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HOLOMORPHIC OPERATOR VALUED FUNCTIONS GENERATED BY PASSIVE SELFADJOINT SYSTEMS

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Dedicated to Professor Joseph Ball on the occasion of his 70-th birthday

ABSTRACT. Let $\mathfrak{M}$ be a Hilbert space. In this paper we study a class $\mathcal{RS}(\mathfrak{M})$ of operator functions that are holomorphic in the domain $\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$ and whose values are bounded linear operators in $\mathfrak{M}$. The functions in $\mathcal{RS}(\mathfrak{M})$ are Schur functions in the open unit disk $\mathbb{D}$ and, in addition, Nevanlinna functions in $\mathbb{C}_+\cup\mathbb{C}_-$. Such functions can be realized as transfer functions of minimal passive selfadjoint discrete-time systems. We give various characterizations for the class $\mathcal{RS}(\mathfrak{M})$ and obtain an explicit form for the inner functions from the class $\mathcal{RS}(\mathfrak{M})$ as well as an inner dilation for any function from $\mathcal{RS}(\mathfrak{M})$. We also consider various transformations of the class $\mathcal{RS}(\mathfrak{M})$, construct realizations of their images, and find corresponding fixed points.

1. Introduction

Throughout this paper we consider separable Hilbert spaces over the field $\mathbb{C}$ of complex numbers and certain classes of operator valued functions which are holomorphic on the open upper/lower half-planes $\mathbb{C}_+/\mathbb{C}_-$ and/or on the open unit disk $\mathbb{D}$. A $\mathbf{B}(\mathfrak{M})$-valued function M is called a *Nevanlinna function* if it is holomorphic outside the real axis, symmetric $M(\lambda)^*=M(\bar\lambda)$, and satisfies the inequality $\operatorname{Im}\lambda\operatorname{Im}M(\lambda)\ge 0$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. This last condition is equivalent to the nonnegativity of the kernel

$$\frac{M(\lambda)-M(\mu)^*}{\lambda-\bar\mu},\quad \lambda,\mu\in\mathbb{C}_+\cup\mathbb{C}_-.$$

On the other hand, a $\mathbf{B}(\mathfrak{M})$-valued function $\Theta(z)$ belongs to the *Schur class* if it is holomorphic on the unit disk $\mathbb{D}$ and contractive, $||\Theta(z)||\le 1\ \forall z\in\mathbb{D}$ or, equivalently, the kernel

$$\frac{I-\Theta^*(w)\Theta(z)}{1-z\bar w},\quad z,w\in\mathbb{D}$$

is nonnegative. Functions from the Schur class appear naturally in the study of linear discrete-time systems; we briefly recall some basic terminology here; cf. D.Z. Arov [7, 8]. Let T be a bounded operator given in the block form

$$T=\begin{bmatrix}D & C\\ B & A\end{bmatrix}:\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathcal{K}\end{array}\to\begin{array}{c}\mathfrak{N}\\ \oplus\\ \mathcal{K}\end{array}\tag{1.1}$$

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with separable Hilbert spaces $\mathfrak{M}, \mathfrak{N}$, and $\mathfrak{K}$. The system of equations

$$\left\{ \begin{array}{l} h_{k+1} = Ah_k + B\xi_k, \\ \sigma_k = Ch_k + D\xi_k, \end{array} \right. \qquad k \geq 0, \tag{1.2}$$

describes the evolution of a *linear discrete time-invariant system* $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathfrak{K}\}$. Here $\mathfrak{M}$ and $\mathfrak{N}$ are called the input and the output spaces, respectively, and $\mathfrak{K}$ is the state space. The operators A, B, C, and D are called the main operator, the control operator, the observation operator, and the feedthrough operator of τ, respectively. The subspaces

$$\mathfrak{K}^c = \overline{\operatorname{span}}\,\{\, A^n B\mathfrak{M} :\, n \in \mathbb{N}_0\} \quad \text{and} \quad \mathfrak{K}^o = \overline{\operatorname{span}}\,\{\, A^{*n} C^*\mathfrak{N} :\, n \in \mathbb{N}_0\} \tag{1.3}$$

are called the controllable and observable subspaces of $\tau = \{T, \mathfrak{M}, \mathfrak{N}, \mathfrak{K}\}$, respectively. If $\mathfrak{K}^c = \mathfrak{K}$ ($\mathfrak{K}^o = \mathfrak{K}$) then the system τ is said to be *controllable* (*observable*), and *minimal* if τ is both controllable and observable. If $\mathfrak{K} = \operatorname{clos}\{\mathfrak{K}^c + \mathfrak{K}^o\}$ then the system τ is said to be a *simple*. Closely related to these definitions is the notion of $\mathfrak{M}$-simplicity: given a nontrivial subspace $\mathfrak{M} \subset \mathfrak{H}$ the operator T acting in $\mathfrak{H}$ is said to be $\mathfrak{M}$-*simple* if

$$\overline{\operatorname{span}}\,\{\, T^n\mathfrak{M},\, n \in \mathbb{N}_0\} = \mathfrak{H}.$$

Two discrete-time systems $\tau_1 = \{T_1, \mathfrak{M}, \mathfrak{N}, \mathfrak{K}_1\}$ and $\tau_2 = \{T_2, \mathfrak{M}, \mathfrak{N}, \mathfrak{K}_2\}$ are *unitarily similar* if there exists a unitary operator U from $\mathfrak{K}_1$ onto $\mathfrak{K}_2$ such that

$$A_2 = UA_1U^*, \quad B_2 = UB_1, \quad C_2 = C_1U^*, \text{ and } D_2 = D_1. \tag{1.4}$$

If the linear operator T is contractive (isometric, co-isometric, unitary), then the corresponding discrete-time system is said to be *passive* (*isometric, co-isometric, conservative*). With the passive system τ in (1.2) one associates the *transfer function* via

$$\Omega_\tau(z) := D + zC(I - zA)^{-1}B, \quad z \in \mathbb{D}. \tag{1.5}$$

It is well known that the transfer function of a passive system belongs to the *Schur class* $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ and, conversely, that every operator valued function $\Theta(\lambda)$ from the Schur class $\mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be realized as the transfer function of a passive system, which can be chosen as observable co-isometric (controllable isometric, simple conservative, passive minimal). Notice that an application of the Schur-Frobenius formula (see Appendix A) for the inverse of a block operator gives with $\mathfrak{M} = \mathfrak{N}$ the relation

$$P_{\mathfrak{M}}(I - zT)^{-1}{\upharpoonright}\,\mathfrak{M} = (I_{\mathfrak{M}} - z\Omega_\tau(z))^{-1}, \quad z \in \mathbb{D}. \tag{1.6}$$

It is known that two isometric and controllable (co-isometric and observable, simple conservative) systems with the same transfer function are unitarily similar. However, D.Z. Arov [7] has shown that two minimal passive systems τ_1 and τ_2 with the same transfer function $\Theta(\lambda)$ are only weakly similar; weak similarity neither preserves the dynamical properties of the system nor the spectral properties of its main operator A. Some necessary and sufficient conditions for minimal passive systems with the same transfer function to be (unitarily) similar have been established in [9, 10].

By introducing some further restrictions on the passive system τ it is possible to preserve unitary similarity of passive systems having the same transfer function. In particular, when the main operator A is normal such results have been obtained in [5]; see in particular Theorem 3.1 and Corollaries 3.6–3.8 therein. A stronger condition on τ where main operator is selfadjoint naturally yields to a class of systems which preserve such a unitary similarity property. A class of such systems appearing in [5] is the class of *passive quasi-selfadjoint systems*, in short *pqs-systems*, which is defined as follows: a collection

$$\tau = \{T, \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$$

is a *pqs*-system if the operator T determined by the block formula (1.1) with the input-output space $\mathfrak{M} = \mathfrak{N}$ is a contraction and, in addition,

$$\operatorname{ran}(T - T^*) \subseteq \mathfrak{M}.$$

Then, in particular, $F = F^*$ and $B = C^*$ so that T takes the form

$$T = \begin{bmatrix} D & C \\ C^* & F \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{matrix},$$

i.e., T is a quasi-selfadjoint contraction in the Hilbert space $\mathfrak{H} = \mathfrak{M} \oplus \mathcal{K}$. The class of *pqs*-systems gives rise to transfer functions which belong to the subclass $\mathcal{S}^{qs}(\mathfrak{M})$ of Schur functions. The class $\mathcal{S}^{qs}(\mathfrak{M})$ admits the following intrinsic description; see [5, Definition 4.4, Proposition 5.3]: a $\mathbf{B}(\mathfrak{M})$-valued function Ω belongs to $\mathcal{S}^{qs}(\mathfrak{M})$ if it is holomorphic on $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ and has the following additional properties:

(S1) $W(z) = \Omega(z) - \Omega(0)$ is a Nevanlinna function;
(S2) the strong limit values $W(\pm 1)$ exist and $W(1) - W(-1) \leq 2I$;
(S3) $\Omega(0)$ belongs to the operator ball

$$\mathcal{B}\left(-\frac{W(1) + W(-1)}{2},\ I - \frac{W(1) - W(-1)}{2}\right)$$

with the center $-\dfrac{W(1) + W(-1)}{2}$ and with the left and right radii $I - \dfrac{W(1) - W(-1)}{2}$.

It was proved in [5, Theorem 5.1] that the class $\mathcal{S}^{qs}(\mathfrak{M})$ coincides with the class of all transfer functions of *pqs*-systems with input-output space $\mathfrak{M}$. In particular, every function from the class $\mathcal{S}^{qs}(\mathfrak{M})$ can be realized as the transfer function of a *minimal pqs*-system and, moreover, two minimal realization are unitarily equivalent; see [3, 5, 6]. For *pqs*-systems the controllable and observable subspaces $\mathcal{K}^c$ and $\mathcal{K}^o$ as defined in (1.3) necessarily coincide. Furthermore, the following equivalences were established in [6]:

$$\begin{aligned} T \text{ is } \mathfrak{M}\text{-simple} &\iff \text{the operator } F \text{ is } \overline{\operatorname{ran}}\, C^* - \text{simple in } \mathcal{K} \\ &\iff \text{the system} \quad \tau = \left\{ \begin{bmatrix} D & C \\ C^* & F \end{bmatrix}, \mathfrak{M}, \mathfrak{M}, \mathcal{K} \right\} \text{ is minimal.} \end{aligned}$$

We can now introduce one of the main objects to be studied in the present paper.

Definition 1.1. *Let $\mathfrak{M}$ be a Hilbert space. A $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function Ω which is holomorphic on $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ is said to belong to the class $\mathcal{RS}(\mathfrak{M})$ if*

$$-I \leq \Omega(x) \leq I, \quad x \in (-1, 1).$$

The class $\mathcal{RS}(\mathfrak{M})$ will be called the combined Nevanlinna-Schur class of $\mathbf{B}(\mathfrak{M})$-valued operator functions.

If $\Omega \in \mathcal{RS}(\mathfrak{M})$, then $\Omega(x)$ is non-decreasing on the interval $(-1, 1)$. Therefore, the strong limit values $\Omega(\pm 1)$ exist and satisfy the following inequalities

$$-I_{\mathfrak{M}} \leq \Omega(-1) \leq \Omega(0) \leq \Omega(1) \leq I_{\mathfrak{M}}. \tag{1.7}$$

It follows from (S1)–(S3) that the class $\mathcal{RS}(\mathfrak{M})$ is a subclass of the class $\mathcal{S}^{qs}(\mathfrak{M})$.

In this paper we give some new characterizations of the class $\mathcal{RS}(\mathfrak{M})$, find an explicit form for inner functions from the class $\mathcal{R}(\mathfrak{M})$, and construct a bi-inner dilation for an arbitrary function from $\mathcal{RS}(\mathfrak{M})$. For instance, in Theorem 4.1 it is proven that a $\mathbf{B}(\mathfrak{M})$-valued

Nevanlinna function defined on $\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$ belongs to the class $\mathcal{RS}(\mathfrak{M})$ if and only if

$$K(z,w):=I_{\mathfrak{M}}-\Omega^*(w)\Omega(z)-\frac{1-\bar{w}z}{z-\bar{w}}\,(\Omega(z)-\Omega^*(w))$$

defines a nonnegative kernel on the domains

$$\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\},\quad \operatorname{Im} z>0\quad\text{and}\quad \mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\},\quad \operatorname{Im} z<0.$$

We also show that the transformation

$$\mathcal{RS}(\mathfrak{M})\ni\Omega\mapsto\mathbf{\Phi}(\Omega)=\Omega_{\mathbf{\Phi}},\quad \Omega_{\mathbf{\Phi}}(z):=(zI-\Omega(z))(I-z\Omega(z))^{-1},\tag{1.8}$$

with $z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$ is an automorphism of $\mathcal{RS}(\mathfrak{M})$, $\mathbf{\Phi}^{-1}=\mathbf{\Phi}$, and that $\mathbf{\Phi}$ has a unique fixed point, which will be specified in Proposition 6.6.

It turns out that the set of inner functions from the class $\mathcal{RS}(\mathfrak{M})$ can be seen as the image $\mathbf{\Phi}$ of constant functions from $\mathcal{RS}(\mathfrak{M})$: in other words, the inner functions from $\mathcal{RS}(\mathfrak{M})$ are of the form

$$\Omega_{\rm in}(z)=(zI+A)(I+zA)^{-1},\ A\in[-I_{\mathfrak{M}},I_{\mathfrak{M}}].$$

In Theorem 6.3 it is proven that every function $\Omega\in\mathcal{RS}(\mathfrak{M})$ admits the representation

$$\Omega(z)=P_{\mathfrak{M}}\widetilde{\Omega}_{in}(z)\!\upharpoonright\!\mathfrak{M}=P_{\mathfrak{M}}(zI+\widetilde{A})(I+z\widetilde{A})^{-1}\!\upharpoonright\!\mathfrak{M},\quad \widetilde{A}\in[-I_{\widetilde{\mathfrak{M}}},I_{\widetilde{\mathfrak{M}}}],\tag{1.9}$$

where $z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$ and $\widetilde{\mathfrak{M}}$ is a Hilbert space containing $\mathfrak{M}$ as a subspace and such that $\overline{\operatorname{span}}\,\{\widetilde{A}^n\mathfrak{M}:\ n\in\mathbb{N}_0\}=\widetilde{\mathfrak{M}}$ (i.e., $\widetilde{A}$ is $\mathfrak{M}$-simple). Equality (1.9) means that an arbitrary function of the class $\mathcal{RS}(\mathfrak{M})$ admits a bi-inner dilation (in the sense of [8]) that belongs to the class $\mathcal{RS}(\widetilde{\mathfrak{M}})$.

In Section 6 we also consider the following transformations of the class $\mathcal{RS}(\mathfrak{M})$:

$$\Omega\left(\frac{z+a}{1+za}\right)=:\Omega_a(z)\leftarrowtail\Omega(z)\rightarrowtail\widehat{\Omega}_a(z):=(aI+\Omega(z))(I+a\Omega(z))^{-1},\tag{1.10}$$
$$a\in(-1,1),z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}.$$

These are analogs of the Möbius transformation

$$w_a(z)=\frac{z+a}{1+az},\quad z\in\mathbb{C}\setminus\{-a^{-1}\}\ (a\in(-1,1),\ a\neq0)$$

of the complex plane. The mapping w_a is an automorphism of $\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$ and it maps $\mathbb{D}$ onto $\mathbb{D}$, $[-1,1]$ onto $[-1,1]$, $\mathbb{T}$ onto $\mathbb{T}$, as well as $\mathbb{C}_+/\mathbb{C}_-$ onto $\mathbb{C}_+/\mathbb{C}_-$.

The mapping

$$\mathcal{RS}(\mathfrak{M})\ni\Omega\mapsto\Omega_a(z)=\Omega\left(\frac{z+a}{1+za}\right)\in\mathcal{RS}(\mathfrak{M})$$

can be rewritten as

$$\Omega\mapsto\Omega\circ w_a.$$

In Proposition 6.13 it is shown that the fixed points of this transformation consist only of the constant functions from $\mathcal{RS}(\mathfrak{M})$: $\Omega(z)\equiv A$ with $A\in[-I_{\mathfrak{M}},I_{\mathfrak{M}}]$.

One of the operator analogs of w_a is the following transformation of $\mathbf{B}(\mathfrak{M})$:

$$W_a(T)=(T+aI)(I+aT)^{-1},\quad a\in(-1,1).$$

The inverse of W_a is given by

$$W_{-a}(T)=(T-aI)(I-aT)^{-1}.$$

The class $\mathcal{RS}(\mathfrak{M})$ is stable under the transform W_a:

$$\Omega \in \mathcal{RS}(\mathfrak{M}) \Longrightarrow W_a \circ \Omega \in \mathcal{RS}(\mathfrak{M}).$$

If T is selfadjoint and unitary (a fundamental symmetry), i.e., $T = T^* = T^{-1}$, then for every $a \in (-1, 1)$ one has

$$W_a(T) = T \tag{1.11}$$

Conversely, if for a selfadjoint operator T the equality (1.11) holds for some $a : -a^{-1} \in \rho(T)$, then T is a fundamental symmetry and (1.11) is valid for all $a \neq \{\pm 1\}$.

