies a different estimate on this interval.

4 Example: Fast bimolecular reaction with monomolecular slow reaction

In this section we apply our results to the following differential system which describes a fast bimolecular reaction including slow monomolecular reactions (see [11] and references therein)

$$\begin{aligned} \varepsilon\ \frac{dy}{dt} &= \varepsilon\ (I_a(t) - g_1(y)) - r(y,z), \\ \varepsilon\ \frac{dz}{dt} &= \varepsilon\ (I_b(t) - g_2(z)) - r(y,z). \end{aligned} \tag{4.1}$$

To (4.1) we consider the initial value problem

$$y(t_0,\varepsilon) = y^0,\ \ z(t_0,\varepsilon) = z^0, \qquad t_0 < t \le t_e. \tag{4.2}$$

Concerning the inputs I_a and I_b we assume that they are nonnegative and twice continuously differentiable for $t > t_0$; for g_1, g_2, and r we consider the special case

$$g_1(y) \equiv y,\ \ g_2(z) \equiv z,\ \ r(y,z) \equiv yz. \tag{4.3}$$

By means of the coordinate transformation $y = y, z = y - x$ we get from (4.1) and (4.3) the singularly perturbed system

$$\begin{aligned} \varepsilon\ \frac{dy}{dt} &= \varepsilon\ (I_a(t) -\ y) - y\ (y - x) \equiv g(x,y,t,\varepsilon), \\ \frac{dx}{dt} &= I_a(t) - I_b(t) - x \equiv f(x,y,t,\varepsilon) \end{aligned} \tag{4.4}$$

and the initial condition

$$y(t_0,\varepsilon) = y^0,\ \ x(t_0,\varepsilon) = x^0 = y^0 - z^0. \tag{4.5}$$

The last equation in (4.4) can be integrated. Taking into account (4.5) we obtain

$$\tilde{x}(t,x^0) = e^{-(t-t_0)}\left(x^0 + \int_{t_0}^{t} e^{(s-t_0)}(I_a(s) - I_b(s))ds\right) \tag{4.6}$$

such that (4.4) and (4.5) are equivalent to

$$\varepsilon\ \frac{dy}{dt} = \varepsilon(I_a(t) - y) - y(y - \tilde{x}(t,x^0)), \quad y(t_0,\varepsilon) = y^0. \tag{4.7}$$

The corresponding degenerate equation has the solutions $y = y^{(1)}(t) \equiv 0$, $y = y^{(2)}(t) \equiv \tilde{x}(t,x^0)$. It can be easily checked that $y^{(1)}(t) \equiv 0$ is an asymptotically stable equilibrium of the associated system

$$\frac{dy}{d\tau} = -y(y - \tilde{x}(t,x^0))$$

for $t \in [t_0,t_e]$ provided $\tilde{x}(t,x^0)$ is negative for $t \in [t_0,t_e]$. Consequently, in this case we may apply Theorem 5.3 .

If $\tilde{x}(t,x^0)$ changes its sign at $t = t_c \in (t_0, t_e)$ then we have the case of exchange of stability which was considered in Theorem 3.1. To obtain an explicit expression for the corresponding composed stable solution we consider the special case

$$I_a(t) \equiv 1\,,\ I_b(t) \equiv 1 + \cos t,\ t_0 = 0,\ t_1 = \pi/4.$$

Then (4.6) reads

$$\tilde{x}(t,x^0) \;=\; \left(x^0 + \frac{1}{2}\right) e^{-t} - \frac{\cos t + \sin t}{2}. \tag{4.8}$$

For $0 < x^0 < \frac{1}{2}\left(\sqrt{2}\, e^{\pi/4} - 1\right)$ the equation

$$\left(x^0 + \frac{1}{2}\right) e^{-t} - \frac{\cos t + \sin t}{2} = 0 \tag{4.9}$$

has a unique solution $t = t_c$ in $\left(0, \frac{\pi}{4}\right)$. It can be easily shown that $y^{(2)}(t) \equiv \tilde{x}(t,x^0)$ is stable for $[0, t_c)$ and $y^{(1)}(t) \equiv 0$ is stable for $(t_c, \frac{\pi}{4}]$. Consequently, the composed stable solution reads

$$\hat{y}(t) = \begin{cases} \left(\frac{1}{2} + x^0\right) e^{-t} - \frac{\sin t + \cos t}{2} & \text{for} \quad 0 \le t \le t_c, \\ 0 & \text{for} \quad t_c \le t \le \frac{\pi}{4}. \end{cases}$$

Now we check the hypotheses of Theorem 3.1. In our case it is easy to see that the hypotheses $(A_1) - (A_5)$ are satisfied. From (4.4) we get that g is nondecreasing in x if we replace y by any nonnegative function $\tilde{y}(t)$, additionally we have $-g_{yy} = 2$. Thus, the assumptions (A_6) and (A_8) are valid. Since the derivative of $\hat{y}$ is strictly negative for $0 \le t \le t_c$ and I_a is nonnegative it can be easily verified that $\hat{y}$ fulfills assumption (A_7). Consequently, Theorem 3.1 can be applied to the initial value problem (4.1), (4.2) or equivalently to (4.7).

