On the rational functions in non-commutative random variables

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Abstract

This thesis is devoted to some problems on non-commutative rational functions in non-commutative random variables that come from free probability theory and from random matrix theory.

First, we will consider the non-commutative random variables in tracial W^*-probability spaces, such as freely independent semicircular and Haar unitary random variables. A natural question on rational functions in these random variables is the well-definedness question. Namely, how large is the family of rational functions that have well-defined evaluations for a given tuple X of random variables? Note that for a fixed rational function r, the well-definedness of its evaluation $r(X)$ depends on the interpretation of the invertibility of random variables. This is because the invertibility of a random variable in a tracial W^*-probability space (or an operator in a finite von Neumann algebra) can be also considered in a larger algebra, i.e., the $*$-algebra of affiliated operators. One of our goals in this thesis to show some criteria that characterize the well-definedness of all rational functions in the framework of affiliated operators. In particular, one of these criteria is given by a homological-algebraic quantity on non-commutative random variables. We will also show that some notions provided by free probability are related to this quantity. So we can finally answer the well-definedness question via these related notions from free probability.

Those criteria for the well-definedness of rational functions are actually intrinsic connected to the Atiyah conjecture or Atiyah property. We will explore these connections between the Atiyah property and our question on the well-definedness of rational functions. In particular, we will present a result to show a connection between the so-called strong Atiyah property and the invertibility of evaluations of rational functions. In this result, the evaluation of a rational function at a tuple of random variables may not be well-defined, but it is always invertible as an affiliated operator once it is well-defined.

In the last part of this thesis, we will turn to the questions on rational functions in random matrices. Besides the well-definedness problem for rational functions in random matrices, we will also address the convergence problem for rational functions in random matrices. Due to the unstableness of the convergence in distribution, we will limit our random matrices to the ones that strongly converge in distribution and our rational functions to the ones that have bounded evaluations. We will show that both the well-defineness and the convergence problem have an affirmative answer under such conditions.

Abstrakt

Diese Doktorarbeit widmet sich Problemen aus der freien Wahrscheinlichkeitstheorie und der Zufallsmatrizentheorie über nicht-kommutative rationale Funktionen in nicht-kommutativen Zufallsvariablen.

Zunächst betrachten wir nicht-kommutative Zufallsvariable in endlichen (d.h. tracial) W^*-Wahrscheinlichkeitsräumen, wie z.B. freie Halbkreiselemente oder freie Haar unitäre Zufallsvariable. Eine natürliche Frage über rationale Funktionen in solchen Zufallsvariablen ist die nach der Wohldefiniertheit. Genauer gesagt, wie groß ist die Familie von rationalen Funktionen, die eine wohldefinierte Auswertung für ein gegebenes Tupel X von Zufallsvariablen haben? Man muss beachten, dass für eine feste rationale Funktion r die Wohldefiniertheit der Auswertung $r(X)$ von der Interpretation der Invertierbarkeit von Zufallsvariablen abhängt. Dies liegt daran, dass die Invertierbarkeit einer Zufallsvariablen in einem endlichen W^*-Wahrscheinlichkeitsraum (oder eines Operators in einer endlichen von Neumann Algebra) in einer größeren Algebra, nämlich der $*$-Algebra der affiliierten Operatoren, betrachtet werden kann. Eines der Ziele dieser Doktorarbeit ist es Kriterien zu finden, welche die Wohldefiniertheit von allen rationalen Funktionen im Rahmen von affiliierten Operatoren charakterisieren. Insbesondere wird eines dieser Kriterien durch eine homologisch-algebraische Größe von nicht-kommutativen Zufallsvariablen gegeben sein. Wir werden auch zeigen, dass diese Größe mit verschiedenen Größen aus der freien Wahrscheinlichkeitstheorie zusammenhängt. So werden wir schließlich die Wohldefiniertheitsfrage durch diese Größen aus der freien Wahrscheinlichheitstheorie beschreiben.

Diese Kriterien für die Wohldefiniertheit von rationalen Funktionen hängen inhärent mit der Atiyah Vermutung/Eigenschaft zusammen. Wir werden dieser Verbindung zwischen der Atiyah Eigenschaft und unserer Frage nach der Wohldefiniertheit von rationalen Funktionen auf den Grund gehen. Insbesondere werden wir den Zusammenhang zwischen der sogenannten starken Atiyah Eigenschaft und der Wohldefiniertheitsfrage klären. Dabei mag die Auswertung einer rationalen Funktion an einem Tupel von Zufallsvariablen nicht wohldefiniert sein, aber sofern sie wohldefiniert ist, ist sie immer invertierbar als affiliierter Operator.

Im letzten Teil der Doktorarbeit wenden wir uns Fragen zu rationalen Funktionen in Zufallsmatrizen zu. Neben dem Wohldefiniertheitsproblem für rationale Funktionen in Zufallsmatrizen werden wir auch das Konvergenzproblem in dem Rahmen ansprechen. Wegen der Instabilität der Konvergenz in Verteilung schränken wir uns dabei auf Zufallsmatrizen ein, welche stark in Verteilung konvergieren, und betrachen nur rationale Funktionen, welche beschränkte Auswertungen besitzen. Wir werden zeigen, dass under solchen Voraussetzungen sowohl die Wohldefiniertheitsfrage als auch die Konvergenzfrage eine positive Antwort hat.

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Contents

Introduction

In this thesis we address the behaviors of non-commutative random variables toward non-commutative rational functions, in free probability theory and random matrix theory.

Free probability theory was established by Voiculescu in the 1980s during his investigation on the isomorphism problem of free group factors. In order to attack this isomorphism problem, he introduced a probabilistic perspective to regard operators as random variables. Moreover, a non-commutative analogue of the notion of independence in classical probability theory was abstracted out of certain operators, for example, the operators in the reduced free products of C^*-algebras (see [**Voi85**]). This notion of independence was named free independence, which models a relation for non-commuting operators in the free products. It turns out that free independence can be defined in the more general algebraic framework of non-commutative probability spaces. A non-commutative probability space is a pair $(\mathcal{A}, \varphi)$ consisting of a unital complex algebra $\mathcal{A}$ and a unital linear functional φ on $\mathcal{A}$. Elements in the algebra $\mathcal{A}$ are regarded as random variables and the linear functional φ is regarded as the expectation on random variables. In other words, we regard the algebra of random variables and their expectations as foundational objects instead of the underlying probability space. In this way it is possible that random variables may be non-commuting. In that case free independence describes a relation between joint distributions and marginal distributions of non-commuting random variables with the help of moments. Free independence can also be understood as a rule to compute the mixed moments of random variables from their respective moments.

In particular, free independence allows us to describe the sum and the product of freely independent random variables. For example, the free additive convolution $\boxplus$ (see [**Voi86**]) and respectively the free multiplicative convolution $\boxtimes$ (see [**Voi87**]), as analogues of the convolution of probability measures in probability theory, describe the measure for the sum and respectively the product of freely independent random variables. Moreover, one can go further to understand non-commutative polynomials in freely independent random variables. For example, [**BMS17, BSS18**] show that one can compute the distribution of a given polynomial in freely independent random variables out of the distributions of these random variables. Besides quantitative results like the above, qualitative results can also be shown for the polynomials in freely independent random variables. For example, freely independent random variables have no polynomial relations if each distribution of them has no atoms. Moreover, this absence of polynomial relations can be shown to hold locally for freely independent random variables with non-atomic distributions (see [**SS15, CS16, MSW17**]).

There is a possible extension of the above results by considering another basic arithmetic operation—division—on random variables. This leads us to consider the so-called non-commutative rational functions. Non-commutative rational functions constitute a universal skew field (or division ring) containing the free associative algebra of polynomials. We call this skew field the free field. It has a more complicated structure in comparison to its commutative counterpart, i.e., the field of fractions of commutative polynomials. For example, we know that a commutative rational function can always be represented by two polynomials (as a fraction), but a non-commutative rational function usually needs a matrix of polynomials to be represented. Actually, matrices over polynomials (in particular over linear polynomials) are also objects that are studied in the context of free probability. For example, a similar idea under the name of the linearization trick was used in [**BMS17, BSS18**] to reduce a polynomial problem to a matrix-valued linear problem. This suggests many aforementioned results may be extended to the rational function case.

Moreover, this linearization trick is used not only in free probability theory but also in random matrix theory. A random matrix is a matrix-valued random variable, that is, a matrix whose entries are classical random variables. The appearance of random matrices is much earlier than free probability and goes back to the 1920s in statistics (see [**Wis28**]). But there exist deep connections between random matrix theory and free probability theory. One such connection was revealed for the first time by Voiculescu [**Voi91**], who showed that independent Wigner random matrices are asymptotically freely independent as their dimension goes to infinity. Many asymptotic phenomena of Wigner random matrices were known for the case of one Wigner random matrix since the discovery of Wigner's semicircle law [**Wig55**] in the 1950s. But Voiculescu's result teaches us that operators, as non-commutative random variables, can perfectly well describe the limiting distributions of many random matrices even in the multi-variable case. This shows that free probability theory is not merely a non-commutative probability theory in parallel to classical probability theory. Later on a new connection between independent random matrices and freely independent random variables appeared in [**HT05, HST06**]. The usual convergence of random matrices was promoted to a stronger convergence with operator norms involved. The aforementioned linearization trick for polynomials plays an important role in the proof for this convergence. This also suggests that a similar result may hold for rational functions in some random matrices.

Our goal in this thesis is thus to present these extensions mentioned in the above two paragraphs. Our presentation will be based on [**Yin18, MSY18, MSY19**]. The motivation for [**Yin18**] is to extend the result in [**HT05, HST06**] to the rational function case. This extension addresses the convergence problem for rational functions in random matrices. Actually, this convergence problem also naturally arose in [**HMS18**], where the algorithm in [**BMS17**] was developed further to the rational function case. A solution for this convergence problem was given in [**Yin18**] under some conditions on the evaluation of rational functions and on random matrices. As an attempt to remove one of these conditions, it requires a generalization of results in [**SS15, CS16, MSW17**] to the rational function case. The demand on such a generalization finally provides a motivation for [**MSY18, MSY19**]. Since the linearization trick works equally well for rational functions, we may also reduce our rational function problem to a matrix-valued polynomial problem. This idea leads us to the so-called strong Atiyah property, which addresses the matrices over polynomials in random variables (see [**SS15**]). Actually, a solution for our problem was already provided in [**Lin93**] for a specific case, in the context of Atiyah conjecture for groups.

In the following two sections, we will explain our questions as well as their solutions in more details. Their full treatments will be given in Chapter V, VI and VII. Moreover, we will also outline some unexpected discoveries during our investigation on the second question. This will be discussed in details in Chapter V.

The remaining part of this thesis is organized as follows: in Chapter I we will give an introduction to free probability theory. It will be focused on the preliminaries that are needed in this thesis. Then the necessary preliminaries on random matrix theory will be given in Chapter II. We will focus on the connections between free probability theory and random matrix theory. In Chapter III we will give an introduction to the inner rank and related concepts. The material is majorly collected from [**Coh06**] and organized to fit our aim of this thesis. Chapter IV is an introduction to the free field. Besides rational functions, rational closures and division closures will also be introduced in this chapter.

Convergence problem for rational functions

Since the seminal work [**Voi91**] of Voiculescu, many random matrix models are known to have non-commutative random variables from free probability as their limits when their dimension tends to infinity. In particular, we know that independent GUE random matrices converge in distribution to freely independent semicircular random variables; and independent Haar unitary random matrices converge in distribution to freely independent Haar unitary random variables. These two examples are the basic examples that demonstrate the connection between free probability theory and random matrix theory. They are also the guiding examples that we will use to test our theorems in this thesis. Let us denote by $X^{(N)} = (X_1^{(N)}, \cdots, X_d^{(N)})$ random matrices that converge in distribution to some tuple of non-commutative random variables $X = (X_1, \ldots, X_d)$, where N stands for the dimension of matrices $X_i^{(N)}$ $(i = 1, \ldots, d)$. Let us denote by $(\mathcal{A}, \varphi)$ the non-commutative probability space where $X_1, \ldots, X_d$ live in. Then this convergence in particular says that for every non-commutative polynomial p we have

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}_N(p(X^{(N)}))] = \varphi(p(X)),$$

where $\mathbb{E}$ is the expectation and tr_N is the normalized trace on $N \times N$ matrices. Moreover, such a convergence can be promoted to hold almost surely for many random matrices, like GUE and Haar unitary random matrices (see [**HP00a**]).

Now we want to replace the polynomial p by a rational function r. First, there exist simple examples showing that $r(X)$ and $r(X^{(N)})$ may not be well-defined in general. Secondly, there also exist examples telling us that $(r(X^{(N)}))_{N=1}^{\infty}$ may not converge to $r(X)$ in the trace even if they are well-defined and $(X^{(N)})_{N=1}^{\infty}$ converges in distribution to X almost surely. So naturally we ask the following questions:

- when can $r(X)$ be well-defined?
- when can $r(X^{(N)})$ be well-defined?
- suppose that $r(X)$ and $r(X^{(N)})$ are well-defined. Do we have the convergence

$$\lim_{N\to\infty} \mathrm{tr}_N(r(X^{(N)})) = \varphi(r(X)) \quad \text{almost surely?}$$

In particular, do we know the answer for random matrices like GUE and Haar unitary random matrices?

Actually, the distribution of $r(X)$ can be calculated under suitable conditions by an algorithm provided in [**HMS18**]. Moreover, the examples in [**HMS18**, Section 4.7] provide histograms of random matrices that match perfectly the distributions of rational functions in corresponding random variables. This suggests that there is an affirmative answer to the above questions under some suitable conditions on X, $X^{(N)}$ and r.

In Chapter VII we will provide such conditions. We will state our theorem in a more general framework, which in particular covers the random matrix case. In this framework, the conditions on X, $X^{(N)}$ and r as well as their consequences read as follows.

THEOREM 1. *([**Yin18**]) Let $(\mathcal{A}_N, \varphi_N)$ ($N \in \mathbb{N}$) be a family of C^*-probability spaces with faithful states. Let $X^{(N)}$ ($N \in \mathbb{N}^+$) and X respectively be d-tuples of random variables in $(\mathcal{A}_N, \varphi_N)$ ($N \in \mathbb{N}$) and $(\mathcal{A}_0, \varphi_0)$ respectively. We assume the following two conditions on $X^{(N)}$ and X.*

(i) *$(X^{(N)})_{N=1}^\infty$ strongly converges in distribution to X, namely, for each polynomial p, $\lim_{N\to\infty} \varphi_N(p(X^{(N)})) = \varphi(p(X))$ and $\lim_{N\to\infty}\|p(X^{(N)})\| = \|p(X)\|$.*

(ii) *Let r be a rational function such that its evaluation $r(X)$ is well-defined as bounded operator in $\mathcal{A}_0$.*

Then we have the following conclusions.

(i) *$r(X^{(N)})$ is well-defined in $\mathcal{A}_N$ for N large enough.*

(ii) *$\lim_{N\to\infty} \varphi_N(r(X^{(N)})) = \varphi(r(X))$.*

(iii) *$\lim_{N\to\infty} \|r(X^{(N)})\| = \|r(X)\|$.*

There are two requirements in the assumption in the above theorem. The first one is that $(X^{(N)})_{N=1}^\infty$ strongly converges in distribution to X. Its almost sure version is precisely the aforementioned convergence proved in [**HT05, HST06**] for GUE random matrices. This convergence also holds for many other random matrix models like Haar unitary and Wigner random matrices (see [**Sch05, CD07, CM14, And13**]). The second condition is that the evaluation $r(X)$ is a bounded operator. We will show that those are reasonable requirements on X, $X^{(N)}$ and r that cover many examples.

One can try to relax these requirements. For example, $r(X)$ is well-defined as an unbounded operator for every rational function r if X is a tuple of freely independent Haar unitary random variables, due to a result in [**Lin93**]. But the method used in [**Lin93**] probably does not very directly fit many other random variables like semicircular random variables, since their distributions can be very different than the Haar unitary case. In the following section, we will turn to this question, that is, when can $r(X)$ be well-defined as an unbounded operator for every rational function r?

Atiyah property and zero divisors

As we have seen at the end of the last section, the very same question on rational functions shows up in disguise in two different mathematical topics. The connection between the Atiyah conjecture and free probability theory was actually noticed before. In [**SS15**], the strong Atiyah property was introduced as an analogue of the strong Atiyah conjecture for torsion-free groups. Moreover, they proved that freely independent random variables with non-atomic distributions satisfy the strong Atiyah property. This strong

Atiyah property, in the context of free probability, implies in particular that any non-constant polynomial in these random variables is not a zero divisor. This result on the absence of zero divisors is exactly the result we want to extend to the rational function case.

The absence of zero divisors for polynomials was extended to a weaker condition on random variables later in [**CS16, MSW17**]. The proof in [**MSW17**] was based on a non-commutative derivative that was used to reduce the degree of the polynomial in question. Clearly, this idea cannot directly work on the rational function case since the non-commutative derivative will not reduce the complexity of a rational function. However, an extension of this non-commutative derivative method to matrix-valued polynomials can be introduced to solve our problem. In that case we changed the recursive argument on the degree of polynomials in [**MSW17**] to a recursive argument on the dimension of matrices over polynomials, see [**MSY18**]. So a matrix version of the absence of zero divisors was actually proved in [**MSY18**]. Moreover, it turns out that this result is equivalent to the absence of zero divisors for all rational functions. The proof of this equivalence relies on some algebraic techniques that lead us to a rank equality. This rank equality in particular implies the strong Atiyah property. In other words, we found that the absence of zero divisors for all rational functions is some kind of Atiyah property since it is equivalent to our rank equality. Note that this property can only hold for free groups in the context of Atiyah conjecture because it excludes any rational relation for the generators of groups. So it can be understood as a "free Atiyah property". Now let us spell out all the equivalent statements of this property.

THEOREM 2. *(*[**MSY19**]*) Let $(\mathcal{M},\varphi)$ be a tracial W^*-probability space. We denote by $L^0(\mathcal{M},\varphi)$ the $*$-algebra of densely defined and closed operators affiliated with $\mathcal{M}$. For a given tuple $X=(X_1,\dots,X_d)$ in $\mathcal{M}^n$ the following properties for X are equivalent.*

(i) *For any $n\in\mathbb{N}^+$ and linear full matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $A(X)$ is not a zero divisor in $M_n(L^0(\mathcal{M},\varphi))$.*
(ii) *For any $n\in\mathbb{N}^+$ and full matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $A(X)$ is not a zero divisor in $M_n(L^0(\mathcal{M},\varphi))$.*
(iii) *For any $n\in\mathbb{N}^+$ and matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $\operatorname{rank}(A(X))=\rho(A)$.*
(iv) *The free field $\mathbb{C}\langle\!\langle x_1,\dots,x_d\rangle\!\rangle$ is isomorphic to the division closure $\mathbb{C}\langle\!\langle X_1,\dots,X_d\rangle\!\rangle$ (which is also the rational closure) in $L^0(\mathcal{M},\varphi)$ with the isomorphism given by the evaluation map on the free field.*
(v) *The quantity $\Delta(X)=d$.*

In Item (iii), ρ stands for the inner rank over non-commutative polynomials, which is an algebraic rank function on the matrices over polynomials. While $\operatorname{rank}(\cdot)$ in Item (iii) stands for an analytic rank function on matrices over a von Neumann algebra that measures the size of the image of an operator. This rank equality in Item (iii) in particular says that $\operatorname{rank}(A(X))$ always takes values in $\mathbb{N}$, which implies the strong Atiyah property for X. As we have mentioned, when we restrict the choice of X to random variables that come from groups, this equality only holds for the free group case. But in free probability, a lot of random variables satisfy this equality, for example, freely independent semicircular random variables. The condition on random variables that implies the equivalent properties in the above theorem was given by the maximality of a free entropy dimension in [**MSY18**]. Moreover, the condition on random variables was further weakened to Item (v) in [**MSY19**], which turns out to be an equivalent property. This quantity Δ in Item (v) was introduced in [**CS05**] and was shown to be a homological-algebraic analogue of the free entropy dimension. The detailed discussion on this theorem will be given in Section V.1 and V.2. The explanation that this theorem generalizes [**SS15, CS16, MSW17**] on the zero divisor (and atom) problem will be given in Section VI.1.

Moreover, the rank equality in Item (iii) allows us to extract information on the point spectrum of $A(X)$ for any matrix A over polynomials when X satisfies any one of these equivalent properties. Actually, in Section VI.2 we will see that the point spectrum of $A(X)$ agrees with the set of central eigenvalues of A. Note that A is an algebraic object while $A(X)$ is a matrix-valued random variable living in a W^*-probability space. We will deduce some interesting consequences from this correspondence in Section VI.2.

Quite unexpectedly, we also found that the argument used in the proof of the above theorem can be adapted to show a similar list of equivalent properties for the strong Atiyah property.

THEOREM 3. *(*[**MSY18, MSY19**]*) Let $(\mathcal{M},\varphi)$ be a tracial W^*-probability space and $L^0(\mathcal{M},\varphi)$ the $*$-algebra of densely defined and closed operators affiliated with $\mathcal{M}$. For a given tuple $X=(X_1,\dots,X_d)$ in $\mathcal{M}^n$ the following properties for X are equivalent.*

(i) *For any $n\in\mathbb{N}^+$ and $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, if $A(X)$ is full over $\mathcal{R}$, then $A(X)\in M_n(L^0(\mathcal{M},\varphi))$ is not a zero divisor in $\mathcal{M}_n(L^0(\mathcal{M},\varphi))$.*

(ii) *For any $n\in\mathbb{N}^+$ and $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $\operatorname{rank}(A(X))=\rho_{\mathcal{R}}(A(X))$.*

(iii) *The rational closure $\mathcal{R}$ (or the division closure $\mathbb{C}\langle\!\langle X_1,\dots,X_d\rangle\!\rangle$) in $L^0(\mathcal{M},\varphi)$ is a division ring.*

(iv) *X has the strong Atiyah property, i.e., for any $n\in\mathbb{N}^+$ and $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $\operatorname{rank}(A(X))\in\mathbb{N}$.*

Here $\mathcal{R}$ stands for the rational closure of X in $L^0(\mathcal{M},\varphi)$ and $\rho_{\mathcal{R}}$ stands for the inner rank over $\mathcal{R}$. The detailed discussion on this theorem will be given in Section V.3.

CHAPTER I

An introduction to free probability theory

This chapter serves as an introduction to free probability theory. This theory studies the objects called non-commutative random variables which are the non-commutative analogues of random variables in classical probability theory. Here non-commutative random variables do not mean that they are necessarily non-commuting with each other. But when they are in a non-commuting situation, free probability theory provides a notion of independence to describe their relations that can be considered as a counterpart of the notion of independence in classical probability theory. This notion of independence in free probability is called "free independence".

These freely independent non-commutative random variables appear frequently as non-commuting operators from the theory of operator algebras. In other words, free probability theory provides a probabilistic perspective viewing operators as random variables. This perspective is actually the starting point of free probability theory, which was initiated by Voiculescu in the 1980s during his research on the isomorphism problem of free group factors. Free group factors are von Neumann algebras generated by free groups. The isomorphism problem then asks whether the free group factors are the same or not for different numbers of generators. The notion of free independence was abstracted out of these free group factors as a tool to understand their structure. But it turns out the free independence makes perfect sense for non-commuting random variables as an analogue of the independence in classical probability theory.

Later on, a lot of concepts—paralleling their counterparts in classical probability theory—were developed around the free independence, such as the free convolution, free central limit theorem and various notions of free entropy. But free probability actually offers us more than a non-commutative probability theory paralleling classical probability theory. One reason is that there exist deep connections—also discovered by Voiculescu—between free probability theory and random matrix theory. Roughly speaking, Voiculescu observed that the asymptotic distribution of random matrices can be described by freely independent random variables. This discovery led to very fruitful interactions that benefit both sides. Many notions and tools developed in free probability can be used to answer questions raised by random matrix theory. In turn random matrices can be used to prove results in free probability or operator algebras, for instance, [**Dyk93, HT05, HST06**]. We will introduce random matrices in details in Chapter II.

Nowadays free probability is a very active research field with a broad scope on both theoretical and application aspects. Our goal in this chapter is not trying to provide any complete introduction to free probability. Instead we will focus on the notions that will fit our need in later chapters. We refer the interested reader to the monographs [**VDN92, Voi00, HP00b, NS06, MS17**] for a more detailed introduction.

This chapter is organized as follows. In Section I.1 we will introduce the notion of non-commutative probability spaces. They are the basic frameworks for the study of non-commutative random variables. In Section I.2 the distributions of non-commutative random variables will be introduced. Besides the basic notions, we will introduce two important examples of non-commutative random variables—Haar unitary and semicircular random variables. In Section I.3 we will discuss the atoms of distributions. In particular, some operator algebraic characterizations of atoms will be introduced. These characterizations by kernels and zero divisors are the foundational notions that will be used for the investigation in Chapter VI. Then the notion of free independence will be introduced in Section I.4. The first four sections are based on the non-commutative probability spaces whose random variables have finite moments. In Section I.5 we will introduce unbounded random variables that may not have moments. Such an enlarged framework of non-commutative probability spaces is necessary for our investigation in Chapter V.

