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by Michael Müller and Clemens Puppe

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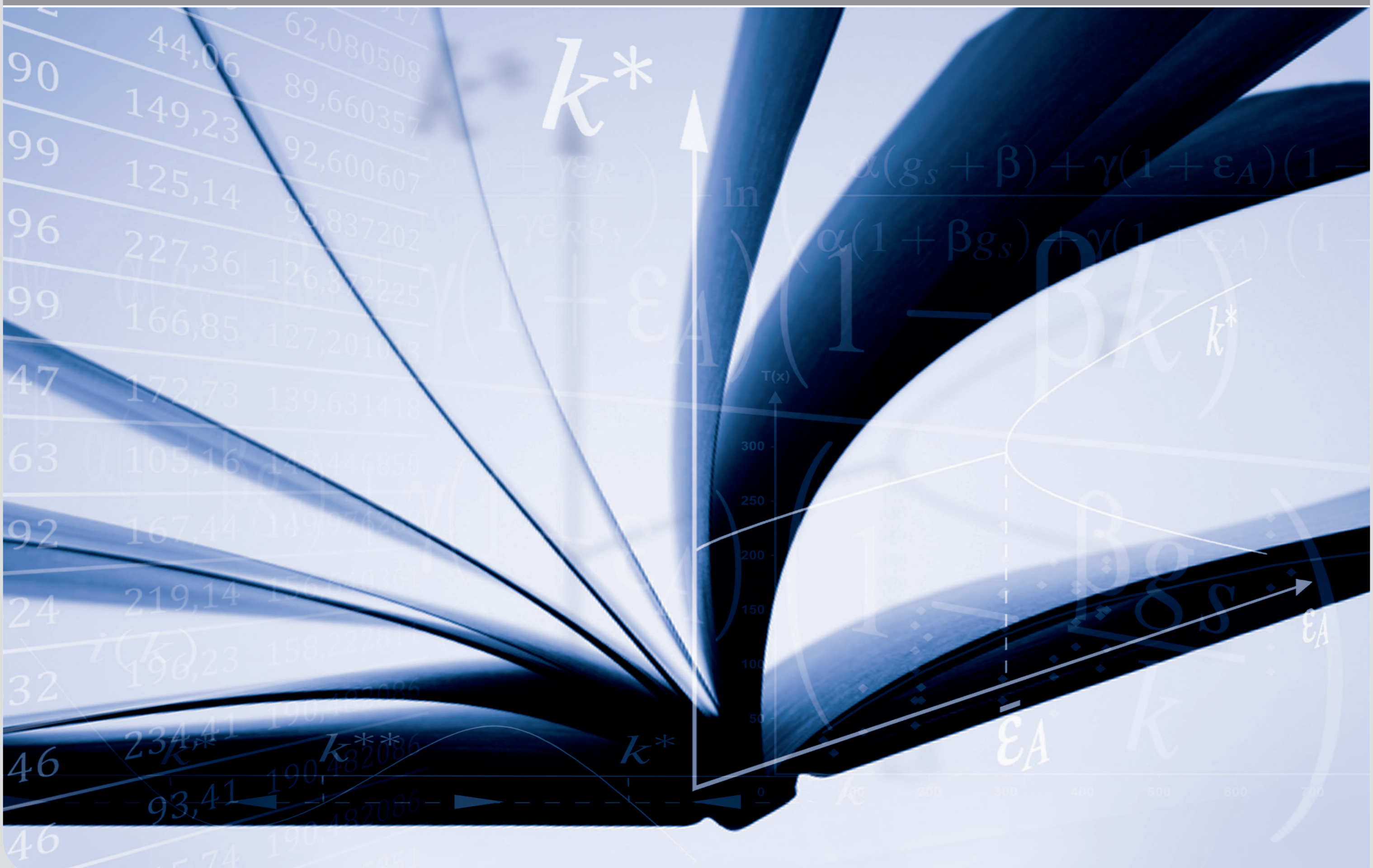

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Strategy-Proofness and Responsiveness imply Minimal Participation*

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Abstract

We study a model in which agents with single-peaked preferences can participate in a costly voting procedure to determine the value of a one-dimensional variable. We show that, for all positive participation cost and all profiles of individual preferences, there exists a (generically) unique equilibrium with (at most) one single participant whenever the voting mechanism is strategy-proof, anonymous, and responsive in the sense that the outcome reacts to a unanimous move of the votes of all agents in the same direction.

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1 Introduction

The paper aims at uncovering a tension between two fundamental kinds of incentive properties of voting mechanisms: strategy-proofness, i.e. the incentive to reveal preferences truthfully, and participation, i.e. the incentive to invest the cost of casting one's vote. Strategy-proofness is desirable because without it the social outcome cannot be assessed vis-à-vis the true preferences of voters.[1] But for the same reason, participation is crucial because one cannot assess the merit of an alternative as a *social* outcome if only few of the agents in fact provide information about their respective preferences even if these signals are truthful.