5 Appendix. Standard results of the asymptotic theory of singularly perturbed systems

Let D_x and D_y be open bounded regions in R^k and R^l respectively, let I_{ε^*} be the interval $I_{\varepsilon^*} := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon^* \ll 1\}$, let $D := D_x \times D_y \times I_t \times I_{\varepsilon^*}$. Concerning the smoothness of f and g we suppose

(T_1). $f : D \to R^k,\ g : D \to R^l$ are continuous and continuously differentiable with respect to the first three variables, where all derivatives are continuous in $\bar{D}$.

It is obvious that the asymptotic behavior of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (2.1), (2.2) with respect to ε depends on the solution set of the equation

$$g(x,y,t,0) = 0. \tag{5.1}$$

The first result in this direction is due to A.N. Tikhonov [16, 17]. To formulate his result we introduce the assumptions:

(T_2). Equation (5.1) has an isolated solution $y = \varphi(x,t)$ defined for $(x,t) \in D_x^0 \times \overline{I}_t$ where D_x^0 is a closed simply connected subset of D_x , and $\overline{I}_t := [t_0, t_1]$.

(T_3). The initial value problem

$$\frac{dx}{dt} = f(x, \varphi(x,t), t, 0) \ , \ x(t_0) = x^0 \in D_x^0 \tag{5.2}$$

has a unique solution $\tilde{x}(t; t_0)$ defined on I_t.

(T_4). $y = \varphi(x,t)$ is an asymptotically stable equilibrium point of the associated system

$$\frac{dy}{d\tau} = g(x, y, t, 0) \tag{5.3}$$

uniformly for $(x,t) \in D_x^0 \times \overline{I}_t$ (x and t are considered as parameters in (5.3)).

(T_5). The initial value problem

$$\frac{dy}{d\tau} = g(x^0, y, t_0, 0), \quad y(0) = y^0 \tag{5.4}$$

has a unique solution $\tilde{y}(\tau, y^0)$ which exists for $\tau \geq 0$ and tends to $\varphi(x^0, t_0)$ as $\tau \to \infty$.

Hypothesis (T_5) says that y^0 is in the basin of attraction of the equilibrium point $\varphi(x^0, t_0)$ of (5.4).

A.N. Tikhonov has got essentially the result

Theorem 5.1 *Suppose the hypotheses $(T_1) - (T_5)$ hold. Then there exists a sufficiently small positive ε_0 such that for $0 < \varepsilon \leq \varepsilon_0$ the initial value problem* (2.1), (2.2) *has a unique solution $(x(t,\varepsilon), y(t,\varepsilon))$ satisfying*

$$\begin{aligned} \lim_{\varepsilon \to 0} x(t,\varepsilon) &= \tilde{x}(t, x^0) && \textit{for} \ \ t_0 \leq t \leq t_1, \\ \lim_{\varepsilon \to 0} y(t,\varepsilon) &= \varphi(\tilde{x}(t, x^0), t) && \textit{for} \ \ t_0 < t \leq t_1. \end{aligned}$$

In order to formulate the next theorem which is due to A.B. Vasil'eva [18], we introduce the concept of an asymptotic expansion of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (2.1), (2.2).

Definition 5.2

An asymptotic expansion of the solution $(x(t,\varepsilon), y(t,\varepsilon))$ of (2.1), (2.2) *is a representation of $x(t,\varepsilon)$ and $y(t,\varepsilon)$ in the form*

$$z_a(t,\varepsilon) = Rz(t,\varepsilon) + \Pi z(\tau, \varepsilon) \tag{5.5}$$

where z is a placeholder for x and y respectively, $Rz(t,\varepsilon)$ is the regular part of the asymptotics, that is,

$$Rz(t,\varepsilon) := \sum_{i=0}^{\infty} \varepsilon^i R_i z(t), \tag{5.6}$$

and $\Pi z(\tau,\varepsilon)$ is the boundary layer correction near $t=t_0$,

$$\Pi z(\tau,\varepsilon) := \sum_{i=0}^{\infty} \varepsilon^i \Pi_i z(\tau) \tag{5.7}$$

where τ is the stretched variable $\tau=(t-t_0)/\varepsilon$. We denote by $Z_k(t,\varepsilon)$ the truncated part of (5.5)

$$Z_k(t,\varepsilon) = \sum_{i=0}^{k} \varepsilon^i (R_i z(t) + \Pi_i z(\tau)).$$