I.1. Non-commutative probability spaces

It is well-known that a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of three foundational objects: a sample space Ω, a σ-algebra $\mathcal{F}$ on Ω, and a probability measure $\mathbb{P} : \mathcal{F} \to [0,1]$. A random variable is then defined as a measurable function on Ω. The first step to introduce free probability theory, as a non-commutative probability theory, is to view the algebra of random variables and their expectations as foundational objects instead of the underlying probability space. This leads us to a notion of non-commutative probability spaces, which generalizes the notion of classical probability spaces. Actually, similar ideas—viewing the algebra of functions as foundational objects rather than the underlying space—are known for many mathematical theories. For example, the theory of C^*-algebras is usually considered as a non-commutative topology theory and the theory of von Neumann algebras is considered as a non-commutative measure theory. In both cases, operators are regarded as "non-commutative functions" that carry the foundational information. In Section I.5 we will see that the theory of von Neumann algebras provide us the foundational objects to consider unbounded random variables.

Now let us begin to introduce non-commutative probability spaces.

Definition I.1.1. A *non-commutative probability space* $(\mathcal{A}, \varphi)$ consists of a unital complex algebra $\mathcal{A}$ and a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying $\varphi(1) = 1$. Elements in $\mathcal{A}$ are called *non-commutative random variables*, or simply *random variables*.

Note that the algebra $\mathcal{A}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$ is not necessarily a non-commutative one. So this notion of non-commutative probability spaces actually generalizes the notion of classical probability spaces. Let us put the classical probability space (with bounded random variables) as an example of a non-commutative probability space.

Example I.1.2. We take the unital complex algebra $\mathcal{A}$ as the algebra $L^\infty(\Omega, \mathbb{P})$ of bounded complex-valued random variables. Then we let the linear functional φ be the expectation $\mathbb{E} : L^\infty(\Omega, \mathbb{P}) \to \mathbb{C}$ defined by

$$\mathbb{E}[X] := \int_\Omega X(\omega) d\mathbb{P}(\omega) \quad \text{for all } X \in L^\infty(\Omega, \mathbb{P}).$$

It is clear that $\mathbb{E}$ is a unital linear functional since $\mathbb{P}$ is a probability measure. So the pair $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ is indeed a non-commutative probability space.

Apparently every two random variables in $L^\infty(\Omega,\mathbb{P})$ commute. So $L^\infty(\Omega,\mathbb{P})$ is indeed a commutative algebra though $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ is an example of a non-commutative probability space. In particular, this implies that the expectation $\mathbb{E}$ is tracial. Functionals that are tracial play very important roles as we will go further into the world of non-commutative random variables.

DEFINITION I.1.3. For a non-commutative probability space $(\mathcal{A},\varphi)$, the unital linear functional φ is called *tracial* or a *trace* if it satisfies

$$\varphi(ab)=\varphi(ba),\quad \forall a,b\in\mathcal{A}.$$

Then we also say that the non-commutative probability space $(\mathcal{A},\varphi)$ is *tracial*.

Now we give the first example of a non-commutative probability space that consists of random variables that may not commute.

EXAMPLE I.1.4. Let n be a positive integer. We regard the algebra $M_n(\mathbb{C})$ of all $n\times n$ matrices over $\mathbb{C}$ as an algebra of non-commutative random variables. There is a well-known trace Tr_n defined on $M_n(\mathbb{C})$, that is,

$$\mathrm{Tr}_n(A):=\sum_{i=1}^{n}X_{ii},\quad \forall X=(X_{ij})_{i,j=1}^{n}\in M_n(\mathbb{C}).$$

In linear algebra, it is well-known that $\mathrm{Tr}_n(XY)=\mathrm{Tr}_n(YX)$ holds for all $X,Y\in M_n(\mathbb{C})$. So a normalized version of Tr_n is unital and thus becomes a trace in the sense of Definition I.1.3. Precisely, we define $\mathrm{tr}_n:=\frac{1}{n}\mathrm{Tr}_n$ and call it the *normalized trace* on $M_n(\mathbb{C})$. Therefore, $(M_n(\mathbb{C}),\mathrm{tr}_n)$ provides us an example of a tracial non-commutative probability space.

An important structure carried by many non-commutative probability spaces, such as Example I.1.2 and I.1.4, is the $*$-structure. In this case, we require the linear functional to be compatible with this additional structure.

DEFINITION I.1.5. Let $(\mathcal{A},\varphi)$ be a non-commutative probability space. Suppose additionally that $\mathcal{A}$ is a $*$-algebra, i.e., there is an antilinear map $*:\mathcal{A}\to\mathcal{A}$ such that $(a^*)^*=a$ and $(ab)^*=b^*a^*$ for all $a,b\in\mathcal{A}$.

(i) We say $(\mathcal{A},\varphi)$ is a *$*$-probability space* if we have

$$\varphi(a^*a)\geq 0,\quad \forall a\in\mathcal{A}.$$

Such a linear factional φ is called *positive*. A positive and unital linear functional is also known as a *state*.

(ii) Let $(\mathcal{A},\varphi)$ be a $*$-probability space. If for all $a\in\mathcal{A}$ we have

$$\varphi(a^*a)=0 \implies a=0,$$

then φ is called *faithful* and $(\mathcal{A},\varphi)$ is called a *faithful* $*$-probability space.

Example I.1.6. The previous examples $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ and $(M_n(\mathbb{C}),\mathrm{tr}_n)$ are indeed $*$-probability spaces. For $L^\infty(\Omega,\mathbb{P})$, the $*$-structure is naturally given by the complex conjugate of complex numbers. For $(M_n(\mathbb{C}),\mathrm{tr}_n)$, the $*$-structure is given by the conjugate transpose, i.e.,

$$X^* := \big(\overline{X_{ji}}\big)_{i,j=1}^n, \quad \forall X = (X_{ij})_{i,j=1}^n \in M_n(\mathbb{C}).$$

Moreover, the positivity and the faithfulness of $\mathbb{E}$ and respectively tr_n simply comes from the basics of calculus and respectively linear algebra.

A very rich source that can provide examples of $*$-probability spaces are (discrete) groups. Namely, we can interpret group elements as non-commutative random variables in a $*$-probability space given by the group algebra.

Example I.1.7. Let G be a discrete group. We denote by $\mathbb{C}G$ the group algebra of G, that is, the linear span of the indicator functions $\{\delta_g \mid g \in G\}$ in the vector space of complex-valued functions on G. Namely,

$$\mathbb{C}G := \Big\{ \sum_{g\in G} \alpha_g \delta_g \;\Big|\; \alpha_g \in \mathbb{C}, \alpha_g = 0 \text{ except for finitely many } g \Big\}.$$

It is a $*$-algebra with the multiplication and $*$-operation determined by

$$\delta_g \cdot \delta_h := \delta_{gh} \quad \text{and} \quad (\delta_g)^* := \delta_{g^{-1}}$$

for all $g,h \in G$. Let e be the identity element of G. The linear functional φ on $\mathbb{C}G$ defined by

$$\varphi\Big(\sum_{g\in G} \alpha_g\delta_g\Big) := \alpha_e$$

is called the *canonical trace* on $\mathbb{C}G$. One can verify that $(\mathbb{C}G,\varphi)$ is a faithful tracial $*$-probability space.

The previous two examples, $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ as in Example I.1.2 and $(M_n(\mathbb{C}),\mathrm{tr}_n)$ as in Example I.1.4, carry another important structure. Moreover, the example $(\mathbb{C}G,\varphi)$ as in Example I.1.7 can also be extended to carry this structure, which will be presented in Section I.2. This structure is the C^*-algebra structure, which endows a $*$-probability space with some topological structure. It will provide us an analytic framework to study random variables as we will see in Section I.2 and I.3.

Definition I.1.8. A *C^*-probability space* is a $*$-probability space $(\mathcal{A},\varphi)$ with $\mathcal{A}$ being a unital C^*-algebra.

Alternatively, we can also say that a C^*-probability space is a C^*-algebra with a state φ on $\mathcal{A}$. For the basics of the theory of C^*-algebras, we refer to [**KR83, Con90, Bla06**]. Before we move forward, there is an important fact on C^*-algebras that we want to remark here. Namely, each C^*-algebra can be faithfully realized as a (concrete) algebra of bounded operators on some Hilbert space, though it is usually abstractly defined by axioms. Such a representation of a C^*-algebra by bounded operators is usually done by the GNS construction (or GNS representation), see, for instance, [**Bla06**, II.6.4]. For a C^*-probability space $(\mathcal{A},\varphi)$, we usually represent $\mathcal{A}$ as a subalgebra of $B(L^2(\mathcal{A},\varphi))$

by a $*$-homomorphism π. Here $L^2(\mathcal{A},\varphi)$ stands for the Hilbert space induced by the sesquilinear form

$$\langle a,b\rangle := \varphi(b^*a), \quad \forall a,b \in \mathcal{A}$$

(see [**NS06**, Lecture 7] for more details). Moreover, this representation π becomes injective (namely, one-to-one onto its image in $B(L^2(\mathcal{A},\varphi))$) when φ is faithful. In conclusion, we can always regard a random variable in a (faithful) C^*-probability space $(\mathcal{A},\varphi)$ as an operator on some Hilbert space. This allows us to talk about the kernels and images of random variables. These terminologies will offer us operator-algebraic ways to address some question on random variables in Section I.3.

Moreover, with the help of representations of C^*-algebras, one can prove that a matrix algebra over a C^*-algebra is also a C^*-algebra (see [**Bla06**, II.6.6]). This allows us to construct a matricial amplification of a C^*-probability space for each $n \in \mathbb{N}^+$. In this thesis, $\mathbb{N}^+$ always stands for the set of all positive integers; and $\mathbb{N}$ stands for the set of non-negative integers.

Example I.1.9. Let $(\mathcal{A},\varphi)$ be a C^*-probability space whose elements are represented on a Hilbert space H. Then for each $n \in \mathbb{N}^+$, $M_n(\mathcal{A})$ is a C^*-algebra whose elements are in $B(H^n) \cong M_n(B(H))$. Moreover, $(M_n(\mathcal{A}), \mathrm{tr}_n \circ \varphi^{(n)})$ is also a C^*-probability space. Here for a map φ we always denote by $\varphi^{(n)}$ the matricial amplification of φ via applying φ entrywisely. Namely,

$$\varphi^{(n)}(A) := (\varphi(A_{ij}))_{i,j=1}^n \in M_n(\mathbb{C}) \quad \text{for all } A = (A_{ij})_{i,j=1}^n \in M_n(\mathcal{A}).$$

Note that $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ given in Example I.1.2 is a C^*-probability space. We in particular have $(M_n(L^\infty(\Omega,\mathbb{P})), \mathrm{tr}_n \circ \mathbb{E}^{(n)})$ as a C^*-probability space.

It is well-known that a von Neumann algebra is a C^*-algebra but with some special topology. Thus we also have a special subclass of C^*-probability spaces where underlying algebras are von Neumann algebras. Usually we will use $\mathcal{M}$ instead of $\mathcal{A}$ to indicate that the underlying algebra is a von Neumann algebra.

Definition I.1.10. A *W^*-probability space* is $*$-probability space $(\mathcal{M},\varphi)$ such that $\mathcal{M}$ is a von Neumann algebra and φ is a faithful normal state.

Recall that a state φ on $\mathcal{M}$ is called *normal* when $\lim_{\lambda\in\Lambda}\varphi(X_\lambda) = \varphi(X)$ for each monotone increasing net $(X_\lambda)_{\lambda\in\Lambda}$ of operators in $\mathcal{M}$ with least upper bound X. For more detailed and precise description of normal states, we refer to [**KR86**, Chapter 7] and [**Bla06**, III.2].

A tracial W^*-probability space $(\mathcal{M},\varphi)$ is then a von Neumann algebra $\mathcal{M}$ with a faithful normal trace, which is also known as a *finite* von Neumann algebra. We will majorly work in a framework of tracial W^*-probability spaces in this thesis. More precisely, the framework of tracial W^*-probability spaces is necessary for our theorems in Chapter V and VI. It is because that in these two chapters the projections onto kernels and images as well as affiliated operators are necessary tools for our investigation on atoms and zero divisors. While C^*-probability spaces will be enough to fulfill our need for a framework in Chapter VII, where the norm convergence is discussed.

Finally, let us remark that previous examples $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ and $(M_n(\mathbb{C}),\mathrm{tr}_n)$ are W^*-probability spaces. Moreover, similar to the situation in Example I.1.9, $(M_n(\mathcal{M}), \mathrm{tr}_n \circ\varphi^{(n)})$

is also a W^*-probability space for a given W^*-probability space $(\mathcal{M}, \varphi)$. So in particular $(M_n(L^\infty(\Omega, \mathbb{P})), \mathrm{tr}_n \circ \mathbb{E}^{(n)})$ is a W^*-probability space.

I.2. Non-commutative distributions

In this section, we will focus on non-commutative random variables. A basic concept we want to introduce for non-commutative random variables is the concept of distribution. In classical probability theory, the joint probability distribution of a tuple of random variables is the push forward measure of the probability measure by these random variables. However, in the world of non-commutative random variables, we do not have an exact counterpart of such a joint distribution. Instead, we use a non-commutative version of mixed moments to encode the information of a tuple of random variables.

First let us introduce some notation. For succinctness, we usually let the index set be finite. But for many notions and results in this thesis it is easy to see that they could be stated for an infinite index set.

Definition I.2.1. Let $\{x_1, \dots, x_d\}$ be an alphabet for some positive integer d.

(i) We will always use $\mathbb{C}\langle x_1, \dots, x_d\rangle$ to denote the free (associative) unital complex algebra on $\{x_1, \dots, x_d\}$, i.e., the free $\mathbb{C}$-module with a basis consisting of all words over the alphabet $\{x_1, \dots, x_d\}$ and a multiplication defined by the concatenation of words. Usually we will call an element in $\mathbb{C}\langle x_1, \dots, x_d\rangle$ a non-commutative polynomial, or simply a polynomial when it is clear that we refer to $\mathbb{C}\langle x_1, \dots, x_d\rangle$.

(ii) The commutative counterpart of $\mathbb{C}\langle x_1, \dots, x_d\rangle$, namely, the free (associative) commutative unital complex algebra over $\{x_1, \dots, x_d\}$ is known as the (commutative) polynomial ring. We will denote it by $\mathbb{C}[x_1, \dots, x_d]$.

(iii) Let $X = (X_1, \dots, X_d)$ be a d-tuple of elements in a complex unital algebra $\mathcal{A}$. Then we define the *evaluation homomorphism* of $\mathbb{C}\langle x_1, \dots, x_d\rangle$ as the homomorphism uniquely determined by $1 \mapsto \mathbf{1}_{\mathcal{A}}$ and $x_i \mapsto X_i$ for each $i = 1, \dots, d$. We denote this homomorphism by

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_d\rangle \to \mathcal{A}.$$

Given any non-commutative polynomial $p \in \mathbb{C}\langle x_1, \dots, x_d\rangle$, we usually abbreviate $p(X) := \mathrm{ev}_X(p)$ and call it the *evaluation* of p at the tuple X. We denote the image $\mathrm{ev}_X(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ by $\mathbb{C}\langle X_1, \dots, X_d\rangle$. It can also be understood as the subalgebra of $\mathcal{A}$ generated by $\{X_1, \dots, X_d\}$.

Similarly, we can consider a tuple $X = (X_1, \dots, X_d)$ of elements in a commutative unital complex algebra, or equivalently a tuple $X = (X_1, \dots, X_d)$ of commuting elements in a unital complex algebra. Then for each commutative polynomial $p \in \mathbb{C}[x_1, \dots, x_d]$ its *evaluation* $p(X)$ can be defined.

(iv) There is a natural way to endow $\mathbb{C}\langle x_1, \dots, x_d\rangle$ with a $*$-algebra structure. That is, for each $i = 1, \dots, d$, we can simply define $(x_i)^* := x_i$ and $1^* := 1$. This is usually the case when we want to evaluate $\mathbb{C}\langle x_1, \dots, x_d\rangle$ at a tuple $X = (X_1, \dots, X_d)$ consisting of self-adjoint random variables. (A random variable Y is *self-adjoint* if $Y = Y^*$.)

(v) Let $\mathcal{A}$ be a $*$-algebra and $X \in \mathcal{A}^d$ a tuple whose entries may not be self-adjoint. We can enlarge the algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ to enable the evaluation at X which also encodes the $*$-structure information. Namely, we denote by $x_1^*, \dots, x_d^*$ another d formal variables, then we have the algebra $\mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle$ on which we can define an antilinear map by $x_i \mapsto x_i^*$ and $x_i^* \mapsto x_i$ for each $i = 1, \dots, d$. In this case, we define the evaluation map by $x_i \mapsto X_i$ and $x_i^* \mapsto X_i^*$ and denote it by $\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle \to \mathcal{A}$. The evaluation $\mathrm{ev}_X(p)$ for a polynomial p is also abbreviated as $p(X)$ in this case.

Now we give the definition for the non-commutative joint distribution and the non-commutative moments. Their existence is simply due to the definition of the $*$-probability space. Clearly there are many random variables in classical probability theory that do not have finite moments. These random variables without finite moments will be dealt with in Section I.5.

Definition I.2.2. Let $X = (X_1, \dots, X_d)$ be a tuple of non-commutative random variables in some $*$-probability space $(\mathcal{A}, \varphi)$. We define the *non-commutative joint distribution* (or simply *joint distribution*) of X as the linear functional $\mu_X := \varphi \circ \mathrm{ev}_X$, i.e.,

$$\mu_X : \mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle \to \mathbb{C}, \quad p \mapsto \varphi(p(X)).$$

An expression of the form

$$\mu_X(x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}) = \varphi(X_{i_1}^{\varepsilon_1} \dots X_{i_k}^{\varepsilon_k}),$$

with $k \in \mathbb{N}^+$, $i_1, \dots, i_k \in \{1, \dots, d\}$ and $\varepsilon_1, \dots, \varepsilon_k \in \{1, *\}$, is called a *(non-commutative) moment of order* k of X.

Example I.2.3. In Example I.1.2, we have seen that a non-commutative probability space $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$ can always be constructed out of a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A bounded random variable $X \in L^\infty(\Omega, \mathbb{P})$ always has finite moments that are given by

$$\mathbb{E}(X^k (X^*)^l) = \int_\Omega (X(\omega))^k (\overline{X(\omega)})^l d\mathbb{P}(\omega), \quad k, l \in \mathbb{N}^+.$$

These are exactly the moments of X in the sense of Definition I.2.2. Clearly, for a tuple of random variables in $L^\infty(\Omega, \mathbb{P})$, its moments are the same as their counterparts in classical probability theory.

As we have mentioned in Section I.1, in the framework of a C^*-probability space, it is possible to have more analytic tools to study random variables. One of them is the following analytic distribution that is determined by the moments.

Definition I.2.4. Let X be a random variable in a C^*-probability space $(\mathcal{A}, \varphi)$. If X is *normal*, i.e., $XX^* = X^*X$, then there exists a unique regular Borel probability measure μ_X supported on its spectrum $\sigma(X) := \{\lambda \in \mathbb{C} \mid \lambda - X \text{ is not invertible in } \mathcal{A}\}$ such that

$$\varphi(p(X, X^*)) = \int_{\mathbb{C}} p(z, \overline{z}) \mu_X(z)$$

for any polynomial $p \in \mathbb{C}[x, x^*]$. We call this measure the *analytic distribution* of X.

EXAMPLE I.2.5. We know that $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$ is indeed a C^*-probability space. Let $X\in L^\infty(\Omega,\mathbb{P})$ be a classical random variable, which is always normal. Then its analytic distribution μ_X is determined by

$$\int_{\mathbb{C}} z^k\overline{z}^l d\mu_X(z)=\mathbb{E}(X^k(X^*)^l)=\int_\Omega (X(\omega))^k(\overline{X(\omega)})^l d\mathbb{P}(\omega),\quad k,l\in\mathbb{N}^+$$

according to Definition I.2.4. One can show that μ_X is the push-forward measure of the probability measure $\mathbb{P}$ along the random variable X. Namely, μ_X is exactly the probability distribution (or the law of) of X in classical probability theory.

EXAMPLE I.2.6. Let X be a normal matrix in the C^*-probability space $(M_n(\mathbb{C}),\mathrm{tr}_n)$ defined in Example I.1.4. Then its analytic distribution μ_X is given by

$$\mu_X=\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i},$$

where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of X and $\delta_\cdot$ stands for the Dirac measure at a point. This is due to

$$\mathrm{tr}_n(X^k(X^*)^l)=\frac{1}{n}\sum_{i=1}^n\lambda_i^k\overline{\lambda_i}^l,$$

which can be deduced by the spectral theorem for normal matrices. We call this measure μ_X the *eigenvalue distribution* of X. One can see that for a given Borel set $B\subseteq\mathbb{C}$

$$\mu_X(B)=\frac{\#\{\lambda\in\sigma(X)\mid\lambda\in B\}}{n},$$

that is, $\mu(B)$ is the ratio of the number of eigenvalues lying in B relative to the total number of eigenvalues.

Moreover, we have a closely related example by adapting Example I.1.9 to $L^\infty(\Omega,\mathbb{P})$, i.e. the C^*-probability space $(M_n(L^\infty(\Omega,\mathbb{P})),\mathrm{tr}_n\circ\mathbb{E}^{(n)})$. We consider a normal random variable in this case, that is, $X=(X(\omega)_{ij})_{i,j=1}^n$ is a normal matrix for almost every $\omega\in\Omega$. Then the analytic distribution μ_X of X is determined by

$$\begin{aligned}\int_{\mathbb{C}} z^k\overline{z}^l d\mu_X(z)&=(\mathrm{tr}_n\circ\mathbb{E}^{(n)})(X^k(X^*)^l)=\mathbb{E}\big[\,\mathrm{tr}_n\left((X(\omega))^k((X(\omega))^*)^l\right)\big]\\&=\mathbb{E}\left[\int_{\mathbb{C}} z^k\overline{z}^l d\mu_{X(\omega)}(z)\right]\end{aligned}$$

So one can understand μ_X as a measure-valued integral of $\omega\mapsto\mu_{X(\omega)}$, i.e.,

$$\mu_X=\mathbb{E}[\mu_{X(\cdot)}]=\int_\Omega\mu_{X(\omega)}d\mathbb{P}.$$

We call this measure μ_X the *averaged eigenvalue distribution* of X. Then for a Borel set $B\subset\mathbb{C}$, $\mu_X(B)$ is the averaged ratio of the number of eigenvalues lying in B to relative n.

These random variables in $(M_n(L^\infty(\Omega,\mathbb{P})),\mathrm{tr}_n\circ\mathbb{E}^{(n)})$ are the so-called random matrices. They will be introduced in Chapter II in more details. In the following, we will turn to two basic examples in the context of free probability. They will provide the models for the asymptotic distributions of some random matrices in Chapter II.

Example I.2.7. Let U be a random variable in a C^*-probability space $(\mathcal{A}, \varphi)$. If U is a *unitary* (i.e. $UU^* = U^*U = 1$) that satisfies

$$\varphi(U^k) = \varphi((U^*)^k) = 0, \quad \forall k \in \mathbb{N}^+,$$

then U is called a *Haar unitary* random variable. It is easy to see that the moments of U are given by

$$\varphi(U^k(U^*)^l) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

for all $k, l \in \mathbb{N}^+$. Note that these moments agree with the moments of the normalized Haar measure on the circle $\mathbb{T} \subseteq \mathbb{C}$. Namely, for all $k, l \in \mathbb{N}^+$

$$\int_{\mathbb{T}} z^k \overline{z}^l dz = \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-l)t} dt = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

where dz stands for the normalized Haar measure on $\mathbb{T}$. So, according to Definition I.2.4, the analytic distribution of a Haar unitary random variable U is the Haar measure on its spectrum $\mathbb{T}$.

A basic way to construct a Haar unitary is regarding a non-torsion group element as a random variable in the $*$-probability space given in Example I.1.7. To be more precise, let us consider a discrete group G. Let $(\mathbb{C}G, \varphi)$ be the $*$-probability space defined in Example I.1.7. Suppose that $g \in G$ is not a torsion element, i.e. $g^k \neq e$ for all $k \in \mathbb{N}^+$ (where e is the identity element of G). Following Example I.1.7, we denote by $\delta_\cdot$ the indicator function at a point. Then we see for all $k \in \mathbb{Z}$

$$\varphi(\delta_g^k) = \varphi(\delta_{g^k}) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

So δ_g has to be a Haar unitary random variable if we can extend $(\mathbb{C}G, \varphi)$ to a C^*-probability space. For that purpose, let $l^2(G)$ be the Hilbert space consisting of square-integrable functions on G with respect to the counting measure. That is, we have

$$l^2(G) := \{ \sum_{g \in G} \alpha_g \delta_g \mid \sum_{g \in G} |\alpha_g|^2 < \infty \}$$

and its inner product is determined by

$$\langle \delta_g, \delta_h \rangle := \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

So $l^2(G)$ has a canonical orthonormal basis $(\delta_g)_{g \in G}$. The map $\lambda : G \to B(l^2(G))$ determined by

$$\lambda(g)\delta_h = \delta_{gh}, \quad \forall g, h \in G,$$

is called the *left regular representation* of G. Actually, λ sends each group element $g \in G$ to a unitary operator acting on $l^2(G)$ and $\lambda(g^{-1}) = (\lambda(g))^*$. It can be extended to a $*$-homomorphism defined on the group algebra $\mathbb{C}G$. Moreover, the operator norm closure of the image of $\mathbb{C}G$ under λ is a C^*-algebra. This C^*-algebra is usually called the *reduced*

C^*-algebra of G and is denoted by $C^*_{\text{red}}(G)$. The *vector state* $\varphi : B(l^2(G)) \to \mathbb{C}$ with respect to δ_e, which is given by

$$\varphi(X) := \langle X\xi_e, \xi_e \rangle, \quad \forall X \in B(l^2(G)),$$

is a faithful trace on $C^*_{\text{red}}(G)$ and agrees with the φ defined in Example I.1.7 on $\mathbb{C}G$. So $(C^*_{\text{red}}(G), \varphi)$ is a C^*-probability space extending $(\mathbb{C}G, \varphi)$. Furthermore, the strong operator topological closure (or equivalently, the bicommutant) of $\lambda(\mathbb{C}G)$ is a von Neumann algebra. We usually call this von Neumanna algebra the *group von Neumann algebra* and denote it by $\mathcal{L}(G)$. With the vector state defined above, $(\mathcal{L}(G), \varphi)$ is then a faithful tracial W^*-probability space. Then $\lambda(g)$ is a Haar unitary random variable for each non-torsion element $g \in G$ in both $(C^*_{\text{red}}(G), \varphi)$ and $(\mathcal{L}(G), \varphi)$.