The *pivotal voter model* (Downs, 1957; Palfrey and Rosenthal, 1983) posits that rational agents will engage in a voting procedure if and only if the expected benefit of doing so exceeds its costs. Since the expected benefit can be positive only if an agent is indeed able to influence the outcome, the possibility of being pivotal is essential for a positive participation decision whenever participation comes even at a small but positive cost. Strategy-proofness, on the other hand, limits the extent to which an agent can influence the outcome: if at all, a voter can change the outcome only to a less preferred outcome. One may thus wonder to what extent strategy-proofness is compatible with participation when voting is costly.

We study this question in a simple voting model with complete information. Our main departure from most of the literature on the pivotal voter model is that we assume a rich one-dimensional space of alternatives in which individuals generically have different top alternatives. As we shall see, this radically changes the properties of equilibria. Of course, our assumption requires that there are more alternatives than voters, therefore our model applies to decisions in small committees and not to 'large' elections in which there are many more voters than alternatives.

Agents take two decisions: whether or not to participate in the voting procedure, and if so, which vote to cast. We consider two versions of this general set-up: a sequential model and a simultaneous model. The sequential model has two stages: in the first stage agents simultaneously decide whether or not to participate in a committee, and in the second stage a simultaneous vote is cast by the committee members on the level of a one-dimensional variable. In the simultaneous model, the participation and voting decisions are made simultaneously by all agents. In either model, we assume that agents incur positive cost if and only if they in fact cast a vote. We shall see that while the two models may lead to different predictions in general, for our main result the timing of decisions is in fact not relevant.

In order to ensure the existence of strategy-proof voting mechanisms, we will assume that voters' preferences are single-peaked. In this case, the class of all anonymous and strategy-proof mechanisms has been characterized by Moulin (1980) as the *generalized median mechanisms*, or the median rule with 'phantom voters.'

As a simple example, think of a faculty meeting on a Friday afternoon at which the yearly expenditure shares, say, for research and teaching have to be determined (given a fixed budget). Each faculty member deliberates about whether or not to participate in the voting procedure. In the sequential model, one can think of the participation decision as being taken before the actual meeting. In the simultaneous model, there could be an announcement during

[1] For a closely related recent argument for strategy-proofness, see Dasgupta and Maskin (2019).

the meeting that a vote would be taken after extensive discussion, and committee members may decide to leave early thereby abstaining from the collective decision. The assumption of complete information is strong but does not seem to be unrealistic in such a scenario; in fact, since all strategy-proof mechanisms only depend on the top alternative of each voter, it is sufficient to know each colleagues' top choice.

Our main result shows that in either version of the model there exists a generically unique equilibrium in which (at most) one single individual participates provided that the voting mechanism satisfies, in addition to strategy-proofness and anonymity, a condition of 'responsiveness' in the sense that the collective outcome reacts to a uniform strict increase (or strict decrease) of *all* votes. The identity of the single participant depends on the voting mechanism, but it is always either the agent with the lowest, or the agent with the highest top alternative.

The conclusion is in stark contrast to other voting mechanisms that are not strategy-proof. For instance, if the collective outcome is determined to be the *average* of the individual votes (Renault and Trannoy, 2005), every voter can shift the outcome by a positive amount and full participation is indeed an equilibrium if participation costs are sufficiently small. But while anonymous and responsive, taking the average does evidently not define a strategy-proof mechanism.

The intuition behind the single participation equilibrium under strategy-proofness is easily explained by looking at the case of two voters. Strategy-proofness and the responsiveness condition jointly imply that in the case of two voters the outcome must coincide with one of the two voters' top alternative; anonymity implies that it cannot depend on the identity of the voter, hence it must be either the lower or the higher top alternative. In the first case, if the agent with the lowest top alternative participates, no other agent has an incentive to participate (since costs are positive); in the latter case, the same holds if the agent with the highest top alternative participates. The proof that there no other equilibria is more involved (see below).

Relation to the Literature

The question of participation when voting is costly has been extensively discussed in the literature ever since the first formal formulation of the pivotal voter model by Palfrey and Rosenthal (1983), see among many others, Börgers (2004) for a seminal theoretical contribution and Levin and Palfrey (2007) for empirical results based on experimental data. The vast majority of the contributions in the literature studies the case of majority voting among two alternatives. Under complete information, a key issue is to analyze the equilibrium consequences of the fact that a large fraction of voters shares the same preferences. The resulting coordination problem is typically addressed by an analysis of mixed equilibria, see Nöldeke and Peña (2016); Mavridis and Serena (2018) for recent contributions. In the present paper, we set this problem aside by assuming that voters have distinct preferences (and preference tops) which is the generic case in our framework with a rich set of alternatives.

The paper closest to ours is Osborne *et al.* (2000). These authors also study a complete information environment with individuals that have single-peaked preferences. In their model, the authors assume that agents vote truthfully and show that 'extreme' voters are more likely to participate than 'moderate' voters. Our analysis complements theirs by providing a game-theoretic foundation for the assumption of sincere voting. Indeed, in our model this is justified by the strategy-proofness of the underlying voting mechanism. An important consequence is that in the case of an even number of participants, strategy-proofness and the responsiveness condition force asymmetric treatment of the two middle ('median') votes. Our main result does not strictly contradict the intuition of Osborne *et al.* (2000), but qualifies it in an important way. Under the responsiveness condition, the single participant is indeed always an 'extreme' voter: either the agent with the lowest or the agent with the highest top alternative; but as explained above, the rationale is not that the moderate voters cancel each other out. Moreover, without the responsiveness property, equilibria can occur in which only 'moderate' agents participate, see Section 4 below.