One can interpret the mappings in (1.10) as $\Omega \circ w_a$ and $W_a \circ \Omega$, where $\Omega \in \mathcal{RS}(\mathfrak{M})$. Theorem 6.18 states that inner functions from $\mathcal{RS}(\mathfrak{M})$ are the only fixed points of the transformation

$$\mathcal{RS}(\mathfrak{M}) \ni \Omega \mapsto W_{-a} \circ \Omega \circ w_a.$$

An equivalent statement is that the equality

$$\Omega \circ w_a = W_a \circ \Omega$$

holds only for inner functions Ω from the class $\mathcal{RS}(\mathfrak{M})$. On the other hand, it is shown in Theorem 6.19 that the only solutions of the functional equation

$$\Omega(z) = \left(\Omega\left(\frac{z-a}{1-az}\right) - a\, I_{\mathfrak{M}}\right)\left(I_{\mathfrak{M}} - a\,\Omega\left(\frac{z-a}{1-az}\right)\right)^{-1}$$

in the class $\mathcal{RS}(\mathfrak{M})$, where $a \in (-1, 1)$, $a \neq 0$, are constant functions Ω, which are fundamental symmetries in $\mathfrak{M}$.

To introduce still one further transform, let

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ H \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ H \end{matrix}$$

be a selfadjoint contraction and consider the mapping

$$\mathcal{RS}(H) \ni \Omega \;\mapsto\; \Omega_{\mathbf{K}}(z) := K_{11} + K_{12}\Omega(z)(I - K_{22}\Omega(z))^{-1}K_{12}^*,$$

where $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. In Theorem 6.8 we prove that if $||K_{22}|| < 1$, then $\Omega_{\mathbf{K}} \in \mathcal{RS}(\mathfrak{M})$ and in Theorem 6.9 we construct a realization of $\Omega_{\mathbf{K}}$ by means of realization of $\Omega \in \mathcal{RS}(H)$ using the so-called *Redheffer product*; see [17, 21]. The mapping

$$\mathbf{B}(H) \ni T \mapsto K_{11} + K_{12}T(I - K_{22}T)^{-1}K_{21} \in \mathbf{B}(\mathfrak{M})$$

can be considered as one further operator analog of the Möbius transformation, cf. [18].

Finally, it is emphasized that in Section 6 we will systematically construct *explicit realizations* for each of the transforms $\mathbf{\Phi}(\Omega)$, Ω_a, and $\widehat{\Omega}_a$ as transfer functions of minimal passive selfadjoint systems using a minimal realization of the initially given function $\Omega \in \mathcal{RS}(H)$.

Basic notations. We use the symbols $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ for the domain, the range, and the kernel of a linear operator T. The closures of $\operatorname{dom} T$, $\operatorname{ran} T$ are denoted by $\overline{\operatorname{dom}}\, T$, $\overline{\operatorname{ran}}\, T$, respectively. The identity operator in a Hilbert space $\mathfrak{H}$ is denoted by I and sometimes by $I_{\mathfrak{H}}$. If $\mathfrak{L}$ is a subspace, i.e., a closed linear subset of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. The notation $T{\upharpoonright}\mathfrak{L}$ means the restriction of a linear operator T on the set $\mathfrak{L} \subset \operatorname{dom} T$. The resolvent set of T is denoted by $\rho(T)$. The linear space of bounded operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is denoted by $\mathbf{B}(\mathfrak{H}, \mathfrak{K})$ and the Banach algebra $\mathbf{B}(\mathfrak{H}, \mathfrak{H})$ by $\mathbf{B}(\mathfrak{H})$. For a contraction $T \in \mathbf{B}(\mathfrak{H}, \mathfrak{K})$ the defect operator $(I - T^*T)^{1/2}$ is denoted by D_T and $\mathfrak{D}_T := \overline{\operatorname{ran}}\, D_T$. For defect operators one has the commutation relations

$$TD_T = D_{T^*}T, \quad T^*D_{T^*} = D_T T^* \tag{1.12}$$

and, moreover,

$$\operatorname{ran} TD_T = \operatorname{ran} D_{T^*}T = \operatorname{ran} T \cap \operatorname{ran} D_{T^*}. \tag{1.13}$$

In what follows we systematically use the Schur-Frobenius formula for the resolvent of a block-operator matrix and parameterizations of contractive block operators, see Appendices A and B.

2. The combined Nevanlinna-Schur class $\mathcal{RS}(\mathfrak{M})$

In this section some basic properties of operator functions belonging to the combined Nevanlinna-Schur class $\mathcal{RS}(\mathfrak{M})$ are derived. As noted in Introduction every function $\Omega \in \mathcal{RS}(\mathfrak{M})$ admits a realization as the transfer function of a passive selfadjoint system. In particular, the function $\Omega{\upharpoonright}\mathbb{D}$ belongs to the Schur class $\mathcal{S}(\mathfrak{M})$.

It is known from [1] that, if $\Omega \in \mathcal{RS}(\mathfrak{M})$ then for every $\beta \in [0, \pi/2)$ the following implications are satisfied:

$$\begin{array}{l} \left\{\begin{array}{l} |z\sin\beta + i\cos\beta| \le 1 \\ \quad z \ne \pm 1 \end{array}\right. \Longrightarrow \|\Omega(z)\sin\beta + i\cos\beta\, I\| \le 1 \\ \left\{\begin{array}{l} |z\sin\beta - i\cos\beta| \le 1 \\ \quad z \ne \pm 1 \end{array}\right. \Longrightarrow \|\Omega(z)\sin\beta - i\cos\beta\, I\| \le 1 \end{array}. \tag{2.1}$$

In fact, in Section 4 these implications will be we derived once again by means of some new characterizations for the class $\mathcal{RS}(\mathfrak{M})$.

To describe some further properties of the class $\mathcal{RS}(\mathfrak{M})$ consider a passive selfadjoint system given by

$$\tau = \left\{ \begin{bmatrix} D & C \\ C^* & F \end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathcal{K} \right\}, \tag{2.2}$$

with $D = D^*$ and $F = F^*$. It is known, see Proposition B.1 and Remark B.2 in Appendix B, that the entries of the selfadjoint contraction

$$T = \begin{bmatrix} D & C \\ C^* & F \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{array} \to \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{array} \tag{2.3}$$

admit the parametrization

$$C = KD_F, \quad D = -KFK^* + D_{K^*}YD_{K^*}, \tag{2.4}$$

where $K \in \mathbf{B}(\mathfrak{D}_F, \mathfrak{M})$ is a contraction and $Y \in \mathbf{B}(\mathfrak{D}_{K^*})$ is a selfadjoint contraction. The minimality of the system τ means that the following equivalent equalities hold:

$$\overline{\operatorname{span}}\, \{F^n D_F K^*,\ n \in \mathbb{N}_0\} = \mathcal{K} \Longleftrightarrow \bigcap_{n\in\mathbb{N}_0} \ker(KF^nD_F) = \{0\}. \tag{2.5}$$

Notice that if τ is minimal, then necessarily $\mathcal{K} = \mathfrak{D}_F$ or, equivalently, $\ker D_F = \{0\}$.

Recall from [20] the Sz.-Nagy – Foias characteristic function of the selfadjoint contraction F, which for every $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ is given by

$$\begin{aligned} \Delta_F(z) &= \big(-F + zD_F(I - zF)^{-1}D_F\big){\upharpoonright}\mathfrak{D}_F \\ &= \big(-F + z(I - F^2)(I - zF)^{-1}\big){\upharpoonright}\mathfrak{D}_F \\ &= (zI - F)(I - zF)^{-1}{\upharpoonright}\mathfrak{D}_F. \end{aligned}$$

Using the above parametrization one obtains the representations, cf. [5, Theorem 5.1],

$$\begin{aligned}\Omega(z) &= D + zC(I - zF)^{-1}C^* = D_{K^*}YD_{K^*} + K\Delta_F(z)K^* \\ &= D_{K^*}YD_{K^*} + K(zI - F)(I - zF)^{-1}K^*.\end{aligned} \tag{2.6}$$

Moreover, this gives the following representation for the limit values $\Omega(\pm 1)$:

$$\Omega(-1) = -KK^* + D_{K^*}YD_{K^*}, \quad \Omega(1) = KK^* + D_{K^*}YD_{K^*}. \tag{2.7}$$

The case $\Omega(\pm 1)^2 = I_{\mathfrak{M}}$ is of special interest and can be characterized as follows.

Proposition 2.1. *Let $\mathfrak{M}$ be a Hilbert space and let $\Omega \in \mathcal{RS}(\mathfrak{M})$. Then the following statements are equivalent:*

(i) $\Omega(1)^2 = \Omega(-1)^2 = I_{\mathfrak{M}}$;

(ii) *the equalities*

$$\begin{aligned}\left(\frac{\Omega(1) - \Omega(-1)}{2}\right)^2 &= \frac{\Omega(1) - \Omega(-1)}{2}, \\ \left(\frac{\Omega(1) + \Omega(-1)}{2}\right)^2 &= I_{\mathfrak{M}} - \frac{\Omega(1) - \Omega(-1)}{2}\end{aligned} \tag{2.8}$$

hold;

(iii) *if $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ is a passive selfadjoint system* (2.2) *with the transfer function Ω and if the entries of the block operator T are parameterized by* (2.4), *then the operator $K \in \mathbf{B}(\mathfrak{D}_F, \mathfrak{M})$ is a partial isometry and $Y^2 = I_{\ker K^*}$.*

Proof. From (2.7) we get for all $f \in \mathfrak{M}$

$$||f||^2 - ||\Omega(\pm 1)f||^2 = ||f||^2 - ||(D_{K^*}YD_{K^*} \pm KK^*)f||^2 = ||(K^*(I \mp Y)D_{K^*}f||^2 + ||D_Y D_{K^*}f||^2;$$

cf. [4, Lemma 3.1]. Hence

$$\Omega(1)^2 = \Omega(-1)^2 = I_{\mathfrak{M}} \Longleftrightarrow \left\{\begin{array}{l} K^*(I - Y)D_{K^*} = 0 \\ K^*(I + Y)D_{K^*} = 0 \\ D_Y D_{K^*} = 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} K^*D_{K^*} = D_K K^* = 0 \\ K^*YD_{K^*} = 0 \\ D_Y D_{K^*} = 0 \end{array}\right.$$

$$\Longleftrightarrow \left\{\begin{array}{l} K \quad \text{is a partial isometry} \\ Y^2 = I_{\mathfrak{D}_{K^*}} = I_{\ker K^*} \end{array}\right. .$$

Thus (i)$\Longleftrightarrow$(iii).

Since K is a partial isometry, i.e., KK^* is an orthogonal projection, the formulas (2.7) imply that

$$K \quad \text{is a partial isometry} \Longleftrightarrow \left(\frac{\Omega(1) - \Omega(-1)}{2}\right)^2 = \frac{\Omega(1) - \Omega(-1)}{2},$$

and in this case $D_{K^*}Y = Y$, which implies that

$$Y^2 = I_{\mathfrak{D}_{K^*}} = I_{\ker K^*} \Longleftrightarrow \left(\frac{\Omega(1) + \Omega(-1)}{2}\right)^2 = I_{\mathfrak{M}} - \frac{\Omega(1) - \Omega(-1)}{2}.$$

Thus (iii) $\Longleftrightarrow$(ii). □

By interchanging the roles of the subspaces $\mathcal{K}$ and $\mathfrak{M}$ as well as the roles of the corresponding blocks of T in (2.3) leads to the passive selfadjoint system

$$\eta = \left\{ \begin{bmatrix} D & C \\ C^* & F \end{bmatrix}, \mathcal{K}, \mathcal{K}, \mathfrak{M} \right\}$$

now with the input-output space $\mathcal{K}$ and the state space $\mathfrak{M}$. The transfer function of η is given by

$$B(z) = F + zC^*(I - zD)^{-1}C, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

By applying Appendix B again one gets for (2.4) the following alternative expression to parameterize the blocks of T:

$$C = D_D N^*, \quad F = -NDN^* + D_{N^*} X D_{N^*}, \tag{2.9}$$

where $N : \mathfrak{D}_D \to \mathcal{K}$ is a contraction and X is a selfadjoint contraction in $\mathfrak{D}_{N^*}$. Now, similar to (2.7) one gets

$$B(1) = NN^* + D_{N^*} X D_{N^*}, \quad B(-1) = -NN^* + D_{N^*} X D_{N^*}.$$

For later purposes, define the selfadjoint contraction $\widehat{F}$ by

$$\widehat{F} := D_{N^*} X D_{N^*} = \frac{B(-1) + B(1)}{2}. \tag{2.10}$$

The statement in the next lemma can be checked with a straightforward calculation.

Lemma 2.2. *Let the entries of the selfadjoint contraction*

$$T = \begin{bmatrix} D & C \\ C^* & F \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{array} \to \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{array}$$

be parameterized by the formulas (2.9) *with a contraction $N : \mathfrak{D}_D \to \mathcal{K}$ and a selfadjoint contraction X in $\mathfrak{D}_{N^*}$. Then the function $W(\cdot)$ defined by*

$$W(z) = I + zDN^* \left(I - z\widehat{F} \right)^{-1} N, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}, \tag{2.11}$$

where $\widehat{F}$ is given by (2.10)*, is invertible and*

$$W(z)^{-1} = I - zDN^*(I - zF)^{-1}N, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}. \tag{2.12}$$

The function $W(\cdot)$ is helpful for proving the next result.

Proposition 2.3. *Let $\Omega \in \mathcal{RS}(\mathfrak{M})$. Then for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ the function $\Omega(z)$ can be represented in the form*

$$\Omega(z) = \Omega(0) + D_{\Omega(0)} \Lambda(z) \left(I + \Omega(0)\Lambda(z) \right)^{-1} D_{\Omega(0)} \tag{2.13}$$

with a function $\Lambda \in \mathcal{RS}(\mathfrak{D}_{\Omega(0)})$ for which $\Lambda(z) = z\Gamma(z)$, where Γ is a holomorphic $\mathbf{B}(\mathfrak{D}_{\Omega(0)})$-valued function such that $\|\Gamma(z)\| \le 1$ for $z \in \mathbb{D}$. In particular, $\|\Lambda(z)\| \le |z|$ when $z \in \mathbb{D}$.

Proof. To prove the statement, let the function Ω be realized as the transfer function of a passive selfadjoint system $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ as in (2.2), i.e. $\Omega(z) = D + zC(I - zF)^{-1}C^*$. Using (2.9) rewrite Ω as

$$\Omega(z) = D + zD_D N^*(I - zF)^{-1} N D_D = \Omega(0) + zD_{\Omega(0)} N^*(I - zF)^{-1} N D_{\Omega(0)}.$$

The definition of $\widehat{F}$ in (2.10) implies that the block operator

$$\begin{bmatrix} 0 & N^* \\ N & \widehat{F} \end{bmatrix} : \begin{array}{c} \mathfrak{D}_{\Omega(0)} \\ \oplus \\ \mathcal{K} \end{array} \to \begin{array}{c} \mathfrak{D}_{\Omega(0)} \\ \oplus \\ \mathcal{K} \end{array}$$

is a selfadjoint contraction (cf. Appendix B). Consequently, the $\mathbf{B}(\mathfrak{D}_D)$-valued function

$$\Lambda(z) := zN^* \left(I_{\mathcal{K}} - z\widehat{F}\right)^{-1} N, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}, \tag{2.14}$$

is the transfer function of the passive selfadjoint system

$$\tau_0 = \left\{ \begin{bmatrix} 0 & N^* \\ N & \widehat{F} \end{bmatrix} ; \mathfrak{D}_{\Omega(0)}, \mathfrak{D}_{\Omega(0)}, \mathcal{K} \right\}$$

Hence Λ belongs the class $\mathcal{RS}(\mathfrak{D}_{\Omega(0)})$. Furthermore, using (2.11) and (2.12) in Lemma 2.2 one obtains

$$I + \Omega(0)\Lambda(z) = I + zDN^* \left(I - z\widehat{F}\right)^{-1} N = W(z)$$

and

$$(I + \Omega(0)\Lambda(z))^{-1} = W(z)^{-1} = I - zDN^*(I - zF)^{-1}N$$

for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Besides, in view of (2.9) one has $\widehat{F} - F = NDN^*$. This leads to the following implications

$$\begin{aligned} &N^* \left(I - \widehat{F}\right)^{-1} N - N^*(I - zF)^{-1}N = zN^* \left(I - \widehat{F}\right)^{-1} NDN^*(I - zF)^{-1}N \\ &\Longleftrightarrow zN^* \left(I - \widehat{F}\right)^{-1} N \left(I - zDN^*(I - zF)^{-1}N\right) = zN^*(I - zF)^{-1}N \\ &\Longleftrightarrow \Lambda(z) \left(I + \Omega(0)\Lambda(z)\right)^{-1} = zN^*(I - zF)^{-1}N \\ &\Longrightarrow \Omega(z) = \Omega(0) + D_{\Omega(0)}\Lambda(z) \left(I + \Omega(0)\Lambda(z)\right)^{-1} D_{\Omega(0)}. \end{aligned}$$

Since $\Lambda(0) = 0$, it follows from Schwartz's lemma that $||\Lambda(z)|| \le |z|$ for all z with $|z| < 1$. In particular, one has a factorization $\Lambda(z) = z\Gamma(z)$, where Γ is a holomorphic $\mathbf{B}(\mathfrak{D}_{\Omega(0)})$-valued function such that $\|\Gamma(z)\| \le 1$ for $z \in \mathbb{D}$; this is also obvious from (2.14). □

One can verify that the following relation for $\Lambda(z)$ holds

$$\Lambda(z) = D^{(-1)}_{\Omega(0)}(\Omega(z) - \Omega(0))(I - \Omega(0)\Omega(z))^{-1}D_{\Omega(0)}, \tag{2.15}$$

where $D^{(-1)}_{\Omega(0)}$ stands for the Moore-Penrose inverse of $D_{\Omega(0)}$.