We want to point out that for a torsion element $g \in G$, we can also regard it as a non-commutative random variable via the left regular representation λ. In this case, the random variable $\lambda(g)$ is called *p-Haar unitary* where p is the order of g. It has the analytic distribution

$$\mu_{\lambda(g)} = \frac{1}{p}\sum_{i=1}^{p} \delta_{\lambda_i},$$

where $\delta.$ stands for the Dirac measure at a point and $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$ are the roots of order p of unity.

Example I.2.8. Let S be a self-adjoint random variable in a C^*-probability space $(\mathcal{A}, \varphi)$. Suppose that the moments of X are given by

$$\varphi(S^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ C_{\frac{k}{2}} & \text{if } k \text{ is even,} \end{cases}$$

where

$$C_p := \frac{1}{p+1}\binom{2p}{p} = \frac{(2p)!}{p!(p+1)!}$$

stands for the p-th *Catalan number*. Then S is called a *(standard) semicircular* random variable. According to Definition I.2.4, there exists a unique probability measure μ_S such that

$$\int_{\mathbb{R}} t^k d\mu_S(t) = \varphi(S^k).$$

Actually, one can prove that

$$\frac{1}{2\pi}\int_{-2}^{2} t^k\sqrt{4-t^2}dt = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ C_{\frac{k}{2}} & \text{if } k \text{ is even,} \end{cases}.$$

So we see that $d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}\mathbb{1}_{[-2,2]}(t)dt$, where $\mathbb{1}_{[-2,2]}$ is the indicator function of the interval $[-2, 2]$.

Now, we will present a concrete construction of a standard semicircular random variable via the one-sided shift operator. Consider the Hilbert space $l^2(\mathbb{N})$ in which a vector is a sequence $\xi = (\xi_i)_{i=0}^{\infty}$ of complex numbers satisfying $\sum_{i=0}^{\infty}|\xi_i|^2 < \infty$. The inner product of two vectors $\xi = (\xi_i)_{i=1}^{\infty}$ and $\eta = (\eta_i)_{i=1}^{\infty}$ is $\langle \xi, \eta \rangle := \sum_{i=0}^{\infty} \xi_i\overline{\eta_i}$. Then $(B(l^2(\mathbb{N})), \varphi)$

becomes a C^*-probability space with the vector state $\varphi : B(l^2(\mathbb{N})) \to \mathbb{C}$ with respect to e_0, that is,

$$\varphi(X) := \langle Xe_0, e_0 \rangle\,, \quad \forall X \in B(l^2(\mathbb{N})).$$

Note that $l^2(\mathbb{N})$ has an orthonormal basis $(e_i)_{i=0}^{\infty}$, where for each $i \in \mathbb{N}$ e_i stands for the sequence whose components are all zeros except that the i-th component is 1. We define the *right-shift operator* R as the operator in $B(l^2(\mathbb{N}))$ determined by

$$Re_i = e_{i+1}, \quad \forall i \in \mathbb{N}.$$

Then its adjoint R^* is the *left-shift operator* determined by

$$R^*e_0 = 0 \quad \text{and} \quad R^*e_i = e_{i-1},\ \forall i \in \mathbb{N}^+.$$

We set

$$S := R + R^* \in B(l^2(\mathbb{N})).$$

This operator S is indeed a standard semicircular random variable in $(B(l^2)(\mathbb{N}), \varphi)$. This can be seen by showing that

$$\varphi(S^k) = \sum_{\varepsilon_i,\dots,\varepsilon_k \in \{1,*\}} \langle R^{\varepsilon_1} \cdots R^{\varepsilon_k} e_0, e_0 \rangle$$

and the sum on the right hand side is actually the cardinality of *Dyck paths* of length k. This cardinality is known to be 0 if k is odd and to be the Catalan number $C_{k/2}$ if k is even. We refer the interested reader to [**NS06**, Lecture 2] for more details.

I.3. Atoms and zero divisors

In this section, we will introduce the notion of atoms for the analytic distribution of a random variable as well as their algebraic avatars—zero divisors. Atoms and zero divisors of non-commutative random variables will be the main topic of Chapter VI.

Let us first introduce atoms for a probability measure.

Definition I.3.1. Let μ be a Borel probability measure on $\mathbb{C}$. A number $\lambda \in \mathbb{C}$ is called an *atom* of μ if $\mu(\{\lambda\}) \neq 0$.

Atoms can be detected by the Cauchy transform, which is a very important and powerful tool in free probability. We will not need this tool in our later investigation on atoms. But let us introduce it here and show how it can used for the study of atoms.

Definition I.3.2. Let μ be a Borel probability measure on $\mathbb{R}$. The *Cauchy transform* of μ is the analytic function G_μ defined on the upper half plane $\mathbb{C}^+ := \{z \in \mathbb{C} \bigm| \operatorname{im}(z) > 0\}$ by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t), \quad \forall z \in \mathbb{C}^+.$$

Remark I.3.3. Let μ be a Borel probability measure on $\mathbb{R}$ with its Cauchy transform G_μ. Then the existence of an atom of μ at $\lambda \in \mathbb{R}$ can be examined by taking the limit of the Cauchy transform along a sequence of points that is approaching λ from the complex upper half-plane. More precisely, we have the formula as follows:

$$\lim_{\measuredangle z \to \lambda} (z-\lambda) G_\mu(z) = \mu(\{\lambda\}), \quad \forall \lambda \in \mathbb{R}, \tag{I.1}$$

where $\lim_{\measuredangle z\to\lambda}$ stands for the *non-tangential limit*. For the precise definition of the non-tangential limit and the proof of this formula, we refer to [**MS17**, Proposition 3.1.8].

Now let us consider a normal random variable X in a faithful W^*-probability space $(\mathcal{M},\varphi)$. Then we can ask whether its analytic distribution μ_X given in Definition I.2.4 has atoms or not. Instead of considering its Cauchy transform, we will introduce a more algebraic (or operator-algebraic) method to detect its atoms. Namely, we will interpret its atoms through eigenspaces and zero divisors. For that purpose, the framework of W^*-probability spaces is necessary. First, let us recall some notions in the theory of operator algebras.

Definition I.3.4. [**Con90**, Definition IX.1.1] Let Σ be a subset of $\mathbb{C}$ and H a Hilbert space. A *projection-valued measure* over Σ is a map E sending each Borel subset of Σ to a projection in $B(H)$ such that:

(i) $E(\emptyset)=0$ and $E(\Sigma)=1$.
(ii) $E(A\cap B)=E(A)E(B)$ for any Borel subsets A and B of Σ.
(iii) If $\{B_i\}_{i=1}^\infty$ are pairwise disjoint Borel subsets of Σ, then $E(\bigcup_{i=1}^\infty B_i)=\sum_{i=1}^\infty E(B_i)$.

Next, let us recall the spectral theorem (see [**Con90**, Theorem IX.2.2]) and Borel functional calculus (see [**Con90**, Theorem IX.2.3]). Let $(\mathcal{M},\varphi)$ be a W^*-probability space. We denote $L^2(M,\varphi)$ by H and regard random variables in M as operators in $B(H)$. For any normal random variable X in $\mathcal{M}$, the spectral theorem states that there exists a projection-valued measure E_X over $\sigma(X)$ such that

$$X=\int_{\sigma(X)} z dE_X(z),$$

where the integral can be interpreted as an operator-valued Lebesgue integral over $\sigma(X)$. Then we can define the *Borel functional calculus* for X with the help of this projection-valued measure. That is, for any bounded Borel measurable function f on $\sigma(X)$, we can define an element $f(X)\in\mathcal{M}$ as follows:

$$f(X):=\int_{\sigma(X)} f(z)dE_X(z).$$

Moreover, the map $f\mapsto f(X)$ is a $*$-homomorphism that extends the evaluation homomorphism of $\mathbb{C}[z,\overline{z}]$ as well as the *continuous functional calculus* (see [**Con90**, Section VIII.2]).

Remark I.3.5. With the help of these notions, we can have the following interpretations of the analytic distribution and Cauchy transform for random variables. Let X be a random variable in a W^*-probability space $(\mathcal{M},\varphi)$.

(i) If X is normal, then its analytic distribution μ_X can be given as $\mu_X=\varphi\circ E_X$. More precisely, for any Borel subset B of $\sigma(X)$, the projection $E_X(B)$ lies in the abelian von Neumann subalgebra $\mathrm{vN}(X)\subseteq\mathcal{M}$ generated by X and

$$\mu_X(B)=\varphi(E_X(B)).$$

(ii) If X is self-adjoint, then the Cauchy transform of its analytic distribution μ_X satisfies

$$G_{\mu_X}(z)=\varphi\big((z-X)^{-1}\big),\quad \forall z\in\mathbb{C}^+,$$

which can be seen from the functional calculus.

In the following, let X be a normal random variable in a faithful W^*-probability space $(\mathcal{M}, \varphi)$ with its projection-valued measure E_X. We restrict the Borel subsets to singletons to give alternative descriptions for atoms with the help of E_X. Let $\lambda \in \sigma(X)$ be given. First, we have $E_X(\{\lambda\}) = p_{\ker(\lambda - X)}$, where $p_{\ker(\lambda-X)}$ stands for the orthogonal projection onto the kernel of $\lambda - X$, i.e., the eigenspace of X associated with λ. Therefore, $\lambda \in \sigma(X)$ is an atom of μ_X if and only if $p_{\ker(\lambda-X)}$ is a non-zero projection (or equivalently, $\ker(\lambda - X)$ is non-trivial). Moreover, we actually have $\mu_X(\{\lambda\}) = \varphi(p_{\ker(\lambda-X)})$.

Remark I.3.6. If X is self-adjoint, we have

$$\lim_{\measuredangle z \to \lambda} (z - \lambda)(z - X)^{-1} = p_{\ker(\lambda - X)}, \quad \forall \lambda \in \mathbb{R},$$

where the non-tangential limit is taken under the strong operator topology. For a proof of this formula, see [**BV98**, Lemma 7.1]. Moreover, one can actually see that Equation I.1 follows from this formula.

Let us go a bit further along the operator algebraic interpretation of atoms. Note that $\lambda - X$ is a zero divisor in $\mathcal{M}$ when λ is an atom since $(\lambda - X)p_{\ker(\lambda-X)} = 0$ and $p_{\ker(\lambda-X)} \neq 0$. (Recall that a non-zero element X is called a *zero divisor* in $\mathcal{M}$ if there exists another non-zero element Y in $\mathcal{M}$ such that $XY = 0$.) Actually, the converse is also true. That is, if $\lambda - X$ is a zero divisor in $\mathcal{M}$, then λ is an atom of μ_X. For seeing that, let $Y \neq 0$ be another random variable in M such that $(\lambda - X)Y = 0$. Then we have $\mathrm{im}(Y) \neq \{0\}$ and $\mathrm{im}(Y) \subseteq \ker(\lambda - X)$, which immediately yields that $p_{\ker(\lambda-X)} \neq 0$. We record these observations as the following lemma.

Lemma I.3.7. *Let X be a normal random variable in a W^*-probability space $(\mathcal{M}, \varphi)$. Then $\lambda \in \sigma(X)$ is an atom of the analytic distribution μ_X of X if and only if one of the following equivalent conditions holds.*

(i) $p_{\ker(\lambda-X)} \neq 0$.
(ii) $\lambda - X$ *is a zero divisor in* $\mathcal{M}$.

Moreover, we have

$$\mu_X(\{\lambda\}) = \varphi(p_{\ker(\lambda-X)})$$

for each atom λ of μ_X.

Remark I.3.8. So far we have seen that for a normal random variable, the algebraic notion of zero divisors can be used to characterize atoms of its analytic distribution. But let us point out that $\lambda - X$ is a zero divisor is equivalent to $p_{\ker(\lambda-X)} \neq 0$ even if X is not normal.

So one may ask whether a zero divisor $\lambda - X$ can be interpreted through the notion of atoms even for a non-normal random variable X. However, even defining an appropriate analytic measure for a non-normal random variable is not an easy task. Fortunately, *Brown measures* provide us a consistent generalization of the analytic distribution from the normal case to more general case; see [**Bro86, HL00, BSS18**].

But we will not go further into the exploration of Brown measures. Instead we will focus on the investigation of zero divisors for general random variables. When the discussion comes to atoms, we will restrict our random variables to normal ones. See Chapter VI for these discussion.

I.4. Free independence

In this section we will introduce the notion of free independence, which distinguishes free probability theory from non-commutative probability theories. Let us first recall the notion of independence in classical probability theory. Consider two real-valued random variables X and Y in $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$. Suppose that they are independent, then we know that the joint probability distribution $\mu_{X,Y}$ is the product measure of the densities μ_X and μ_Y. So in particular we have

$$\mathbb{E}[X^k Y^l] = \mathbb{E}[X^k]\mathbb{E}[Y^l], \quad \forall k, l \in \mathbb{N}^+.$$

It tells us that the independence can be interpreted as a rule to compute the mixed moments of two random variables from their respective moments. Similarly, the following definition of free independence can be understood as a rule to calculate the joint distribution of a tuple of random variables from their respective distribution.

Definition I.4.1. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space.

(i) Let $(\mathcal{A}_1, \ldots, \mathcal{A}_d)$ be a tuple of subalgebras of $\mathcal{A}$ containing the unit 1 of $\mathcal{A}$. We say the subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_d$ are *freely independent* if

$$\varphi(X_1 \cdots X_k) = 0$$

whenever we have:

- $k \in \mathbb{N}^+$ and there are indices $i_1, \ldots, i_k \in \{1, \ldots, d\}$ such that
$$i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k;$$
- for $j = 1, \ldots, k$, $X_j \in \mathcal{A}_{i_j}$ and $\varphi(X_j) = 0$.

(ii) Let $(X_1, \ldots, X_d)$ be a tuple of random variables in $\mathcal{A}$. The random variables $X_1, \ldots, X_d$ are called *freely independent* if the unital $*$-subalgebras of $\mathcal{A}$ generated by X_i $(i = 1, \ldots, d)$ are freely independent.

We have mentioned that the notion of free independence was motivated by the free groups in the introduction of this chapter. Actually, there is a notion of freeness for groups and this algebraic freeness is equivalent to free independence if we regard group elements as random variables as in Example I.1.7. Let us make it precise in the following example.

Example I.4.2. Let G be a group. We denote the identity element of G by e. Let $(G_1, \ldots, G_d)$ be a tuple of subgroups of G. We say the subgroups $G_1, \ldots, G_d$ are *free* if

$$g_1 \cdots g_k \neq e$$

whenever we have:

- $k \in \mathbb{N}^+$ and there are indices $i_1, \ldots, i_k \in \{1, \ldots, d\}$ such that
$$i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k;$$
- for $j = 1, \ldots, k$, $g_j \in G_{i_j} \backslash \{e\}$.

Then this algebraic freeness for subgroups is actually equivalent to the free independence for group algebras. Namely, the following are equivalent.

(i) The subgroups $G_1, \ldots, G_d$ are free in G.

(ii) The group subalgebras $\mathbb{C}G_1, \ldots, \mathbb{C}G_d$ of $\mathbb{C}G$ are freely independent in the $*$-probability space $(\mathbb{C}G, \varphi)$ defined in Example I.1.7.

We refer the interested reader to [**NS06**, Proposition 5.11] for a proof of this equivalence.

A special case of the above example is given by free groups $\mathbb{F}_d$, $d \in \mathbb{N}^+$.

EXAMPLE I.4.3. Let $\mathbb{F}_d$ be the *free group* with d generators. Namely, $\mathbb{F}_d$ consists of all words built from $\{g_1, \ldots, g_d\}$ and $\{g_1^{-1}, \ldots, g_d^{-1}\}$ up to the relations following from the group axioms, where $\{g_1, \ldots, g_d\}$ is the set of generators. Note that each element in $\mathbb{F}_d$ can be uniquely written in the form $g_{i_1}^{m_i} \cdots g_{i_k}^{m_k}$ with some $k \in \mathbb{N}$, $m_1, \ldots, m_k \in \mathbb{Z}\backslash\{0\}$ and $i_1, \ldots, i_k \in \{1, \ldots, d\}$ such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$. Therefore, the subgroups generated by g_i $(i = 1 \ldots, d)$ are free in $\mathbb{F}_d$. In other words, $g_1, \ldots, g_d$, regarded as random variables, are freely independent in $(\mathbb{C}\mathbb{F}_d, \varphi)$ according to Example I.4.2. In particular, we see that each g_i is a non-torsion element in $\mathbb{F}_d$ for $i = 1 \ldots, d$. So $U_i := \lambda(g_i)$ $(i = 1 \ldots, d)$ are all Haar unitary random variables defined as in Example I.2.7. Thus they form an example of freely independent Haar unitary random variables. Moreover, they are also freely independent in the C^*-probability space $(C^*_{\mathrm{red}}(G), \varphi)$ and in the W^*-probability space $(\mathcal{L}(G), \varphi)$ due to the following remark.

REMARK I.4.4. Let $(\mathcal{A}, \varphi)$ be a C^*-probability space with a tuple $(\mathcal{A}_1, \ldots, \mathcal{A}_d)$ of freely independent subalgebras of $\mathcal{A}$. For each $i = 1, \ldots, d$, let $\mathcal{B}_i$ be the norm closure of $\mathcal{A}_i$ in $\mathcal{A}$. Then $\mathcal{B}_1, \ldots, \mathcal{B}_d$ are also freely independent in $(\mathcal{A}, \varphi)$ due to the continuity of φ. Similarly result also holds for the case of W^*-probability space. Namely, if $(\mathcal{A}, \varphi)$ is a W^*-probability space and $\mathcal{B}_i$ is the von Neumann subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_i$ for each $i = 1, \ldots, d$. Then $\mathcal{B}_1, \ldots, \mathcal{B}_d$ are freely independent in $(\mathcal{A}, \varphi)$. See [**MS17**, Proposition 6.5] for a proof.

In group theory, one can actually define a big group G for any given tuple $(G_1, \ldots, G_d)$ of groups such that $G_1, \ldots, G_d$ are free as subgroups of G. We denote this group by $*_{i=1}^d G_i$ and call it the *free product* of $G_1, \ldots, G_d$. Similarly, let $(\mathcal{A}_1, \ldots, \mathcal{A}_d)$ be a tuple of unital $*$-algebras. Then one can construct a *free product* $*_{i=1}^d \mathcal{A}_i$ out of them. It is a unital $*$-algebra that contains each $\mathcal{A}_i$ as a subalgebra with the units identified and satisfies some universal property. Moreover, one can endow these algebras with states such that this construction becomes the free product of $*$-probability spaces. We refer to [**NS06**, Lecture 6] for the details of this construction.

DEFINITION I.4.5. Let $(\mathcal{A}_i, \varphi_i)_{i=1}^d$ be a tuple of $*$-probability spaces.

(i) There is a $*$-probability space $(\mathcal{A}, \varphi) := *_{i=1}^d (\mathcal{A}_i, \varphi_i)$, called the *free product* of $(\mathcal{A}_i, \varphi_i)_{i=1}^d$ such that
   - each $\mathcal{A}_i$ can be regarded as a subalgebra of $\mathcal{A}$;
   - φ agrees with φ_i on each $\mathcal{A}_i$;
   - $(\mathcal{A}_i)_{i=1}^d$ are freely independent in $(\mathcal{A}, \varphi)$.

(ii) Suppose that additionally $(\mathcal{A}_i, \varphi_i)_{i=1}^d$ are faithful C^*-probability spaces. Then there is also a faithful C^*-probability space $(\mathcal{A}, \varphi) := *_{i=1}^d (\mathcal{A}_i, \varphi_i)$, called the *reduced free product* of $(\mathcal{A}_i, \varphi_i)_{i=1}^d$ such that $(\mathcal{A}_i, \varphi_i)_{i=1}^d$ are freely independent in $(\mathcal{A}, \varphi)$. See [**NS06**, Lecture 7] and [**VDN92**, Chapter 1].

With the help of the above free product construction, one can always construct a d-tuple of freely independent copies of one random variable. In particular, if U is a Haar unitary random variable in the $*$-probability space $(\mathbb{C}\mathbb{Z}, \varphi)$, then we can have d freely independent copies of U in the free product $(\mathbb{C}\mathbb{Z}, \varphi)^{*d}$. This free product is indeed isomorphic to $(\mathbb{F}_d, \varphi)$ given in Example I.4.3. Naturally, we can also consider the semicircular random variable defined and constructed in Example I.2.8. Then we can construct d freely independent copies of semicircular random variables in the free product $(\mathcal{A}, \varphi)^{*d}$, where $\mathcal{A}$ stands for the C^*-algebra generated by one semicircular random variable. But let us give another construction of freely independent semicircular random variables via the full Fock space. We can see that the free independence shows up also in a very natural way in this construction.

EXAMPLE I.4.6. Let H be a Hilbert space. The *full Fock space* over H is a Hilbert space defined by

$$\mathcal{F}(H) := \mathbb{C}\Omega \oplus \bigoplus_{k=1}^{\infty} H^{\otimes k},$$

where Ω is a vector of norm one, called the *vacuum vector*, in a one-dimensional Hilbert space denoted by $\mathbb{C}\Omega$. Then there is a vector state φ on $B(\mathcal{F}(H))$ defined by $\varphi(X) := \langle X\Omega, \Omega\rangle$. For each vector $\xi \in H$, we can define a *creation operator* $L(\xi) \in B(\mathcal{F}(H))$ determined by $L(\xi)\Omega := \xi$ and

$$L(\xi)(\xi_1 \otimes \cdots \otimes \xi_k) := \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k, \quad \forall k \in \mathbb{N}^+, \forall \xi_1, \cdots, \xi_k \in H.$$

Its adjoint operator $L(\xi)^*$ is called *annihilation* , which is determined by $L(\xi)^*\Omega = 0$ and

$$\begin{cases} L(\xi)^*\xi_1 = \langle \xi_1, \xi\rangle\, \Omega, & \forall \xi_1 \in H, \\ L(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_k) = \langle \xi_1, \xi\rangle\, \xi_2 \otimes \cdots \otimes \xi_k, & \forall k \in \mathbb{N}^+, \forall \xi_1, \cdots, \xi_k \in H. \end{cases}$$

For any $\xi \in H$ with $\|\xi\| = 1$, $S(\xi) := L(\xi) + L(\xi)^*$ is actually a semicircular random variable. Moreover, if $(\xi_1, \ldots, \xi_d)$ is a tuple of orthonormal vectors in H, then $(S_1, \cdots, S_d)$ is a tuple of freely independent semicircular random variables. For a reference, we refer to [**NS06**, Corollary 7.17].

There are several more notions related to free independence that we haven't introduced so far are . For example, *free cumulants* (as the counterpart of their analogues in classical probability theory) are defined by the moment-cumulant formula with the Möbius function defined on the lattice of noncrossing partitions. They characterize the free independence by the vanishing of mixed free cumulants. We refer to [**NS06**, Lecture 11] for its precise definition and related theorems. Free cumulants are intrinsically related to Voiculescu's *R-transform*, which is the analogue of the logarithm of the Fourier transform for random variables in classical probability theory. Similar, R-transforms also provide a characterization of free independence by their additivity. But we will not go further into these subjects since they will be not needed in our later investigations in this thesis. One of our later investigations is about atoms for random variables. So let us record here a structure theorem on the von Neuamnn algebra generated by freely independent random variables without atoms. We refer the interested reader to [**MS17**, Theorem 6.6] for a proof of this theorem.

THEOREM I.4.7. *Let $(\mathcal{M}, \varphi)$ be a $W*$-probability space. Suppose that $\mathcal{M}$ is generated by random variables $X_1, \ldots, X_d \in \mathcal{M}$. Let $(X_1, \ldots, X_d)$ satisfy the following two conditions:*

(i) $X_1, \dots, X_d$ *are freely independent in* $(\mathcal{M}, \varphi)$*;*
(ii) *for each* $i = 1, \dots, d$*,* X_i *is normal and its analytic distribution* μ_{X_i} *has no atoms.*

Then $\mathcal{M} \cong \mathcal{L}(\mathbb{F}_d)$ *as von Neumann algebras.*

I.5. Unbounded random variables

In this section, we set $(\mathcal{M}, \varphi)$ to be a tracial W^*-probability space. The condition that φ is a trace is necessary since we are going to consider closed and densely defined operators affiliated with a von Neumann algebra. In general, these operators might not well-behave under either addition or composition. However, in the case of tracial W^*-probability space, they will form a $*$-algebra. This $*$-algebra will provide us a framework of unbounded random variables. In particular, this allows us to consider random variables that may not have finite moments. Note that in the framework of Definition I.1.1, random variables in a $*$-probability space always have finite moments of all orders. We also refer the interested reader to [**Ber16**] and [**AGZ10**, Section 5.2.3 and 5.3.5] for introductions on unbounded random variables.

Our introduction of unbounded random variables will be focused on an operator algebraic level. This is because we are particularly interested in the atoms and zero divisor for random variables. We will see that the eigenspaces and zero divisors are still applicable for the study of atoms for unbounded random variables. Moreover, we will show that invertibility within the enlarged algebra of unbounded random variables can also be used to detect atoms or zero divisors.