2 The model

We denote the set of agents by $N = \{1, 2, ..., n\}$. Each agent i is characterized by a single-peaked (ordinal) preference relation $\succcurlyeq_i$ over a compact interval in the reals which we assume to be normalized to unity, i.e. $[0, 1] \subseteq \mathbb{R}$. One possible interpretation is that each $0 \leq x \leq 1$ represents the expenditure share for a public project; but there are other, purely ordinal interpretations as well (e.g. potential positions on a political spectrum). In fact, none of our results depends on the assumption of a *continuous* space of alternatives; what is important is that there are sufficiently many more alternatives than agents.

Single-peakedness means that each agent i has a unique top alternative $0 \leq p_i \leq 1$ (the 'peak') such that, for all $x, y \in [0, 1]$, we have $x \succ_i y$ whenever $y < x \leq p_i$ or $p_i \leq x < y$.[2]

Agents decide whether or not to participate in a committee that decides on the level of the one-dimensional variable $x \in [0, 1]$ by a voting procedure. Each agent i faces a positive participation cost $c_i > 0$. This cost may vary from agent to agent, it may depend on the finally chosen outcome, and even on the set of the other participating agents; in fact, all what matters for our purpose is that these costs are strictly positive for all agents. In particular, we could allow the cost to be private information. For each possible non-empty set $K \subseteq N$ of participants, there is a social choice function $F^K(\succcurlyeq_1, ..., \succcurlyeq_{\#K}) \in [0, 1]$ that maps every profile of preferences of the agents in K to an outcome in $[0, 1]$. The collection $F = \{F^K\}_{\emptyset \neq K \subseteq N}$ of these social choice functions is referred to as a *voting mechanism*. The employed voting mechanism is common knowledge among the agents.

In our model with endogenous participation, we need to specify agents' preferences $\widehat{\succcurlyeq}_i$ over pairs (x, K) of outcomes and sets of participants K who actually cast a vote. We denote by x_0 the (exogenously determined) outcome if nobody participates in the voting process, and will make the following assumptions. For all $i \in N$,

(i) the outcome x_0 is strictly worse than the most preferred outcome with single own participation, i.e.

$$(p_i, \{i\}) \ \widehat{\succ}_i \ (x_0, \emptyset),$$

[2]Note that we do not need to make any assumptions about the comparisons of alternatives on different sides of the peak; in fact, the preference relation may even be incomplete and refrain from making such comparisons.

(ii) for every fixed set $K \neq \emptyset$ the preference over outcomes is given by $\succcurlyeq_i$, i.e.

$$(x, K) \; \widehat{\succcurlyeq}_i \; (y, K) \iff x \succcurlyeq_i y,$$

(iii) for every fixed $x \in [0,1]$, non-participation is strictly preferred to participation (and indifference with respect to the composition of the set of participants otherwise), i.e. for all $K, K' \neq \emptyset$,

$$\begin{aligned} \{i \notin K \quad \text{or } i \in K'\} &\implies (x, K) \; \widehat{\succcurlyeq}_i \; (x, K'), \\ \{i \notin K \text{ and } i \in K'\} &\implies (x, K) \; \widehat{\succ}_i \; (x, K'). \end{aligned}$$

Observe that no assumptions are made about agents' preferences comparing an outcome without participation to a *strictly better* outcome with own participation; indeed, such trade-offs would determine the particular magnitude of participation cost.

We will consider two variants of the model, a simultaneous and a sequential version. In the sequential version, agents first simultaneously decide whether or not to participate and vote simultaneously in a second stage after having observed who the other participants are. By contrast, in the simultaneous version, both the voting and participation decisions are made simultaneously. While the equilibria in general differ in the two models (see the discussion section below), the main conclusions of the present paper are robust with respect to the timing of decision.

In our main result, we will require the voting mechanism to be anonymous and strategy-proof. The *anonymity* condition has two components: first, for each given set of participants K, the outcome under F^K is invariant with respect to permutations of the agents in K; secondly, the employed social choice function F^K should depend only on the number of agents in K. Using the latter condition, we can write F^k for all social choice functions F^K with $\#K = k$, and describe the voting mechanism $F = \{F^k\}_{1 \leq k \leq n}$ in terms of n social choice functions, one for each number of participants.

Strategy-proofness requires that truth-telling be a (weakly) dominant strategy for all participating agents: for all K, $i \in K$, $\succcurlyeq_i$, $\succcurlyeq_i'$, $\succcurlyeq_{K-i}$,

$$F^k(\succcurlyeq_i, \succcurlyeq_{K-i}) \; \succcurlyeq_i \; F^k(\succcurlyeq_i', \succcurlyeq_{K-i}),$$

where $k = \#K$ and $\succcurlyeq_{K-i}$ denotes any profile of preferences for the agents in K other than i.