It should be noted that the formula (2.13) holds for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. A general Schur class function $\Omega \in \mathbf{S}(\mathfrak{M}, \mathfrak{N})$ can be represented in the form

$$\Omega(z) = \Omega(0) + D_{\Omega(0)^*}\Lambda(z) \left(I + \Omega(0)^*\Lambda(z)\right)^{-1} D_{\Omega(0)}, \quad z \in \mathbb{D}.$$

This is called a Möbius representation of Ω and it can be found in [12, 14, 18].

3. Inner functions from the class $\mathcal{RS}(\mathfrak{M})$

An operator valued function from the Schur class is called *inner/co-inner* (or $*$-*inner*) (see e.g. [20]) if it takes isometric/co-isometric values almost everywhere on the unit circle $\mathbb{T}$, and it is said to be *bi-inner* when it is both inner and co-inner.

Observe that if $\Omega \in \mathcal{RS}(\mathfrak{M})$ then $\Omega(z)^* = \Omega(\bar{z})$. Since $\mathbb{T} \setminus \{-1, 1\} \subset \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$, one concludes that $\Omega \in \mathcal{RS}(\mathfrak{M})$ is inner (or co-inner) precisely when it is bi-inner. Notice also that every function $\Omega \in \mathcal{RS}(\mathfrak{M})$ can be realized as the transfer function of a minimal passive selfadjoint system τ as in (2.2); cf. [5, Theorem 5.1].

The next statement contains a characteristic result for transfer functions of conservative selfadjoint systems.

Proposition 3.1. *Assume that the selfadjoint system $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ in* (2.2) *is conservative. Then its transfer function $\Omega(z) = D + zC(I_\mathcal{K} - zF)^{-1}C^*$ is bi-inner and it takes the form*

$$\Omega(z) = (zI_\mathfrak{M} + D)(I_\mathfrak{M} + zD)^{-1}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}. \tag{3.1}$$

On the other hand, if τ is a minimal passive selfadjoint system whose transfer function is inner, then τ is conservative.

Proof. Let the entries of T in (2.3) be parameterized as in (2.9). By assumption T is unitary and hence $N \in \mathbf{B}(\mathfrak{D}_D, \mathcal{K})$ is isometry and X is selfadjoint and unitary in the subspace $\mathfrak{D}_{N^*} = \ker N^*$; see Remark B.3 in Appendix B. Thus NN^* and D_{N^*} are orthogonal projections and $NN^* + D_{N^*} = I_\mathcal{K}$ which combined with (2.9) leads to

$$\begin{aligned}(I_\mathcal{K} - zF)^{-1} &= (N(I + zD)N^* + D_{N^*}(I - zX)D_{N^*})^{-1} \\ &= N(I + zD)^{-1}N^* + D_{N^*}(I - zX)^{-1}D_{N^*},\end{aligned}$$

and, consequently,

$$\begin{aligned}\Omega(z) &= D + zC(I_\mathcal{K} - zF)^{-1}C^* \\ &= D + zD_DN^*\left(N(I + zD)^{-1}N^* + D_{N^*}(I - zX)^{-1}D_{N^*}\right)ND_D \\ &= D + z(I + zD)^{-1}D_D^2 = (zI_\mathfrak{M} + D)(I_\mathfrak{M} + zD)^{-1},\end{aligned}$$

for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. This proves (3.1) and this clearly implies that $\Omega(z)$ is bi-inner.

To prove the second statement assume that the transfer function of a minimal passive selfadjoint system τ is inner. Then it is automatically bi-inner. Now, according to a general result of D.Z. Arov [8, Theorem 1] (see also [10, Theorem 1], [4, Theorem 1.1]), if τ is a passive simple discrete-time system with bi-inner transfer function, then τ is conservative and minimal. This proves the second statement. □

The formula (3.1) in Proposition 3.1 gives a one-to-one correspondence between the operators D from the operator interval $[-I_\mathfrak{M}, I_\mathfrak{M}]$ and the inner functions from the class $\mathcal{RS}(\mathfrak{M})$. Recall that for $\Omega \in \mathcal{RS}(\mathfrak{M})$ the strong limit values $\Omega(\pm 1)$ exist as selfadjoint contractions; see (1.7). The formula (3.1) shows that if $\Omega \in \mathcal{RS}(\mathfrak{M})$ is an inner function, then necessarily these limit values are also unitary:

$$\Omega(1)^2 = \Omega(-1)^2 = I_\mathfrak{M}. \tag{3.2}$$

However, these two conditions do not imply that $\Omega \in \mathcal{RS}(\mathfrak{M})$ is an inner function; cf. Proposition 2.1 and Remark B.3 in Appendix B.

The next two theorems offer some sufficient conditions for $\Omega \in \mathcal{RS}(\mathfrak{M})$ to be an inner function. The first one shows that by shifting $\xi \in \mathbb{T}$ ($|\xi| = 1$) away from the real line then existence of a unitary limit value $\Omega(\xi)$ at a single point implies that $\Omega \in \mathcal{RS}(\mathfrak{M})$ is actually a bi-inner function.

Theorem 3.2. *Let Ω be a nonconstant function from the class $\mathcal{RS}(\mathfrak{M})$. If $\Omega(\xi)$ is unitary for some $\xi_0 \in \mathbb{T}$, $\xi_0 \neq \pm 1$. Then Ω is a bi-inner function.*

Proof. Let $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ in (2.2) be a minimal passive selfadjoint system whose transfer function is Ω and let the entries of T be parameterized as in (2.4). Using the representation (2.6) one can derive the following formula for all $\xi \in \mathbb{T} \setminus \{\pm 1\}$:

$$\left\|D_{\Omega(\xi)}h\right\|^2 = \|D_{\Delta_F(\xi)}K^*h\|^2 + \|D_Y D_{K^*}h\|^2 + \|(D_K\Delta_F(\xi)K^* - K^*YD_{K^*})\,h\|^2;$$

cf. [4, Theorem 5.1], [5, Theorem 2.7]. Since $\Delta_F(\xi)$ is unitary for all $\xi \in \mathbb{T}\setminus\{\pm 1\}$ and $\Omega(\xi_0)$ is unitary, one concludes that Y is unitary on $\mathfrak{D}_{K^*}$ and $(D_K\Delta_F(\xi_0)K^* - K^*YD_{K^*})\,h = 0$ for all $h \in \mathfrak{M}$.

Suppose that there is $h_0 \neq 0$ such that $D_K\Delta_F(\xi_0)K^*h_0 \neq 0$ and $K^*YD_{K^*}h_0 \neq 0$. Then, due to $D_K\Delta_F(\xi_0)K^*h_0 = K^*YD_{K^*}h_0$, the equalities $D_KK^* = K^*D_{K^*}$, and

$$\operatorname{ran} D_K \cap \operatorname{ran} K^* = \operatorname{ran} D_KK^* = \operatorname{ran} K^*D_{K^*},$$

see (1.12), (1.13), one concludes that there exists $\varphi_0 \in \mathfrak{D}_{K^*}$ such that

$$\begin{cases} \Delta_F(\xi_0)K^*h_0 = K^*\varphi_0 \\ YD_{K^*}h_0 = D_{K^*}\varphi_0 \end{cases}.$$

Furthermore, the equality $D_{\Omega(\xi_0)^*} = D_{\Omega(\bar{\xi}_0)} = 0$ implies $\left(D_K\Delta_F(\bar{\xi}_0)K^* - K^*YD_{K^*}\right)h = 0$ for all $h \in \mathfrak{M}$. Now $YD_{K^*}h_0 = D_{K^*}\varphi_0$ leads to $\Delta_F(\bar{\xi}_0)K^*h_0 = K^*\varphi_0$. It follows that

$$\Delta_F(\xi_0)K^*h_0 = \Delta_F(\bar{\xi}_0)K^*h_0.$$

Because $\Delta_F(\bar{\xi}_0) = \Delta_F(\xi_0)^* = \Delta_F(\xi_0)^{-1}$, one obtains $(I - \Delta_F(\xi_0)^2)\,K^*h_0 = 0$. From

$$\Delta_F(\xi_0) = (\xi_0 I - F)(I - \xi_0 F)^{-1}$$

it follows that

$$(1 - \xi_0^2)(I - \xi_0F)^{-2}(I - F^2)K^*h_0 = 0.$$

Since $\ker D_F = \{0\}$ (because the system τ is minimal), we get $K^*h_0 = 0$. Therefore, $D_K\Delta_F(\xi_0)K^*h_0 = 0$ and $K^*YD_{K^*}h_0 = 0$. One concludes that

$$\begin{cases} D_K\Delta_F(\xi_0)K^*h = 0 \\ K^*YD_{K^*}h = 0 \end{cases} \quad \forall h \in \mathfrak{M}.$$

The equality $\operatorname{ran} Y = \mathfrak{D}_{K^*}$ implies $K^*D_{K^*} = D_KK^* = 0$. Therefore K is a partial isometry. The equality $D_K\Delta_F(\xi_0)K^* = 0$ implies $\operatorname{ran}(\Delta_F(\xi_0)K^*) \subseteq \operatorname{ran} K^*$. Representing $\Delta_F(\xi_0)$ as

$$\Delta_F(\xi_0) = (\xi_0I - F)(I - \xi_0F)^{-1}K^* = \left(\bar{\xi}_0 I + (\xi_0 - \bar{\xi}_0)(I - \xi_0F)^{-1}\right)K^*,$$

we obtain that $F(\operatorname{ran} K^*) \subseteq \operatorname{ran} K^*$. Hence $F^nD_F(\operatorname{ran} K^*) \subseteq \operatorname{ran} K^*$ for all $n \in \mathbb{N}_0$. Because the system τ is minimal it follows that $\operatorname{ran} K^* = \mathfrak{D}_F = \mathcal{K}$, i.e., K is isometry and hence T is unitary (see Appendix B). This implies that $D_{\Omega(\xi)} = 0$ for all $\zeta \in \mathbb{T} \setminus \{-1, 1\}$, i.e., Ω is inner and, thus also bi-inner. □

Theorem 3.3. *Let $\Omega \in \mathcal{RS}(\mathfrak{M})$. If the equalities* (3.2) *hold and, in addition, for some $a \in (-1, 1)$, $a \neq 0$, the equality*

$$(\Omega(a) - aI_{\mathfrak{M}})(I_{\mathfrak{M}} - a\Omega(a))^{-1} = \Omega(0) \tag{3.3}$$

is satisfied, then Ω is bi-inner.

Proof. Let $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ be a minimal passive selfadjoint system as in (2.2) with the transfer function Ω and let the entries of T in (2.3) be parameterized as in (2.4). According to Proposition 2.1 the equalities (3.2) mean that K is a partial isometry and $Y^2 = I_{\ker K^*}$.

Since D_{K^*} is the orthogonal projection, $\operatorname{ran} Y \subseteq \operatorname{ran} D_{N^*}$, from (2.6) we have

$$\Omega(z) = YD_{K^*} + K(zI - F)(I - zF)^{-1}K^*.$$

Rewrite (3.3) in the form

$$\Omega(0)(I_{\mathfrak{M}} - a\Omega(a)) = \Omega(a) - aI_{\mathfrak{M}}. \tag{3.4}$$

This leads to

$$\begin{aligned}(-KFK^* + YD_{K^*})\left(I_{\mathfrak{M}} - a\left(YD_{K^*} + K(aI - F)(I - aF)^{-1}K^*\right)\right)\\ = YD_{K^*} + K(aI - F)(I - aF)^{-1}K^* - aI_{\mathfrak{M}},\end{aligned}$$

$$\begin{aligned}(-KFK^* + YD_{K^*})\left((I - aY)D_{K^*} + K\big(I - a\,(aI - F)\,(I - aF)^{-1}\big)\,K^*\right)\\ = (Y - aI)D_{K^*} + K\left((aI - F)(I - aF)^{-1} - aI\right)K^*,\end{aligned}$$

$$\begin{aligned}-KFK^*K\left(I - a(aI - F)(I - aF)^{-1}\right)K^* + Y(I - aY)D_{K^*}\\ = (Y - aI)D_{K^*} + K\left((aI - F)(I - aF)^{-1} - aI\right)K^*.\end{aligned}$$

Let P be an orthogonal projection from $\mathcal{K}$ onto $\operatorname{ran} K^*$. Since K is a partial isometry, one has $K^*K = P$. The equality $Y^2 = I_{\mathfrak{D}_{K^*}}$ implies $Y(I - aY)D_{K^*} = (Y - aI)D_{K^*}$. This leads to the following identities:

$$\begin{aligned}&K\Big(-FP\,(I - a(aI - F)(I - aF)^{-1}) - (aI - F)(I - aF)^{-1} + aI\Big)K^* = 0,\\ &KF(I_{\mathfrak{M}} - P)(I - aF)^{-1}K^* = 0,\\ &PF(I_{\mathfrak{M}} - P)(I - aF)^{-1}P = 0.\end{aligned}$$

Represent the operator F in the block form

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{12}^* & F_{22}\end{bmatrix} : \begin{matrix}\operatorname{ran} P \\ \oplus \\ \operatorname{ran}(I - P)\end{matrix} \to \begin{matrix}\operatorname{ran} P \\ \oplus \\ \operatorname{ran}(I - P)\end{matrix}.$$

Define

$$\Theta(z) = F_{11} + zF_{12}(I - zF_{22})^{-1}F_{12}^*.$$

Since F is a selfadjoint contraction, the function Θ belongs to the class $\mathcal{RS}(\operatorname{ran} P)$. From the Schur-Frobenius formula (A.1) it follows that

$$(I - P)(I - aF)^{-1}P = a(I - aF_{22})^{-1}F_{12}^*(I - a\Theta(a))^{-1}P.$$

This equality yields the equivalences

$$\begin{aligned}PF(I_{\mathfrak{M}} - P)(I - aF)^{-1}P = 0 \Longleftrightarrow F_{12}(I - aF_{22})^{-1}F_{12}^*(I - a\Theta(a))^{-1}P = 0\\ \Longleftrightarrow F_{12}(I - aF_{22})^{-1}F_{12}^* = 0 \Longleftrightarrow (I - aF_{22})^{-1/2}F_{12}^* = 0 \Longleftrightarrow F_{12}^* = 0.\end{aligned}$$

It follows that the subspace $\operatorname{ran} K^*$ reduces F. Hence $\operatorname{ran} K^*$ reduces D_F and, therefore $F^n D_F \operatorname{ran} K^* \subseteq \operatorname{ran} K^*$ for an arbitrary $n \in \mathbb{N}_0$. Since the system τ is minimal, we get

$\operatorname{ran} K^* = \mathcal{K}$ and this implies that K is an isometry. Taking into account that $Y^2 = I_{\mathfrak{D}_{K^*}}$, we get that the block operator T is unitary. By Proposition 3.1 Ω is bi-inner. □

For completeness we recall the following result on the limit values $\Omega(\pm 1)$ of functions $\Omega \in \mathbf{S}^{qs}(\mathfrak{M})$ from [5, Theorem 5.8].

Lemma 3.4. *Let $\mathfrak{M}$ be a Hilbert space and let $\Omega \in \mathbf{S}^{qs}(\mathfrak{M})$. Then:*

(1) *if $\Omega(\lambda)$ is inner then*

$$\left(\frac{\Omega(1)-\Omega(-1)}{2}\right)^2 = \frac{\Omega(1)-\Omega(-1)}{2},\tag{3.5}$$
$$(\Omega(1)+\Omega(-1))^*(\Omega(1)+\Omega(-1)) = 4I_{\mathfrak{M}} - 2\left(\Omega(1)-\Omega(-1)\right);$$

(2) *if Ω is co-inner then*

$$\left(\frac{\Omega(1)-\Omega(-1)}{2}\right)^2 = \frac{\Omega(1)-\Omega(-1)}{2},\tag{3.6}$$
$$(\Omega(1)+\Omega(-1))(\Omega(1)+\Omega(-1))^* = 4I_{\mathfrak{M}} - 2\left(\Omega(1)-\Omega(-1)\right);$$

(3) *if (3.5)/(3.6) holds and $\Omega(\xi)$ is isometric/co-isometric for some $\xi \in \mathbb{T}$, $\xi \neq \pm 1$, then Ω is inner/co-inner.*

Proposition 3.5. *If $\Omega \in \mathcal{RS}(\mathfrak{M})$ is an inner function, then*

$$\Omega(z_1)\Omega(z_2) = \Omega(z_2)\Omega(z_1), \quad \forall z_1, z_2 \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}.$$

In particular, $\Omega(z)$ is a normal operator for each $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$.