First, we recall some basic notions on unbounded operators; see [**Bla06**, Section I.7] for a more detailed treatment.

Definition I.5.1. Let H be a Hilbert space.

(i) An *unbounded operator* or a *partially defined operator* X on H is a linear map $X : D \to H$ where its domain D is a vector subspace of H. We usually denote its domain D by $D(X)$.
(ii) Let X and Y be two partially defined operators on H. We say $X = Y$ if $D(X) = D(Y)$ and they agree on the domain. We write $X \subseteq Y$ if $D(X) \subseteq D(Y)$ and they agree on $D(X)$.
(iii) A *densely defined operator* X on H is a partially defined operator whose domain $D(X)$ is dense in H.
(iv) A *closed operator* X on H is a partially defined operator such that its *graph* $\Gamma(X) := \{(\xi, X\xi) \mid \xi \in D(X)\}$ is closed in $H \oplus H$.
(v) A *closable* or *preclosed operator* X on H is a partially defined operator such that the closure of $\Gamma(X)$ is the graph of a (necessarily closed) operator. We call this closed operator the *closure* of X and denote it by $[X]$.
(vi) Let X be a densely defined operator on H. We define its *adjoint* X^* as follows. First, its domain is defined as

$$D(X^*) := \{\eta \in H \mid \exists \zeta \in H \text{ such that } \langle \xi, \zeta \rangle = \langle X\xi, \eta \rangle \text{ for all } \xi \in D(X)\}.$$

Then we define $X^*\eta = \zeta$.

Let X and Y be two partially defined operators on H, then we define their sum $X+Y$ and product XY as partial defined operators with domains $D(X+Y) := D(X) \cap D(Y)$ and $D(XY) := \{\xi \in D(Y) \mid Y\xi \in D(X)\}$. It may happen that $X+Y$ or XY is not closed or densely defined even when X and Y are closed and densely defined. So we usually consider the *strong sum* $[X+Y]$ (respectively the *strong product* $[XY]$), namely, the closures of $X+Y$ (respectively XY), if $X+Y$ (respectively XY) is closable and densely defined. But for seeing that they can form a $*$-algebra, we need some additional information—they are related to a tracial W^*-probability space—which we haven't used so far.

In the following, we set $H = L^2(M, \varphi)$. Then $\mathcal{M} \subseteq B(H)$, whose elements can be regarded as closed and everywhere defined operator on H.

Definition I.5.2. Let X be a closed and densely defined operator on H. We say X is *affiliated* with M if for any $Y \in \mathcal{M}'$, $YX \subseteq XY$.

It was originally observed by Murray and von Neumann in [**Mv36**] that these affiliated operators can form a $*$-algebra. But we want to introduce another approach via measurable operators. In [**Seg53**], Segal introduced measurable operators as an analogue of measurable functions. He showed that these measurable operators form a $*$-algebra with the strong sum and product. Moreover, a notion of convergence in measure was developed by Nelson [**Nel74**] for this theory of measurable operators. We will follow the definition in [**Ter81**] for these measurable operators. See [**Ter81**, Proposition 23 and 24] for the fact that these measurable operators form a $*$-algebra. This $*$-algebra was also shown to be a complete Hausdorff topological $*$-algebra with the measure topology due to Nelson.

Definition I.5.3. (See [**Ter81**, Definition 14]) Let X be a closed and densely defined operator affiliated with $\mathcal{M}$. We say X is *φ-measurable* if for any real number $\delta > 0$, there exists a projection $p \in \mathcal{M}$ such that

$$p(H) \subseteq D(X) \quad \text{and} \quad \varphi(1-p) \leq \delta.$$

We denote by $L^0(\mathcal{M}, \varphi)$ the $*$-algebra of all φ-measurable closed densely defined operators affiliated with $\mathcal{M}$ under the strong sums and products. We will simply call an element in $L^0(\mathcal{M}, \varphi)$ an *unbounded random variable*.

Actually, for tracial W^*-probability spaces closed and densely defined operators are automatically measurable, see [**Ter81**, Examples after Theorem 28]. So the notion of measurable operators is not really necessary for the mere purpose of seeing that these unbounded random variable form a $*$-algebra. However, in a perspective of free probability, this notion of measurable operators is tailor-made for us to regard these unbounded operators as unbounded random variables.

Moreover, for each normal unbounded random variable, we can also associate an analytic distribution to it. For that purpose, we use the spectral theorem for normal closed densely defined operators.

An unbounded random variable X is called *normal* if $X^*X = XX^*$. Note that this implicitly requires that $D(X^*X) = D(XX^*)$. We refer the interested reader to [**Con90**, Section 4] for basic properties of these normal operators.

Definition I.5.4. Let X be an unbounded random variable in $L^0(\mathcal{M},\varphi)$.

(i) Its *spectrum* is defined as

$$\sigma(X) := \{\lambda \in \mathbb{C} \mid \lambda - X \text{ is not a bijection of } D(X) \text{ onto } H\}.$$

Note that $\lambda \notin \sigma(X)$ means $\lambda - X$ has its inverse as a bounded operator.

(ii) Its *point spectrum* (of *eigenvalues*) is defined as

$$\sigma_p(X) := \{\lambda \in \mathbb{C} \mid \ker(\lambda - X) \neq \{0\}\} \subseteq \sigma(X).$$

Now the spectral theorem (see [**Con90**, Theorem X.4.11]) says that for any normal unbounded random variable X there exists a projection-valued measure E_X over $\sigma(X)$ such that

$$X = \int_{\sigma(X)} z dE_X(z).$$

Therefore, we can define:

Definition I.5.5. The *analytic distribution* of a normal unbounded random variable X is $\mu_X := \varphi \circ E_X$.

Since $\sigma(X)$ may be non-compact in general, μ_X can be a measure without compact support. Therefore, it could happen that X has no finite moments. Note that for each bounded random variable in $\mathcal{M}$, its moments are finite and determine its analytic distribution. Nevertheless, the Cauchy transform introduced in Definition I.3.2 is still available for self-adjoint unbounded random variables. Namely, for a self-adjoint unbounded random variable X with its analytic distribution μ_X, we have

$$G_{\mu_X}(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t) = \varphi\big((z-X)^{-1}\big), \quad \forall z \in \mathbb{C}^+,$$

where $(z-X)^{-1}$ can be shown to be a bounded operator in $\mathcal{M}$.

Moreover, for a normal random variable, the projections $E_X(B)$ for Borel subsets B of $\sigma(X)$ are the analogues of identity functions on the Borel sets in the σ-algebra generated by a random variable. So we can actually use these projections to define the free independence. Namely, two normal unbounded random variables X and Y are *freely independent* if the subalgebras generated by $\{E_X(B) | B \text{ is a Borel set of } \sigma(X)\}$ and $\{E_Y(B) | B \text{ is a Borel set of } \sigma(Y)\}$ are freely independent in $(\mathcal{M},\varphi)$. Moreover, for arbitrary unbounded random variables, we can also define the free independence for them with the help of the polar decomposition.

Lemma I.5.6. *Let X be a closed and densely defined operator on H. Then we have $X = U|X|$, where $|X| = (X^*X)^{1/2}$ is a positive self-adjoint closed and densely defined operator and U is a partial isometry such that $U^*U = p_{(\ker(X))^\perp}$ and $UU^* = p_{\overline{\mathrm{im}(X)}}$. Moreover, X is affiliated with $\mathcal{M}$ if and only if $U \in \mathcal{M}$ and $|X|$ is affiliated with $\mathcal{M}$.*

For a reference, see [**SZ79**, Section 9.29]. We refer the interested reader to [**MS17**, Definition 8.15] for a definition of the free independence of unbounded random variables.

In the remaining part of this section, we turn our attention to the atoms and zero divisors for unbounded random variables. Let X be a normal unbounded random variable in $L^0(\mathcal{M},\varphi)$. Similar to the bounded case, for each $\lambda \in \sigma(X)$ the eigenspace $\ker(\lambda - X)$ is given by $E_X(\{\lambda\})$. That is, $p_{\ker(\lambda - X)} = E_X(\{\lambda\})$; see the paragraph after [**KR83**, Remark 5.6.32]. Hence again we can interpret atoms of μ_X with help of zero divisors as in Lemma I.3.7.

REMARK I.5.7. Let X be a normal unbounded random variable in $L^0(\mathcal{M},\varphi)$. Then $\lambda \in \sigma(X)$ is an atom of the analytic distribution μ_X of X if and only if one of the following equivalent conditions holds.

(i) $p_{\ker(\lambda - X)} \neq 0$.
(ii) $\lambda - X$ is a zero divisor in $L^0(\mathcal{M},\varphi)$.

Moreover, we have
$$\mu_X(\{\lambda\}) = \varphi(p_{\ker(\lambda-X)})$$
for each atom λ of μ_X.

Brown measures can also be defined in the non-normal unbounded case, for example, see [**HS07**]. But again we will not go further into the atoms of Brown measures. Instead we will discuss the zero divisors for unbounded random variables in general and limit to normal case when the discussion comes to atoms.

Now we want to show that there is another algebraic notion that can be used to detect the atoms or zero divisors for unbounded random variables. For that purpose, let us first record here a useful lemma on kernels and images.

LEMMA I.5.8. *Let X be an unbounded random variable in $L^0(\mathcal{M},\varphi)$, where $(\mathcal{A},\varphi)$ is a tracial W^*-probability space. Then we have*

(i) $\varphi(p_{\ker(X)}) = \varphi(p_{\ker(X^*)})$,
(ii) $\varphi(p_{\ker(X)}) + \varphi(p_{\overline{\mathrm{im}(X)}}) = 1$.

This lemma follows from the polar decomposition. Let X be an unbounded random variable with its polar decomposition $X = U|X|$. Since φ is a trace on $\mathcal{M}$, we have in this situation that
$$\varphi(p_{(\ker(X))^\perp}) = \varphi(U^*U) = \varphi(UU^*) = \varphi(p_{\overline{\mathrm{im}(X)}}).$$
This implies Item (ii) of the above lemma. Item (i) follows from the relation $\ker(X^*) = (\mathrm{im}(X))^\perp$.

A consequence of Item (ii) in the above lemma is the following. Let $X \in L^0(\mathcal{M},\varphi)$ be an unbounded random variable with $\ker(X) = \{0\}$. Then $p_{\ker(X)} = 0$, and thus, by Item (ii) of Lemma I.5.8, $\varphi(p_{\overline{\mathrm{im}(X)}}) = 1$. It implies, by the faithfulness of φ, that $p_{\overline{\mathrm{im}(X)}} = 1$, which means that the image of X is dense in H. Hence the inverse of X exists as an unbounded operator. So in the tracial W^*-setting the invertibility of an unbounded random variable relies only on whether it is injective or not. Therefore, we have one more interpretation—through the invertibility—for the point spectrum of X (respectively atoms of μ_X if X is normal). In conclusion, we have the following lemma that extends Lemma I.3.7 and Remark I.5.7.

LEMMA I.5.9. *Let X be an unbounded random variable in $L^0(\mathcal{M},\varphi)$. Then the following are equivalent.*

(i) $p_{\ker(\lambda - X)} \neq 0$.
(ii) $\lambda - X$ *is a zero divisor in* $L^0(\mathcal{M},\varphi)$.
(iii) $\lambda - X$ *is not invertible in* $L^0(\mathcal{M},\varphi)$.

(iv) $\varphi(p_{\overline{\mathrm{im}(\lambda-X)}}) < 1$.

In the last item of the above lemma, we actually view $\varphi(p_{\overline{\mathrm{im}(\cdot)}}) : L^0(\mathcal{M}, \varphi) \to [0, 1]$ as a rank function. Namely, it measures the size of images of operators on a Hilbert space. In particular, if this function takes the maximal value on an unbounded random variable X, then X is invertible in $L^0(\mathcal{M}, \varphi)$.

Moreover, we have a matricial amplification of this rank function. For each $n \in \mathbb{N}^+$ we know that $(M_n(\mathcal{M}), \mathrm{tr}_n \circ\varphi^{(n)})$ is again a W^*-probability space. Then the matrix algebra $M_n(L^0(\mathcal{M}, \varphi))$ over $L^0(\mathcal{M}, \varphi)$ is the $*$-algebra of closed and densely defined operators affiliated to $(M_n(\mathcal{M}), \mathrm{tr}_n \circ\varphi^{(n)})$. Thus an element $A \in M_n(L^0(\mathcal{M}, \varphi))$ is invertible in $M_n(L^0(\mathcal{M}, \varphi))$ if and only if $\mathrm{tr}_n \circ\varphi^{(n)}(p_{\ker(A)}) = 0$, i.e., if $\mathrm{tr}_n \circ\varphi^{(n)}(p_{\overline{\mathrm{im}(A)}}) = 1$. So this leads us to a rank function for matrices over $L^0(\mathcal{M}, \varphi)$ that can detect their invertibility.

Definition I.5.10. For every $A \in M_n(L^0(\mathcal{M}, \varphi))$, we define its *rank* as

$$\mathrm{rank}(A) = \mathrm{Tr}_n \circ\varphi^{(n)}(p_{\overline{\mathrm{im}(A)}}) \in [0, n].$$

So, with this analytic notion of rank we can rephrase the statement on the invertibility of elements in $M_n(L^0(\mathcal{M}, \varphi))$.

Lemma I.5.11. *$A \in M_n(L^0(\mathcal{M}, \varphi))$ is invertible in $M_n(L^0(\mathcal{M}, \varphi)$ if and only if* $\mathrm{rank}(A) = n$.

For later use, we point out that it holds true that

$$\mathrm{rank}(A) = n - \mathrm{Tr}_n \circ\varphi^{(n)}(p_{\ker(A)}) \tag{I.2}$$

due to Item (ii) of Lemma I.5.8.

Moreover, this rank stays invariant by invertible matrices over $L^0(\mathcal{M}, \varphi)$.

Lemma I.5.12. *If P is invertible in $M_n(L^0(\mathcal{M}, \varphi))$, then* $\mathrm{rank}(A) = \mathrm{rank}(AP) = \mathrm{rank}(PA)$ *for any $A \in M_n(L^0(\mathcal{M}, \varphi))$.*

This fact is probably well-known to experts. We include a proof here for reader's convenience. See also [**Lin93**, Lemma 2.3] for an alternative proof based on the $*$-regularity of $L^0(\mathcal{M}, \varphi)$.

Proof of Lemma I.5.12. Let $A, P \in M_n(L^0(\mathcal{M}, \varphi))$ with P being invertible. We have to show that $\varphi^{(n)}(p_{ker(A)}) = \varphi^{(n)}(p_{ker(PA)}) = \varphi^{(n)}(p_{ker(AP)})$ according to (I.2). First, it is clear that $\ker(A) \subseteq \ker(PA)$. Actually, if a vector $\xi \in \ker(PA)$, then $0 = P^{-1}(PA\xi) = A\xi$ as P is a bijection from $D(P)$ to $\mathrm{im}(P)$. So we have $\ker(A) = \ker(PA)$ and thus $p_{ker(A)} = p_{ker(PA)}$. This implies that $\varphi^{(n)}(p_{ker(A)}) = \varphi^{(n)}(p_{ker(PA)})$ immediately. Then with the help of Item (i) of Lemma I.5.8 we have

$$\varphi^{(n)}(p_{ker(AP)}) = \varphi^{(n)}(p_{ker(PA^*)}) = \varphi^{(n)}(p_{ker(A^*)}) = \varphi^{(n)}(p_{ker(A)}).$$

Here $\varphi^{(n)}(p_{ker(PA^*)}) = \varphi^{(n)}(p_{ker(A^*)})$ since $p_{ker(PA^*)} = p_{ker(A^*)}$ by the same argument as for A. Hence the proof is completed. □

CHAPTER II

Asymptotic limits of Random Matrices

Random matrix theory studies the matrix-valued random variables, or alternatively, the matrices whose entries are (classical) random variables. In statistics, the research of random matrices goes back to Wishart [**Wis28**] in the 1920s. Later in nuclear physics, Wigner introduced random matrices as statistical models for the nuclei of heavy atoms. In his seminal work [**Wig55**], a certain random matrix model was shown to have deterministic behavior asymptotically. These random matrices are now called Wigner random matrices (see Example II.1.2 and II.1.3 below) and have semicircle distribution as their limiting distribution. The semicircle distribution is nothing else but the analytic distribution of a semicircular random variable (see Example I.2.8). This coincidence was observed by Voiculescu and finally turned out to be a surprising and exciting connection between random matrix theory and free probability theory. Namely, Voiculescu found in [**Voi91**] that freely independent random variables naturally arise as limits of random matrices when their dimension tends to infinity.

Our introduction of random matrices in this chapter will be devoted to these connections between free probability and random matrix theory. In Section II.1, we will show that the non-commutative distributions introduced in Section I.2 are natural frameworks to describe the convergence of the eigenvalue distributions of Wigner random matrices. This leads to the notion of convergence in distribution. Then in Section II.2, convergence in distribution will be extended to the multi-variable case. It will be used to phrase Voiculescu's discovery on the asymptotic freeness as well as many its generalizations. Namely, we will see that random matrices can almost surely converge in distribution to freely independent non-commutative random variables.

Moreover, a stronger convergence with the norm involved will be also introduced in Section II.2. This convergence was first considered by Haagerup and Thorbjørnsen [**HT05**] and then was extended to many random matrix models. It strengthens further the connection of random matrix theory and free probability theory. Another reason that we introduce this notion of convergence is that we will use them to extend the convergence of random matrices to a larger class of functions than polynomials. These functions are the so-called rational functions and will be introduced in Chapter IV. In Chapter VII we will show a convergence result for rational functions in random matrices.

II.1. Asymptotic behavior of random matrices

In this section, we will first introduce several basic and important examples of random matrices. Our goal of this section is to describe the limiting (or asymptotic) distributions of these random matrices when their dimension tends to infinity. We will see that their limiting distributions are actually the $*$-distributions of the examples that we have seen in Chapter I.

We have already introduced a framework of C^*-probability space (in Example I.2.6) whose elements are random matrices. However, that framework is very limited because we ask the entries to be bounded (classical) random variables. This in particular exclude the Gaussian random variables. In order to include a larger class of random variables as entries for our random matrices, we will consider the following $*$-probability space as the underlying $*$-probability space.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. We define

$$L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \le p < \infty} L^p(\Omega, \mathbb{P}).$$

It is the complex unital algebra consisting of all random variables that have finite moments of all orders. Let $\mathbb{E}$ be the expectation as in Example I.1.2. One can prove that $(L^{\infty-}(\Omega, \mathbb{P}), \mathbb{E})$ is a $*$-probability space. Then $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \operatorname{tr}_N \circ \mathbb{E}^{(N)})$ is also a $*$-probability space for all $N \in \mathbb{N}^+$.

Definition II.1.1. Let N be a positive integer. An element X in the $*$-probability space $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \operatorname{tr}_N \circ \mathbb{E}^{(N)})$ is called an $N \times N$ *random matrix*.

Of course we can actually consider the algebra $L^0(\Omega, \mathbb{P})$ of all measurable functions and take elements in $M_N(L^0(\Omega, \mathbb{P}))$ as our random matrices. But in order to streamline our introduction to the connection of random matrices and non-commutative random variables as their limit, we will simply consider random matrices in $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \operatorname{tr}_N \circ \mathbb{E}^{(N)})$ to avoid complicated technical issues.

Example II.1.2. Let $X = (\frac{1}{\sqrt{N}} X_{ij})_{i,j=1}^N$ be in $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \operatorname{tr}_N \circ \mathbb{E}^{(N)})$ satisfying

(i) $X_{ij} = X_{ji}$ for all $i, j = 1, \dots, N$,
(ii) $\{X_{ij} \mid 1 \le i < j \le N\}$ are identically distributed real-valued random variables with $\mathbb{E}[X_{12}] = 0$ and $\mathbb{E}[X_{12}^2] = 1$,
(iii) $\{X_{ii} \mid i = 1, \dots, N\}$ are identically distributed real-valued random variables with $\mathbb{E}[X_{11}] = 0$,
(iv) $\{X_{ij} \mid 1 \le i \le j \le N\}$ are independent.

Then X is usually called a *real Wigner* random matrix. If the entries of X are Gaussian, then we call X a *real Gaussian Wigner* random matrix. A special case of Gaussian Wigner random matrices is that we additionally have $\mathbb{E}(X_{11}^2) = 2$. Such a random matrix is called a *GOE (Gaussian orthogonal ensemble)* random matrix.

These random matrix models have complex analogues as follows.

Example II.1.3. Let $X = (\frac{1}{\sqrt{N}} (X_{ij})_{i,j=1}^N$ be in $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \operatorname{tr}_N \circ \mathbb{E}^{(N)})$ satisfying

(i) $X_{ij} = \overline{X_{ji}}$ for all $i, j = 1, \dots, N$,
(ii) $\{X_{ij} \mid 1 \le i < j \le N\}$ are identically distributed complex-valued random variables with $\mathbb{E}[X_{12}] = 0$ and $\mathbb{E}[|X_{12}|^2] = 1$,
(iii) $\{X_{ii} \mid i = 1, \dots, N\}$ are identically distributed real-valued random variables with $\mathbb{E}[X_{11}] = 0$,
(iv) $\{\Re(X_{ij}) \mid 1 \le i < j \le N\} \cup \{\Im(X_{ij}) \mid 1 \le i < j \le N\} \cup \{X_{ii} \mid i = 1, \dots, N\}$ are independent.

Then X is called a *complex Wigner* random matrix. If $\Re(X_{12})$, $\Re(X_{12})$ and X_{11} are Gaussian, then we call X a *complex Gaussian Wigner* random matrix. Moreover, if $\mathbb{E}(X_{11}^2) = 1$, then X is called a *GUE (Gaussian unitary ensemble)* random matrix.

Clearly, for a random matrix we can forget about its matrix structure and consider it as a random vector. Then we can ask what is the joint density of this random vector. It turns out GOE and GUE random matrices have highly symmetric densities on the vector space of symmetric matrices and respectively Hermitian matrices. In both case, the density function of a GOE or GUE random matrix is a function of matrices that is only dependent on the trace of matrices. So the density of GOE (and respectively GUE) is invariant under orthogonal (and respectively unitary) matrices. For details of this fact, we refer the interested reader to [**AGZ10**, Section 2.5.1].

Now, if we know there is a probability density on a subset of matrices, then we realize a random matrix by randomly choosing matrices according to this density. For example, Haar unitary random matrices can be defined in this way.

Example II.1.4. Let $\mathcal{U}(N) \subseteq M_N(\mathbb{C})$ be the group of unitary matrices. It is well-known that $\mathcal{U}(N)$ is a compact group. So there is a probability measure μ_N on $\mathcal{U}(N)$ deduced from the Haar measure on $\mathcal{U}(N)$. A *Haar unitary* random matrix is a matrix $U^{(N)}$ chosen randomly with respect to this probability measure μ_N on $\mathcal{U}(N)$.

Now we turn to the eigenvalue distributions of random matrices. In Example I.2.6 we have already seen that for each normal random matrix X in $M_N(L^\infty(\Omega,\mathbb{P}), \mathrm{tr}_n \circ\mathbb{E}^{(N)})$, the analytic distribution μ_X (in the sense of Definition I.2.4) of X is given by the average of eigenvalue distributions of $(X(\omega))_{\omega\in\Omega}$. But Gaussian Wigner random matrices do not fit this framework $M_N(L^\infty(\Omega,\mathbb{P}), \mathrm{tr}_n \circ\mathbb{E}^{(N)})$ since a Gaussian random variable is not bounded. Nevertheless for a given random matrix, we always have the eigenvalue distribution of $X(\omega)$ for a fixed $\omega \in \Omega$. Then we have an averaged measure of them, which is actually the analytic distribution of X in $M_N(L^0(\Omega,\mathbb{P}))$ in the sense of Definition I.5.5.

Definition II.1.5. Let X be a normal random matrix in $(M_N(L^{\infty-}(\Omega,\mathbb{P})), \mathrm{tr}_N \circ\mathbb{E}^{(N)})$. The *empirical eigenvalue distribution* of X is a random measure $\omega \in \Omega \mapsto \mu_{X(\omega)}$ given by

$$\mu_{X(\omega)} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(\omega)},$$

where $\lambda_1(\omega), \dots, \lambda_n(\omega)$ are the eigenvalues of $X(\omega)$ and $\delta.$ stands for the Dirac measure at a point. The *averaged eigenvalue distribution* of X is defined by

$$\mu_X := \mathbb{E}[\mu_{X(\cdot)}] = \int_\Omega \mu_{X(\omega)} d\mathbb{P}.$$

The averaged eigenvalue distribution of a random matrix will be a deterministic measure once we specify the random matrix model. For example, for a GUE random matrix its averaged eigenvalue distribution can be expressed with the help of *Hermite polynomials*. See, for example, [**Kem13**, Theorem 15.13] for the precise formula of the averaged eigenvalue distribution of GUE random matrices. But the empirical eigenvalue distribution of a GUE random matrix, as a random measure, also has some deterministic behavior when its dimension is going to infinity. Before we give the precise mathematical theorem, let us see this phenomenon via the histograms of eigenvalues.