By a famous result of Moulin (1980), the conditions of anonymity and strategy-proofness jointly imply that all social choice functions F^k are 'generalized medians' with $k+1$ so-called 'phantom voters.' Specifically, for each $k \in \{2, ..., n\}$, F^k only depends on the individual peaks, i.e. for some function $f^k : [0,1]^k \to [0,1]$

$$F^k(\succcurlyeq_1, ..., \succcurlyeq_k) \; = \; f^k(p_1, ..., p_k),$$

and there exist fixed values $\alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k \in [0,1]$ such that

$$f^k(p_1, ..., p_k) \; = \; med\{p_1, ..., p_k, \alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k\}, \tag{2.1}$$

where *med* denotes the usual median operator and the p_i are the peaks of $\succcurlyeq_i$ for each participating agent i; note that there are $2k+1$, i.e. an odd number of values in (2.1). An important example is the standard median rule with an odd number of participants; in this case, half of the phantom voters are placed at 0 and half are placed at 1.

We will say that F^k, respectively f^k, satisfies *responsiveness* if, for all $p_1, ..., p_k$, $p'_1, ..., p'_k$,

$$p'_i > p_i \text{ for all } i \in K \quad \Rightarrow \quad f^k(p'_1, ..., p'_k) \neq f^k(p_1, ..., p_k).$$

Responsiveness can be viewed as a condition of 'local non-imposition:' if *every* agent desires a strictly higher (lower) outcome, the chosen alternative should move at least minimally.[3]

While arguably a weak and plausible condition, responsiveness does restrict the set of admissible voting mechanisms, as follows.

Observation. *The generalized median functions f^k in (2.1) satisfy responsiveness if and only if all 'phantom voters' are either at* 0 *or at* 1*, i.e. for all $j = 1, ..., k+1$, $\alpha_j^k \in \{0, 1\}$, and neither are all phantom voters located at* 0 *nor at* 1*. In particular, in this case the generalized median always coincides with one of the peaks of the agents and the corresponding voting mechanism is efficient.*

Proof. To verify this, suppose that, for some k and $j_0 \leq k+1$, one has $0 < \alpha_{j_0}^k < 1$. Clearly, for any given set of the other phantom voters α_j^k, $j \neq j_0$, one can choose peaks $p_i \in (0, 1)$ all distinct from $\alpha_{j_0}^k$ such that $med\{p_1, ..., p_k, \alpha_1^k, ..., \alpha_{k+1}^k\} = \alpha_{j_0}^k$. But then the generalized median does not react to a sufficiently small uniform move of all peaks. The same is evidently true if all $k+1$ phantom voters are located either at 0 or at 1. Conversely, if all $k+1$ phantom voters are either at 0 or at 1, but not all of them at the same location, the generalized median must be one of the k peaks of the real agents; hence the underlying mechanism is efficient and must react to a uniform move of all peaks. □

It is well-known (Moulin, 1980) that under efficiency, the generalized median functions f^k in (2.1) can be assumed to have $k-1$ instead of $k+1$ phantom voters. Generalized medians for which all $k-1$ phantom voters are at one of the two extremes are also known as the *order statistics*. Specifically, the choice of the i-th lowest value of the $\{p_1, ..., p_k\}$ is referred to as the i-th order statistic, and corresponds to the generalized median in which $k-i$ phantom voters are at 0 and $i-1$ phantom voters are at 1, see Caragiannis *et al.* (2016) for further discussion.

In the present work, we are not focusing on the coordination problem that arises if several agents have the same top alternative. We therefore assume in all what follows that agents' peaks are in generic position, i.e. that no two peaks coincide: $p_i \neq p_j$ for all pairs $i, j \in N$ of distinct agents. If all social choice functions F^k are strategy-proof, voting truthfully is the unique (weakly) dominant strategy for every participant in the simultaneous game, and in every second-stage voting subgame of the sequential game. We will therefore assume that all participants who actually cast a vote submit their true peak. This assumption could be

[3]Intuitively, it should clearly move in the *same* direction; in the present context, this slightly stronger requirement is redundant because it follows from responsiveness plus strategy-proofness.

further justified by an appeal to an equilibrium refinement concept such as perfectness (Selten, 1975), or strong equilibrium (Aumann, 1959)[4].

3 Main result

Theorem 1. *Suppose that the voting mechanism is anonymous, strategy-proof and responsive, and that all individuals' voting costs are positive.*

a) The simultaneous voting game has a unique perfect equilibrium in which exactly one agent participates.

b) The sequential voting game has a unique subgame perfect equilibrium in which all agents choose their unique (weakly) dominant strategy in the second stage; in this equilibrium, again exactly one agent participates.

In either model, the participating agent is either the individual with the highest peak, or the individual with the lowest peak.

Proof. The assumptions on the voting mechanism imply that, for all non-empty sets $K \subseteq N$ of participating agents, the outcome is determined by a generalized median with $\#K-1$ phantom voters. Moreover, by anonymity, the set of phantom voters only depends on $k = \#K$.