Proof. The commutativity property follows from (3.1), where $D = \Omega(0)$. Normality follows from commutativity and symmetry $\Omega(z)^* = \Omega(\bar{z})$ for all z. □

4. Characterization of the class $\mathcal{RS}(\mathfrak{M})$

Theorem 4.1. *Let Ω be an operator valued Nevanlinna function defined on $\mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$. Then the following statements are equivalent:*

(i) *Ω belongs to the class $\mathcal{RS}(\mathfrak{M})$;*

(ii) *Ω satisfies the inequality*

$$I - \Omega^*(z)\Omega(z) - (1-|z|^2)\frac{\operatorname{Im}\Omega(z)}{\operatorname{Im} z} \geq 0, \quad \operatorname{Im} z \neq 0;\tag{4.1}$$

(iii) *the function*

$$K(z,w) := I - \Omega^*(w)\Omega(z) - \frac{1-\bar{w}z}{z-\bar{w}}\left(\Omega(z)-\Omega^*(w)\right)$$

is a nonnegative kernel on the domains

$$\mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\},\ \operatorname{Im} z > 0 \quad \textit{and} \quad \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\},\ \operatorname{Im} z < 0;$$

(iv) *the function*

$$\Upsilon(z) = \left(zI - \Omega(z)\right)\left(I - z\Omega(z)\right)^{-1}, \quad z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\},\tag{4.2}$$

is well defined and belongs to $\mathcal{RS}(\mathfrak{M})$.

Proof. (i)$\Longrightarrow$(ii) and (i)$\Longrightarrow$(iii). Assume that $\Omega \in \mathcal{RS}(\mathfrak{M})$ and let Ω be represented as the the transfer function of a passive selfadjoint system $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ as in (2.2) with the selfadjoint contraction T as in (2.4). According to (2.6) we have

$$\Omega(z) = D_{K^*} Y D_{K^*} + K\Delta_F(z)K^*, \ z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

Taking into account that, see [20, Chapter VI],

$$((I - \Delta_F^*(w)\Delta_F(z))\varphi, \psi) = (1 - \bar{w}z)((I - zF)^{-1}D_F\varphi, (I - wF)^{-1}D_F\psi)$$

and

$$((\Delta_F(z) - \Delta_F^*(w))\varphi, \psi) = (z - \bar{w})((I - zF)^{-1}D_F\varphi, (I - wF)^{-1}D_F\psi),$$

we obtain

$$\begin{aligned} ||h||^2 - ||\Omega(z)h||^2 &= ||K^*h||^2 - ||\Delta_F(z)K^*h||^2 \\ &\quad + ||D_Y D_{K^*} h||^2 + ||(K^*YD_{K^*} - D_K\Delta_F(z)K^*)h||^2 \\ &= (1 - |z|^2)||(I - zF)^{-1}D_F K^* h||^2 + ||D_Y D_{K^*}h||^2 \\ &\quad + ||(K^*YD_{K^*} - D_K\Delta_F(z)K^*)h||^2. \end{aligned}$$

Moreover,

$$\mathrm{Im}\,(\Omega(z)h, h) = \mathrm{Im}\,z||(I - zF)^{-1}D_F K^* h||^2$$

and

$$\begin{aligned} &\mathrm{Im}\,z(||h||^2 - ||\Omega(z)h||^2) - (1 - |z|^2)\mathrm{Im}\,(\Omega(z)h, h) \\ &\qquad\qquad = \mathrm{Im}\,z\left(||D_Y D_{K^*}h||^2 + ||(K^*YD_{K^*} - D_K\Delta_F(z)K^*)h||^2\right). \end{aligned}$$

Similarly,

$$\begin{aligned} (4.3)\quad (K(z, w)f, g) &= ((I - \Omega^*(w)\Omega(z))f, g) - \frac{1 - \bar{w}z}{z - \bar{w}}((\Omega(z) - \Omega^*(w))f, g) \\ &= (D_Y^2 D_{K^*}f, D_{K^*}g) + ((D_K\Delta_F(z)K^* - K^*YD_{K^*})f, (D_K\Delta_F(w)K^* - K^*YD_{K^*})g). \end{aligned}$$

It follows from (4.3) that for arbitrary complex numbers $\{z_k\}_{k=1}^m \subset \mathbb{C}\setminus\{(-\infty, -1]\cup[1, +\infty)\}$, $\mathrm{Im}\,z_k > 0$, $k = 1, \ldots, n$ or $\{z_k\}_{k=1}^m \subset \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$, $\mathrm{Im}\,z_k < 0$, $k = 1, \ldots, n$ and for arbitrary vectors $\{f_k\}_{k=1}^\infty \subset \mathfrak{M}$ the relation

$$\sum_{k=1}^n (K(z_k, z_m)f_k, f_m) = \left\|D_Y D_{K^*}\sum_{k=1}^\infty f_k\right\|^2 + \left\|\sum_{k=1}^\infty (D_K\Delta_F(z_k)K^* - K^*YD_{K^*})f_k\right\|^2$$

holds. Therefore $K(z, w)$ is a nonnegative kernel.

(iii)$\Longrightarrow$(ii) is evident.

(ii)$\Longrightarrow$(iv) Because $\mathrm{Im}\,z > 0$ $(\mathrm{Im}\,z < 0) \Longrightarrow \mathrm{Im}\,\Omega(z) \ge 0$ $(\mathrm{Im}\,\Omega(z) \le 0)$, the inclusion $1/z \in \rho(\Omega(z))$ is valid for z with $\mathrm{Im}\,z \ne 0$. In addition $1/x \in \rho(\Omega(x))$ for $x \in (-1, 1)$, $x \ne 0$, because $\Omega(x)$ is a contraction. Hence $\Upsilon(z)$ is well defined on $\mathfrak{M}$ and $\Upsilon^*(z) = \Upsilon(\bar{z})$ for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Furthermore, with $\mathrm{Im}\,z \ne 0$ one has

$$\mathrm{Im}\,\Upsilon(z) = (I - \bar{z}\Omega^*(z))^{-1}\left[\mathrm{Im}\,z(I - \Omega^*(z)\Omega(z)) - (1 - |z|^2)\mathrm{Im}\,\Omega(z)\right](I - z\Omega(z))^{-1},$$

while for $x \in (-1, 1)$

$$I - \Upsilon^2(x) = (1 - x^2)(I - x\Omega(x))^{-1}(I - \Omega^2(x))(I - x\Omega(x))^{-1}.$$

Thus, $\Upsilon \in \mathcal{RS}(\mathfrak{M})$.

(iv)$\Longrightarrow$(i) It is easy to check that if Υ is given by (4.2), then

$$\Omega(z) = (zI - \Upsilon(z))\,(I - z\Upsilon(z))^{-1}\,,\ z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

Hence, this implication reduces back to the proven implication (i)$\Longrightarrow$(ii). □

Remark 4.2. *1) Inequality* (4.1) *can be rewritten as follows*

$$((I - \Omega^*(z)\Omega(z))f, f) - \frac{1 - |z|^2}{|\mathrm{Im}\, z|}|\mathrm{Im}\,(\Omega(z)f, f)| \ge 0, \quad \mathrm{Im}\, z \ne 0,\ f \in \mathfrak{M}.$$

Let $\beta \in [0, \pi/2]$. *Taking into account that*

$$|z \sin\beta \pm i\cos\beta|^2 = 1 \Longleftrightarrow 1 - |z|^2 = \pm 2\cot\beta\, \mathrm{Im}\, z$$

one obtains, see (2.1),

$$\begin{array}{l} \left\{\begin{array}{c} |z\sin\beta + i\cos\beta| = 1 \\ z \ne \pm 1 \end{array}\right. \Longrightarrow \|\Omega(z)\sin\beta + i\cos\beta\, I\| \le 1 \\ \left\{\begin{array}{c} |z\sin\beta - i\cos\beta| = 1 \\ z \ne \pm 1 \end{array}\right. \Longrightarrow \|\Omega(z)\sin\beta - i\cos\beta\, I\| \le 1 \end{array}.$$

2) Inequality (4.1) *implies*

$$I - \Omega^*(x)\Omega(x) - (1 - x^2)\Omega'(x) \ge 0, \quad x \in (-1, 1).$$

3) Formula (3.1) *implies that if* $\Omega \in \mathcal{RS}(\mathfrak{M})$ *is an inner function, then*

$$I - \Omega^*(w)\Omega(z) - \frac{1 - \bar{w}z}{z - \bar{w}}\,(\Omega(z) - \Omega^*(w)) = 0,\ z \ne \bar{w}.$$

In particular,

$$\begin{array}{l} \dfrac{\Omega(z) - \Omega(0)}{z} = I - \Omega(0)\Omega(z), \quad z \in \mathbb{C} \setminus \{-\infty, -1] \cup [1, +\infty)\},\ z \ne 0, \\ \Omega'(0) = I - \Omega(0)^2. \end{array}$$

This combined with (2.15) *yields* $\Lambda(z) = zI_{\mathfrak{D}_{\Omega(0)}}$ *in the representation* (2.13) *for an inner function* $\Omega \in \mathcal{RS}(\mathfrak{M})$.

5. Compressed resolvents and the class $\mathbf{N}^0_{\mathfrak{M}}[-1, 1]$

Definition 5.1. *Let* $\mathfrak{M}$ *be a Hilbert space. A* $\mathbf{B}(\mathfrak{M})$*-valued Nevanlinna function* M *is said to belong to the class* $\mathbf{N}^0_{\mathfrak{M}}[-1, 1]$ *if it is holomorphic outside the interval* $[-1, 1]$ *and*

$$\lim_{\xi \to \infty} \xi M(\xi) = -I_{\mathfrak{M}}.$$

It follows from [3] that $M \in \mathbf{N}^0_{\mathfrak{M}}[-1, 1]$ if and only if there exist a Hilbert space $\mathfrak{H}$ containing $\mathfrak{M}$ as a subspace and a selfadjoint contraction T in $\mathfrak{H}$ such that T is $\mathfrak{M}$-simple and

$$M(\xi) = P_{\mathfrak{M}}(T - \xi I)^{-1}{\upharpoonright}\,\mathfrak{M}, \quad \xi \in \mathbb{C} \setminus [-1, 1].$$

Moreover, formula (1.6) implies the following connections between the classes $\mathbf{N}^0_{\mathfrak{M}}[-1, 1]$ and $\mathcal{RS}(\mathfrak{M})$ (see also [3, 5]):

$$\begin{array}{rl} M(\xi) \in \mathbf{N}^0_{\mathfrak{M}}[-1, 1] & \Longrightarrow \Omega(z) := M^{-1}(1/z) + 1/z \in \mathcal{RS}(\mathfrak{M}), \\ \Omega(z) \in \mathcal{RS}(\mathfrak{M}) & \Longrightarrow M(\xi) := (\Omega(1/\xi) - \xi)^{-1} \in \mathbf{N}^0_{\mathfrak{M}}[-1, 1]. \end{array} \tag{5.1}$$

Let $\Omega(z)=(zI+D)(I+zD)^{-1}$ be an inner function from the class $\mathcal{RS}(\mathfrak{M})$, then by (5.1)

$$\Omega(z)=(zI+D)(I+zD)^{-1}\Longrightarrow M(\xi)=\frac{\xi I+D}{1-\xi^2},\quad \xi\in\mathbb{C}\setminus[-1,1].$$

The identity $\Omega(z)^*\Omega(z)=I_{\mathfrak{M}}$ for $z\in\mathbb{T}\setminus\{\pm1\}$ is equivalent to

$$2\mathrm{Re}\,(\xi M(\xi))=-I_{\mathfrak{M}},\quad \xi\in\mathbb{T}\setminus\{\pm1\}.$$

The next statement is established in [2]. Here we give another proof.

Theorem 5.2. *If $M(\xi)\in\mathbf{N}^0_{\mathfrak{M}}[-1,1]$, then the function*

$$\frac{M^{-1}(\xi)}{\xi^2-1},\quad \xi\in\mathbb{C}\setminus[-1,1],$$

belongs to $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ as well.

Proof. Let $M(\xi)\in\mathbf{N}^0_{\mathfrak{M}}[-1,1]$. Then due to (5.1) the function $\Omega(z)=M^{-1}(1/z)+1/z$ belongs to $\mathcal{RS}(\mathfrak{M})$. By Theorem 4.1 the function

$$\Upsilon(z)=(zI-\Omega(z))\left(I-z\Omega(z)\right)^{-1},\quad z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$$

belongs to $\mathcal{RS}(\mathfrak{M})$. From the equality

$$I-z\Upsilon(z)=(1-z^2)\left(I-z\Omega(z)\right)^{-1},\quad z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$$

we get

$$\left(I-z\Upsilon(z)\right)^{-1}=\frac{I-z\Omega(z)}{1-z^2}.$$

Simple calculations give

$$\left(\Upsilon(1/\xi)-\xi\right)^{-1}=\frac{M^{-1}(\xi)}{\xi^2-1},\quad \xi\in\mathbb{C}\setminus[-1,1].$$

Now in view of (5.1) the function $\dfrac{M^{-1}(\xi)}{\xi^2-1}$ belongs to $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$. □

6. Transformations of the classes $\mathcal{RS}(\mathfrak{M})$ and $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$

We start by studying transformations of the class $\mathcal{RS}(\mathfrak{M})$ given by (1.8), (1.10):

$$\mathcal{RS}(\mathfrak{M})\ni\Omega\mapsto\mathbf{\Phi}(\Omega)=\Omega_{\mathbf{\Phi}}(z):=(zI-\Omega(z))(I-z\Omega(z))^{-1},$$

$$\mathcal{RS}(\mathfrak{M})\ni\Omega\mapsto\mathbf{\Xi}_a(\Omega)=\Omega_a(z):=\Omega\left(\frac{z+a}{1+za}\right),\quad a\in(-1,1),$$

and the transform

$$\mathcal{RS}(H)\ni\Omega\mapsto\mathbf{\Pi}(\Omega)=\Omega_{\mathbf{\Pi}}(z):K_{11}+K_{12}\Omega(z)(I-K_{22}\Omega(z))^{-1}K_{12}^*,\tag{6.1}$$

which is determined by the selfadjoint contraction K of the form

$$\mathbf{K}=\begin{bmatrix}K_{11}&K_{12}\\K_{12}^*&K_{22}\end{bmatrix}:\begin{array}{c}\mathfrak{M}\\\oplus\\H\end{array}\to\begin{array}{c}\mathfrak{M}\\\oplus\\H\end{array};$$

in all these transforms $z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$.

A particular case of (6.1) is the transformation $\mathbf{\Pi}_a$ determined by the block operator

$$\mathbf{K}_a = \begin{bmatrix} aI & \sqrt{1-a^2}I \\ \sqrt{1-a^2} & -aI \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}\end{array}, \quad a \in (-1,1),$$

i.e., see (1.10),

$$\mathcal{RS}(\mathfrak{M}) \ni \Omega(z) \mapsto \widehat{\Omega}_a(z) := (aI + \Omega(z))(I + a\Omega(z))^{-1}.$$

By Theorem 4.1 the mapping $\mathbf{\Phi}$ given by (1.8) is an automorphism of the class $\mathcal{RS}(\mathfrak{M})$, $\mathbf{\Phi}^{-1} = \mathbf{\Phi}$. The equality (3.1) shows that the set of all inner functions of the class $\mathcal{RS}(\mathfrak{M})$ is the image of all constant functions under the transformation $\mathbf{\Phi}$. In addition, for $a, b \in (-1,1)$ the following identities hold:

$$\mathbf{\Pi}_b \circ \mathbf{\Pi}_a = \mathbf{\Pi}_a \circ \mathbf{\Pi}_b = \mathbf{\Pi}_c, \quad \Xi_b \circ \Xi_a = \Xi_a \circ \Xi_b = \Xi_c, \quad \text{where } c = \frac{a+b}{1+ab}.$$

The mapping $\mathbf{\Gamma}$ on the class $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$ (see Theorem 5.2) defined by

$$\mathbf{N}^0_{\mathfrak{M}}[-1,1] \ni M(\xi) \overset{\mathbf{\Gamma}}{\mapsto} M_{\mathbf{\Gamma}}(\xi) := \frac{M^{-1}(\xi)}{\xi^2 - 1} \in \mathbf{N}^0_{\mathfrak{M}}[-1,1] \tag{6.2}$$

has been studied recently in [2]. It is obvious that $\mathbf{\Gamma}^{-1} = \mathbf{\Gamma}$.

Using the relations (5.1) we define the transform $\mathbf{U}$ and its inverse $\mathbf{U}^{-1}$ which connect the classes $\mathcal{RS}(\mathfrak{M})$ and $\mathbf{N}^0_{\mathfrak{M}}[-1,1]$:

$$\mathcal{RS}(\mathfrak{M}) \ni \Omega(z) \overset{\mathbf{U}}{\mapsto} M(\xi) := (\Omega(1/\xi) - \xi)^{-1} \in \mathbf{N}^0_{\mathfrak{M}}[-1,1], \quad \xi \in \mathbb{C} \setminus [-1,1]. \tag{6.3}$$

$$\mathbf{N}^0_{\mathfrak{M}}[-1,1] \ni M(\xi) \overset{\mathbf{U}^{-1}}{\mapsto} \Omega(z) := M^{-1}(1/z) + 1/z \in \mathcal{RS}(\mathfrak{M}), \tag{6.4}$$

where $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$. The proof of Theorem 5.2 contains the following commutation relations

$$\mathbf{U}\mathbf{\Phi} = \mathbf{\Gamma}\mathbf{U}, \quad \mathbf{\Phi}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{\Gamma}. \tag{6.5}$$

One of the main aims in this section is to solve the following realization problem concerning the above transforms: given a passive selfadjoint system $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ with the transfer function Ω, construct a passive selfadjoint systems whose transfer function coincides with $\Phi(\Omega)$, $\Xi_a(\Omega)$, $\mathbf{\Pi}(\Omega)$, and $\mathbf{\Pi}_a(\Omega)$, respectively. We will also determine the fixed points of all the mappings $\mathbf{\Phi}$, $\mathbf{\Gamma}$, Ξ_a, and $\mathbf{\Pi}_a$.