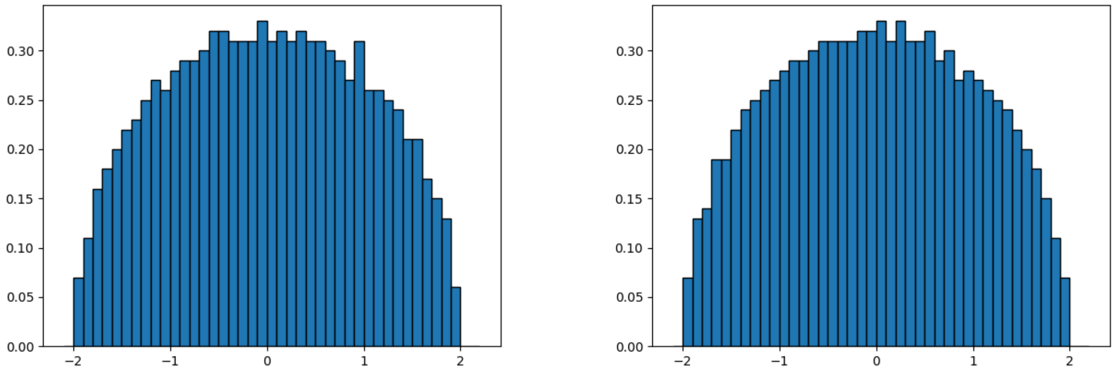


FIGURE II.1. Two samples of 1000×1000 GUE random matrices

In Figure II.1 we show the histograms of the eigenvalues of two samples of GUE random matrices. That is, for each case we take 10^6 samples from a real Gaussian distribution and arrange them as a 1000×1000 Hermitian matrix. This matrix has 1000 real eigenvalues, which are divided into intervals of size 0.1 on the real line. In the histogram, the height of each bar on an interval is the ratio of number of eigenvalues that lie in the corresponding interval. Namely, the height is

$$\frac{\#\{\lambda \in \sigma(X(\omega)) \mid \lambda \in I\}}{N} = \mu_{X(\omega)}(I), \quad N = 1000,$$

where $X(\omega)$ stands for the sample of the random matrix, I stands for an interval and $\mu_{X(\omega)}$ stands for the eigenvalue distribution of $X(\omega)$. In other words, the histogram gives a graphic approximation of the eigenvalue distribution $X(\omega)$. Clearly the left histogram in Figure II.1 is different from the right one since they are two different random sample. But they also share a highly similar shape of the histogram. This suggests that the eigenvalue distributions of both samples are close to a common measure when N is large. This common measure is exactly the semicircle distribution $d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}\mathbb{1}_{[-2,2]}(t)dt$ that showed up in Example I.2.8.

Moreover, this phenomenon is actually shared by a large class of random matrices. For example, Figure II.2 shows the histograms of two sample of a Wigner random matrices with its entries are i.i.d Bernoulli random variables taking values in $\{-1, 1\}$.

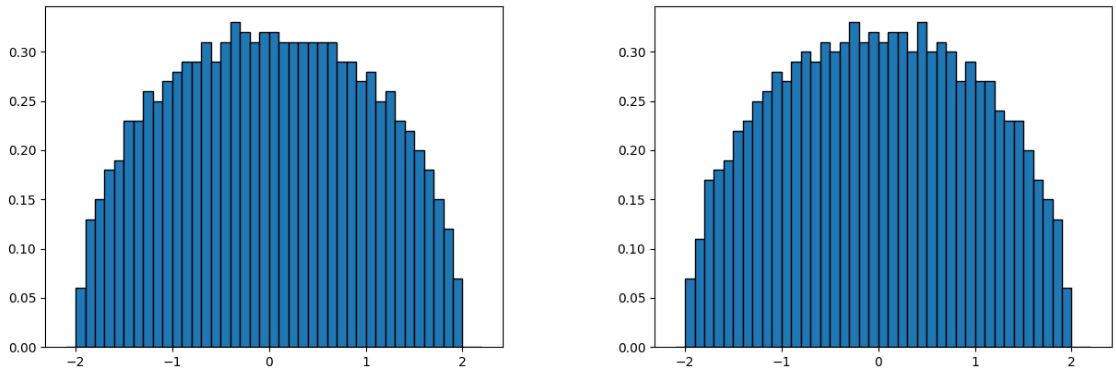


FIGURE II.2. Two samples of 1000×1000 Wigner random matrices with Bernoulli entries

These asymptotic behavior of Wigner random matrices are described by the following theorem. We will simply call a random matrix a Wigner random matrix if it a real or complex one, since the result holds for both cases.

THEOREM II.1.6. *Let $X^{(N)}$ be a Wigner random matrix in $(M_N(L^{\infty-}(\Omega,\mathbb{P})), \mathrm{tr}_N \circ \mathbb{E}^{(N)})$ for each $N \in \mathbb{N}^+$. Then for almost every $\omega \in \Omega$, we have*

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(t) d\mu_{X^{(N)}(\omega)}(t) = \int_{\mathbb{R}} f(t) d\mu_S(t),$$

for any bounded continues function f on $\mathbb{R}$, where $d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}\mathbb{1}_{[-2,2]}(t)dt$ is the semicircle distribution.

For a proof, see, for example, [**AGZ10**, Chapter 2]. Note the convergence in the above theorem is the weak convergence for probability measures. It is known that when the limiting measure is compactly supported this convergence is equivalent to the convergence for all moments. Therefore, let us introduce the following notion of convergence in distribution. Then we can rephrase Wigner's semicircle law in a way that highlights its connection to a semicircular random variable.

DEFINITION II.1.7. Let $(\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N}^+)$ be a family of $*$-probability spaces. For each $N \in \mathbb{N}^+$, let $X^{(N)}$ be a random variable in $(\mathcal{A}_N, \varphi_N)$. Let X be a random variable in some $*$-probability space $(\mathcal{A}, \varphi)$. We say the sequence $(X^{(N)})_{N=1}^{\infty}$ *converges in distribution* to X if

$$\lim_{N\to\infty} \varphi_N(p(X^{(N)})) = \varphi(p(X))$$

for all polynomials $p \in \mathbb{C}\langle x, x^* \rangle$.

Now let us rephrased Wigner's semicircle law as follows.

THEOREM II.1.8. *Let $X^{(N)}$ be a Wigner random matrix in $(M_N(L^{\infty-}(\Omega,\mathbb{P})), \mathrm{tr}_N \circ \mathbb{E}^{(N)})$ for each $N \in \mathbb{N}^+$. Let S be a standard semicircular random variable in a C^*-probability* *space* $(\mathcal{A}, \varphi)$. *Then the sequence* $(X^{(N)})_{N=1}^{\infty}$ *almost surely converges in distribution to* S, *that is, for almost every* $\omega \in \Omega$,

$$\lim_{N\to\infty} \mathrm{tr}_N((X^{(N)}(\omega))^k) = \varphi(S^k) \quad \textit{for all } k \in \mathbb{N}^+.$$

So far we have seen that a semicircular random variable can be regarded as a mode that describes the limiting eigenvalue distribution of Wigner matrices as their dimension goes to infinity. This connection between Wigner matrices and semicircular random variables can be also seen from other asymptotic phenomena. For example, we also know that almost surely

$$\lim_{N\to\infty} \|X^{(N)}(\omega)\|_{M_N(\mathbb{C})} = 2.$$

For a reference, see, for example, [**BY88**]. On the other hand, we also know that $\|S\| = 2$ for a standard semicircular random variable S.

In Section II.2, more connections between Wigner random matrices and semicircular random variables will be presented. Of course, these connections between random matrices and non-commutative random variables hold actually for a large family of random matrix models and their corresponding limiting operators. In particular, let us mention that Haar unitary random matrices almost surely converge in distribution to a Haar unitary random variable when their dimension tends to infinity.

II.2. Asymptotic freeness and strong asymptotic freeness

In this section, we will strengthen the connection between random matrices and non-commutative random variables to the multi-variable case. In particular, we will see that independent copies of Wigner random matrices converge in distribution to freely independent semicircular random variables.

First, let us extend the convergence in distribution to the multi-variable case. It naturally leads us to the notion of asymptotic free independence.

Definition II.2.1. Let $(\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N}^+)$ be a family of $*$-probability spaces. For each $N \in \mathbb{N}^+$, let $X^{(N)} = (X_1^{(N)}, \ldots, X_d^{(N)})$ be a d-tuple of random variables in $(\mathcal{A}_N, \varphi_N)$. Let $X = (X_1, \ldots, X_d)$ denote a d-tuple of random variables in some $*$-probability space $(\mathcal{A}, \varphi)$.

(i) We say the sequence $(X^{(N)})_{N=1}^{\infty}$ *converges in distribution* to X if

$$\lim_{N\to\infty} \varphi_N(p(X^{(N)})) = \varphi(p(X))$$

for any polynomial $p \in \mathbb{C}\langle x_1, \ldots, x_d, x_1^* \ldots, x_d^*\rangle$.

(ii) Suppose that the sequence $(X^{(N)})_{N=1}^{\infty}$ converges in distribution to X. If X is a tuple of freely independent random variables in $(\mathcal{A}, \varphi)$, then we say $(X^{(N)})_{N=1}^{\infty}$ is *asymptotically freely independent*.

Asymptotic free independence for random matrices is one of the fundamental discoveries of Voiculescu. This discovery tells us that Wigner's semicircle law for Wigner random matrices is not a mere coincidence. It reveals a deep link between free probability theory and random matrix theory.

Now let us state the asymptotic freeness for Gaussian Wigner random matrices. We say random matrices $X_1^{(N)}, \dots, X_d^{(N)}$ are *independent* if their entries (divided into real and imaginary parts for the complex case) form a set of independent random variables.

Theorem II.2.2. *(See* [**Voi91**]*) Let $X^{(N)} = (X_1^{(N)}, \dots, X_d^{(N)})$ be a tuple of independent Gaussian Wigner random matrices in $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \mathrm{tr}_N \circ \mathbb{E}^{(N)})$ for each $N \in \mathbb{N}^+$. Then the sequence $(X^{(N)})_{N=1}^\infty$ converges in distribution to a tuple $S = (S_1, \dots, S_d)$ of freely independent semicircular random variables in some C^*-probability space, that is,*

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}_N(p(X^{(N)}(\omega)))] = \varphi(p(S)),$$

for every $p \in \mathbb{C}\langle x_1, \dots, x_d\rangle$.

The theorem indeed can be strengthened to the almost surely convergence in distribution; see [**HP00a, Tho00**] and [**MS17**, Remark 5.14]. Moreover, the assumption that the entries of our random matrices are Gaussian can be weakened; see [**Dyk93**] for the almost surely convergence of independent Wigner random matrices.

For independent Haar unitary random matrices, naturally we would expect that they almost surely converge in distribution to freely independent Haar unitary random variables. We refer to [**HP00a**] for a proof of this theorem.

Theorem II.2.3. *Let $U^{(N)} = (U_1^{(N)}, \dots, U_d^{(N)})$ be a tuple of independent Haar unitary random matrices in $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \mathrm{tr}_N \circ \mathbb{E}^{(N)})$ for each $N \in \mathbb{N}^+$. Then the sequence $(U^{(N)})_{N=1}^\infty$ almost surely converges in distribution to a tuple $U = (U_1, \dots, U_d)$ of freely independent Haar unitary random variables in some C^*-probability space, that is, for almost every $\omega \in \Omega$*

$$\lim_{N\to\infty} \mathrm{tr}_N(p(U^{(N)}(\omega))) = \varphi(p(U)),$$

for every $p \in \mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle$.

Recall that we have mentioned at the end of Section II.1 that we have almost surely

$$\lim_{N\to\infty} \|X^{(N)}(\omega)\|_{M_N(\mathbb{C})} = \|S\|$$

for Wigner random matrices $X^{(N)}$ and a semicircular random variable S. Such a result can also be strengthened to the multi-variable case and also to other random matrix models. Let us first introduce the notion of strong convergence in distribution to describe this phenomenon.

Definition II.2.4. Let $X^{(N)} = (X_1^{(N)}, \dots, X_d^{(N)})$ be a tuple of random matrices in $(M_N(L^{\infty-}(\Omega, \mathbb{P})), \mathrm{tr}_N \circ \mathbb{E}^{(N)})$ for each $N \in \mathbb{N}^+$. Let $X = (X_1, \dots, X_d)$ be a tuple of random variables in a C^*-probability space $(\mathcal{A}, \varphi)$. We say the sequence $(X^{(N)})_{N=1}^\infty$ *almost surely strongly converges in distribution* to X if

(i) $(X^{(N)})_{N=1}^\infty$ almost surely converges in distribution to X, i.e. for almost every $\omega \in \Omega$.

$$\lim_{N\to\infty} \mathrm{tr}_N(p(X^{(N)}(\omega))) = \varphi(p(X))$$

for every $p \in \mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle$.

(ii) we also have for almost every $\omega \in \Omega$

$$\lim_{N\to\infty} \|p(X^{(N)}(\omega))\|_{M_N(\mathbb{C})} = \|p(X)\|_{\mathcal{A}}$$

for every $p \in \mathbb{C}\langle x_1, \ldots, x_d, x_1^*, \ldots, x_d^*\rangle$.

For succinctness, we will abuse the notation and write the norms $\|\cdot\|_{M_N(\mathbb{C})}$ and $\|\cdot\|_{\mathcal{A}}$ as $\|\cdot\|$ in the following.

THEOREM II.2.5. *(See* [**HT05, HST06**]*) Let $X^{(N)} = (X_1^{(N)}, \ldots, X_d^{(N)})$ be a tuple of independent GUE random matrices in $(M_N(L^{\infty-}(\Omega,\mathbb{P})), \mathrm{tr}_N \circ \mathbb{E}^{(N)})$ for each $N \in \mathbb{N}^+$. Then the sequence $(X^{(N)})_{N=1}^{\infty}$ almost surely strongly converges in distribution to a tuple $S = (S_1, \ldots, S_d)$ of freely independent semicircular random variables in some C^*-probability space.*

For GOE and GSE (*Gaussian symplectic ensemble*) random matrices, their strong convergence in distribution toward freely independent semicircular random variables was proven in [**Sch05**]. For the Wigner case, this result was proven in [**CD07, And13**]. A Haar unitary version of this theorem also holds, see [**CM14**].

In Chapter VII, we will see that these connections between random matrices and non-commutative random variables can be further strengthened. But our strengthening goes in a different way than the results above. In Definition II.2.1 and II.2.4, we note that the convergence is always stated for non-commutative polynomials. In Chapter VII, we will see that polynomials can be extended to a larger class of non-commutative functions that involve the inverse of variables.

CHAPTER III

Inner rank

In this chapter we focus on a purely algebraic concept—the inner rank—and some related algebraic concepts, for example, central eigenvalues. The analytic counterpart of the inner rank over polynomials was introduced in Definition I.5.10 as an analytic rank over von Neumann algebras. An equality will be established in Section V.2 between this analytic rank and the inner rank over polynomials. Moreover, we will also correspond the central eigenvalues with their analytic counterparts in Section VI.2. These counterparts were described in Section I.3 under the name of zero divisors or atoms.

The inner rank is a generalization of the usual notion of rank in linear algebra where the complex numbers are replaced by more general rings. In Section III.1, we will introduce the inner rank for matrices over a unital (not necessarily commutative) complex algebra. Then we will pay special attention to the case when the algebra is stably finite or a division ring.

In Section III.2, we will limit our discussion of the inner rank to the case of Sylvester domains. Sylvester domains are of particular interest for us because they characterize rings that can be embedded into division rings with inner rank preserving homomorphisms (see Theorem IV.2.5 in the next chapter). The embeddability of non-commutative polynomials into a division ring with the inner rank preserved is a crucial property which will be used in Section V.2. So the algebras of polynomials give us the practical example among Sylvester domains. After that we will focus on the case of polynomials until the end of this chapter.

In Section III.3, some zero block structure of linear matrices over non-commutative polynomials will be presented. Such a structure will allow us later to do an inductive argument on the dimension of matrices in the proof of Theorem V.1.1.

Section III.4 is a brief introduction to central eigenvalues. It is well-known that eigenvalues can be defined by the invertibility of matrices (which is equivalent to the fullness of the usual rank). Here we replace the invertibility by the maximality of the inner rank to define these central eigenvalues. It is a natural concept if we treat the inner rank as a generalization of the usual rank. Two facts on central eigenvalues will be given: one about the cardinality of central eigenvalues of a given matrix and one about the possible positions of central eigenvalues when the given matrix is linear. These two facts can be used to predict the cardinality and positions of atoms for distributions of some matrix-valued random variables, since we have the aforementioned correspondence between central eigenvalues and atoms.

III.1. Some basics of the inner rank

Let $\mathcal{A}$ be a general unital (not necessarily commutative) complex algebra. By $M_{m,n}(\mathcal{A})$ we denote the $m \times n$ matrices with entries in $\mathcal{A}$; and we set $M_n(\mathcal{A}) := M_{n,n}(\mathcal{A})$.

Definition III.1.1. For any non-zero $A \in M_{m,n}(\mathcal{A})$, the *inner rank* of A is defined as the least positive integer r such that there are matrices $P \in M_{m,r}(\mathcal{A})$, $Q \in M_{r,n}(\mathcal{A})$ satisfying $A = PQ$. We denote this number by $\rho(A)$, and we call any such factorization with $r = \rho(A)$ a *rank factorization*. Additionally, if A is a zero matrix, we define $\rho(A) = 0$.

For example, it is easy to verify the factorization

$$\begin{pmatrix} y^2 & yxy \\ yxy & yx^2y \end{pmatrix} = \begin{pmatrix} y \\ yx \end{pmatrix} \begin{pmatrix} y & xy \end{pmatrix}$$

in matrices over $\mathbb{C}\langle x, y\rangle$. So, this matrix has inner rank 1: its inner rank has to be less than or equal to 1 by the above factorization and it is not a zero matrix. Moreover, any element in $\mathcal{A}$, regarded as a 1×1 matrix, has inner rank 1 unless it is 0.

This notion of inner rank was defined in [**Ber67**], but goes back much further according to a commment in [**Coh06**, Chapter 0]: "almost any pre-1914 book on matrix theory defines the rank of a matrix A as the least number of terms in the expression of A as a sum of *dyads*, i.e. products of a column by a row, which is a matrix of inner rank 1".

It is clear that the inner rank of an $m \times n$ matrix is less than or equal to $\min\{m, n\}$. Those matrices maximizing their inner ranks play a very important role and thus we have the following definition for them.

Definition III.1.2. A non-zero matrix $A \in M_{m,n}(\mathcal{A})$ is called *full* if $\rho(A) = \min\{m, n\}$, namely, if there is no rank factorization with $\rho(A) < \min\{m, n\}$.

When we set $\mathcal{A}$ to be a unital complex algebra, $M_n(\mathbb{C})$ always becomes a subalgebra of $M_n(\mathcal{A})$ for any $n \in \mathbb{N}^+$. Moreover, it is not difficult to check that the inner rank agrees with the usual notion of rank for matrices over $\mathbb{C}$, so the inner rank generalizes consistently the notion of rank from $M_n(\mathbb{C})$ to $M_n(\mathcal{A})$.

In the following remarks, we record some basic properties for the inner rank to depict more analogies between the inner rank and the usual rank. These properties follow directly by the definition of inner rank.

Remark III.1.3. Let $A \in M_{m,n}(\mathcal{A})$ be given.

(i) For any invertible $B \in M_m(\mathcal{A})$ and $C \in M_n(\mathcal{A})$, we have

$$\rho(A) = \rho(BAC).$$

(ii) For any $B \in M_{n,s}(\mathcal{A})$, we have

$$\rho(AB) \leq \min\{\rho(A), \rho(B)\}.$$

(iii) If we write $A = (B\ C)$, where B is the block of the first r columns of A and C consists of the remaining columns, then

$$\rho(A) \geq \max\{\rho(B), \rho(C)\}.$$

Additionally, if $C = \mathbf{0}$, then $\rho(A) = \rho(B)$. Moreover, similar inequality also holds if we partition A by rows.

(iv) For any $B \in M_{p,q}(\mathcal{A})$, the *diagonal sum* of A and B, i.e.,

$$A \oplus B := \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \in M_{m+p,n+q}(\mathcal{A})$$

satisfies $\rho(A \oplus B) \leq \rho(A) + \rho(B)$. Combining this inequality with Item (iii), we have in particular

$$\max\{\rho(A), \rho(B)\} \leq \rho(A \oplus B) \leq \rho(A) + \rho(B).$$

(v) For any ring homomorphism φ defined on $\mathcal{A}$,

$$\rho(\varphi(A)) \leq \rho(A),$$

where $\varphi(A)$ stands for the entrywise image of A under φ.

According to the definition, the inner rank measures some "non-degenerateness" of matrices via their factorizations. So one may be tempted to believe that full square matrices are invertible or vice versa. However, neither is true without any further assumption on the algebra $\mathcal{A}$. For example, any non-zero element is a full 1×1 matrix but may not be invertible in general. To see that an invertible matrix might not be full, one can consider the following example.

Example III.1.4. For any integers m and n satisfying $1 \leq m < n$, let $\mathcal{A}$ be the unital algebra $\mathcal{A}$ generated by $2mn$ generators, arranged as an $n \times m$ matrix A and an $m \times n$ matrix B, and by the defining relations $AB = \mathbf{1}_n$. Clearly the identity matrix $\mathbf{1}_n$ is not full by the defining relations of $\mathcal{A}$ while it is always invertible.

Fortunately, we will not encounter such algebras over which the inner rank behaves badly. Instead we always consider algebras with the following property.

Definition III.1.5. A unital algebra $\mathcal{A}$ is called *stably finite* (or *weakly finite*) if for all $n \in \mathbb{N}^+$ and all $A, B \in M_n(\mathcal{A})$, $AB = \mathbf{1}_n$ implies $BA = \mathbf{1}_n$.

Over a stably finite algebra we have that every invertible matrix is full. Actually, we have a more general result as follows.

Proposition III.1.6. *(*[**Coh06**, Proposition 5.4.6]*) Suppose that $\mathcal{A}$ is stably finite. Let $A \in M_{m+n}(\mathcal{A})$ be of the form*

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where $B \in M_m(\mathcal{A})$, $C \in M_{m,n}(\mathcal{A})$, $D \in M_{n,m}(\mathcal{A})$ and $E \in M_n(\mathcal{A})$. If B is invertible, then $\rho(A) \geq m$, with equality if and only if $E = DB^{-1}C$.

Now if we assume that all full matrices are invertible over an algebra $\mathcal{A}$, then in particular all non-zero elements are invertible in $\mathcal{A}$. That is, $\mathcal{A}$ has to be a *division ring* or *skew field*. Actually, this condition is not only necessary, but also sufficient. We record this fact here by the following lemma.

Lemma III.1.7. *If $\mathcal{A}$ is a division ring, then every full square matrix over $\mathcal{A}$ is invertible.*

Proof. It is clear that the result holds for matrices of dimension 1. We proceed by mathematical induction on the dimension of matrices. Suppose that the result holds for matrices of dimension n and A is a full matrix in $M_{n+1}(\mathcal{A})$. On the one hand, A is non-zero, so at least one entry of A is non-zero. On the other hand, permutations of rows and columns preserve the fullness of A. Thus we can assume the $(1,1)$-entry of A is non-zero. That is, we can assume that A has the form

$$A = \begin{pmatrix} a & b \\ c & D \end{pmatrix}$$

with $a \neq 0$. With the invertible matrices

$$U := \begin{pmatrix} 1 & \mathbf{0} \\ -ca^{-1} & \mathbf{1}_n \end{pmatrix}, \; V := \begin{pmatrix} a^{-1} & -a^{-1}b \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix} \in M_{n+1}(\mathcal{A}),$$

we obtain another full matrix

$$UAV = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & D - ca^{-1}b \end{pmatrix},$$

where the block $D - ca^{-1}b$ has to be a full matrix by Item (iv) of Remark III.1.3. Therefore, by induction hypothesis we see that $D - ca^{-1}b$ is invertible. It follows that UAV is invertible and thus A is also invertible. □

Actually, we see that a non-commutative version of the Gaussian elimination method works equally well over a division ring (which may not be commutative in general), from the proof of the above lemma. The matrices U and V in the above proof play a role of row and column operations. Moreover, we will obtain a diagonal matrix from which the inner rank of the original matrix can be directly read off if we continue this algorithm to its end. We record this fact here as the following proposition.

PROPOSITION III.1.8. *Suppose that $\mathcal{A}$ is a division ring. Then for any matrix A over $\mathcal{A}$, there exist two invertible matrices U and V over $\mathcal{A}$ such that*

$$UAV = \begin{pmatrix} \mathbf{1}_{\rho(A)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

PROOF. If $A \neq \mathbf{0}$, then we implement the non-commutative Gaussian elimination method to A. That means there are invertible matrices U and V such that

$$UAV = \begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

for some $r \in \mathbb{N}^+$. So it remains to show that $\rho(A) = r$, or equivalently,

$$\rho(UAV) = \rho \begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = r.$$

If we recall the inequality in Item (iv) of Remark III.1.3, we see that $\rho(UAV) = \rho(\mathbf{1}_r)$. Therefore, we only need to show $\rho(\mathbf{1}_r) = r$, i.e., $\mathbf{1}_r$ is full. As an invertible matrix, $\mathbf{1}_r$ is indeed full if $\mathcal{A}$ is stably finite. Hence the proof is completed by the well-known fact that a division algebra is stably finite. □

III.2. Sylvester domains

Our main goal of this section is to present some properties of the algebra $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ of non-commutative polynomials. At this point, we want to introduce these properties at ring theoretic level to highlight them. At the end of this section, we will come back to the algebra $\mathbb{C}\langle x_1, \ldots, x_d\rangle$.

These properties are actually shared by more general rings known as Sylvester domains. So we will first introduce Sylvester domains as well as some basic properties of them. Then we will give the reason why $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ is a Sylvester domain.

Moreover, a difference appears here between the commutative and the non-commutative polynomials. That is, the algebra $\mathbb{C}[x_1, \ldots, x_d]$ of commutative polynomials with $d > 2$ is not a Sylvester domain, which has some consequence during our investigations in Section V.2.