Let F be some function defined on $R^k \times R \times I_{\varepsilon^*}$. By means of the representation (5.5) we may rewrite $F(z_a(t,\varepsilon),t,\varepsilon)$ in the form

$$\begin{aligned} F(z_a(t,\varepsilon),t,\varepsilon) &= F(Rz(t,\varepsilon),t,\varepsilon) + F(z_a(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon) \\ &\quad -F(Rz(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon) =: RF + \Pi F \end{aligned} \tag{5.8}$$

where

$$RF := F(Rz(t,\varepsilon),t,\varepsilon),\ \Pi F := F(z_a(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon) - F(Rz(\tau\varepsilon,\varepsilon),\tau\varepsilon,\varepsilon). \tag{5.9}$$

In order to compute the coefficients $R_i z(t)$ and $\Pi_i z(\tau)$ we substitute (5.5)$-$(5.7) into (2.1), (2.2) and use the representation (5.8), (5.9). By equating expressions with the same power of ε (separately for t and τ) we obtain equations which determine the unknown coefficients of the asymptotic expansion. In particular, by assumption (T_2), $R_0 x(t)$ and $R_0 y(t)$ are uniquely determined by the degenerate system (5.1) and the initial value $x^0 : R_0 x(t) = \tilde{x}(t,x^0)$, $R_0 y(t) = \varphi(R_0 x(t),t)$. Note that $\Pi x_0(\tau)$ and $\Pi y_0(\tau)$ are determined by the initial value problems (see [18])

$$\begin{aligned} \frac{d\Pi_0 y}{d\tau} &= \Pi_0 g(R_0 x(t_0) + \Pi_0 x(\tau), R_0 y(t_0) + \Pi_0 y(\tau), t_0, 0), \quad \Pi_0 y(t_0) = y^0 - R_0 y(t_0), \\ \frac{d\Pi_0 x}{d\tau} &= 0, \quad \Pi_0 x(t_0) = 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} \Pi_0 x(\tau) &\equiv 0 \\ \frac{d\Pi_0 y}{d\tau} &= \Pi_0 g(x^0, \varphi(x^0,t_0) + \Pi_0 y(\tau), t_0, 0). \end{aligned} \tag{5.10}$$

Finally, we strengthen the assumptions (T_1) and (T_4) as follows.

$(\tilde{T_1})$. *The functions $f : D \to R^k$ and $g : D \to R^l$ are $(k+2)$-times continuously differentiable with respect to all variables, where all derivatives are continuous in $\bar{D}$.*

$(\tilde{T_4})$. *All eigenvalues $\lambda_i(t)$ of the Jacobian $g_y(\tilde{x}(t,x^0), \varphi(\tilde{x}(t,x^0),t),t,0)$ satisfy*

$$\mathrm{Re}\ \lambda_i(t) < 0 \quad \text{for} \quad t \in \overline{I}_t, \quad 1 \le i \le m.$$

Theorem 5.3 *We assume the hypotheses* $(\tilde{T}_1), (T_2), (T_3), (\tilde{T}_4), (T_5)$ *to hold. Let* $(X_k(t,\varepsilon),\, Y_k(t,\varepsilon))$ *be the truncated parts of the asymptotic expansion of the solution of problem* (2.1), (2.2) *obtained by the method of boundary layer functions* (see [18], [19] for details). *Then there exists a sufficiently small* ε_0 *and a constant* $c = c(\varepsilon_0)$ *such that for* $0 < \varepsilon \leq \varepsilon_0$ *the initial value problem* (2.1), (2.2) *has a unique solution* $(x(t,\varepsilon), y(t,\varepsilon))$ *for* $t \in I_t$ *satisfying*

$$\begin{aligned} |x(t,\varepsilon) - X_k(t,\varepsilon)| &\leq c\,\varepsilon^{k+1}, \\ |y(t,\varepsilon) - Y_k(t,\varepsilon)| &\leq c\,\varepsilon^{k+1}. \end{aligned}$$

In particular, we have for $k = 0$:

$$x(t,\varepsilon) = R_0 x(t) + O(\varepsilon),\;\; y(t,\varepsilon) = R_0 y(t) + \Pi_0 y(\tau) + O(\varepsilon)$$

where $\Pi_0 y$ *is defined by (5.10).*

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