We first show that this implies the existence of an equilibrium with a single participant. As before, denote by $p_1, ..., p_n$ the preference peaks of the agents, and assume without loss of generality that $p_1 < p_2 < ... < p_n$. For $\#K = 2$, there is one phantom voter α_1^2, and by the Observation in the previous section, we have either $\alpha_1^2 = 0$, or $\alpha_1^2 = 1$. Suppose the former, then the single participation of agent 1 (who reports truthfully) constitutes an equilibrium. Indeed, the outcome then is p_1 which by assumption is preferred by agent 1 to x_0 (the outcome if nobody participates). Every other agent has a higher peak and can thus not unilaterally change the outcome because $\alpha_1^2 = 0$; hence, if costs are positive each other agent prefers not to participate. The argument is completely symmetric if $\alpha_1^2 = 1$, in which case single participation of individual n is an equilibrium. Note that the argument applies to the dynamic model in the same way as to the simultaneous game.[5]

It remains to show that there are no other equilibria. By contradiction, suppose we have an equilibrium with the set $K \subseteq N$ of participants where $\#K > 1$. If this situation constitutes an equilibrium, it is optimal for all $i \in K$ to participate; we will show that this is not possible. By re-numbering agents, we may assume that $K = \{1, ..., k\}$ and $p_1 < p_2 < ... < p_k$. By the Observation above, there exists $j \in K$ with $f^k(p_1, ..., p_k) = p_j$. First assume that $j = 1$, i.e. that the voter with the lowest peak gets her most preferred alternative. Then, voter k

[4] In the implementation literature, there has been some discussion on the fact that that the median rule (as well as generalized medians) may have other Nash equilibria, in which agents do not follow their unique weakly dominant strategy (see Yamamura and Kawasaki (2013)). For instance, if $k \geq 3$ and all agents cast exactly the same (non-truthful) vote nobody is pivotal, and hence such vote profile constitutes a Nash equilibrium. However, such equilibria are evidently neither robust against trembles, nor against deviations by coalitions of agents.

[5] Of course, the complete strategy in the dynamic game also specifies, for each non-participant, truth-telling in all counterfactual participation situations.

(the one with the highest peak among the participants) has an incentive to abstain; indeed, by the efficiency of f^{k-1} the outcome without voter k cannot be smaller than p_1. By a similar argument, one can show that $j \neq k$.

Thus, we must have that $1 < j < k$ for the individual j who receives her peak p_j. In this case, the assumed optimality of participation by agent 1 implies that

$$f^{k-1}(p_2, ..., p_k) \; = \; med\{p_2, ..., p_k, \alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\} > p_j, \tag{3.2}$$

since otherwise agent 1 would prefer not to participate thereby saving the associated cost. Similarly, the assumed participation of agent k implies that

$$f^{k-1}(p_1, ..., p_{k-1}) \; = \; med\{p_1, ..., p_{k-1}, \alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\} < p_j. \tag{3.3}$$

Without agent 1 there are $j-1$ peaks that are below or equal to p_j. By (3.2), the generalized median $f^{k-1}(p_2, ..., p_k)$ with $k-1$ participants (i.e. agents 1 to $k-1$) is strictly above p_j; this implies that at most $(k-1-j)$ of the $k-2$ phantom voters $\{\alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\}$ can be located at 0. Similarly, without agent k there are $k-j$ peaks above or equal to p_j. By (3.3), the generalized median $f^{k-1}(p_1, ..., p_{k-1})$ with $k-1$ participants (i.e. agents 2 to k) is strictly below p_j; this implies that at most $j-2$ of the $k-2$ phantom voters $\{\alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\}$ can be located at 1. By the responsiveness, all of the $k-2$ phantom voters have to be located either at 0 or at 1. But we have just shown that under conditions (3.2) and (3.2) this is not possible since

$$(k-1-j) \; + \; (j-2) = k-3 \; < \; k-2.$$

Thus, there can be no equilibrium in which more that one agent participates. This concludes the proof of Theorem 1.

□

4 Discussion

In order to assess the scope and the robustness of Theorem 1, we now consider each of its assumptions. We explain why they are necessary for the conclusion and we discuss what happens if they were dropped.

Anonymity

Our anonymity condition has two components. First, it requires the voting mechanism not to depend on the 'names' of voters *for any given set of participants*; secondly, it requires that the same voting mechanism is employed for all subsets with the *same number* of participants. Arguably, both conditions are natural in the present context. The first part is a standard assumption in voting theory, and in fact Moulin's characterization of all strategy-proof mechanisms for single-peaked preferences in terms of phantom voters needs this assumption. In our present variable electorate context the second part also appears to be highly plausible. Importantly, it also guarantees the existence of an equilibrium in pure strategies (and its uniqueness). We show this by means of two simple examples, as follows.