Definition III.2.1. ([**DS78**]) A non-zero ring $\mathcal{A}$ is called a *Sylvester domain* if for any $A \in M_{m,n}(\mathcal{A})$, $B \in M_{n,r}(\mathcal{A})$ such that $AB = 0$, it follows that

$$\rho(A) + \rho(B) \leq n.$$

In particular, if two elements a and b satisfy $ab = 0$, then $\rho(a) + \rho(b) \leq 1$ according to the definition of Sylvester domain. This implies that $a = 0$ or $b = 0$. So Sylvester domains are indeed domains. The reason for naming these domains after Sylvester is that they are actually equivalently characterized by *Sylvester's law of nullity* or *rank inequality* with respect to the inner rank, that is, for any $A \in M_{m,n}(\mathcal{A})$, $B \in M_{n,r}(\mathcal{A})$,

$$\rho(A) + \rho(B) \leq \rho(AB) + n.$$

Apparently the defining property in Definition III.2.1 is a weaker form of Sylvester's law of nullity. A proof for that it implies the law of nullity can be found in [**DS78**] or [**Coh06**, Section 5.5].

Clearly, (commutative) fields satisfy Sylvester's law of nullity due to Sylvester, and therefore they are Sylvester domains. More generally, division rings are also Sylvester domains. To see this, let A and B be two matrices over a division ring $\mathcal{A}$ satisfying $AB = 0$. By Proposition III.1.8 we can find invertible matrices U and V such that

$$UAV = \begin{pmatrix} \mathbf{1}_{\rho(A)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

By writing

$$V^{-1}B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where $B_1 \in M_{\rho(A)}(\mathcal{A})$ and other blocks are of appropriate size, we have

$$\begin{pmatrix} \mathbf{1}_{\rho(A)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = (UAV)(V^{-1}B) = UAB = \mathbf{0}.$$

This enforces that $B_1 = B_2 = \mathbf{0}$, which implies that

$$\rho(V^{-1}B) = \rho \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B_3 & B_4 \end{pmatrix} = \rho \begin{pmatrix} B_3 & B_4 \end{pmatrix}$$

according to Item (iii) of Remark III.1.3. Note $(B_3\ B_4)$ is an $(n-\rho(A))\times n$ matrix, we conclude that

$$\rho(V^{-1}B)\leq n-\rho(A).$$

Hence $\rho(B)\leq n-\rho(A)$ follows.

An immediate consequence of Sylvester's law of nullity is that the product of full square matrices is again full over a Sylvester domain. However, this may not be true in general. Instead we only have an upper bound for the inner rank of a product of full matrices by Item (ii) of Remark III.1.3.

Moreover, for any matrices A and B the inequality $\rho(A\oplus B)\leq\rho(A)+\rho(B)$ in Item (iv) of Remark III.1.3 becomes an equality for the case of $\mathcal{A}=\mathbb{C}$, which is well-known in linear algebra. However, this equality also fails in general. For example, if we choose $m=1$ and $n=2$ for the algebra in Example III.1.4, then $\mathbf{1}_2$ is the diagonal sum of two copies of 1 but has inner rank 1. However, this equality indeed holds for Sylvester domains, which we record as the following lemma.

Lemma III.2.2. *(See* [**DS78**] *or* [**Coh06**, Lemma 5.5.3]*) Let $\mathcal{A}$ be a Sylvester domain. Then for any matrices A, B over $\mathcal{A}$,*

$$\rho(A\oplus B)=\rho(A)+\rho(B).$$

A well-known characterization of the usual rank of a scalar-valued matrix is given by submatrices. Now we can state its analogue for the inner rank over Sylvester domains. That is, the inner rank of a matrix is given by the maximal dimension of its full submatrices. This characterization was first proven by Cohn for a special class of Sylvester domains called *semifir* in [**Coh74**]. But it actually holds under weaker assumptions. These assumptions are exactly the two properties of Sylvester domain as explained above.

Theorem III.2.3. *(See* [**DS78**] *or* [**Coh06**, Theorem 5.4.9]*) Suppose that the set of all full square matrices over $\mathcal{A}$ is closed under products and diagonal sums. Then for any $A\in M_{m,n}(\mathcal{A})$, there exists a square submatrix of A which is a full matrix over $\mathcal{A}$ of dimension $\rho(A)$. Moreover, $\rho(A)$ is the maximal dimension for such submatrices.*

The most important example of Sylvester domains for us is the algebra $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ of non-commutative polynomials. However, it is far from obvious that $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ is a Sylvester domain. Briefly, the reason is that $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ has the following property: For any matrices A, B over $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ satisfying $AB=\mathbf{0}$, there exists an invertible square matrix U over $\mathbb{C}\langle x_1,\ldots,x_d\rangle$ such that

$$AU=\begin{pmatrix}C & \mathbf{0}\end{pmatrix},\ U^{-1}B=\begin{pmatrix}\mathbf{0}\\ D\end{pmatrix},$$

where the number of C's columns is the same as the number of rows of the zero block in $U^{-1}B$. Then it is easy to see that $\rho(A)+\rho(B)=\rho(AU)+\rho(BU^{-1})\leq n$, where n denotes the number of columns of A.

A matrix relation expressed as $AB=\mathbf{0}$ is called *trivial* if for each $i=1,\ldots,n$ either the i-th column of A or the i-th row of B is $\mathbf{0}$. So the property in the above paragraph says that every matrix relation $AB=\mathbf{0}$ for two matrices A and B can be trivialized by invertible matrices over $\mathbb{C}\langle x_1,\ldots,x_d\rangle$. For example, $A=(x,xy)$ and $B=(yz,-z)^T$ gives a relation $AB = 0$ which is not trivial over $\mathbb{C}\langle x, y, z\rangle$, but it can be trivialized by an invertible matrix

$$U = \begin{pmatrix} 1 & y \\ 0 & -1 \end{pmatrix}$$

because $AU = (x, 0)$ and $U^{-1}B = (0, z)^T$.

Moreover, this property, the trivialization for any matrix relation $AB = \mathbf{0}$, is shared by a general class of Sylvester domains, namely, semifirs. Semifirs have been mentioned before Theorem III.2.3 without definition. Again we will not go further into the details of semifirs here. Instead we refer the interested reader to [**Coh06**, Section 2.3] for its precise definition. But we want to point out here that this trivialization property is actually one of the many equivalent characterizations of semifirs.

Then it remains to check that $\mathbb{C}\langle x_1, \dots, x_d\rangle$ is a semifir in order to see it is a Sylvester domain. This is still not an easy task but it turns out to be sufficient to check the matrix relation $AB = 0$ for the special case that A is a row vector and B a column vector (see [**Coh06**, Section 2.3] for the details). So let us take a closer look at an n-term relation $\sum_{i=1}^{n} p_i q_i = 0$, where $p_i, q_i \in \mathbb{C}\langle x_1, \dots, x_d\rangle, i = 1 \dots, n$. This particular relation says that $(p_1, \dots, p_n)$ is right linear dependent over $\mathbb{C}\langle x_1, \dots, x_d\rangle$. So if we can find an invertible matrix which reduces $(p_1, \dots, p_n)$, by acting on the right, to a tuple whose non-zero entries are right linear independent over $\mathbb{C}\langle x_1, \dots, x_d\rangle$, then the relation is trivialized by this invertible matrix. This reduction is indeed feasible by an algorithm. However, describing such an algorithm requires some extra effort and deviates a lot from our original quest here. So we leave out the details here and refer the interested reader to [**Coh06**, Theorem 2.5.1]. Actually, in [**Coh06**, Section 2.4], a notion called *weak algorithm* was developed (as a generalization of the division algorithm) to do this job. Any ring with weak algorithm was also proven to be a semifir; see [**Coh06**, Theorem 2.4.4]. At last but not least, [**Coh06**, Corollary 2.5.2] and the paragraph thereafter tell us that the free algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ indeed satisfies the weak algorithm.

Therefore, we are finally able to record:

EXAMPLE III.2.4. The algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ of non-commutative polynomials in d variables is a Sylvester domain for any $d \in \mathbb{N}^+$. Hence, by Theorem III.2.3, the inner rank of a matrix over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ is given by the maximal dimension of its submatrices.

So far we have seen that Sylvester domains are domains on which the inner rank behaves very nicely. Furthermore, they include important examples such as algebras of non-commutative polynomials. However, they do not include commutative polynomials if there are more than two variables. We record this fact by the following remark.

REMARK III.2.5. The algebra $\mathbb{C}[x_1, \dots, x_d]$ of commutative polynomials is a Sylvester domain if and only if $d = 1$ or $d = 2$.

The fact that $\mathbb{C}[x_1, \dots, x_d]$ is a Sylvester domain when $d = 1, 2$ can be extracted from [**Coh06**, Theorem 5.5.4]. To see that the other cases are not Sylvester domains, it is enough to find two matrices over $\mathbb{C}[x, y, z]$ that fail the defining property. The following example, taken from [**Coh06**, Section 5.5], provides us an example that fails the defining property. Let us consider

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \ B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then it is clear that $AB = \mathbf{0}$ and $\rho(B) = 1$. Moreover, it can be proven that A is full, i.e., $\rho(A) = 3$. Therefore, $\rho(A) + \rho(B) = 4 > 3$, which tells us that $\mathbb{C}[x, y, z]$ is not a Sylvester domain.

The purpose of introducing Sylvester domains in [**DS78**] was to study rings that can be embedded into division rings by inner rank-preserving homomorphisms. This property was first proven for semifirs by Cohn. It turned out to be exactly an equivalent description of Sylvester domain; see Theorem IV.2.5 for the precise statement. Therefore, as a Sylvester domain, $\mathbb{C}\langle x_1, \dots, x_d\rangle$ can be embedded into some division ring. Let us recall that commutative polynomials can be embedded into the field of rational functions. With that in mind, it is not surprising that the division ring of non-commutative rational functions also exists and thus extends $\mathbb{C}\langle x_1, \dots, x_d\rangle$. But we postpone the related discussion to the next chapter since these non-commutative rational functions are very important for our investigation in Section V.2 and deserve a whole chapter.

III.3. Linear matrices

From now on, we will focus on the algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ of non-commutative polynomials. This short section is devoted to special matrices whose entries are linear polynomials, i.e., polynomials of degree at most 1. These matrices have some zero block structure as mentioned in the introduction of this chapter. Our goal of this section is to present this structure in details.

In order to introduce this structure, let us first consider a matrix of the form

$$A = \begin{pmatrix} B & \mathbf{0} \\ C & D \end{pmatrix} \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle),$$

which has a zero block of size $r \times s$ and blocks B, C, D of sizes $r \times (n-s)$, $(n-r) \times (n-s)$, $(n-r) \times s$, respectively. Then we have the factorization

$$A = \begin{pmatrix} B & \mathbf{0} \\ C & D \end{pmatrix} = \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n-r} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n-s} & \mathbf{0} \\ C & D \end{pmatrix}.$$

So A has been expressed as a product of an $n \times (2n-r-s)$ matrix and an $(2n-r-s) \times n$ matrix. This allows us to conclude that $\rho(A) \leq 2n - r - s$. Therefore, if the size of the zero block of A satisfies $r + s > n$, we have $\rho(A) < n$, which means that A is not full. Such matrices are called hollow matrices.

Definition III.3.1. A matrix in $M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ is called *hollow* if it has an $r \times s$ block of zeros with $r + s > n$.

In general, a non-full $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ may not have any zero blocks or submatrices. However, we can say more for the following special matrices.

Definition III.3.2. A matrix $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ is called *linear* if it can be written in the form $A = A_0 + A_1x_1 + \cdots + A_dx_d$, where $A_0, A_1 \dots, A_d$ are $n \times n$ matrices over $\mathbb{C}$. Note that we allow also a constant term in a general linear matrix. We call the non-constant part $A - A_0 = A_1x_1 + \cdots + A_dx_d$ the *homogeneous part of* A.

For linear matrices we have the following theorem to reveal their zero block structure. That is, for a linear matrix A it is always possible to bring A into a form with some possible zero block. Then we can see that A is not full if and only if A is hollow up to multiplying by scalar-valued invertible matrices.

Theorem III.3.3. *Let A be a linear matrix in $M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$. Then there exist invertible matrices $U, V \in M_n(\mathbb{C})$ and an integer $s \in [0, \rho(A)]$ such that*

$$UAV = \begin{pmatrix} B & \mathbf{0} \\ C & D \end{pmatrix} \tag{III.1}$$

where $B \in M_{n-s,\rho(A)-s}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ is full, that is, $\rho(B) = \rho(A) - s$.

Actually, we see that the zero block has size $(n - s) \times (n - \rho(A) + s)$ from the above block structure. Therefore, if A is not full, i.e., $\rho(A) < n$, then the right hand side of (III.1) is hollow since $2n - \rho(A) > n$.

For a proof of the above theorem, we refer to [**Coh95**, Corollary 6.3.6]. Alternatively, the same proof can also be read off from the proof of [**Coh06**, Theorem 5.8.8], though it is aimed to prove something different. The original statement of the above theorem as well as these two proofs do not address the inner rank of the block B of UAV explicitly. But this additional information on B can be extracted as follow. Let $B = PQ$ be a rank factorization with $P \in M_{n-s,r}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ and $r := \rho(B)$. Then we have

$$\begin{pmatrix} B & \mathbf{0} \\ C & D \end{pmatrix} = \begin{pmatrix} PQ & \mathbf{0} \\ C & D \end{pmatrix} = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_s \end{pmatrix} \begin{pmatrix} Q & \mathbf{0} \\ C & D \end{pmatrix},$$

which gives a factorization of UAV. It follows that $\rho(A) = \rho(UAV) \leq r + s = \rho(B) + s$, which implies immediately that $\rho(B) = \rho(A) - s$.

III.4. Central eigenvalues

In this section, we want to introduce a notion which generalizes the usual notion of eigenvalues for scalar-valued matrices. Recall that for a matrix $A \in M_n(\mathbb{C})$ its spectrum $\sigma(A)$ is given as the finite set

$$\{\lambda \in \mathbb{C} \mid A - \lambda\mathbf{1}_n \text{ is not invertible in } M_n(\mathbb{C})\}.$$

Or equivalently,

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \rho_{\mathbb{C}}(A - \lambda\mathbf{1}_n) < n\},$$

where $\rho_{\mathbb{C}}$ denotes the (inner) rank of A over $\mathbb{C}$. This is simply because the invertibility of A is equivalent to the maximality of its rank.

Now, let us consider the inner rank defined for matrices over a unital complex algebra $\mathcal{A}$. Though a full matrix over $\mathcal{A}$ may not be invertible, as we have seen in Section III.1, we can still have a generalized notion of eigenvalues by the inner rank as follows.

Definition III.4.1. For every square matrix A over a unital complex algebra $\mathcal{A}$, say $A \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}^+$, we define
$$\sigma_{\mathcal{A}}^{\text{full}}(A) := \big\{\lambda \in \mathbb{C} \bigm| \rho_{\mathcal{A}}(A - \lambda \mathbf{1}_n) < n\big\},$$
where $\rho_{\mathcal{A}}$ denotes the inner rank over $\mathcal{A}$. The numbers $\lambda \in \sigma_{\mathcal{A}}^{\text{full}}(A)$ are called *central eigenvalues* of A.

This concept of central eigenvalues was introduced in [**Coh85**, Section 8.4].[1] These eigenvalues are called central because they are values taken in the centre of the algebra $\mathcal{A}$. The centre of $\mathcal{A}$ is, in our special case, exactly given by complex numbers $\mathbb{C}$. Section 8.4 in Cohn's book is actually based on [**Jac37**], which is on *pseudo-linear transforms*. Central eigenvalues were introduced therein from a very different perspective than ours. Instead of studying anything related to the topic of pseudo-linear transforms, we simply introduce central eigenvalues as a generalization of eigenvalues here.

Actually, for any scalar-valued matrix in $M_n(\mathcal{A})$, its central eigenvalues are nothing but its usual eigenvalues. Moreover, Lemma III.1.7 also teaches us that the fullness, i.e., the maximality of inner rank over $\mathcal{A}$ is indeed equivalent to the invertibility if $\mathcal{A}$ is additionally a division ring. So central eigenvalues are reasonable generalization of the usual eigenvalues with respect to the inner rank.

However, our motivation to investigate central eigenvalues is more than generalizing a notion. In Section VI.2, we will show that these central eigenvalues can become exactly atoms of some probability distributions under certain conditions. Therefore, central eigenvalues come to us very naturally when we want to study these atoms in the probabilistic context.

We specify our considerations now to the relevant case $\mathcal{A} = \mathbb{C}\langle x_1, \dots, x_d\rangle$ and will in the following write $\sigma^{\text{full}}(A)$ for $\sigma^{\text{full}}_{\mathbb{C}\langle x_1,\dots,x_d\rangle}(A)$ for any square matrix A over $\mathbb{C}\langle x_1, \dots, x_d\rangle$. In this case, a matrix $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ has at most n central eigenvalues. We state this formally as the following proposition.

Proposition III.4.2. *Let A be an $n \times n$ matrix over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ for some $n \in \mathbb{N}^+$. Then A has at most n central eigenvalues.*

This proposition can be deduced from Cohn's result [**Coh85**, Proposition 8.4.1]. Actually, the result in his book is stated for a division ring $\mathcal{K}$ rather than our specific case $\mathbb{C}\langle x_1, \dots, x_d\rangle$. It covers the case $\mathbb{C}\langle x_1, \dots, x_d\rangle$ since $\mathbb{C}\langle x_1, \dots, x_d\rangle$ can be embedded into a division ring with the inner rank preserved; see Theorem IV.2.5. His proof relied on some involved algebraic considerations, which is, very briefly, considering the given matrix over the field of fraction $\mathcal{K}(t)$ in one variable t. He showed that for a matrix $A \in M_n(\mathcal{K})$, $\rho_{\mathcal{K}(t)}(A - t) = \sup_{\alpha \in \mathcal{C}} \rho_{\mathcal{K}(t)}(A - \alpha) = n$, where the supremum is attained for all but at most n values of α in the centre $\mathcal{C}$ of $\mathcal{K}$. From this the above proposition follows.

In Section VI.2, we will give an alternative proof of Proposition III.4.2, based on our results. An interesting point of this new proof is that it has a very analytic or probabilistic nature though we aim to prove a purely algebraic proposition. Such a proof is doable because we found certain analytic objects behave very much the same as formal variables. Thus we can prove algebraic results using some analytic concepts and tools. More detailed discussion will be given in Section VI.2.

[1] We thank Konrad Schrempf for bringing this reference to our attention.

Conversely, algebraic results like [**Coh85**, Proposition 8.4.1] will give us information about the corresponding analytic objects. So here we give another algebraic result on central eigenvalues. Unfortunately, for the following proposition, we do not know at moment if there is an analytic or probabilistic proof or explanation.

PROPOSITION III.4.3. *Let any linear $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ of the form $A = A_0 + A_1 x_1 + \dots + A_d x_d$ with $A_0, A_1, \dots, A_d \in M_n(\mathbb{C})$ be given. Then the following statements hold:*

(i) *We have that $\sigma^{\text{full}}(A) \subset \sigma(A_0)$, where $\sigma(A_0)$ is the usual spectrum of A_0 consisting of all eigenvalues of A_0.*
(ii) *If the homogeneous part $A - A_0$ of A is full, then $\sigma^{\text{full}}(A) = \emptyset$.*

PROOF. Let $\lambda \in \sigma^{\text{full}}(A)$ be given. This means that $A - \lambda \mathbf{1}_n$ is not full. So Theorem III.3.3 guarantees the existence of invertible matrices $U, V \in M_n(\mathbb{C})$ such that

$$U(A - \lambda \mathbf{1}_n)V = U(A_0 - \lambda \mathbf{1}_n)V + \sum_{j=1}^{d} (UA_jV)x_j$$

is hollow. Due to linearity, this enforces both $U(A_0 - \lambda \mathbf{1}_n)V$ and $\sum_{j=1}^{d}(UA_jV)x_j$ to be hollow. Now, on the one hand, it follows that neither $U(A_0 - \lambda \mathbf{1}_n)V$ nor $A_0 - \lambda \mathbf{1}_n$, thanks to the invertibility of U and V, can be invertible. Thus, we infer that $\lambda \in \sigma(A_0)$, which shows the validity of (i). On the other hand, we see that neither $\sum_{j=1}^{d}(UA_jV)x_j$ nor $\sum_{j=1}^{d} A_j x_j$, by the invertibility of U and V, can be full. Thus, if the homogeneous part of A is assumed to be full, that contradiction rules out the existence of $\lambda \in \sigma^{\text{full}}(A)$, which proves (ii). $\square$

CHAPTER IV

Free field

In this chapter, our first goal is to give an introduction to the free field, i.e., the universal skew field of fractions of non-commutative polynomials. It is a follow-up introduction of Sylvester domains in the last chapter. Actually, universal skew field of fractions can be constructed for any Sylvester domain with the inner rank preserved. This fundamental result will be record as Theorem IV.2.5 (see [**DS78**] or [**Coh06**][Section 7.5]). Our application of this result is going to the algebra $\mathbb{C}\langle x_1, \ldots, x_d\rangle$. In this particular case, the resulting universal field of fractions, the free field, gives us the core objects—non-commutative rational functions—that will be investigated in the context of free probability. Moreover, the free field provides us a much larger underlying algebra to consider matrices over polynomials. This will allow us to diagonalize a matrix of polynomials to recover its inner rank. This diagonalization idea will play an essential role for establishing the equality between the inner rank over polynomials and the analytic rank over von Neumann algebra in Section V.2.

We will discuss universal skew fields of fractions in Section IV.2. Section IV.1 will be devoted to an alternative approach to the free field. This approach provides us a more intuitive way to understand the free field, in comparison to the one in Section IV.2. Instead of building a skew field of fractions to extend $\mathbb{C}\langle x_1, \ldots, x_d\rangle$, we will build rational expressions from $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ by arithmetic operations in this approach. When it comes to define the evaluation for rational expressions they are also much easier to define than for rational functions. The evaluation of rational functions will be discussed in Section IV.4. At the end of Section IV.1, we will point out a link between rational expressions and rational functions with the help of the evaluation of rational expressions.

An important concept related to rational functions, which will be given in Section IV.3, is the linear representation. That is, each rational function can be associated with some full matrix over linear polynomials. Similar concepts are known as the "linearization trick" in many different realms of mathematics. We will present an algorithm taken from [**HMS18**] to show how a linear representation can be build very explicitly for a rational expression.

In Section IV.4, we will handle the technical issue when we want to define the evaluation of rational functions. To achieve a reasonable definition of the evaluation, assumptions like stable finiteness can not be avoided. We will give a self-contained proof for the well-definedness of our definition of the evaluation.

Rational closure and division closure will be introduced Section IV.5. They will replace the role of rational functions when we discuss the strong Atiyah property in Section V.3. Actually, the rational closure is given by all well-defined evaluations of rational functions. The division closure usually coincides with the rational closure in our consideration.

IV.1. Non-commutative rational expressions

In this section we will introduce non-commutative rational expressions. Theoretically, they are independent notions from the non-commutative rational functions. But they complete the practical side of rational functions and thus compensate the abstractness of rational function. Moreover, in Section IV.3, they will go in parallel with rational functions for a better understanding of linear representations.

A non-commutative rational expression, intuitively speaking, is obtained by taking repeatedly sums, products and inverses, starting from scalars and some formal non-commuting variables, without taking care about possible cancellations or resulting mathematical inconsistencies. For example, we allow 0^{-1} and $(x-x)^{-1}$ as valid and different expressions, though they don't make any sense when we try to treat them as functions. We will take care of this problem, when we talk about the domain of such expressions.

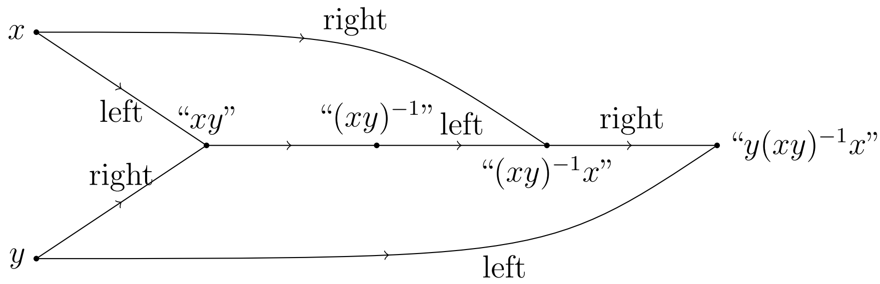
A formal definition for rational expressions can be achieved in terms of graphs as follows.

Definition IV.1.1. A *rational expression* in variables $x_1, \ldots, x_d$ is a finite directed graph with labels on vertices and edges, which satisfies the following rules.

(i) For each vertex, its *in-degree* is the number of edges directed to it. The *out-degree* of a vertex is the number of edges directed from it.
(ii) Each vertex of in-degree zero is labelled by an element from the set $\mathbb{C}$ or $\{x_1, \ldots, x_d\}$, that is, each vertex of in-degree zero represents a complex number or some variable in $\{x_1, \ldots, x_d\}$. All other vertices are of in-degree 1 or 2.
(iii) For any pair of edges directed to a vertex of in-degree 2, they can be labelled by *left* and *right*. Such a pair of edges represents the product. Secondly, a pair of edges directed to a vertex of in-degree 2 without labels represents a sum. Finally, an edge directed to a vertex of in-degree 1 represents the inverse.
(iv) There is only one vertex that has out-degree zero, which represents the final expression we want to build.

In the following, we denote the set of all non-commutative rational expressions in $x_1, \ldots, x_d$ by $\mathfrak{R}_{\mathbb{C}}(x_1, \ldots, x_d)$.