6.1. The mappings $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ and inner dilations of the functions from $\mathcal{RS}(\mathfrak{M})$.

Theorem 6.1. (1) *Let $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ be a passive selfadjoint system and let Ω be its transfer function. Define*

$$T_{\mathbf{\Phi}} := \begin{bmatrix} -P_{\mathfrak{M}}T{\upharpoonright}\mathfrak{M} & P_{\mathfrak{M}}D_T \\ D_T{\upharpoonright}\mathfrak{M} & T \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{D}_T\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{D}_T\end{array}. \tag{6.6}$$

Then $T_{\mathbf{\Phi}}$ is a selfadjoint contraction and $\Omega_{\mathbf{\Phi}}(z) = (zI - \Omega(z))(I - z\Omega(z))^{-1}$ is the transfer function of the passive selfadjoint system of the form

$$\tau_{\mathbf{\Phi}} = \{T_{\mathbf{\Phi}}; \mathfrak{M}, \mathfrak{M}, \mathfrak{D}_T\}.$$

Moreover, if the system τ is minimal, then the system $\tau_{\mathbf{\Phi}}$ is minimal, too.

(2) *Let T be a selfadjoint contraction in $\mathfrak{H}$, let $\mathfrak{M}$ be a subspace of $\mathfrak{H}$ and let*

$$M(\xi) = P_{\mathfrak{M}}(T - \xi I)^{-1}{\upharpoonright}\,\mathfrak{M}. \tag{6.7}$$

Consider a Hilbert space $\widehat{\mathfrak{H}} := \mathfrak{M} \oplus \mathfrak{H}$ and let $\widehat{P}_{\mathfrak{M}}$ be the orthogonal projection in $\widehat{\mathfrak{H}}$ onto $\mathfrak{M}$. Then

$$\frac{M^{-1}(\xi)}{\xi^2 - 1} = \widehat{P}_{\mathfrak{M}}(T_{\mathbf{\Phi}} - \xi I)^{-1}{\upharpoonright}\,\mathfrak{M},$$

where $T_{\mathbf{\Phi}}$ is defined by (6.6).

(3) *The function*

$$\widetilde{\Omega}(z) = (zI - T_{\mathbf{\Phi}})(I - zT_{\mathbf{\Phi}})^{-1}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$$

satisfies

$$\Omega(z) = P_{\mathfrak{M}}\widetilde{\Omega}(z){\upharpoonright}\,\mathfrak{M}.$$

Proof. (1) According to (1.6) one has

$$P_{\mathfrak{M}}(I - zT)^{-1}{\upharpoonright}\,\mathfrak{M} = (I_{\mathfrak{M}} - z\Omega(z))^{-1}$$

for $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Let

$$\Omega_{\mathbf{\Phi}}(z) = (zI - \Omega(z))(I - z\Omega(z))^{-1}.$$

Now simple calculations give

$$\Omega_{\mathbf{\Phi}}(z) = \left(z - \frac{1}{z}\right)(I - z\Omega(z))^{-1} + \frac{I_{\mathfrak{M}}}{z} = P_{\mathfrak{M}}(zI - T)(I - zT)^{-1}{\upharpoonright}\,\mathfrak{M}. \tag{6.8}$$

Observe that the subspace $\mathfrak{D}_T$ is invariant under T; cf. (1.12). Let $\mathfrak{H} := \mathfrak{M} \oplus \mathfrak{D}_T$ and let $T_{\mathbf{\Phi}}$ be given by (6.6). Since T is a selfadjoint contraction in $\mathfrak{M} \oplus \mathcal{K}$, the inequalities

$$\left(\begin{bmatrix}\varphi\\ f\end{bmatrix}, \begin{bmatrix}\varphi\\ f\end{bmatrix}\right) \pm \left(\begin{bmatrix}\varphi\\ f\end{bmatrix}, T_{\mathbf{\Phi}}\begin{bmatrix}\varphi\\ f\end{bmatrix}\right) = \left\|(I \mp T)^{1/2}\varphi \pm (I \pm T)^{1/2} f\right\|^2$$

hold for all $\varphi \in \mathfrak{M}$ and $f \in \mathfrak{D}_T$. Therefore $T_{\mathbf{\Phi}}$ is a selfadjoint contraction in the Hilbert space $\mathfrak{H}$ and the system

$$\tau_{\mathbf{\Phi}} = \left\{\begin{bmatrix} -P_{\mathfrak{M}}T{\upharpoonright}\,\mathfrak{M} & P_{\mathfrak{M}}D_T \\ D_T{\upharpoonright}\,\mathfrak{M} & T\end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{D}_T\right\}$$

is passive selfadjoint. Suppose that τ is minimal, i.e.,

$$\overline{\operatorname{span}}\,\{T^n\mathfrak{M},\ n \in \mathbb{N}_0\} = \mathfrak{M} \oplus \mathcal{K} \Longleftrightarrow \bigcap_{n=0}^{\infty} \ker(P_{\mathfrak{M}}T^n) = \{0\}.$$

Since

$$\mathfrak{D}_T \ominus \{\overline{\operatorname{span}}\,\{T^n D_T\mathfrak{M},\ n \in \mathbb{N}_0\}\} = \bigcap_{n=0}^{\infty} \ker(P_{\mathfrak{M}}T^n D_T),$$

we get $\overline{\operatorname{span}}\,\{T^n D_T\mathfrak{M} :\ n \in \mathbb{N}_0\} = \mathfrak{D}_T$. This means that the system $\tau_{\rm T}$ is minimal.

For the transfer function $\Upsilon(z)$ of $\tau_{\mathbf{\Phi}}$ we get

$$\begin{aligned}\Upsilon(z) &= (-P_{\mathfrak{M}}T + zP_{\mathfrak{M}}D_T(I - zT)^{-1}D_T){\upharpoonright}\,\mathfrak{M} \\ &= P_{\mathfrak{M}}\left(-T + zD_T^2(I - zT)^{-1}\right){\upharpoonright}\,\mathfrak{M} \\ &= P_{\mathfrak{M}}(zI - T)(I - zT)^{-1}{\upharpoonright}\,\mathfrak{M},\end{aligned}$$

with $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Comparison with (6.8) completes the proof.

(2) The function $M(\xi) = P_{\mathfrak{M}}(T - \xi I)^{-1} \upharpoonright \mathfrak{M}$ belongs to the class $\mathbf{N}^0_{\mathfrak{M}}[-1, 1]$. Consequently, $\Omega(z) := M^{-1}(1/z) + 1/z \in \mathcal{RS}(\mathfrak{M})$. The function Ω is the transfer function of the passive selfadjoint system

$$\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\},$$

where $\mathcal{K} = \mathfrak{H} \ominus \mathfrak{M}$. Let $\Upsilon = \mathbf{\Phi}(\Omega)$ and $\widehat{M} = \mathbf{U}(\Upsilon)$. From (6.2)–(6.5) it follows that

$$\widehat{M}(\xi) = \frac{M^{-1}(\xi)}{\xi^2 - 1}, \quad \xi \in \mathbb{C} \setminus [-1, 1].$$

As was shown above, the function Υ is the transfer function of the passive selfadjoint system

$$\tau_{\mathbf{\Phi}} = \{T_{\mathbf{\Phi}}; \mathfrak{M}, \mathfrak{M}, \mathfrak{H}\},$$

where $T_{\mathbf{\Phi}}$ is given by (6.6). Then again the Schur-Frobenius formula (1.6) gives

$$\widehat{M}(\xi) = \widehat{P}_{\mathfrak{M}}(T_{\mathbf{\Phi}} - \xi I)^{-1} \upharpoonright \mathfrak{M}, \quad \xi \in \mathbb{C} \setminus [-1, 1].$$

(3) For all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ one has

$$\widetilde{\Omega}(z) = \left(z - \frac{1}{z}\right)(I - zT_{\mathbf{\Phi}})^{-1} + \frac{1}{z} I.$$

Then

$$\begin{aligned} P_{\mathfrak{M}} \widetilde{\Omega}(z) \upharpoonright \mathfrak{M} &= \left(z - \frac{1}{z}\right)(I_{\mathfrak{M}} - z\Upsilon(z))^{-1} + \frac{1}{z} I_{\mathfrak{M}} \\ &= (zI_{\mathfrak{M}} - \Upsilon(z))(I_{\mathfrak{M}} - z\Upsilon(z))^{-1} = \Omega(z). \end{aligned}$$

This completes the proof. □

Notice that if $\Omega(z) \equiv const = D$, then $\Upsilon(z) = (zI - D)(I - zD)^{-1}$, $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. This is the transfer function of the conservative and selfadjoint system

$$\Sigma = \left\{ \begin{bmatrix} -D & D_D \\ D_D & D \end{bmatrix}, \mathfrak{M}, \mathfrak{M}, \mathfrak{D}_D \right\}.$$

Remark 6.2. *The block operator $T_{\mathbf{\Phi}}$ of the form* (6.6) *appeared in* [2] *and relation* (6.7) *is also established in* [2].

Theorem 6.3. *1) Let $\mathfrak{M}$ be a Hilbert space and let $\Omega \in \mathcal{RS}(\mathfrak{M})$. Then there exist a Hilbert space $\widetilde{\mathfrak{M}}$ containing $\mathfrak{M}$ as a subspace and a selfadjoint contraction $\widetilde{A}$ in $\widetilde{\mathfrak{M}}$ such that for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ the equality*

$$\Omega(z) = P_{\mathfrak{M}}(zI_{\widetilde{\mathfrak{M}}} + \widetilde{A})(I_{\widetilde{\mathfrak{M}}} + z\widetilde{A})^{-1} \upharpoonright \mathfrak{M} \tag{6.9}$$

holds. Moreover, the pair $\{\widetilde{\mathfrak{M}}, \widetilde{A}\}$ can be chosen such that $\widetilde{A}$ is $\mathfrak{M}$-simple, i.e.,

$$\overline{\operatorname{span}}\,\{\widetilde{A}^n \mathfrak{M} : \ n \in \mathbb{N}_0\} = \widetilde{\mathfrak{M}}. \tag{6.10}$$

The function Ω is inner if and only if $\widetilde{\mathfrak{M}} = \mathfrak{M}$ in the representation (6.10).

If there are two representations of the form (6.9) *with pairs $\{\widetilde{\mathfrak{M}}_1, \widetilde{A}_1\}$ and $\{\widetilde{\mathfrak{M}}_2, \widetilde{A}_2\}$ that are $\mathfrak{M}$-simple, then there exists a unitary operator $\widetilde{U} \in \mathbf{B}(\widetilde{\mathfrak{M}}_1, \widetilde{\mathfrak{M}}_2)$ such that*

$$\widetilde{U} \upharpoonright \mathfrak{M} = I_{\mathfrak{M}}, \quad \widetilde{A}_2 \widetilde{U} = \widetilde{U} \widetilde{A}_1. \tag{6.11}$$

2) The formula

$$\Omega(z)=\int\limits_{-1}^{1}\frac{z+t}{1+zt}\,d\sigma(t),\quad z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\},\tag{6.12}$$

gives a one-one correspondence between functions Ω *from the class* $\mathcal{RS}(\mathfrak{M})$ *and nondecreasing left-continuous* $\mathbf{B}(\mathfrak{M})$*-valued functions* σ *on* $[-1,1]$ *with* $\sigma(-1)=0$, $\sigma(1)=I_{\mathfrak{M}}$.

Proof. 1) Realize Ω as the transfer function of a minimal passive selfadjoint system $\tau=\{T;\mathfrak{M},\mathfrak{M},\mathcal{K}\}$. Let the selfadjoint contraction $T_{\mathbf{\Phi}}$ be given by (6.6) and let $\widetilde{\mathfrak{M}}:=\mathfrak{M}\oplus\mathfrak{D}_T$ and $\widetilde{A}:=-T_{\mathbf{\Phi}}$. Then the relations (6.9) and (6.10) are obtained from Theorem 6.1. Using Proposition 3.1 one concludes that Ω is inner precisely when $\widetilde{\mathfrak{M}}=\mathfrak{M}$ in the righthand side of (6.10). Since

$$\begin{aligned}&P_{\mathfrak{M}}(zI_{\widetilde{\mathfrak{M}}_1}+\widetilde{A}_1)(I_{\widetilde{\mathfrak{M}}_1}+z\widetilde{A}_1)^{-1}\upharpoonright\mathfrak{M}=P_{\mathfrak{M}}(zI_{\widetilde{\mathfrak{M}}_2}+\widetilde{A}_2)(I_{\widetilde{\mathfrak{M}}_2}+z\widetilde{A}_2)^{-1}\upharpoonright\mathfrak{M}\\ \iff\ &P_{\mathfrak{M}}(I_{\widetilde{\mathfrak{M}}_1}+z\widetilde{A}_1)^{-1}\upharpoonright\mathfrak{M}=P_{\mathfrak{M}}(I_{\widetilde{\mathfrak{M}}_2}+z\widetilde{A}_2)^{-1}\upharpoonright\mathfrak{M},\end{aligned}$$

the $\mathfrak{M}$-simplicity with standard arguments (see e.g. [3, 6]) yields the existence of unitary $\widetilde{U}\in\mathbf{B}(\widetilde{\mathfrak{M}}_1,\widetilde{\mathfrak{M}}_2)$ satisfying (6.11).

2) Let (6.9) be satisfied and let $\sigma(t)=P_{\mathfrak{M}}\widetilde{E}(t)\upharpoonright\mathfrak{M}$, $t\in[-1,1]$, where $E(t)$ is the spectral family of the selfadjoint contraction $\widetilde{A}$ in $\widetilde{\mathfrak{M}}$. Then clearly (6.12) holds.

Conversely, let σ be a nondecreasing left-continuous $\mathbf{B}(\mathfrak{M})$-valued function $[-1,1]$ with $\sigma(-1)=0$, $\sigma(1)=I_{\mathfrak{M}}$. Define Ω by the right-hand side of (6.12). Then, the function Ω in (6.12) belongs to the class $\mathcal{RS}(\mathfrak{M})$. □

Remark 6.4. *If* Ω *is represented in the form* (6.9)*, then the proof of Theorem 6.1 shows that the transfer function of the passive selfadjoint system* $\widetilde{\sigma}_{\mathbf{\Phi}}=\{(-\widetilde{A})_{\mathbf{\Phi}};\mathfrak{M},\mathfrak{M},\mathfrak{D}_{\widetilde{A}}\}$ *coincides with* Ω*. Moreover, if* $\widetilde{A}$ *is* $\mathfrak{M}$*-simple, then* $\widetilde{\sigma}_{\mathbf{\Phi}}$ *is minimal.*

Remark 6.5. *The functions from the class* $\mathcal{S}^{qs}(\mathfrak{M})$ *admits the following integral representations, see* [5]*:*

$$\Theta(z)=\Theta(0)+z\int_{-1}^{1}\frac{1-t^2}{1-tz}\,dG(t),$$

where $G(t)$ *is a nondecreasing* $\mathbf{B}(\mathfrak{M})$*-valued function with bounded variation,* $G(-1)=0$, $G(1)\le I_{\mathfrak{M}}$*, and*

$$\left|\left(\left(\Theta(0)+\int_{-1}^{1}t\,dG(t)\right)f,g\right)\right|^2\le((I-G(1))\,f,f)\ ((I-G(1))\,g,g)\,,\quad f,g\in\mathfrak{M}.$$

Proposition 6.6 (cf. [2])**.** *1) The mapping* $\mathbf{\Phi}$ *of* $\mathcal{RS}(\mathfrak{M})$ *has a unique fixed point*

$$\Omega_0(z)=\frac{zI_{\mathfrak{M}}}{1+\sqrt{1-z^2}},\quad\text{with}\quad\Omega_0(i)=\frac{iI_{\mathfrak{M}}}{1+\sqrt{2}}.\tag{6.13}$$

2) The mapping $\mathbf{\Gamma}$ *has a unique fixed point*

$$M_0(\xi)=-\frac{I_{\mathfrak{M}}}{\sqrt{\xi^2-1}}\quad\text{with}\quad M_0(i)=\frac{iI_{\mathfrak{M}}}{\sqrt{2}}.\tag{6.14}$$

3) Define the weight function $\rho(t)$ and the weighted Hilbert space $\mathfrak{H}_0$ as follows
(6.15)

$$\rho_0(t)=\frac{1}{\pi}\frac{1}{\sqrt{1-t^2}},\; t\in(-1,1),$$
$$\mathfrak{H}_0:=L_2([-1,1],\mathfrak{M},\rho_0(t))=L_2([-1,1],\;\rho_0(t))\bigotimes\mathfrak{M}=\left\{f(t):\int\limits_{-1}^{1}\frac{||f(t)||^2_{\mathfrak{M}}}{\sqrt{1-t^2}}dt<\infty\right\}.$$

Then $\mathfrak{H}_0$ is the Hilbert space with the inner product

$$(f(t),g(t))_{\mathfrak{H}_0}=\frac{1}{\pi}\int\limits_{-1}^{1}(f(t),g(t))_{\mathfrak{M}}\,\rho_0(t)\,dt=\frac{1}{\pi}\int\limits_{-1}^{1}\frac{(f(t),g(t))_{\mathfrak{M}}}{\sqrt{1-t^2}}dt.$$

Identify $\mathfrak{M}$ with a subspace of $\mathfrak{H}_0$ of constant vector-functions $\{f(t)\equiv f,\; f\in\mathfrak{M}\}$. Let

$$\mathcal{K}_0:=\mathfrak{H}_0\ominus\mathfrak{M}=\left\{f(t)\in\mathfrak{H}_0:\int\limits_{-1}^{1}\frac{(f(t),h)_{\mathfrak{M}}}{\sqrt{1-t^2}}dt=0\;\forall h\in\mathfrak{M}\right\}$$

and define in $\mathfrak{H}_0$ the multiplication operator by

$$(T_0f)(t)=tf(t),\;f\in\mathfrak{H}_0. \tag{6.16}$$

Then $\Omega_0(z)$ is the transfer function of the simple passive selfadjoint system

$$\tau_0=\{T_0;\mathfrak{M},\mathfrak{M},\mathcal{K}_0\},$$

while

$$M_0(\xi)=P_{\mathfrak{M}}(T_0-\xi I)^{-1}\upharpoonright\mathfrak{M}.$$

Proof. 1)–2) Let $\Omega_0(z)$ be a fixed point of the mapping $\mathbf{\Phi}$ of $\mathcal{RS}(\mathfrak{M})$, i.e.,

$$\Omega_0(z)=(zI-\Omega_0(z))\left(I-z\Omega_0(z)\right)^{-1},\quad z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}.$$

Then

$$(I-z\Omega_0(z))^2=(1-z^2)I_{\mathfrak{M}}.$$

Using $\Omega_0\in\mathcal{RS}(\mathfrak{M})$ and the Taylor expansion $\Omega_0(z)=\sum_{n=0}^\infty C_kz^k$ in the unit disk, it is seen that Ω_0 is of the form (6.13).