Assume that all conditions of Theorem 1 are satisfied except the second part of the anonymity condition, and consider the following examples with $N = \{1, 2, 3\}$. Suppose that if the set of participants consists of agents 1 and 2, the outcome function $f^{\{1,2\}}$ chooses the higher peak, i.e. we have $\alpha_1^{\{1,2\}} = 1$ for the corresponding phantom voter; if the set of participants consists of agents 1 and 3, the outcome function $f^{\{1,3\}}$ chooses the lower peak, i.e. $\alpha_1^{\{1,3\}} = 0$ for the corresponding phantom voter; and finally, if the set of participants consists of agents 2 and 3, the outcome function $f^{\{2,3\}}$ chooses again the higher peak, i.e. $\alpha_1^{\{2,3\}} = 1$ for the corresponding phantom voter. Evidently, this specification violates the (second part of the) anonymity condition. Suppose that agents are ordered so that $p_1 < p_2 < p_3$. For no agent single participation is an equilibrium: if agent 1 is the single voter, agent 2 has an incentive to join; if agent 2 is the single voter, agent 3 has an incentive to join; and if 3 is the single voter, agent 1 has an incentive to join. A situation with two participants cannot be an equilibrium either because, by the responsiveness condition, one of the two gets her peak in which case the other has an incentive to abstain and save the voting costs. Finally, full participation cannot be an equilibrium either. Indeed, suppose that all agents participate; again by the responsiveness condition, one of the agents must receive her peak. If agent 1 receives her peak, agent 2 has an incentive to abstain, because this would not change the outcome and agent 2 would save the participation cost; similarly, if agent 2 receives her peak, agent 3 has an incentive to abstain, and if agent 3 receives her peak, agent 1 has an incentive to abstain. Hence, in this example there is no equilibrium in pure strategies.

Here is an example in which there are several equilibria, including one with *full participation.* If all agents participate, the social choice function $f^{\{1,2,3\}}$ chooses the standard median, in other words, the corresponding phantom voters are given by $\alpha_1^{\{1,2,3\}} = 0$ and $\alpha_2^{\{1,2,3\}} = 1$; if agents 1 and 2 participate, the social choice function $f^{\{1,2\}}$ chooses the lower peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{1,2\}} = 0$; if agents 2 and 3 participate, the social choice function $f^{\{2,3\}}$ chooses the higher peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{2,3\}} = 1$. No matter how we specify the outcome in the case that the set of participants consists of agents 1 and 3, this already implies that for sufficiently small participation costs full participation is an equilibrium. Indeed, if all agents participate the agent with the median peak gets her peak and hence has no incentive to abstain if her participation costs are sufficiently small; for either of the other two agents, unilateral non-participation would move the outcome further away from their respective peak, so neither of them has an incentive to abstain as well. There also exists an additional single participation equilibrium. Indeed, for the set of participants $\{1, 3\}$ we either have $\alpha_1^{\{1,3\}} = 0$ or $\alpha_1^{\{1,3\}} = 1$. In the first case, single participation of the agent with the lowest peak is an equilibrium (since none of the other two agents can unilaterally change the outcome); in the second case, single participation of the agent with the highest peak is an equilibrium.

Responsiveness

Above, we have justified the responsiveness condition by an appeal to a 'local non-imposition' property: if all agents uniformly move in one direction, the outcome should not remain unchanged. We have also shown that this condition is equivalent to the property that all phantom voters should be at the two extreme points 0 or 1. There may an even deeper justification for the responsiveness condition in purely ordinal contexts. Indeed, if the set of alternatives is linearly ordered but in a purely ordinal way, any specific location of a phantom voter in the interior of the interval $[0, 1]$ seems arbitrary. On the other hand, if cardinal information is available, such as in the example of the dividing a fixed budget, phantoms may placed in the midpoint (at 0.5), or distributed uniformly in $[0, 1]$ (the latter specification is also known as the 'linear median,' see Balinski and Laraki (2011) based on Jennings (2009)). In any case, Theorem 1 fails without the responsiveness condition. As a simple example, consider the case of an even number of agents $N = \{1, 2, ..., 2m\}$, and suppose that the phantom voters are located as follows: $m-1$ phantom voters are at 0, $m-1$ phantom voters are at 1, and one phantom voter is located in the interior, say at $x \in (0, 1)$. Also, suppose that for any set of $2m-1$ participants the standard median is chosen as the outcome. Consider any distribution such that m agents have their peak below x and m agents have their peak above x. Then, if all individuals' costs are sufficiently small, full participation is an equilibrium. Indeed, the outcome under full participation is x, and for any agent unilateral abstention moves the outcome further away from her peak.

But even without the responsiveness condition, there often also exist profiles of peaks such that only one single agent participates in equilibrium. Specifically, let $p_1 < p_2 < ... < p_N$ be such that (i) $\{\alpha_j^k\}_{j=1,...,k+1}^{k=1,...,N} \cap [p_1, p_N] = \emptyset$, i.e. all phantom voters are either below the smallest peak or above the highest peak, and (ii) for all $k = 1, ..., N$, $\min_j\{\alpha_j^k\} < p_1$ and $p_N < \max_j\{\alpha_j^k\}$. (Note that condition (ii) is implied by efficiency of the voting mechanism.) Then, we obtain single participation as the unique equilibrium by the same logic as in the proof of Theorem 1.

Strategy-proofness

The assumption of strategy-proofness is essential for the conclusion of Theorem 1. Strategy-proofness together with a suitable refinement concept such as robustness against small trembles ('perfectness') implies that all participants will cast their vote truthfully. Evidently, the proof of Theorem 1 hinges on that property. In particular, also the fact that the equilibrium does not depend on individual costs (as long as they are all positive) crucially depends on the strategy-proofness (together with the responsiveness condition).