The definition is more or less self-explanatory. For any given "rational expression", such as $y(xy)^{-1}x$, we can construct such a graph according to the above rules: the variables and coefficients in the expression are given by some vertices of in-degree zero and we add new vertices according to the way how we read off subexpressions. For each $^{-1}$ applied to some subexpression, we add an directed edge from it to a new vertex without any label. For each $+$ applied to two subexpressions, we add two directed edges from them to a new vertex without any labels. For each $\times$ applied to two subexpressions, we add two directed edges from them to a new vertex with left and right labels to determine the order of multiplication. We proceed in such a way until we arrive at the vertex which corresponds to the desired "rational expression". For example, the rational expression $y(xy)^{-1}x$ is given by the following graph:



From this definition, taking sums, products and inverses of rational expressions are clear: if we adjoin two rational expressions by adding two edges from their unique vertices of out-degree zero to a new vertex, then the resulting new graph is the sum, or product if these two edges are labelled by left and right. If we add a new edge from the vertex of out-degree zero to a new vertex, the resulting graph is the inverse. This definition of rational expressions is known as *circuits*, or *non-commutative arithmetic circuits with division*. We refer to [**HW15**] and the references collected therein for this notion and related topics.

Recall that a commutative rational function is a quotient of two commutative polynomials. In other words, a commutative rational function can be represented as an equivalence class of two polynomials. However, to identify a class of rational expressions as one rational function is much more complicated in the non-commutative case. First of all, even deciding which expressions are trivial is diffcult. Some expression like $y(xy)^{-1}x - 1$ can be easily reduced to zero since $y(xy)^{-1}x = yy^{-1}x^{-1}x = 1$, but some expressions like

$$(x - y^{-1})^{-1} - x^{-1} - (xyx - x)^{-1}$$

take more effort to see they are also trivial. One way to overcome this difficulty is to define the equivalence classes by evaluations.

Definition IV.1.2. Let $\mathcal{A}$ be a unital and complex algebra. For each $r \in \mathfrak{R}_{\mathbb{C}}(x_1, \ldots, x_d)$, we define its $\mathcal{A}$-*domain* $\mathrm{dom}_{\mathcal{A}}(r) \subseteq \mathcal{A}^d$ together with its *evaluation* $\mathrm{ev}_X(r)$ for any tuple $X = (X_1, \ldots, X_d) \in \mathrm{dom}_{\mathcal{A}}(r)$ by the following rules:

(i) For any $\lambda \in \mathbb{C}$, we put $\mathrm{dom}_{\mathcal{A}}(r) = \mathcal{A}^d$ and $\mathrm{ev}_X(\lambda) = \lambda 1$, where 1 stands for the unit of algebra $\mathcal{A}$;

(ii) For $i = 1, \ldots, d$, we put $\mathrm{dom}_{\mathcal{A}}(x_i) = \mathcal{A}^d$ and $\mathrm{ev}_X(x_i) = X_i$;

(iii) For any two rational expressions r_1, r_2, we have

$$\mathrm{dom}_{\mathcal{A}}(r_1 \cdot r_2) = \mathrm{dom}_{\mathcal{A}}(r_1 + r_2) = \mathrm{dom}_{\mathcal{A}}(r_1) \cap \mathrm{dom}_{\mathcal{A}}(r_2)$$

and

$$\mathrm{ev}_X(r_1 \cdot r_2) = \mathrm{ev}_X(r_1) \cdot \mathrm{ev}_X(r_2),$$
$$\mathrm{ev}_X(r_1 + r_2) = \mathrm{ev}_X(r_1) + \mathrm{ev}_X(r_2);$$

(iv) For a rational expression r, we have

$$\mathrm{dom}_{\mathcal{A}}(r^{-1}) = \{X \in \mathrm{dom}_{\mathcal{A}}(r) \mid \mathrm{ev}_X(r) \text{ is invertible in } \mathcal{A}\}$$

and

$$\mathrm{ev}_X(r^{-1}) = (\mathrm{ev}_X(r))^{-1}.$$

We also abbreviate $r(X) := \mathrm{ev}_X(r)$ for any given rational expression r and $X \in \mathrm{dom}_{\mathcal{A}}(r)$.

Therefore, we can define equivalence classes of rational expressions by the evaluation when a unital algebra $\mathcal{A}$ is given. That is, two rational expressions r_1 and r_2 are called $\mathcal{A}$-*evaluation equivalent* if $\mathrm{dom}_{\mathcal{A}}(r_1) \cap \mathrm{dom}_{\mathcal{A}}(r_2) \neq \emptyset$ and $r_1(X) = r_2(X)$ for all $X \in \mathrm{dom}_{\mathcal{A}}(r_1) \cap \mathrm{dom}_{\mathcal{A}}(r_2)$. Of course there are many rational expressions that have empty domain, such as 0^{-1} and $(x-x)^{-1}$. If a rational expression has empty domain, we say that it is *degenerate*. But here they are safely discarded when we consider the evaluation equivalence.

Then it remains to choose an appropriate algebra $\mathcal{A}$ if we want to define rational functions as $\mathcal{A}$-evaluation equivalent classes of rational expressions. This approach was first achieved by Amitsur [**Ami66**] by evaluating rational expressions on some large auxiliary skew field. It turns out that the evaluation on matrices of all sizes is also sufficient, which was proved in [**KV12**].

IV.2. Non-commutative rational functions

In this section, we will introduce non-commutative rational functions, which constitute the free field. It was already mentioned at the end of Section IV.1 that rational functions can be constructed as evaluation equivalence classes of rational expressions . That gives us an intuitive way to understand rational functions but still does not answer the question: what are rational functions? In other words, we need a characterizing property of rational functions which can be examined on what we have constructed.

So our first part in this section is to introduce some characterization property of the free field. Recall that in the commutative situation, such a characterization is not difficult to state: the field of rational functions is the smallest field extending the algebra of commutative polynomials. However, non-commutative polynomials can be embedded into skew fields which are non-isomorphic; see [**KV12**] or [**Coh85**, Exercises 7.2] for some examples. Hence it is not enough to characterize the skew field of non-commutative rational functions as the smallest skew field extending non-commutative polynomials.

It turns out that the missing property here is a universal property of the skew field of rational functions. In order to make this notion precise, we follow the terminologies in [**Coh06**, Section 7.2]. From now on, we will focus on the non-commutative case. So we will skip the adjective "skew" when we talk about skew fields.

Definition IV.2.1. Let $\mathcal{A}$ be a ring.

(i) An $\mathcal{A}$-*ring* is a ring $\mathcal{K}$ together with a homomorphism $\phi : \mathcal{A} \to \mathcal{K}$. In particular, if $\mathcal{K}$ is a field, then it will be called an $\mathcal{A}$-*field*.
(ii) An $\mathcal{A}$-field $\mathcal{K}$ with $\phi : \mathcal{A} \to \mathcal{K}$ is called *epic* if $\mathcal{K}$ is generated by the image $\phi(\mathcal{A})$, i.e., there is no proper subfield of $\mathcal{K}$ containing $\phi(\mathcal{A})$.
(iii) An epic $\mathcal{A}$-field $\mathcal{K}$ is called *field of fractions of* $\mathcal{A}$ if the homomorphism $\phi : \mathcal{A} \to \mathcal{K}$ is injective.

Since we want to define some universal property for $\mathcal{A}$-fields, we need to consider homomorphisms which respect the $\mathcal{A}$-field structure. So it is natural to consider an $\mathcal{A}$-*ring homomorphism*, i.e., a homomorphism f from an $\mathcal{A}$-ring $\mathcal{K}$ to another one $\mathcal{L}$ with the homomorphisms $\phi_{\mathcal{K}} : \mathcal{A} \to \mathcal{K}$ and $\phi_{\mathcal{L}} : \mathcal{A} \to \mathcal{L}$ satisfying $f \circ \phi_{\mathcal{K}} = \phi_{\mathcal{L}}$. However, since any non-trivial homomorphism between fields must be injective, for an $\mathcal{A}$-ring homomorphism $f : \mathcal{K} \to \mathcal{L}$ between two $\mathcal{A}$-fields $\mathcal{K}$ and $\mathcal{L}$, $f(\mathcal{K})$ has to be a field containing $(f \circ \phi_{\mathcal{K}})(\mathcal{A}) =$

$\phi_{\mathcal{L}}(\mathcal{A})$. Therefore, an $\mathcal{A}$-ring homomorphism $f : \mathcal{K} \to \mathcal{L}$ has to be an isomorphism whenever $\mathcal{L}$ is epic. This shows that $\mathcal{A}$-ring homomorphisms can not be used for the universal property as there may be non-isomorphic $\mathcal{A}$-fields. Hence we need to consider more general maps.

Definition IV.2.2. Let $\mathcal{K}$ and $\mathcal{L}$ be $\mathcal{A}$-fields with homomorphisms $\phi_{\mathcal{K}} : \mathcal{A} \to \mathcal{K}$ and $\phi_{\mathcal{L}} : \mathcal{A} \to \mathcal{L}$. A *subhomomorphism* is an $\mathcal{A}$-ring homomorphism $f : \mathcal{K}_f \to \mathcal{L}$, where $\mathcal{K}_f$ is an $\mathcal{A}$-subring of $\mathcal{K}$ such that

- $\mathcal{K}_f$ contains $\phi_{\mathcal{K}}(\mathcal{A})$ and its homomorphism $\phi_{\mathcal{K}_f} : \mathcal{A} \to \mathcal{K}_f$ agrees with $\phi_{\mathcal{K}}$,
- any element $x \in \mathcal{K}_f$ satisfying $x \notin \ker f$ is invertible in $\mathcal{K}_f$.

From this definition, it is clear that $\mathcal{K}_f$ is a local ring with the maximal ideal $\ker f$. Hence $\mathcal{K}_f / \ker f$ is a field, which is isomorphic to a subfield of $\mathcal{L}$, namely $\operatorname{im} f$. Therefore, if $\mathcal{L}$ is an epic field, we have $\mathcal{L} = \operatorname{im} f$. That is, any subhomomorphism to an epic field is surjective.

Now consider two subhomomorphisms from an $\mathcal{A}$-field $\mathcal{K}$ to another one $\mathcal{L}$. Then they are said to be *equivalent* if they agree on an $\mathcal{A}$-subring $\mathcal{K}_0$ of $\mathcal{K}$ such that the common restriction to $\mathcal{K}_0$ is again a subhomomorphism, which is called their *intersection*. This suggests the following definition.

Definition IV.2.3. A *specialization* from an $\mathcal{A}$-field $\mathcal{K}$ to another one $\mathcal{L}$ is an equivalence class of subhomomorphisms from $\mathcal{K}$ to $\mathcal{L}$. Moreover, the intersection of all subhomomorphisms defining a given specialization is called its *minimal* subhomomorphism.

With the help of specializations we can now clarify a universal property for epic $\mathcal{A}$-fields.

Definition IV.2.4. An epic $\mathcal{A}$-field $\mathcal{U}$ is called a *universal $\mathcal{A}$-field* if for any epic $\mathcal{A}$-field $\mathcal{K}$ there is a unique specialization $\mathcal{U} \to \mathcal{K}$. If $\mathcal{U}$ is in addition a field of fractions of $\mathcal{A}$, then we call $\mathcal{U}$ the *universal field of fractions of $\mathcal{A}$*.

In other words, the epic $\mathcal{A}$-fields and specializations form a category. An epic $\mathcal{A}$-field $\mathcal{U}$ is universal if it is an initial object in this category, i.e., for any other epic $\mathcal{A}$-field $\mathcal{K}$, the corresponding $\phi_{\mathcal{K}}$ factorizes through a specialization f from $\mathcal{U}$ to $\mathcal{K}$, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\phi_{\mathcal{U}}"] \arrow[dr, "\phi_{\mathcal{K}}"'] & \mathcal{U} \arrow[d, dotted, "f"] \\ & \mathcal{K}. \end{tikzcd}$$

For a specialization $f : \mathcal{U} \to \mathcal{K}$, we also use f to denote its minimal subhomomorphism, then $\ker f$ is the maximal ideal of $\mathcal{U}_f$ and $\mathcal{U}_f / \ker f$ is a field isomorphic to $\mathcal{K}$. Therefore, from a universal $\mathcal{A}$-field one can obtain any other epic $\mathcal{A}$-field by a specialization. By this universal property a universal $\mathcal{A}$-field, if it exists, is unique up to isomorphism.

Cohn provided an approach to construct the free field by generalizing the idea of *localization* to the non-commutative case (see [**Coh06**, Chapter 7] for details). Recall that, for a commutative unital ring $\mathcal{A}$ and a given set $S \subseteq \mathcal{A}$ which is closed under multiplication and contains 1, localization allows us to construct another ring $\mathcal{A}_S$ together with a homomorphism $\phi : \mathcal{A} \to \mathcal{A}_S$ such that all elements in the image $\phi(\mathcal{A})$ are invertible in $\mathcal{A}_S$. Cohn discovered that one can replace the set S by a set of matrices Σ over $\mathcal{A}$ and construct a *universal localization* $\mathcal{A}_\Sigma$, that is, a ring with a homomorphism $\phi : \mathcal{A} \to \mathcal{A}_\Sigma$ such that all elements in the image $\phi(\Sigma)$ are invertible in matrices over $\mathcal{A}_\Sigma$ and any other ring with such a homomorphism can be factorized through $\mathcal{A}_\Sigma$. Moreover, if we take the set Σ to be the set of all full matrices over $\mathcal{A}$ and if Σ satisfies some "multiplicative closure" property, then this universal localization $\mathcal{A}_\Sigma$ turns out to be the universal field of fractions of $\mathcal{A}$. This closure property is called *lower multiplicative*; see the paragraph before [**Coh06**, Proposition 7.1.1] for its definition. Actually, Cohn gives a list of characterizations for rings that can be embedded into universal fields of fractions. We select from this list the following items that are relevant for our purpose:

Theorem IV.2.5. *(See* [**DS78**] *or* [**Coh06**, Theorem 7.5.13]*) Let $\mathcal{A}$ be any non-zero ring. We denote by Σ the set of all full matrices over $\mathcal{A}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a Sylvester domain.*
(ii) *Σ is lower multiplicative.*
(iii) *The universal localization $\mathcal{A}_\Sigma$ is a field, necessarily the universal field of fractions of $\mathcal{A}$.*
(iv) *$\mathcal{A}$ has an inner rank preserving homomorphism to the universal field of fractions of $\mathcal{A}$.*

Recall that in Section III.2, the algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ was shown to be a Sylvester domain; see Example III.2.4 and the discussion before it. Therefore, $\mathbb{C}\langle x_1, \dots, x_d\rangle$ can be embedded into its universal field of fractions. We call this universal field of fractions the *free field* and denote it by $\mathbb{C}\,\langle\!\!\langle x_1, \dots, x_d \rangle\!\!\rangle$. Naturally, an element in $\mathbb{C}\,\langle\!\!\langle x_1, \dots, x_d \rangle\!\!\rangle$ is called a *rational function*. Moreover, for a matrix A over polynomials, we do not need to distinguish between its inner rank over polynomials and its inner rank over rational functions. This common inner rank is denoted by $\rho(A)$ for a given matrix A.

IV.3. Linear representations

The localization in the last section tells us that a full matrix A over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ is invertible as a matrix over $\mathbb{C}\,\langle\!\!\langle x_1, \dots, x_d \rangle\!\!\rangle$. So each entry in A^{-1} is a rational function. Therefore, for any row vector u and any column vector v over $\mathbb{C}$, $uA^{-1}v$ is a rational function since it is a linear combination of some rational functions. Actually, we add new elements more or less in this way to extend $\mathbb{C}\langle x_1, \dots, x_d\rangle$ to its universal localization, which turns out to be the free field $\mathbb{C}\,\langle\!\!\langle x_1, \dots, x_d \rangle\!\!\rangle$. So one may expect that the converse should be true, that is, any rational function r can be written in the form $r = uA^{-1}v$, with some full matrix A over polynomials and two scalar-valued vectors u and v. This expectation is indeed the case. Moreover, this matrix A can be chosen to be linear, though the dimension of A may increase for exchange. This culminates in the following definition borrowed from [**CR99**].

Definition IV.3.1. Let r be a rational function. A *representation* of a rational function r is a tuple $\rho = (u, A, v)$ consisting of a full matrix $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$, a row vector $u \in M_{1,n}(\mathbb{C})$ and a column vector $M_{n,1}(\mathbb{C})$ such that $r = uA^{-1}v$. Moreover, if the matrix A is linear (in the sense of Definition III.3.2), then we say ρ is a *linear representation* of r.

In [**CR99**], such linear representations were used to give an alternative construction of the free field. That indeed each element in the free field admits a linear representation is a direct consequence of the approach of [**CR99**]. But it also follows from the general theory presented in [**Coh06**]. The existence of a linear representation is an important feature for a rational function that will be used in Section V.2. So let us first we record the existence of linear representations for rational functions as a theorem.

Theorem IV.3.2. *Each rational function $r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle$ admits a linear representation in the sense of Definition IV.3.1.*

The idea of realizing rational non-commutative functions by inverses of linear matrices has been known for more than fifty years. It was rediscovered several times in many different mathematical realms, such as automaton theory and non-commutative rational series, as well as computer science and engineering. Under the name "linearization trick", it was introduced to the community of free probability by the work of Haagerup and Thorbjørnsen [**HT05**] and Haagerup, Schultz, and Thorbjørnsen [**HST06**], building on earlier operator space versions; for the latter see in particular the work of Pisier [**Pis18**].

We will not include a detailed proof here for the above theorem. Instead we will shift our discussion to similar representations for rational expressions. In this way, we will present an explicit algorithm to show how a linear matrix over non-commutative polynomials can be constructed step by step to represent a concrete rational expression.

Actually, for the special case of non-commutative polynomials, similar concepts were developed by Anderson [**And12, And13, And15**] and were used in [**BMS17**] in order to study evaluations of non-commutative polynomials in non-commutative random variables by means of operator-valued free probability theory. Later, in [**HMS18**], these methods were generalized to non-commutative rational expressions, based on a variant of Definition IV.3.1. We take this definition from [**HMS18**, Section 5], but with the sign changed for convenience.

Definition IV.3.3. Let r be a rational expression in variables $x_1, \ldots, x_d$. A *formal linear representation* $\rho = (u, A, v)$ of r consists of a linear matrix A over $\mathbb{C}\langle x_1, \ldots, x_d \rangle$, a row vector u and a column vector v over $\mathbb{C}$ such that for any unital algebra $\mathcal{A}$,

$$\mathrm{dom}_{\mathcal{A}}(r) \subseteq \{X \in \mathcal{A}^d \mid A(X) \text{ is invertible in } \mathcal{A}\}$$

and

$$r(X) = u(A(X))^{-1}v$$

for any tuple $X \in \mathrm{dom}_{\mathcal{A}}(r)$.

This definition asks that the inclusion of $\mathrm{dom}_{\mathcal{A}}(r)$ (see Definition IV.1.2) into the set $\{X \in \mathcal{A}^d \mid A(X) \text{ is invertible in } \mathcal{A}\}$ holds for arbitrary algebra $\mathcal{A}$, but drops the requirement on the fullness of matrix A, in comparison to Definition IV.3.1. So clearly these two representations are not the same objects. However, they have certain intrinsic connections which will be exposed in the following.

First, let us give the explicit algorithm that builds a formal linear representation for any rational expression. We refer to [**HMS18**] for the motivation and the background for the establishing of this algorithm.

ALGORITHM IV.3.4. *A formal linear representation $\rho = (u, A, v)$ of a rational expression r can be constructed by using successively the following rules:*

(i) *For scalars $\lambda \in \mathbb{C}$ and the variables x_j, $j = 1, \dots, d$, formal linear representations are given by*

$$\rho_\lambda := (\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} -\lambda & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}),$$

and

$$\rho_{x_j} := (\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} -x_j & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

(ii) *If $\rho_1 = (u_1, A_1, v_1)$ and $\rho_2 = (u_2, A_2, v_2)$ are two formal linear representations for rational expressions r_1 and r_2, respectively, then*

$$\rho_1 \oplus \rho_2 := (\begin{pmatrix} u_1 & u_2 \end{pmatrix}, \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix})$$

gives a formal linear representation of $r_1 + r_2$ and

$$\rho_1 \odot \rho_2 := (\begin{pmatrix} \mathbf{0} & u_1 \end{pmatrix}, \begin{pmatrix} -v_1 u_2 & A_1 \\ A_2 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ v_2 \end{pmatrix})$$

gives a formal linear representation of $r_1 \cdot r_2$.

(iii) *If $\rho = (u, A, v)$ is a formal linear representation for rational expression r, then*

$$\rho^{-1} := (\begin{pmatrix} 1 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} 0 & u \\ v & A \end{pmatrix}, \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix})$$

gives a formal linear representation of r^{-1}.

A detailed proof for showing that this algorithm indeed produces formal linear representations can be found in [**HMS18**, Section 5] or [**Mai17**, Chapter III]. These formal linear representations are closely related to linear representations as introduced in Definition IV.3.1. So this algorithm yields a perfect analogue of Theorem IV.3.2.

It may happen that the linear matrix A for a rational expression is not full, since degenerate rational expressions like 0^{-1} are allowed. However, for non-degenerate rational expressions, their formal linear representations automatically produce linear matrices A that are full. This is explained in [**Mai17**, Chapter III]. Therefore, one can recover the existence of linear representations for rational functions via formal linear representations of rational expressions.

IV.4. Evaluation of rational functions

Let $X = (X_1, \dots, X_d)$ be a tuple of elements in a unital algebra $\mathcal{A}$. Its evaluation map ev_X from $\mathbb{C}\langle x_1, \dots, x_d \rangle$ to $\mathcal{A}$ is well-defined as a homomorphism. We have also seen that the evaluation of rational expressions can be defined naturally with $\mathcal{A}$-domains considered in Definition IV.1.2. Then the question is: how can we define the evaluation for rational functions? Unfortunately, the evaluation cannot be well-defined for all algebras without additional assumptions. Here is an example which illustrates the problem: considering $\mathcal{A} = B(H)$ for some infinite dimensional separable Hilbert space, let l denote the one-sided left-shift operator. Then l^* is the right-shift operator and we have $l \cdot l^* = 1$ but $l^* \cdot l \neq 1$. It is clear that the evaluation of the rational expression $r(x,y) = y(xy)^{-1}x$ is $r(l,l^*) = l^*l \neq 1$. However, since this rational expression also represents the rational function 1, there is no consistent way to define its value for the arguments l and l^*. So it is natural to consider algebras in which a left inverse is also a right inverse to avoid such a problem. Actually, we require algebras to be stably finite in order to make sure that we have a well-defined evaluation.

THEOREM IV.4.1. *Let $\mathcal{A}$ be a stably finite algebra. Then for any rational function r in the free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle$, we have a well-defined $\mathcal{A}$-domain* $\mathrm{dom}_{\mathcal{A}}(r) \subseteq \mathcal{A}^d$ *and an evaluation $r(X)$ for any $X \in \mathrm{dom}_{\mathcal{A}}(r)$.*

Actually, the converse also holds in some sense, see in [**Coh06**, Theorem 7.8.3] (one should note that the terminology there is quite different from ours). When rational functions are treated as equivalence classes of rational expressions evaluated on matrices of all sizes, see also [**HMS18**, Theorem 6.1] for a proof of the same theorem. For the reader's convenience, here we give a proof with the help of representations of rational functions (introduced in Definition IV.3.1). Note we do not require the representations to be linear here.

DEFINITION IV.4.2. For a representation $\rho = (u, A, v)$ of a rational function, we define its $\mathcal{A}$-*domain*

$$\mathrm{dom}_{\mathcal{A}}(\rho) = \{X \in \mathcal{A}^d \mid A(X) \text{ is invertible as a matrix over } \mathcal{A}\}.$$

For a rational function r, we define its $\mathcal{A}$-*domain*

$$\mathrm{dom}_{\mathcal{A}}(r) = \bigcup_{\rho} \mathrm{dom}_{\mathcal{A}}(\rho),$$

where the union is taken over all possible representations of r. Then we define the *evaluation* of r at a tuple $X \in \mathrm{dom}_{\mathcal{A}}(r)$ by

$$\mathrm{Ev}_X(r) = r(X) = u(A(X))^{-1}v$$

for any representation $\rho = (u, A, v)$ satisfying $X \in \mathrm{dom}_{\mathcal{A}}(\rho)$.

Of course, as the choice of the representations for a rational function is not unique, we have to prove that different choices always give the same evaluation.