It follows that the transform $M_0=\mathbf{U}(\Omega_0)$ defined in (6.3) is of the form (6.14) and it is the unique fixed point of the mapping $\mathbf{\Gamma}$ in (6.2); cf. (6.5).

3) For each $h\in\mathfrak{M}$ straightforward calculations, see [13, pages 545–546], lead to the equality

$$-\frac{h}{\sqrt{\xi^2-1}}=\frac{1}{\pi}\int\limits_{-1}^{1}\frac{h}{t-\xi}\frac{1}{\sqrt{1-t^2}}dt.$$

Therefore, if T_0 is the operator of the form (6.16), then

$$M_0(\xi)=P_{\mathfrak{M}}(T_0-\xi I)^{-1}\upharpoonright\mathfrak{M}.$$

It follows that Ω_0 is the transfer function of the system $\tau_0=\{T_0;\mathfrak{M},\mathfrak{M},\mathcal{K}_0\}$. □

As is well known, the Chebyshev polynomials of the first kind given by

$$\widehat{T}_0(t)=1,\ \widehat{T}_n(t):=\sqrt{2}\cos(n\arccos t),\ n\geq 1$$

form an orthonormal basis of the space $L_2([-1,1],\rho_0(t))$, where $\rho_0(t)$ is given by (6.15). These polynomials satisfy the recurrence relations

$$\begin{aligned}
&t\widehat{T}_0(t)=\frac{1}{\sqrt{2}}\widehat{T}_1(t),\quad t\widehat{T}_1(t)=\frac{1}{\sqrt{2}}\widehat{T}_0(t)+\frac{1}{2}\widehat{T}_2(t),\\
&t\widehat{T}_n(t)=\frac{1}{2}\widehat{T}_{n-1}(t)+\frac{1}{2}\widehat{T}_{n+1}(t),\quad n\neq 2.
\end{aligned}$$

Hence the matrix of the operator multiplication by the independent variable in the Hilbert space $L_2([-1,1],\rho_0(t))$ w.r.t. the basis $\{\widehat{T}_n(t)\}_{n=0}^\infty$ (the Jacobi matrix) takes the form

$$J=\begin{bmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdot & \cdot & \cdot\\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot\\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdot & \cdot & \cdot\\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdot & \cdot\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.$$

In the case of vector valued weighted Hilbert space $\mathfrak{H}_0=L_2([-1,1],\mathfrak{M},\rho_0(t))$ the operator (6.16) is unitary equivalent to the block operator Jacobi matrix $\mathbf{J}_0=J\bigotimes I_{\mathfrak{M}}$. It follows that the function Ω_0 is the transfer function of the passive selfadjoint system with the operator T_0 given by the selfadjoint contractive block operator Jacobi matrix

$$T_0=\left[\begin{array}{c|c}
0 & \frac{1}{\sqrt{2}}I_{\mathfrak{M}}\ \ 0\ \ 0\ \ \ldots\\
\hline
\begin{array}{c}\frac{1}{\sqrt{2}}I_{\mathfrak{M}}\\ 0\\ \vdots\end{array} & \mathbf{J_0}
\end{array}\right],\quad \mathbf{J_0}=\begin{bmatrix}
0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & 0 & 0 & \cdot & \cdot & \cdot\\
\frac{1}{2}I_{\mathfrak{M}} & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & 0 & \cdot & \cdot & \cdot\\
0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \cdot & \cdot & \cdot\\
0 & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \frac{1}{2}I_{\mathfrak{M}} & 0 & \cdot & \cdot\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.$$

6.2. **The mapping Π and Redheffer product.**

Lemma 6.7. *Let H be a Hilbert space, let K be a selfadjoint contraction in H and let $\Omega\in\mathcal{RS}(H)$. If $||K||<1$, then $(I-K\Omega(z))^{-1}$ is defined on H and it is bounded for all $z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$.*

Proof. If $|z|\leq 1$, $z\neq\pm 1$, then $||K||<1$ and $||\Omega(z)||\leq 1$ imply that $||K\Omega(z)||<1$. Hence $(I-K\Omega(z))^{-1}$ exists as bounded everywhere defined operator on H.

Now let $|z|>1$ and $z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$. Then there exists $\beta\in(0,\pi/2)$ such that either $|z\sin\beta-i\cos\beta|=1$ or $|z\sin\beta+i\cos\beta|=1$. Suppose that, for instance,

$|z\sin\beta - i\cos\beta| = 1$. Then from (2.1) one obtains $||\Omega(z)\sin\beta - i\cos\beta I_H|| \le 1$. Hence $S := \Omega(z)\sin\beta - i\cos\beta I_H$ satisfies $||S|| \le 1$ and one has

$$\Omega(z) = \frac{S + i\cos\beta\, I_H}{\sin\beta}.$$

Furthermore,

$$\begin{aligned} I - K\Omega(z) &= I - \frac{KS + i\cos\beta\, K}{\sin\beta} = \frac{1}{\sin\beta}\left((\sin\beta\, I - i\cos\beta\, K) - KS\right) \\ &= \frac{1}{\sin\beta}(\sin\beta\, I - i\cos\beta\, K)\left(I - (\sin\beta\, I - i\cos\beta\, K)^{-1}KS\right). \end{aligned}$$

Clearly

$$||(\sin\beta\, I - i\cos\beta\, K)^{-1}K||^2 \le \frac{||K||^2}{\sin^2\beta + ||K||^2\cos^2\beta} < 1,$$

which shows that $||(\sin\beta\, I - i\cos\beta\, K)^{-1}KS|| < 1$. Therefore, the following inverse operator $\left(I - (\sin\beta\, I - i\cos\beta\, K)^{-1}KS\right)^{-1}$ exists and is everywhere defined on H. This implies that

$$(I - K\Omega(z))^{-1} = \sin\beta\left(I - (\sin\beta\, I - i\cos\beta\, K)^{-1}KS\right)^{-1}(\sin\beta\, I - i\cos\beta\, K)^{-1}.$$

□

Theorem 6.8. *Let*

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ H\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ H\end{array}$$

be a selfadjoint contraction. Then the following two assertions hold:

1) If $||K_{22}|| < 1$, then for every $\Omega \in \mathcal{RS}(H)$ the transform

$$\Theta(z) := K_{11} + K_{12}\Omega(z)(I - K_{22}\Omega(z))^{-1}K_{12}^*, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}, \tag{6.17}$$

also belongs to $\mathcal{RS}(\mathfrak{M})$.

2) If $\Omega \in \mathcal{RS}(H)$ and $\Omega(0) = 0$, then again the transform Θ defined in (6.17) belongs to $\mathcal{RS}(\mathfrak{M})$.

Proof. 1) It follows from Lemma 6.7 that $(I - K_{22}\Omega(z))^{-1}$ exists as a bounded operator on H for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$. Furthermore,

$$\begin{aligned} &\Theta(z) - \Theta(z)^* = K_{12}\Omega(z)(I - K_{22}\Omega(z))^{-1}K_{12}^* - K_{12}(I - \Omega(z)^*K_{22})^{-1}\Omega(z)^*K_{12}^* \\ &= K_{12}(I - \Omega(z)^*K_{22})^{-1}\left((I - \Omega(z)^*K_{22})\Omega(z) - \Omega(z)^*(I - K_{22}\Omega(z))\right)(I - K_{22}\Omega(z))^{-1}K_{12}^* \\ &\qquad = K_{12}(I - \Omega(z)^*K_{22})^{-1}\left(\Omega(z) - \Omega(z)^*\right)(I - K_{22}\Omega(z))^{-1}K_{12}^*. \end{aligned}$$

Thus, Θ is a Nevanlinna function on the domain $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$.

Since $\mathbf{K}$ is a selfadjoint contraction, its entries are of the form (again see Proposition B.1 and Remark B.2):

$$K_{12} = ND_{K_{22}},\ K_{12}^* = D_{K_{22}}N^*,\ K_{11} = -NK_{22}N^* + D_{N^*}LD_{N^*},$$

where $N : \mathfrak{D}_{K_{22}} \to \mathfrak{M}$ is a contraction and $L : \mathfrak{D}_{N^*} \to \mathfrak{D}_{N^*}$ is a selfadjoint contraction. This gives

$$\Theta(z) = N\left(-K_{22} + D_{K_{22}}\Omega(z)(I - K_{22}\Omega(z))^{-1}D_{K_{22}}\right)N^* + D_{N^*}LD_{N^*}.$$

Denote

$$\widetilde{\Theta}(z) := -K_{22} + D_{K_{22}}\Omega(z)(I - K_{22}\Omega(z))^{-1}D_{K_{22}}.$$

Then

$$\widetilde{\Theta}(z) = D_{K_{22}}^{-1}(\Omega(z) - K_{22})(I - K_{22}\Omega(z))^{-1}D_{K_{22}} = D_{K_{22}}(I - \Omega(z)K_{22})^{-1}(\Omega(z) - K_{22})D_{K_{22}}^{-1}$$

and

$$\Theta(z) = N\widetilde{\Theta}(z)N^* + D_{N^*}LD_{N^*}.$$

Again straightforward calculations (cf. [18, 4]) show that for all $f \in \mathfrak{D}_{K_{22}}$,

$$||f||^2 - ||\widetilde{\Theta}(z)f||^2 = ||(I - K_{22}\Omega(z))^{-1}D_{K_{22}}f||^2 - ||\Omega(z)(I - K_{22}\Omega(z))^{-1}D_{K_{22}}f||^2,$$

and for all $h \in \mathfrak{M}$,

$$||h||^2 - ||\Theta(z)h||^2$$
$$= ||N^*h||^2 - ||\widetilde{\Theta}(z)N^*h||^2 + ||D_L D_{N^*}h||^2 + ||(D_N\widetilde{\Theta}(z)N^* - N^*LD_{N^*})h||^2.$$

Since $\Omega(z)$ is a contraction for all $|z| \le 1$, $z \ne \pm 1$, one concludes that $\widetilde{\Theta}(z)$ and, thus, also $\Theta(z)$ is a contraction. In addition, the operators $\Theta(x)$ are selfadjoint for $x \in (-1,1)$. Therefore $\Theta \in \mathcal{RS}(\mathfrak{M})$.

2) Suppose that $\Omega(0) = 0$. To see that the operator $(I - K_{22}\Omega(z))^{-1}$ exists as a bounded operator on H for all $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$, realize Ω as the transfer function of a passive selfadjoint system

$$\sigma = \left\{ \begin{bmatrix} 0 & N \\ N^* & S \end{bmatrix}; H, H, \mathcal{K} \right\},$$

i.e., $\Omega(z) = zN(I - zS)^{-1}N^*$. Since

$$T = \begin{bmatrix} 0 & N \\ N^* & S \end{bmatrix} : \begin{matrix} H \\ \oplus \\ \mathcal{K} \end{matrix} \to \begin{matrix} H \\ \oplus \\ \mathcal{K} \end{matrix}$$

is a selfadjoint contraction, the operator $N \in \mathbf{B}(\mathcal{K}, H)$ is a contraction and S is of the form $S = D_{N^*}LD_{N^*}$, where $L \in \mathbf{B}(\mathfrak{D}_{N^*})$ is a selfadjoint contraction. It follows that the operator $N^*K_{22}N + S$ is a selfadjoint contraction for an arbitrary selfadjoint contraction K_{22} in H. Therefore, $(I - z(N^*K_{22}N + S))^{-1}$ exists on $\mathcal{K}$ and is bounded for all $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$. It is easily checked that for all $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$ the equality

$$\left(I - zK_{22}N(I - zS)^{-1}N^*\right)^{-1} = I + zK_{22}N\left(I - z(N^*K_{22}N + S)\right)^{-1}N^*$$

holds. Now arguing again as in item 1) one completes the proof. □

Theorem 6.9. *Let*

$$\mathbf{S} = \begin{bmatrix} A & B \\ B^* & G \end{bmatrix} : \begin{matrix} H \\ \oplus \\ \mathcal{K} \end{matrix} \to \begin{matrix} H \\ \oplus \\ \mathcal{K} \end{matrix}, \quad \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ H \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ H \end{matrix}$$

be selfadjoint contractions. Also let $\sigma = \{\mathbf{S}, H, H, \mathcal{K}\}$ be a passive selfadjoint system with the transfer function $\Omega(z)$. Then the following two assertions hold:

1) Assume that $||K_{22}|| < 1$. Then $\Theta(z)$ given by (6.17) is the transfer function of the passive selfadjoint system

$$\tau = \{\mathbf{T}, \mathfrak{M}, \mathfrak{M}, \mathcal{K}\},$$

where $\mathbf{T} = \mathbf{K} \bullet \mathbf{S}$ *is the Redheffer product (see* [17, 21]*):*

$$\text{(6.18)}\quad \mathbf{T} = \begin{bmatrix} K_{11} + K_{12}A(I - K_{22}A)^{-1}K_{12}^* & K_{12}(I - AK_{22})^{-1}B \\ B^*(I - K_{22}A)^{-1}K_{12}^* & G + B^*K_{22}(I - AK_{22})^{-1}B \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{matrix}.$$

2) Assume that $A = 0$*. Then the Redheffer product* $\mathbf{T} = \mathbf{K} \bullet \mathbf{S}$ *is given by*

$$\mathbf{T} = \begin{bmatrix} K_{11} & K_{12}B \\ B^*K_{12}^* & G + B^*K_{22}B \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathcal{K} \end{matrix}$$

and the transfer function of the passive selfadjoint system $\tau = \{\mathbf{T}, \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ *is equal to the function* Θ *defined in* (6.17)*.*

Proof. By definition

$$\Omega(z) = A + zB(I - zG)^{-1}B^*, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

1) Suppose that $||K_{22}|| < 1$. Since

$$\Theta(z) = K_{11} + K_{12}\Omega(z)(I - K_{22}\Omega(z))^{-1}K_{12}^* = K_{11} + K_{12}(I - \Omega(z)K_{22})^{-1}\Omega(z)K_{12}^*,$$

one obtains

$$\begin{aligned} \Theta(z) - \Theta(0) &= K_{12}(I - \Omega(z)K_{22})^{-1}\left(\Omega(z) - \Omega(0)\right)\left(I - K_{22}\Omega(0)\right)^{-1}K_{12}^* \\ &= zK_{12}\left(I - AK_{22} - zB(I - zG)^{-1}B^*K_{22}\right)^{-1}B(I - zG)^{-1}B^*(I - K_{22}A)^{-1}K_{12}^*. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\left(I - AK_{22} - zB(I - zG)^{-1}B^*K_{22}\right)^{-1}B(I - zG)^{-1} \\ &\quad = (I - AK_{22})^{-1}\left(I - zB(I - zG)^{-1}B^*K_{22}(I - AK_{22})^{-1}\right)^{-1}B(I - zG)^{-1} \\ &\quad = (I - AK_{22})^{-1}B\left(I - z(I - zG)^{-1}B^*K_{22}(I - AK_{22})^{-1}B\right)^{-1}(I - zG)^{-1} \\ &\quad = (I - AK_{22})^{-1}B\left(I - z\left(G + zB^*K_{22}(I - AK_{22})^{-1}B\right)\right)^{-1} \end{aligned}$$

and one has

$$\begin{aligned} \Theta(z) &= K_{11} + K_{12}A(I - K_{22}A)^{-1}K_{12}^* \\ &\quad + zK_{12}(I - AK_{22})^{-1}B\left(I - z\left(G + zB^*K_{22}(I - AK_{22})^{-1}B\right)\right)^{-1}B^*(I - K_{22}A)^{-1}K_{12}^*. \end{aligned}$$

Now it follows from (6.18) that $\Theta(z)$ is the transfer function of the system τ.