To illustrate this, suppose we can use cardinal information and employ the following *symmetric* version of the median rule. Let the peaks of the participants be ordered such that $p_1 < p_2 < ... < p_k$; for an odd number $k = 2m-1$ the outcome is the standard median p_m, and for an even number $k = 2m$ the outcome is the midpoint between the two middle peaks $(p_m + p_{m+1})/2$. This rule is not strategy-proof, and therefore we cannot assume that the reported peaks coincide with the true peaks.[6] Consider the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$. Let $\tilde{p}_i$ be the reported peak by agent i, and assume that the participation costs of the two extreme agents 1 and 3 are small but positive. Then, the equilibrium depends, among other things, on the magnitude of the participation cost (i.e. the

[6]It is well-known that there exist no strategy-proof, anonymous and symmetric ('neutral') social choice functions on the domain of single-peaked preferences for an even number of individuals, see Moulin (1980, 1988).

precise shape of the preferences) of the median agent. If the participation cost of agent 2 is sufficiently small, full participation and truth-telling is an equilibrium in the simultaneous game; on the other hand, if agent 2 prefers the outcome 0.5 without own participation to the outcome 0.45 while participating, $\tilde{p}_1 = 0$ and $\tilde{p}_3 = 1$ is a (non-truthful) equilibrium.

Moreover, it is because of truthful voting in equilibrium that the conclusion of Theorem 1 is robust with respect to the timing of decisions. Indeed, in the sequential model participants' voting strategy may depend on the set of other participants which becomes common knowledge after all agents have made their participation decision. To illustrate this point, consider again the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$, but now assume that agents move sequentially. In this case, full participation is no longer an equilibrium; indeed, if agent 3 does not participate, agents 1's and 2's optimal votes are $\tilde{p}_1 = 0$ and $\tilde{p}_2 = 0.9$, respectively, with the outcome 0.45. Since this is the same outcome as under full participation, agent 3 prefers to abstain whenever she has positive participation costs. However, participation of agents 1 and 2 with outcome 0.45 can also not be an equilibrium since then agent 1 has an incentive to abstain. In this example, if all agents have sufficiently small (but strictly positive) participation cost, the unique subgame perfect equilibrium of the sequential voting game is given by participation of agents 2 and 3 with votes $\tilde{p}_2 = 0$ and $\tilde{p}_3 = 1$, resulting in the outcome 0.5. In general, one has the following result.

Proposition 1. *Suppose that in the sequential move game, the symmetric median rule is used and that all agents have positive participation cost. In all subgame perfect equilibria such that in the second stage a strong Nash equilibrium is played one has either (i) one single participant, or (ii) an even number of participants and the outcome* 0.5.

(Proof in Appendix)

We note that the single equilibrium participant in this result need not be an extreme voter (in the above example with the peak distribution $p_1 = 0.1$, $p_2 = 0.45$, $p_3 = 0.9$ it could be agent 2 provided that the participation cost for agent 3 is sufficiently high); this is at odds with the intuition put forward in Osborne *et al.* (2000) and thus shows that conclusions about voters' participation in equilibrium crucially depend on institutional details. Also observe that, in contrast to the case considered in Theorem 1, under the symmetric median rule an equilibrium in pure strategies may not exist (an example is provided in the Appendix).

If one is prepared to give up strategy-proofness, a large class of possible rules emerges if cardinal information is available. A simple example is the mean rule which takes the average of all votes as the outcome (Renault and Trannoy, 2005). More generally, one can consider the class of *trimmed means* (Louis *et al.*, 2019). While the equilibria of these rules can still be determined for *fixed* sets of participants, the analysis of the participation game becomes quite complex due to the fact that in equilibrium participants will generally not vote truthfully; for an analysis of the mean rule in the simultaneous participation game, see Müller *et al.* (2019).

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Appendix: Proof of Proposition 1 and further examples

The proof of Proposition 1 is split into three lemmata. Lemma 1 shows that in every strong Nash equilibrium with a fixed even number of participants the outcome is the median of the two middle peaks and 0.5. Lemma 2 shows that all equilibria with an even number of participants must have the outcome 0.5 while Lemma 3 shows that there exist no equilibria with an odd number of participants greater than 1.

Lemma 1. *Consider the symmetric median rule with a fixed even number of participants* k *with ordered peaks* $p_1, ..., p_k$*. There always exists a strong Nash equilibrium, and in all strong Nash equilibria the outcome is the median of* $\{p_{\frac{k}{2}}, 0.5, p_{\frac{k}{2}+1}\}$*.*

Proof. Case 1: $p_{\frac{k}{2}} < 0.5 < p_{\frac{k}{2}+1}$:

In this case, it is easily verified that there is a unique strong Nash equilibrium: the $k/2$ agents with peak lower than 0.5 vote for 0, while the $k/2$ agents with peak larger than 0.5 vote for 1, resulting in the outcome 0.5.

Case 2: $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} \leq 50$:

In this case strong Nash equilibrium is not unique (unless $p_{\frac{k}{2}+1} = 50$) but all strong equilibria in fact result in the same outcome. In all strong equilibria the $k/2$ agents with the lowest peaks vote for 0, agent $\frac{k}{2}+1$ votes for $2 \cdot p_{\frac{k}{2}+1}$ (≤ 1), and all other agents submit a vote between $2 \cdot p_{\frac{k}{2}+1}$ and 1, resulting in the outcome $p_{\frac{k}{2}+1}$.