Next it is shown that the selfadjoint operator $\mathbf{T}$ given by (6.18) is a contraction. Let the entries of $\mathbf{S}$ and $\mathbf{K}$ be parameterized by

$$\begin{cases} B^* = UD_A, B = D_AU^* \\ G = -UAU^* + D_{U^*}ZD_{U^*} \end{cases}, \qquad \begin{cases} K_{12} = VD_{K_{22}}, K_{12}^* = D_{K_{22}}V^* \\ K_{11} = -VK_{22}V^* + D_{V^*}YD_{V^*} \end{cases},$$

where V, U, Y, Z are contractions acting between the corresponding subspaces. Also define the operators

$$\Phi_{K_{22}}(A) = -K_{22} + D_{K_{22}}A(I - K_{22}A)^{-1}D_{K_{22}},$$

$$\Phi_A(K_{22}) = -A + D_AK_{22}(I - AK_{22})^{-1}D_A.$$

This leads to the formula

$$\mathbf{T}=\begin{bmatrix}V & 0\\ 0 & U\end{bmatrix}\begin{bmatrix}\Phi_{K_{22}}(A) & D_{K_{22}}(I-AK_{22})^{-1}D_A\\ D_A(I-K_{22}A)^{-1}D_{K_{22}} & \Phi_A(K_{22})\end{bmatrix}\begin{bmatrix}V^* & 0\\ 0 & U^*\end{bmatrix}+\begin{bmatrix}D_{V^*}YD_{V^*} & 0\\ 0 & D_{U^*}ZD_{U^*}\end{bmatrix}.$$

The block operator

$$\mathbb{J}=\begin{bmatrix}\Phi_{K_{22}}(A) & D_{K_{22}}(I-AK_{22})^{-1}D_A\\ D_A(I-K_{22}A)^{-1}D_{K_{22}} & \Phi_A(K_{22})\end{bmatrix}$$

is unitary and selfadjoint. Actually, the selfadjointness follows from selfadjointness of the operators A, K_{22} and $\Phi_{K_{22}}(A), \Phi_A(K_{22})$. Furthermore, one has the equalities

$$||f||^2-||\Phi_{K_{22}}(A)f||^2=||D_A(I-K_{22}A)^{-1}D_{K_{22}}f||^2,$$

$$||g||^2-||\Phi_A(K_{22})g||^2=||D_{K_{22}}(I-AK_{22})^{-1}D_Ag||^2,$$

$$(\Phi_{K_{22}}(A)f, D_{K_{22}}(I-AK_{22})^{-1}D_Ag)=(D_A(I-K_{22}A)^{-1}(A-K_{22})(I-K_{22}A)^{-1}D_{K_{22}}f,g)\,,$$

$$(\Phi_A(K_{22})g, D_A(I-K_{22}A)^{-1}D_{K_{22}}f)=(D_{K_{22}}(I-AK_{22})^{-1}(K_{22}-A)(I-AK_{22})^{-1}D_Ag,f)\,.$$

These equalities imply that $\mathbb{J}$ is unitary.

Denote

$$\mathbb{W}=\begin{bmatrix}V & 0\\ 0 & U\end{bmatrix},\quad \mathbb{X}=\begin{bmatrix}Y & 0\\ 0 & Z\end{bmatrix}.$$

Then

$$\mathbf{T}=\mathbb{W}\mathbb{J}\mathbb{W}^*+D_{\mathbb{W}^*}\mathbb{X}D_{\mathbb{W}^*},$$

and one obtains the equality

$$||h||^2-||\mathbf{T}h||^2=||D_{\mathbb{X}}D_{\mathbb{W}^*}h||^2+||(\mathbb{W}^*\mathbb{X}-D_{\mathbb{W}}\mathbb{J}\mathbb{W}^*)h||^2.$$

Thus, $\mathbf{T}$ is a selfadjoint contraction.

The proof of the statement 2) is similar to the proof of statement 1) and is omitted. □

6.3. **The mapping** $\Omega(z)\mapsto(a\,I+\Omega(z))\,(I+a\,\Omega(z)\,)^{-1}$**.**

Proposition 6.10. *Let*

$$\tau=\left\{\begin{bmatrix}A & B\\ B^* & G\end{bmatrix};\mathfrak{M},\mathfrak{M},\mathcal{K}\right\}$$

be a passive selfadjoint system with transfer function Ω*. Let* $a\in(-1,1)$*. Then the passive selfadjoint system*

$$\sigma_a=\left\{\begin{bmatrix}(aI+A)(I+aA)^{-1} & \sqrt{1-a^2}(I+aA)^{-1}B\\ \sqrt{1-a^2}B^*(I+aA)^{-1} & G-aB^*(I+aA)^{-1}B\end{bmatrix};\mathfrak{M},\mathfrak{M},\mathcal{K}\right\}$$

has transfer function

$$\widehat{\Omega}_a(z)=(a\,I+\Omega(z))(I+a\,\Omega(z))^{-1},\quad z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}.$$

Proof. Let

$$\mathbf{K}_a=\begin{bmatrix}aI & \sqrt{1-a^2}I\\ \sqrt{1-a^2} & -aI\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{M}\end{matrix}\to\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{M}\end{matrix},\ \mathbf{S}=\begin{bmatrix}A & B\\ B^* & G\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathcal{K}\end{matrix}\to\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathcal{K}\end{matrix}.$$

Then the Redheffer product $\mathbf{K}_a \bullet \mathbf{S}$ (cf. (6.18)) takes the form

$$\mathbf{T} = \begin{bmatrix} (aI+A)(I+aA)^{-1} & \sqrt{1-a^2}(I+aA)^{-1}B \\ \sqrt{1-a^2}B^*(I+aA)^{-1} & G-aB^*(I+aA)^{-1}B \end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix} \to \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix}. \tag{6.19}$$

On the other hand, for all $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$ one has

$$\begin{aligned} K_{11} + K_{12}\Omega(z)(I - K_{22}\Omega(z))^{-1}K_{12}^* &= aI + (1-a^2)\Omega(z)(I+a\Omega(z))^{-1} \\ &= (a\,I + \Omega(z))(I + a\,\Omega(z))^{-1}. \end{aligned}$$

This completes the proof. □

6.4. **The mapping** $\Omega(z) \mapsto \Omega\left(\frac{z+a}{1+za}\right)$ **and its fixed points.** For a contraction S in a Hilbert space and a complex number a, $|a|<1$, define, see [20],

$$S_a := (S - aI)(I - \bar{a}S)^{-1}.$$

The operator S_a is a contraction, too. If S is a selfadjoint contraction and $a \in (-1,1)$, then S_a is also selfadjoint. One has $S_a = W_{-a}(S)$ (see Introduction) and, moreover,

$$\begin{aligned} D_{S_a} &= \sqrt{1-a^2}(I-aS)^{-1}D_S, \\ (I - zS_a)^{-1} &= \frac{1}{1+az}(I-aS)\left(I - \frac{z+a}{1+az}S\right)^{-1}, \\ (zI - S_a)(I-zS_a)^{-1} &= \left(\frac{z+a}{1+az}I - S\right)\left(I - \frac{z+a}{1+az}S\right)^{-1}, \end{aligned} \tag{6.20}$$

where $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,\infty\}$. Let the block operator

$$T = \begin{bmatrix} D & C \\ C^* & F \end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix} \to \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix} \tag{6.21}$$

be a selfadjoint contraction and let $\Omega(z) = D + zC(I-zF)^{-1}C^*$. Then from the Schur-Frobenius formula (A.1) and from the relation

$$T_a = (T - aI)(I - aT)^{-1} = \frac{1-a^2}{a}(I - aT)^{-1} - \frac{1}{a}I$$

it follows that T_a has the block form

$$T_a = \begin{bmatrix} (\Omega(a)-aI)(I-a\Omega(a))^{-1} & (1-a^2)(I-a\Omega(a))^{-1}C(I-aF)^{-1} \\ (1-a^2)(I-aF)^{-1}C^*(I-a\Omega(a))^{-1} & F_a + a(1-a^2)(I-aF)^{-1}C^*(I-a\Omega(a))^{-1}C(I-aF)^{-1} \end{bmatrix} \tag{6.22}$$

Theorem 6.11. *Let*

$$\tau = \left\{ \begin{bmatrix} D & C \\ C^* & F \end{bmatrix}, \mathfrak{M}, \mathfrak{M}, \mathcal{K} \right\}$$

be a passive selfadjoint system with the transfer function Ω. *Then for every* $a \in (-1,1)$ *the* $\mathbf{B}(\mathfrak{M})$*-valued function*

$$\Omega\left(\frac{z+a}{1+az}\right)$$

is the transfer function of the passive selfadjoint system

$$\tau_a = \left\{ \begin{bmatrix} \Omega(a) & \sqrt{1-a^2}C(I-aF)^{-1} \\ \sqrt{1-a^2}(I-aF)^{-1}C^* & F_a \end{bmatrix}, \mathfrak{M}, \mathfrak{M}, \mathcal{K} \right\}.$$

Furthermore, if τ is a minimal system then τ_a is minimal, too.

Proof. Let
$$C = KD_F,\ D = -KFK^* + D_{K^*}YD_{K^*},$$
be the parametrization for entries of the block operator T, cf. (2.4), where $K \in \mathbf{B}(\mathfrak{D}_F, \mathcal{K})$ is a contraction and $Y \in \mathbf{B}(\mathfrak{D}_{K^*})$ is a selfadjoint contraction. From (2.6) and (6.20) we get
$$\begin{aligned}\Omega\left(\frac{z+a}{1+az}\right) &= D_{K^*}YD_{K^*} + K\left(\frac{z+a}{1+az}I - F\right)\left(I - \frac{z+a}{1+az}F\right)^{-1}K^*\\ &= D_{K^*}YD_{K^*} + K\left(zI - F_a\right)\left(I - zF_a\right)^{-1}K^*\end{aligned}$$
with $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty\}$. The operator
$$\begin{aligned}\widehat{T}_a &= \begin{bmatrix} -KF_aK^* + D_{K^*}YD_{K^*} & KD_{F_a}\\ D_{F_a}K^* & F_a\end{bmatrix}\\ &= \begin{bmatrix} \Omega(a) & \sqrt{1-a^2}C(I-aF)^{-1}\\ \sqrt{1-a^2}(I-aF)^{-1}C^* & F_a\end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus\\ \mathcal{K}\end{matrix} \to \begin{matrix}\mathfrak{M}\\ \oplus\\ \mathcal{K}\end{matrix}\end{aligned}$$
is a selfadjoint contraction. The formula (2.6) applied to the system τ_a gives
$$\Omega_{\tau_a}(z) = D_{K^*}YD_{K^*} + K\left(zI - F_a\right)\left(I - zF_a\right)^{-1}K^*.$$
Hence $\Omega_{\tau_a}(z) = \Omega\left(\frac{z+a}{1+az}\right)$ for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty\}$.

Suppose τ is the minimal system. This is equivalent to the relations
$$\begin{aligned}&\overline{\operatorname{span}}\,\{F^nD_FK^*\mathfrak{M} :\ n \in \mathbb{N}_0\} = \mathcal{K}\\ \iff\ &\bigcap_{n=0}^{\infty}\ker(KF^nD_F) = \{0\}\\ \iff\ &\textstyle\bigcap_{|z|<1}\ker K(I - zF)^{-1}D_F = \{0\}.\end{aligned}$$
Using the formulas (6.20) one obtains
$$\begin{aligned}&\textstyle\bigcap_{|z|<1}\ker K(I - zF_a)^{-1}D_{F_a} = \bigcap_{|z|<1}\ker K\left(I - \frac{z+a}{1+az}F\right)^{-1}D_F(I - aF)\\ &= (I - aF)\textstyle\bigcap_{|\mu|<1}\ker K(I - \mu F)^{-1}D_F = \{0\}\end{aligned}$$
or, equivalently,
$$\overline{\operatorname{span}}\,\{F_a^nD_{F_a}K^*\mathfrak{M},\ n \in \mathbb{N}_0\} = \mathcal{K}.$$
This shows that the system τ_a is minimal. □

Remark 6.12. *1) Let T in* (6.21) *be represented in the form*
$$T = \begin{bmatrix} K & 0\\ 0 & I\end{bmatrix}\mathbb{J}_F\begin{bmatrix} K^* & 0\\ 0 & I\end{bmatrix} + \begin{bmatrix} D_{K^*}YD_{K^*} & 0\\ 0 & 0\end{bmatrix},$$
see Remark B.3. Then
$$\begin{aligned}\begin{bmatrix} -KF_aK^* + D_{K^*}YD_{K^*} & KD_{F_a}\\ D_{F_a}K^* & F_a\end{bmatrix} &= \begin{bmatrix} \Omega(a) & \sqrt{1-a^2}C(I-aF)^{-1}\\ \sqrt{1-a^2}(I-aF)^{-1}C^* & F_a\end{bmatrix}\\ &= \begin{bmatrix} K & 0\\ 0 & I\end{bmatrix}\mathbb{J}_{F_a}\begin{bmatrix} K^* & 0\\ 0 & I\end{bmatrix} + \begin{bmatrix} D_{K^*}YD_{K^*} & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

2) Let the transformation $\mathbf{V}_a$ *with* $a \in (-1,1)$ *be defined by*

$$\begin{bmatrix} D & C \\ C^* & F \end{bmatrix} \overset{\mathbf{V}_a}{\mapsto} \widehat{T}_a = \begin{bmatrix} \Omega(a) & \sqrt{1-a^2}C(I-aF)^{-1} \\ \sqrt{1-a^2}(I-aF)^{-1}C^* & F_a \end{bmatrix}.$$

Then for all $a, b \in (-1,1)$ *one has the identities*

$$\mathbf{V}_a \circ \mathbf{V}_b = \mathbf{V}_b \circ \mathbf{V}_a = \mathbf{V}_c, \ \textit{where } c = \frac{a+b}{1+ab}.$$

Proposition 6.13. *The fixed points of the mapping* $\Omega(z) \mapsto \Omega\left(\frac{z+a}{1+za}\right)$, $a \in (-1,1)$, $a \neq 0$, *consist only of constant functions.*

Proof. Suppose that for some $a \in (-1,1)$, $a \neq 0$, the equality

$$\Omega\left(\frac{z+a}{1+az}\right) = \Omega(z)$$

is satisfied for all $z \in \mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$. Then, in particular, $\Omega(0) = \Omega(a)$. Therefore from Theorem 6.11 one obtains the equality $KFK^* = KF_aK^*$. Now

$$F - F_a = aD_F^2(I-aF)^{-1}$$

leads to

$$(I-aF)^{-1/2}D_FK^* = 0.$$

Taking into account that $\operatorname{ran} K^* \subseteq \mathfrak{D}_F$, we get $K^* = 0$. This means that $\Omega(z) \equiv \Omega(0)$. So, the fixed points of the mapping $\Omega(z) \mapsto \Omega\left(\frac{z+a}{1+za}\right)$ are the constant functions only. □

Remark 6.14. *A. Filimonov and E. Tsekanovskiĭ* [16] *considered J-unitary operator colligations that are automorphic invariant w.r.t. a subgroup G of the Möbius transformations of the unit disk and its representations in the channel and state spaces. The characteristic function* $W(z)$ *of such a colligation satisfies the condition*

$$W(g(z))V_g = V_gW(z), \quad \forall z \in \mathbb{D} \quad \textit{and} \quad \forall g \in G,$$

where $\{V_g\}$ *is a representation of G in the channel space.*

6.5. The mapping $\Omega(z) \mapsto \left(\Omega\left(\frac{z+a}{1+az}\right) - a\,I\right)\left(I - a\,\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1}$ and its fixed points.

Proposition 6.15. *Let* $\tau = \{T; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$ *be a passive selfadjoint system with transfer function* Ω. *Then the passive selfadjoint system* $\eta_a = \{T_a; \mathfrak{M}, \mathfrak{M}, \mathcal{K}\}$, $a \in (-1,1)$, *has the transfer function*

$$\widetilde{\Omega}_a(z) = \left(\Omega\left(\frac{z+a}{1+az}\right) - a\,I_{\mathfrak{M}}\right)\left(I_{\mathfrak{M}} - a\,\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1}.$$

If τ *is minimal then* η_a *is minimal, too.*

Proof. Let T be a selfadjoint contraction in the Hilbert space $\mathfrak{H}$ and let $a \in (-1,1)$. Due to (6.20) for all $z \in \mathbb{C}\setminus\{(-\infty,-1]\cup[1,\infty\}$ one has

$$(I - zT_a)^{-1} = \frac{1}{1+az}(I-aT)\left(I - \frac{z+a}{1+az}T\right)^{-1}.$$

Moreover,

$$\begin{aligned}(I-aT)\quad\left(I-\frac{z+a}{1+az}T\right)^{-1} &= \left(I-\frac{z+a}{1+az}T\right)^{-1}-aT\left(I-\frac{z+a}{1+az}T\right)^{-1}\\ &=\left(I-\frac{z+a}{1+az}T\right)^{-1}+a\,\frac{1+za}{z+a}I-a\,\frac{1+za}{z+a}\left(I-\frac{z+a}{1+az}T\right)^{-1}\\ &=a\,\frac{1+za}{z+a}I+\frac{z(1-a^2)}{z+a}\left(I-\frac{z+a}{1+az}T\right)^{-1},\end{aligned}$$

and

$$\begin{aligned}(I-zT_a)^{-1} &= \frac{1}{1+az}\left(a\,\frac{1+za}{z+a}I+\frac{z(1-a^2)}{z+a}\left(I-\frac{z+a}{1+az}T\right)^{-1}\right)\\ &=\frac{a}{z+a}I+\frac{z(1-a^2)}{(z+a)(1+az)}\left(I-\frac{z+a}{1+az}T\right)^{-1}.\end{aligned}$$

Let $\mathfrak{H}=\mathfrak{M}\oplus\mathcal{K}$. Since $P_\mathfrak{M}(I-zT)^{-1}\upharpoonright\mathfrak{M}=(I-z\Omega(z))^{-1}$, we get

$$\begin{aligned}P_\mathfrak{M}(I-zT_a)^{-1}\upharpoonright\mathfrak{M} &= \frac{a}{z+a}I_\mathfrak{M}+\frac{z(1-a^2)}{(z+a)(1+az)}\left(I_\mathfrak{M}-\frac{z+a}{1+az}\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1}\\ &=\frac{1}{1+az}\left(I_\mathfrak{M}-a\,\Omega\left(\frac{z+a}{1+az}\right)\right)\left(I_\mathfrak{M}-\frac{z+a}{1+az}\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1}.\end{aligned}$$

Now consider the passive selfadjoint system

$$\eta_a=\{T_a;\mathfrak{M},\mathfrak{M},\mathcal{K}\},\quad T_a=(T-aI)(I-aT)^{-1},$$

and let Ω_{η_a} be the transfer function of η_a. Then from $P_\mathfrak{M}(I-zT_a)^{-1}\upharpoonright\mathfrak{M}=(I_\mathfrak{M}-z\Omega_{\eta_a}(z))^{-1}$ we get

$$(I_\mathfrak{M}-z\Omega_{\eta_a}(z)^{-1}=\frac{1}{1+az}\left(I_\mathfrak{M}-a\,\Omega\left(\frac{z+a}{1+az}\right)\right)\left(I_\mathfrak{M}-\frac{z+a}{1+az}\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1}.$$

Hence,

$$\Omega_{\eta_a}(z)=\left(\Omega\left(\frac{z+a}{1+az}\right)-a\,I_\mathfrak{M}\right)\left(I_\mathfrak{M}-a\,\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1}.$$

Since

$$\begin{aligned}\bigcap_{z\in\mathbb{D}}\ker\left(P_\mathfrak{M}(I-zT_a)^{-1}\right) &= \bigcap_{z\in\mathbb{D}}\ker\left(P_\mathfrak{M}\left(I-\frac{z+a}{1+az}T\right)^{-1}(I-aT)\right)\\ &=(I-aT)^{-1}\bigcap_{\mu\in\mathbb{D}}\ker\left(P_\mathfrak{M}(I-\mu T)^{-1}\right),\end{aligned}$$

we conclude that if τ is minimal then also η_a is minimal. □

Corollary 6.16. *Let $\tau=\{T;\mathfrak{M},\mathfrak{M},\mathcal{K}\}$ be a passive selfadjoint system with transfer function Ω. Let $a\in(-1,1)$ and suppose that $\sigma_a=\{\mathcal{T}(a);\mathfrak{M},\mathfrak{M},\mathcal{K}\}$ is a passive selfadjoint system with transfer function $\Omega\left(\frac{z-a}{1-az}\right)$; see Theorem 6.11. Then the passive selfadjoint system*

$$\zeta_a=\{(\mathcal{T}(a))_a;\mathfrak{M},\mathfrak{M},\mathcal{K}\},\ (\mathcal{T}(a))_a:=(\mathcal{T}(a)-aI)(I-a\mathcal{T}(a))^{-1}$$

has the transfer function

$$\Omega_{\zeta_a}(z)=(\Omega(z)-a\,I)(I-a\,\Omega(z))^{-1},\ z\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}.$$

If τ is minimal then ζ_a is minimal, too.

The next result shows that the Redheffer product $\mathbf{K}_{-a} \bullet \mathbf{V}_a(T)$ coincides with $W_{-a}(T)$.

Proposition 6.17. *Let the block operator T in (6.21) be a selfadjoint contraction, let $\Omega(z) = D + zC(I - zF)^{-1}C^*$, and denote*

$$\widehat{T}_a = \begin{bmatrix} \Omega(a) & \sqrt{1-a^2}C(I-aF)^{-1} \\ \sqrt{1-a^2}(I-aF)^{-1}C^* & F_a \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{array}$$

and

$$\mathbf{K}_{-a} = \begin{bmatrix} -aI & \sqrt{1-a^2}I \\ \sqrt{1-a^2} & aI \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}\end{array}.$$

Then the Redheffer product $\mathbf{K}_{-a} \bullet \widehat{T}_a$ satisfies the equality

$$\mathbf{K}_{-a} \bullet \widehat{T}_a = T_a \left(= (T - aI)(I - aT)^{-1}\right). \tag{6.23}$$

Proof. It follows from (6.19) that the mapping $\mathbf{K}_{-a} \bullet \widehat{T}_a : \mathfrak{M} \oplus \mathcal{K} \to \mathfrak{M} \oplus \mathcal{K}$ has the form

$$\mathbf{K}_{-a} \bullet \widehat{T}_a = \begin{bmatrix} (aI - \Omega(a))(I - a\Omega(a))^{-1} & (1-a^2)(I - a\Omega(a))^{-1}C(I-aF)^{-1} \\ (1-a^2)C^*(I-aF)^{-1}(I - a\Omega(a))^{-1} & F_a + a(1-a^2)(I-aF)^{-1}C^*(I-a\Omega(a))^{-1}C(I-aF)^{-1} \end{bmatrix}.$$

Comparing this with (6.22) leads to (6.23). □

Theorem 6.18. *1) If the function Ω from $\mathcal{RS}(\mathfrak{M})$ is inner, then the equality*

$$\Omega(z) = \left(\Omega\left(\frac{z+a}{1+az}\right) - a\, I_{\mathfrak{M}}\right)\left(I_{\mathfrak{M}} - a\,\Omega\left(\frac{z+a}{1+az}\right)\right)^{-1} \tag{6.24}$$

holds for all $a \in (-1,1)$ and $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$.

2) If $\Omega \in \mathcal{RS}(\mathfrak{M})$ and (6.24) holds for some $a \in (-1,1)$, $a \neq 0$, then Ω is an inner function.

Proof. 1) If $\Omega \in \mathcal{RS}(\mathfrak{M})$ is an inner function, then it takes the form (3.1) and $D = \Omega(0)$. The equality (6.24) can be verified with a straightforward calculation.

2) Suppose that (6.24) holds for some $a \in (-1,1)$. Then the equality

$$\Omega\left(\frac{z+a}{1+az}\right) - a\, I = \Omega(z)\left(I - a\,\Omega\left(\frac{z+a}{1+az}\right)\right)$$

holds for all $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$. Letting $z \to \pm 1$, we get the equalities $\Omega(1)^2 = \Omega(-1)^2 = I_{\mathfrak{M}}$. Moreover, with $z = 0$ we get from (6.24) the equality

$$(\Omega(a) - aI_{\mathfrak{M}})(I_{\mathfrak{M}} - a\Omega(a))^{-1} = \Omega(0).$$

Then by applying Theorem 3.3 one finally concludes that Ω is an inner function. □

6.6. The functional equation $\Omega(z) = \left(\Omega\left(\frac{z-a}{1-az}\right) - a\, I_{\mathfrak{M}}\right)\left(I_{\mathfrak{M}} - a\,\Omega\left(\frac{z-a}{1-az}\right)\right)^{-1}$.

Theorem 6.19. *Let $a \in (-1,1)$, $a \neq 0$. Then the equality*

$$\Omega(z) = \left(\Omega\left(\frac{z-a}{1-az}\right) - a\, I_{\mathfrak{M}}\right)\left(I_{\mathfrak{M}} - a\,\Omega\left(\frac{z-a}{1-az}\right)\right)^{-1} \tag{6.25}$$

holds for all $z \in \mathbb{C} \setminus \{(-\infty,-1] \cup [1,+\infty)\}$ and for some $\Omega \in \mathcal{RS}(\mathfrak{M})$ if and only if Ω is identically equal to a fundamental symmetry in $\mathfrak{M}$.

Proof. We will use the Möbius representation (2.13) for $\Omega \in \mathcal{RS}(\mathfrak{M})$,

$$\Omega(z) = \Omega(0) + D_{\Omega(0)}\Lambda(z)\left(I + \Omega(0)\Lambda(z)\right)^{-1} D_{\Omega(0)}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}, \tag{6.26}$$

with a function $\Lambda \in \mathcal{RS}(\mathfrak{D}_{\Omega(0)})$ such that $\Lambda(z) = z\Gamma(z)$, where Γ is a holomorphic $\mathbf{B}(\mathfrak{D}_{\Omega(0)})$-valued function with $\|\Gamma(z)\| \le 1$ for $z \in \mathbb{D}$; see Proposition 2.3.

Equality (6.25) is equivalent to the equality

$$(\Omega(z) - aI_{\mathfrak{M}})\left(I_{\mathfrak{M}} - a\Omega(z)\right)^{-1} = \Omega\left(\frac{z+a}{1+za}\right) \;\forall z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

Now, with $z = 0$ this gives the equality

$$(\Omega(0) - aI_{\mathfrak{M}})\left(I_{\mathfrak{M}} - a\Omega(0)\right)^{-1} = \Omega(a) \iff \Omega(0) - \Omega(a) = a(I_{\mathfrak{M}} - \Omega(a)\Omega(0)).$$

Denote $\Omega(0) = D$. Assume that $\mathfrak{D}_D \ne \{0\}$ and represent $\Omega \in \mathcal{RS}(\mathfrak{M})$ in the form (6.26). Furthermore, we use that $\Lambda(z) = z\Gamma(z)$. This leads to

$$-aD_D(\Gamma(a)(I + aD\Gamma(a))^{-1}D_D = a\left(I_{\mathfrak{M}} - \left(D + aD_D(\Gamma(a)(I + aD\Gamma(a))^{-1}D_D\right) D\right).$$

It follows that

$$\begin{aligned} & -\Gamma(a)(I + aD\Gamma(a))^{-1} = I - a\Gamma(a)(I + aD\Gamma(a))^{-1}D \\ \iff \quad & (I + a\Gamma(a)D)^{-1}\Gamma(a) = a\Gamma(a)D(I + a\Gamma(a)D)^{-1} - I \\ \iff \quad & (I + a\Gamma(a)D)^{-1}\Gamma(a) = a\Gamma(a)D(I + a\Gamma(a)D)^{-1} - I \\ \iff \quad & (I + a\Gamma(a)D)^{-1}\Gamma(a) = -(I + a\Gamma(a)D)^{-1} \\ \iff \quad & \Gamma(a) = -I. \end{aligned}$$

Since $\Gamma(z)$ belongs to the Schur class in $\mathfrak{M}$, we get

$$\Gamma(z) = -I, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}.$$

Hence for all $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$,

$$\Omega(z) = D - zD_D(I - zD)^{-1}D_D = (D - zI)(I - zD)^{-1}.$$

However, the function $(D - zI)(I - zD)^{-1}$ belongs to the class $\mathcal{RS}(\mathfrak{M})$ if and only if it is a constant function. In other words, one must have $\mathfrak{D}_D = \{0\}$. This means that $\Omega(z) \equiv D$, in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$, and here D is a fundamental symmetry in $\mathfrak{M}$ ($D = D^* = D^{-1}$). □

Appendices

A. The Schur-Frobenius formula for the resolvent

Let

$$\mathcal{U} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{H} \end{matrix}$$

be a bounded block operator. Then the resolvent $R_{\mathcal{U}}(\lambda) = (\mathcal{U} - \lambda I)^{-1}$ of $\mathcal{U}$ (the Schur-Frobenius formula) takes the following block form:

$$R_{\mathcal{U}}(\lambda) = \begin{bmatrix} -V^{-1}(\lambda) & V^{-1}(\lambda)CR_A(\lambda) \\ R_A(\lambda)BV^{-1}(\lambda) & R_A(\lambda)\left(I_{\mathcal{H}} - BV^{-1}(\lambda)CR_A(\lambda)\right) \end{bmatrix}, \quad \lambda \in \rho(\mathcal{U}) \cap \rho(A), \tag{A.1}$$

where

$$V(\lambda) := \lambda I_{\mathfrak{M}} - D + CR_A(\lambda)B,\ \lambda \in \rho(A). \tag{A.2}$$

In particular, $\lambda \in \rho(\mathcal{U}) \cap \rho(A) \iff V^{-1}(\lambda) \in \mathbf{L}(\mathfrak{M})$ and (A.1) and (A.2) imply

$$(P_{\mathfrak{M}} R_U(\lambda)\restriction \mathfrak{M})^{-1} = D - CR_A(\lambda)B - \lambda I_{\mathfrak{M}}.$$

B. Contractive 2×2 block operators

The following well-known result gives the structure of a contractive block operator.

Proposition B.1. [11, 15, 19]. *The block operator 2×2 matrix*

$$T = \begin{bmatrix} D & C \\ B & F \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{array} \to \begin{array}{c}\mathfrak{N}\\ \oplus \\ \mathcal{L}\end{array}.$$

is a contraction if and only if $D \in \mathbf{B}(\mathfrak{M}, \mathfrak{N})$ is a contraction and the entries B,C, and F take the form

$$B = ND_D, \quad C = D_{D^*}G,$$
$$F = -ND^*G + D_{N^*}LD_G,$$

where the operators $N \in \mathbf{B}(\mathfrak{D}_D, \mathcal{L})$, $G \in \mathbf{B}(\mathcal{K}, \mathfrak{D}_{D^*})$ and $L \in \mathbf{B}(\mathfrak{D}_G, \mathfrak{D}_{N^*})$ are contractions. Moreover, the operators N, G, and L are uniquely determined by T. Furthermore, the following equality holds for all $f \in \mathfrak{M}$, $h \in \mathcal{K}$:

$$\left\| \begin{bmatrix} f \\ h \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} D & D_{D^*}G \\ ND_D & -ND^*G + D_{N^*}LD_G \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} \right\|^2$$
$$= \|D_N(D_Df - D^*Gh) - N^*LD_Gh\|^2 + \|D_LD_Gh\|^2.$$

Remark B.2. *If $\mathfrak{N} = \mathfrak{M}$, $\mathcal{L} = \mathcal{K}$, then $T \in \mathbf{B}(\mathfrak{M} \oplus \mathcal{K})$ is a selfadjoint contraction if and only if $D = D^*$, $B = C^*$, $G = N^*$, $L = L^*$.*

Remark B.3. *Let F be a selfadjoint contraction in the Hilbert space $\mathcal{K}$, then the operator given by the block operator*

$$\mathbb{J}_F = \begin{bmatrix} -F & D_F \\ D_F & F \end{bmatrix} : \begin{array}{c}\mathfrak{D}_F\\ \oplus \\ \mathcal{K}\end{array} \to \begin{array}{c}\mathfrak{D}_F\\ \oplus \\ \mathcal{K}\end{array}$$

is selfadjoint and unitary: $\mathbb{J}_F = \mathbb{J}_F = \mathbb{J}_F^{-1}$.

Let $\mathfrak{M}$ be a Hilbert space, let $K \in \mathbf{B}(\mathfrak{D}_F, \mathfrak{M})$ be a contraction and let

$$\begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix} : \begin{array}{c}\mathfrak{D}_F\\ \oplus \\ \mathcal{K}\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{array}.$$

Then for any selfadjoint contraction $Y \in \mathbf{B}(\mathfrak{D}_{K^*})$ the block operator

$$\begin{aligned} T &= \begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -F & D_F \\ D_F & F \end{bmatrix} \begin{bmatrix} K^* & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D_{K^*}YD_{K^*} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -KFK^* + D_{K^*}YD_{K^*} & KD_F \\ D_FK^* & F \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{array} \end{aligned}$$

is selfadjoint contraction. Conversely, any selfadjoint contraction

$$T=\begin{bmatrix} D & C \\ C^* & F\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix}\to\begin{matrix}\mathfrak{M}\\ \oplus \\ \mathcal{K}\end{matrix}$$

has the representation

$$T=\begin{bmatrix} K & 0 \\ 0 & I\end{bmatrix}\mathbb{J}_F\begin{bmatrix} K^* & 0 \\ 0 & I\end{bmatrix}+\begin{bmatrix} D_{K^*}YD_{K^*} & 0 \\ 0 & 0\end{bmatrix}$$

with some contraction $K\in\mathbf{B}(\mathfrak{D}_F,\mathfrak{M})$ *and some selfadjoint contraction* $Y\in\mathbf{B}(\mathfrak{D}_{K^*})$*. Moreover,* T *is unitary if and only if* K *is an isometry and* $Y=Y^*=Y^{-1}$ *in the subspace* $\mathfrak{D}_{K^*}=\ker K^*$*.*

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