Case 3: $50 \leq p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$:

Analogously to Case 2, in all strong equilibria the $k/2$ individuals with the highest peaks vote for 1, individual $k/2$ votes for $2 \cdot p_{\frac{k}{2}} - 1$ (≥ 0), and all other individuals submit a vote between 0 and $2 \cdot p_{\frac{k}{2}} - 1$ resulting in the outcome $p_{\frac{k}{2}}$. □

Lemma 2. *All equilibria of the sequential participation game under the symmetric median rule with an even number of participants have the outcome* 0.5 *provided that in the second stage participants play a strong Nash equilibrium.*

Proof. Assume, by way of contradiction, that there exists an equilibrium of the required sort with an even number of participants k and an outcome that is different from 0.5. We will show that this is not possible. By Lemma 1, the outcome is the median of $p_{\frac{k}{2}}$, 0.5 and $p_{\frac{k}{2}+1}$. As the outcome is assumed to be different from 0.5, we must have either $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} < 0.5$ or $0.5 < p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$.

Case 1: $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} < 0.5$:

In this case, Lemma 1 implies that the outcome is $p_{\frac{k}{2}+1}$. If an agent $i < \frac{k}{2}+1$ (i.e. with a peak below $p_{\frac{k}{2}+1}$) abstains, then there are $\frac{k}{2}-1$ participants with a peak below $p_{\frac{k}{2}+1}$ and $\frac{k}{2}-1$ participants with a peak above $p_{\frac{k}{2}+1}$. Hence $j = \frac{k}{2}+1$ is the median participant and the outcome is $p_{\frac{k}{2}+1}$. Since the outcome is unchanged, agent i has an incentive to abstain whenever her participation costs are positive.

Case 2: $0.5 < p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$:

By a completely symmetric argument, one shows that in this case every agent $i > \frac{k}{2}$ (i.e. with a peak above $p_{\frac{n}{2}}$) has an incentive to abstain since this would again not change the outcome. □

Lemma 3. *Under the assumptions of Proposition 1, there are no equilibria with an odd number of participants greater than* 1*.*

Proof. By contradiction, let $k > 1$ be the number of participants and let k be odd. Then, in every (strong) Nash equilibrium, the median participant $i = \frac{k+1}{2}$ determines the outcome by truthfully revealing her peak $p_{\frac{k+1}{2}}$.

Case 1: $p_{\frac{k+1}{2}} = 0.5$:

Then, there are $\frac{k-1}{2}$ participants with a peak below 0.5 and $\frac{k-1}{2}$ participants with a peak above 0.5. Hence if agent $\frac{k+1}{2}$ abstains, the outcome will be the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+3}{2}}$ by Lemma 1, that is the outcome will be 0.5. Thus, since the outcome would not change, agent $\frac{k+1}{2}$ has an incentive to abstain.

Case 2: $p_{\frac{k+1}{2}} < 0.5$:

If an agent $i > \frac{k+1}{2}$ (i.e. an agent with peak above the outcome) abstains, the outcome becomes the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+1}{2}}$ by Lemma 1. But since $p_{\frac{k-1}{2}} < p_{\frac{k+1}{2}} < 0.5$, this means that the outcome does not change; hence, agent i will rather abstain.

Case 3: $p_{\frac{k+1}{2}} > 0.5$:

This case is symmetric to Case 2. □

We conclude with two examples. The first demonstrates that there could be several equilibria of the sort required in Proposition 1 in the sequential model, the second shows that there could exist no equilibria in pure strategies.

Example 1. *Let $p_1 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.7$. Then there exists an equilibrium of the sort required in Proposition 1 with one participant and an equilibrium with an even number of participants and outcome 0.5.*

If agent 2 is the only participant, the outcome is 0.5. Neither agent 1 nor 3 has an incentive to participate since the outcome would not change. If, on the other hand, agents 1 and 3 participate the outcome is again 0.5, hence agent 2 has no incentive to participate. If cost are sufficiently small, agents 1 and 3 indeed prefer to participate, since otherwise the outcome changes to 0.1 or 0.7 respectively.

Example 2. *Let $p_1 = 0.1$, $p_2 = 0.8$ and $p_3 = 0.9$. Then there is no equilibrium of the sort required in Proposition 1 (provided that costs of participation are small).*

With full participation the outcome would be the median of the votes cast, that is: 0.8. In that case agent 1 has an incentive to abstain, since without her the outcome remains unchanged: indeed, agent 2 would vote $\tilde{p}_2 = 0.6$ and agent 3 would vote of $\tilde{p}_3 = 1$ in the equilibrium of the subgame. But this situation cannot constitute an equilibrium either, since agent 3 would rather abstain. If agents 1 and 2, or agents 1 and 3 are the participants, the outcome is 0.5. But for small participation costs, the respective abstainer would prefers to join and change the result to 0.8. Finally, all single participation cases do not constitute an equilibrium since there always exists someone who abstains and could profitably change the outcome to 0.5 (provided that participation costs are small).

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