PROOF OF THEOREM IV.4.1. Let $\rho_1 = (u_1, A_1, v_1)$ and $\rho_2 = (u_2, A_2, v_2)$ be two representations of a rational function r such that

$$r = u_1 A_1^{-1} v_1 = u_2 A_2^{-1} v_2.$$

We need to prove that for any $X \in \mathrm{dom}_{\mathcal{A}}(\rho_1) \cap \mathrm{dom}_{\mathcal{A}}(\rho_2)$, we have $u_1(A_1(X))^{-1}v_1 = u_2(A_2(X))^{-1}v_2$. It is not difficult to verify that the tuple

$$\left(\begin{pmatrix} u_1 & u_2 \end{pmatrix}, \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & -A_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$$

is a representation of zero in the free field. Hence it suffices to prove that, for any representation $\rho = (u, A, v)$ of zero, we have $u(A(X))^{-1}v = 0$ for any $X \in \mathrm{dom}_{\mathcal{A}}(A)$. Now suppose that $u(A(X))^{-1}v \neq 0$ for some representation $\rho = (u, A, v)$ of the zero function. Then

$$\begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix} \in M_{n+1}(\mathcal{A})$$

has inner rank $n+1$ over $\mathcal{A}$ by Proposition III.1.6 as $\mathcal{A}$ is stably finite. However, we can apply the same proposition to show that

$$\begin{pmatrix} 0 & u \\ v & A \end{pmatrix} \in M_{n+1}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$$

has inner rank n since $uA^{-1}v = 0$ and $\mathbb{C}\langle x_1, \dots, x_d\rangle$ is stably finite. This gives an upper bound n for the inner rank of its evaluation by Item (v) of Remark III.1.3, because the evaluation of polynomials is always well-defined as a homomorphism. Hence we encounter a contradiction and therefore we conclude that $u(A(X))^{-1}v = 0$, as desired. □

We close this section by remarking that this definition of evaluation is consistent with the usual notion of evaluation. That is, given any polynomial p, in order to see that the above definition coincides with the usual one, we should find a representation $\rho = (u, A, v)$ such that $u(A(X))^{-1}v$ equals $p(X)$, the usual evaluation of p at X, for any $X \in \mathcal{A}^d$. Actually such a representation can be constructed by following the first two rules of Algorithm IV.3.4 for formal linear representations. Furthermore, from the last rules of this algorithm, we can also see that the arithmetic operations between rational functions give the corresponding arithmetic operations between their evaluations.

IV.5. Rational closures and division closures

In this section, we will introduce two constructions built by arithmetic operations (including taking inverses) from a given tuple of elements in an algebra. In general they are not necessarily division rings like the free field. But they are algebras that have some property concerning the invertibility. They are called rational closure and division closure. Basically, rational closure is given by the possible evaluations of rational functions at a given tuple. The division closure is the smallest algebra generated by a given tuple with respect to a closure property of inverses. Our purpose of introducing them is to provide the underlying algebra over which the inner rank can be taken when we investigate the strong Atiyah property in Section V.3.

Definition IV.5.1. Let $X = (X_1, \dots, X_d)$ be a tuple of elements in a unital algebra $\mathcal{A}$. We denote by Σ the set of matrices whose evaluations at X are invertible, i.e.,

$$\Sigma = \bigcup_{n=1}^{\infty} \{A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle) \mid A(X) \text{ is invertible in } M_n(\mathcal{A})\}.$$

The *rational closure* of X in $\mathcal{A}$ is the set of all entries of inverses of evaluations of Σ at X. We will denote this set by $\mathcal{A}_X$ in the following.

We want to point out that the rational closure can be defined for more general setting, see [**Coh06**, Section 7.1]. In comparison to the general setting, we limit our consideration to the case that homomorphisms are given by evaluation maps induced by tuples over $\mathcal{A}$.

It is because this setting is exactly what we need later in Section V.3. However, even in the general setting, we have the following lemma that tells us that the rational closure is always an algebra.

LEMMA IV.5.2. *(See* [**Coh06**, Proposition 7.1.1 and Theorem 7.1.2]*) The rational closure $\mathcal{A}_X$ is a subalgebra of $\mathcal{A}$ containing $\mathbb{C}\langle X_1, \ldots, X_d\rangle$, i.e., the image of $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ under the evaluation homomorphism.*

By definition, each element $r \in \mathcal{A}_X$ is an entry of $(A(X))^{-1}$ for some square matrix A over $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ such that $A(X)$ is invertible. Hence we can choose some scalar-valued row and column vectors u and v such that

$$r = u(A(X))^{-1}v.$$

For example, if r is the $(1,1)$-entry of $(A(X))^{-1}$, then we can choose $u = v^T = (1, 0, \ldots, 0)$ whose entries are all zero except the first one. Now if additionally $\mathcal{A}$ is stably finite, then we know that $A(X)$ is a full matrix over $\mathcal{A}$ (by Proposition III.1.6). It follows that A is a full matrix according to Item (v) of Remark III.1.3. Therefore, we can regard (u, A, v) as a representation of the rational function $uA^{-1}v$. Let us denote this rational function by f. So, according to Definition IV.4.2 and Theorem IV.4.1, we see that $r = f(X)$. That is, each element in $\mathcal{A}_X$ is given by the evaluation of some rational function. Conversely, if a rational function f has a well-defined evaluation $f(X)$, then there exist a full matrix A and scalar-valued row and column vectors u and v such that $A(X)$ is invertible and $f(X) = u(A(X))^{-1}v$. That is, we can write $f(X)$ as a linear combination of entries of $(A(X))^{-1}$. So $f(X)$ has to be in the rational closure $\mathcal{A}_X$ since $\mathcal{A}_X$ is an algebra (due to Lemma IV.5.2). In other words, the rational closure of X in $\mathcal{A}$ is the set of all well-defined evaluations of rational functions. We record this fact as the following remark.

REMARK IV.5.3. Let $X = (X_1, \ldots, X_d)$ be a tuple of elements in a stably finite algebra $\mathcal{A}$. Then we have

$$\mathcal{A}_X = \{r(X) \in \mathcal{A} \mid r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_d\rangle\!\rangle,\ r(X) \text{ is well defined}\}.$$

In general, for a given tuple X, the rational closure $\mathcal{A}_X$ may not be a division ring in general. But it has a property concerning inverses: if an element $r \in \mathcal{A}_X$ is invertible in $\mathcal{A}$, then $r^{-1} \in \mathcal{A}_X$. Actually, this can be seen from the last item in Algorithm IV.3.4: if $r = u(A(X))^{-1}v$ for some matrix A over $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ and scalar-valued row and column vectors u and v, then

$$r^{-1} = \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix},$$

where the invertiblity of matrix

$$\begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix}$$

follows from the invertiblity of $r = u(A(X))^{-1}v$ in $\mathcal{A}$ by the following well-known lemma on Schur complements.

LEMMA IV.5.4. *Suppose that $\mathcal{A}$ is a unital algebra. Let m, n be positive integers, $A \in M_m(\mathcal{A})$, $B \in M_{m\times n}(\mathcal{A})$, $C \in M_{n\times m}(\mathcal{A})$ and $D \in M_n(\mathcal{A})$ such that D is invertible.*

Then the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible in $M_{m+n}(\mathcal{A})$ *if and only if the Schur complement* $A - BD^{-1}C$ *is invertible in* $M_m(\mathcal{A})$.

A closely related notion is the smallest subalgebra satisfying this property, see [**Coh85**, Exercises 7.1.4] or [**Coh06**, Exercise 7.1.3].

Definition IV.5.5. Let $\mathcal{B}$ be a subalgebgra of a unital algebra $\mathcal{A}$.

(i) $\mathcal{B}$ is called *division closed* if for every $a \in \mathcal{B}$ that is invertible in $\mathcal{A}$ the inverse a^{-1} lies in $\mathcal{B}$.

(ii) The *division closure* (or *unit-closure*) of $\mathcal{B}$ in $\mathcal{A}$ is the smallest division closed subalgebra of $\mathcal{A}$ containing $\mathcal{B}$. We denote it by $\mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$.
In particular, we denote the division closure of $\mathbb{C}\langle X_1, \ldots, X_d\rangle$ by $\mathbb{C}\{\!\!\langle X_1, \ldots, X_d \rangle\!\!\}$ for a given tuple $X = (X_1, \ldots, X_d)$ over $\mathcal{A}$.

Remark IV.5.6. Let $\mathcal{A}$ be a unital algebra. We have the following remarks in order.

(i) Let $\mathcal{B}$ be a division closed subalgebra of $\mathcal{A}$. If every non-zero $a \in \mathcal{B}$ is invertible in $\mathcal{A}$, then $\mathcal{B}$ is a division ring.

(ii) Let $X = (X_1, \ldots, X_d)$ be a tuple of elements in $\mathcal{A}$. Then the rational closure $\mathcal{A}_X$ contains the division closure $\mathbb{C}\{\!\!\langle X_1, \ldots, X_d \rangle\!\!\}$ since $\mathcal{A}_X$ is division closed.

(iii) In particular, one can set $\mathcal{A} = \mathbb{C}\{\!\!\langle x_1 \ldots, x_d \rangle\!\!\}$ and choose $\mathcal{B} = \mathbb{C}\langle x_1 \ldots, x_d\rangle$. Naturally, one would expect that

$$\mathbb{C}\{\!\!\langle \mathbb{C}\langle x_1 \ldots, x_d\rangle \rangle\!\!\} = \mathbb{C}\{\!\!\langle x_1 \ldots, x_d \rangle\!\!\}.$$

In order to see this, note that $\mathbb{C}\{\!\!\langle \mathbb{C}\langle x_1 \ldots, x_d\rangle \rangle\!\!\}$ becomes a division ring according to the Item (i) of this remark. Then the result follows from the fact that $\mathbb{C}\{\!\!\langle x_1 \ldots, x_d \rangle\!\!\}$ is epic (see Definition IV.2.1).

(iv) Let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Then its division closure $\mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$ has a recursive structure as follows. We denote $\mathcal{R}_0(\mathcal{B}) := \mathcal{B}$ and set

$$\mathcal{R}_0^{-1}(\mathcal{B}) := \{a^{-1} \mid a \in \mathcal{R}_0(\mathcal{B}),\ a \text{ is invertible in } \mathcal{A}\}.$$

Clearly we have $\mathcal{R}_0^{-1}(\mathcal{B}) \subseteq \mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$. Next we define $\mathcal{R}_1(\mathcal{B})$ as the subalgebra of $\mathcal{A}$ generated by $\mathcal{R}_0(\mathcal{B})$ and $\mathcal{R}_0^{-1}(\mathcal{B})$, i.e., the smallest subalgebra of $\mathcal{A}$ containing $\mathcal{R}_0(\mathcal{B}) \cup \mathcal{R}_0^{-1}(\mathcal{B})$. Then $\mathcal{R}_1(\mathcal{B}) \subseteq \mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$ since $\mathcal{R}_0(\mathcal{B}) \cup \mathcal{R}_0^{-1}(\mathcal{B}) \subseteq \mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$. Applying this procedure iteratively yields a sequence $(\mathcal{R}_k(\mathcal{B}))_{k=0}^{\infty}$ of subalgebras of $\mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$ that each $\mathcal{R}_k(\mathcal{B})$ is generated by $\mathcal{R}_{k-1}(\mathcal{B})$ and $\mathcal{R}_{k-1}^{-1}(\mathcal{B})$. Therefore, we define

$$\mathcal{R}_\infty(\mathcal{B}) := \bigcup_{k=0}^{\infty} \mathcal{R}_k(\mathcal{B})$$

which also belongs to $\mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$. Actually, for any division closed subalgebra $\mathcal{D}$ of $\mathcal{A}$ containing $\mathcal{B}$, we have $\mathcal{R}_\infty(\mathcal{B}) \subseteq \mathcal{D}$. Moreover, $\mathcal{R}_\infty(\mathcal{B})$ is also division closed. It is because that for any $r \in \mathcal{R}_k(\mathcal{B})$ we have $r^{-1} \in \mathcal{R}_{k+1}(\mathcal{B})$ if it exists in $\mathcal{A}$. In conclusion, $\mathcal{R}_\infty(\mathcal{B})$ is the smallest division closed subalgebgra of $\mathcal{A}$ containing $\mathcal{B}$, namely, $\mathcal{R}_\infty(\mathcal{B}) = \mathbb{C}\{\!\!\langle \mathcal{B} \rangle\!\!\}$.

In general, the division closure may not equal to the rational closure. However, they could agree with each other in some cases. Here we provide a case that we will meet in Section V.3.

Proposition IV.5.7. *Let $X = (X_1, \dots, X_d)$ be a tuple of elements in $\mathcal{A}$. If the rational closure $\mathcal{A}_X$ or the division closure $\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$ is a division ring, then $\mathcal{A}_X = \mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$.*

Proof. If $\mathcal{A}_X$ is a division ring, then $\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$ is also a division ring due to Item (i) of Remark IV.5.6. Therefore, it suffices to show that if $\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$ is a division ring then $\mathcal{A}_X = \mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$. Let r be an element in $\mathcal{A}_X$. Then there exists a matrix A over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ such that $A(X)$ is invertible as matrix over $\mathcal{A}$ and r is an entry of $(A(X))^{-1}$. So our proposition will follow if $(A(X))^{-1}$ is a matrix over $\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$. Actually, we can prove that for any $n \in \mathbb{N}^+$, $M_n(\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle)$ is division closed in $M_n(\mathcal{A})$. Therefore, the proof will be completed by the following lemma. □

Lemma IV.5.8. *Let $\mathcal{D}$ be a subalgebra of $\mathcal{A}$. If $\mathcal{D}$ is a division ring, then $M_n(\mathcal{D})$ is division closed in $M_n(\mathcal{A})$ for any $n \in \mathbb{N}^+$.*

Proof. We are going to prove this by induction on the size n of matrices. First, the result clearly holds for $n = 1$ since $\mathcal{D}$ is division closed. Now we assume that the result is true for matrices of size $n - 1$. Let $A \in M_n(\mathcal{D})$ be invertible in $M_n(\mathcal{A})$. Then we can write

$$A = \begin{pmatrix} B & c \\ d & e \end{pmatrix},$$

where $B \in M_{n-1}(\mathcal{D})$ and c, d and e are of appropriate size such that $e \neq 0$. (We might need to apply row and column permutations to make sure that $e \neq 0$.) Note that e is invertible in $\mathcal{D}$ since $\mathcal{D}$ is a division ring. We see that $B - ce^{-1}d$ lies in $M_{n-1}(\mathcal{D})$. Moreover, $B - ce^{-1}d$ is invertible in $M_{n-1}(\mathcal{A})$ by Lemma IV.5.4 as A and e are invertible. By the induction hypothesis, it follows that $(B - ce^{-1}d)^{-1} \in M_{n-1}(\mathcal{D})$. Finally, from the equation

$$A^{-1} = \begin{pmatrix} \mathbf{1}_{n-1} & \mathbf{0} \\ -e^{-1}d & 1 \end{pmatrix} \begin{pmatrix} (B - ce^{-1}d)^{-1} & \mathbf{0} \\ \mathbf{0} & e^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n-1} & -ce^{-1} \\ \mathbf{0} & 1 \end{pmatrix}$$

we see that $A^{-1} \in M_n(\mathcal{D})$. This completes the proof. □

As a remark, we point out that the assumption in Proposition IV.5.7 can be weakened. For example, [**Rei06**, Note 13.16] says that if the division closure is a von Neumann regular ring then it coincides with the rational closure.

In Section V.3, we will answer the question when the rational closure $\mathcal{A}_X$ or division closure $\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$ becomes a division ring for a tuple X of non-commutative random variables. In that case, Proposition IV.5.7 tell us that we don't need to distinguish the rational closure and division closure.

CHAPTER V

Atiyah property

In [**Ati76**], Atiyah extended the Atiyah-Singer index theorem to some non-compact manifolds. He showed that using a dimension of a module over a von Neumann algebra can lead to a finite index though the kernel in question is in general infinite dimensional. This version is called the L^2-index theorem as square integrability is imposed to give some growth condition. He also asked whether some analytic L^2-Betti numbers are always rational numbers for certain Riemannian manifolds. An answer can be given by considering the corresponding *strong Atiyah conjecture* for certain groups. We refer the interested reader to [**Lüc02**, Chapter 10] or [**GLSŻ00, DLM$^+$03, PT11**] for its precise statement and relevant literature.

Our exposition of this chapter will convey a perspective from free probability theory, based on [**MSY18, MSY19**]. In the context of free probability, Shlyakhtenko and Skoufranis introduced the strong Atiyah property for $*$-algebras of bounded operators with tracial vector states. We follow their notion with some adaptation:

Definition V.0.1. ([**SS15**, Definition 2.1]) Let $X = (X_1, \dots, X_d)$ be a tuple of random variables from a tracial W^*-probability space $(\mathcal{M}, \varphi)$ and let $\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_d\rangle \to \mathcal{M}$ be the evaluation homomorphism. If for any $n \in \mathbb{N}^+$ and matrix $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$, we have

$$\mathrm{rank}(A(X)) \in \mathbb{N},$$

where $\mathrm{rank} : M_n(\mathcal{M}) \to [0, n]$ is a rank function that measures the size of the image of an element in $M_n(\mathcal{M})$ (see Definition I.5.10), then we say X has the *strong Atiyah property*.

Linnell proved in [**Lin93**] that the free group satisfies the strong Atiyah conjecture. In other words, a tuple of freely independent Haar unitary random variables has the strong Atiyah property in the sense of Definition V.0.1. This result was extended to more general random variables by Shlyakhtenko and Skoufranis in [**SS15**] in the context of free probability. That is, they proved that a tuple of freely independent random variables with non-atomic distributions has the strong Atiyah property.

These random variables turn out to have a stronger property which implies the strong Atiyah property. This property is, roughly speaking, the following: a tuple of freely independent random variables with non-atomic distributions can realize the free field in the $*$-algebra of affiliated operators. In fact, this property was known for free groups in the context of Atiyah conjecture. Linnell showed in [**Lin93**] that the free group can realize the free field in the $*$-algebra of affiliated operators. Restricted to the group case, there are no other groups that can realize the free field besides free groups. But when it comes to freely independent random variables with non-atomic distributions, there is a lot of freedom to choose different distributions (for example, semicircular distributions). However, the choices of distributions might make no differences for the purpose of realizing the free field. This is indeed the case that was confirmed by [**MSY18**]. Detailed discussion on the realization of the free field will be given in Section V.2. But instead of considering freely independent random variables with non-atomic distributions, we will consider random variables satisfying weaker properties. It will be explained in Section VI.1 why freely independent random variables with non-atomic distributions satisfy these properties.

In Section V.1, we will introduce a quantity Δ due to Connes and Shlyakhtenko . The purpose of their article [**CS05**] was to study the L^2-homology for von Neumann algebras. Along the way they found this quantity Δ that behaves very similarly to Voiculescu's free entropy dimension. Moreover, they also showed an inequality between Δ and some variant of the free entropy dimension. From this it can be deduced that freely independent random variables with non-atomic distributions maximize Δ (see Section VI.1 for details). In our investigation, we regard the maximality of Δ as a property that characterizes a tuple of random variables. Our goal of the first section in this chapter is to show that the maximality of Δ is equivalent to the triviality of kernels of linear full matrices.

In Section V.2, we will extend the list of equivalent properties in the first section to a much longer one. In particular, we will include the realization of the free field as one of those equivalent properties in our full list. In order to do that, two key ideas from Chapter III and Chapter IV are applied. First, the linear representation of rational functions will be used to transfer our results about the invertibility from matrices over polynomials to rational functions. Secondly, the diagonalization of matrices over polynomials by rational functions will be used to reveal an equality between the inner rank over over polynomials and the analytic rank over von Neumann algebra. This equality in turn implies that full matrices have trivial kernels.

In Section V.3, we will come back to examine the strong Atiyah property with ideas similar to Section V.2. Namely, we will show an analogue of our list of equivalent properties as in Theorem V.2.2. In particular, we will show that a tuple X has the strong Atiyah property if and only if for any matrix A over $\mathbb{C}\langle x_1,\dots,x_d\rangle$, $\operatorname{rank}(A(X))$ equals to the inner rank of $A(X)$ taken over the rational closure of X. This equality enforces $\operatorname{rank}(A(X))$ to be an integer for any matrix A, so it clearly implies the strong Atiyah property. Note that by definition the strong Atiyah property only asks $\operatorname{rank}(A(X))$ to take values in integers without specification to any quantity. We thus give an algebraic reason why the analytic rank in the strong Atiyah property only takes values in integers. Such an equality for the strong Atiyah property or strong Atiyah conjecture was not known until [**MSY18**] as far as the author's knowledge.

V.1. Maximality of Δ

This section is based on [**MSY19**, Section 3]. In comparison to [**MSY18**], it contains the major new idea that the entropy dimension used in [**MSY18**] can be replaced by the quantity Δ. With the help of Δ, new methods can be introduced (as we will see in this section) to get much stronger results than [**MSY18**].

We will first introduce the quantity $\Delta(X)$ for a tuple $X=(X_1,\dots,X_d)$ of random variables. Our goal is show that $\Delta(X)=d$ if and only if $\ker(A(X))=\{0\}$ for all linear full matrix A over $\mathbb{C}\langle x_1,\dots,x_d\rangle$. To achieve that purpose, we will need a third equivalent property for X. This property drops the notion of the fullness but instead addresses the linear dependence of vectors in kernels. We will actually prove that this third property is equivalent to the previous two properties respectively.

The main proof will be given as the second subsection. In the first subsection, we will present two key lemmas. Moreover, these two lemmas might be of independent interest since they provide explicit constructions that connect Δ and linear matrices with kernel vectors.

Now, let $(\mathcal{M},\varphi)$ be a tracial W^*-probability space and let $X=(X_1,\dots,X_d)$ be a tuple of random variables in $\mathcal{M}$. A quantity $\Delta(X)$ was introduced in [**CS05**] as

$$\text{(V.1)}\quad \Delta(X) := d - \dim_{M\overline{\otimes}M^{\mathrm{op}}} \overline{\left\{(T_1,\dots,T_d)\in(\mathcal{F}(L^2(\mathcal{M},\varphi)))^d \mid \sum_{k=1}^{d}[T_k, JX_kJ]=0\right\}}^{\mathrm{HS}}.$$

Here, we denote by $\mathcal{F}(L^2(\mathcal{M},\varphi))$ the ideal of all finite rank operators on $L^2(\mathcal{M},\varphi)$. *Tomita's conjugation operator* $J : L^2(\mathcal{M},\varphi)\to L^2(\mathcal{M},\varphi)$ is the conjugate-linear map that extends isometrically the conjugation $x\mapsto x^*$ on $\mathcal{M}$. The closure of these tuples of finite rank operators is taken with respect to the Hilbert-Schmidt norm. The dimension function used here is the dimension function introduced by Lück, see [**Lüc02**, Section 6.1].

Our main result is the following list of equivalent descriptions for the maximality of $\Delta(X)$ with the help of linear matrices $\mathbb{C}\langle x_1,\dots,x_d\rangle$.

Theorem V.1.1. *Let $(\mathcal{M},\varphi)$ be a tracial W^*-probability space. For a tuple $X=(X_1,\dots,X_d)$ of random variables in $\mathcal{M}$, the following are equivalent:*

(i) $\Delta(X)=d$.
(ii) *For any $n\in\mathbb{N}^+$ and non-zero linear matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, if $f\in\ker(A(X))$, $e\in\ker((A(X))^*)$, then either e or f has linearly dependent components.*
(iii) *For any $n\in\mathbb{N}^+$ and linear full matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $\ker(A(X))=\{0\}$.*

Remark V.1.2. Before giving the proof, the following remarks are in order.

(i) Suppose that $\Delta(X_1,\dots,X_d)<d$. By definition, this means that there is a tuple $(T_1,\dots,T_d)\neq(0,\dots,0)$ of finite rank operators in $B(L^2(\mathcal{M},\varphi))$ with the property that $\sum_{k=1}^{d}[T_k,JX_kJ]=0$. We infer from the latter that $(JT_1J,\dots,JT_dJ)$, which is again a non-trivial tuple of finite rank operators on $L^2(\mathcal{M},\varphi)$, satisfies $\sum_{k=1}^{d}[JT_kJ,X_k]=0$.
(ii) In Item (ii) of Theorem V.1.1, the random variable $(A(X))^*$ will be frequently rewritten as
$$(A(X))^*=A^*(X^*),$$
where $X^*:=(X_1^*,\dots,X_d^*)$ and A^* is understood in the sense of Item (iv) of Definition I.2.1. Namely, the $*$-structure on $\mathbb{C}\langle x_1,\dots,x_d\rangle$ is determined by $1^*=1$ and $x_k^*=x_k$ for $k=1,\dots,d$. Note that we usually use this $*$-structure on $\mathbb{C}\langle x_1,\dots,x_d\rangle$ if the tuple X consists of self-adjoint random variables. Of course here the random variables $X_1,\dots,X_d$ may not be self-adjoint. But we cannot use the evaluation homomorphism $\mathrm{ev}:\mathbb{C}\langle x_1,\dots,x_d,x_1^*\dots,x_d^*\rangle\to\mathcal{M}$ given in Item (v) of Definition I.2.1. This is because there might be relations between X_k and X_k^* for some k, for example, $X_kX_k^*=X_k^*X_k=1$. However,

$\mathbb{C}\langle x_1,\dots,x_d,x_1^*\dots,x_d^*\rangle$ has no algebraic relations for its indeterminates, so there may be a linear full matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d,x_1^*\dots,x_d^*\rangle)$ such that $\ker(A(X))$ is not trivial. In the next section, we will see that the above equivalent properties actually exclude any algebraic relation for $X_1,\dots,X_d$. But they do not exclude relations between X_k and X_k^*. Nevertheless, by taking the conjugation, the following can also be added to the above list of equivalent properties:

(a) $\Delta(X^*)=d$.

(b) For any $n\in\mathbb{N}^+$ and linear full matrix $A\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $\ker(A(X^*))=\{0\}$.

V.1.1. Two constructive lemmas. Before giving the proof of Theorem V.1.1, we single out the following two lemmas. These lemmas highlight an explicit way how we construct finite rank operators satisfying the commutator relation from linear matrices with vectors in kernels and vice versa.

LEMMA V.1.3. *Let $A=A^{(0)}+A^{(1)}x_1+\cdots+A^{(d)}x_d\in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$ be a linear matrix with vectors $f=(f_1,\dots,f_n)\in\ker(A(X))$ and $e=(e_1,\dots,e_n)\in\ker((A(X))^*)$. We define finite rank operators*

$$T_k:=\sum_{i,j=1}^{n}A_{ij}^{(k)}\langle\cdot,e_i\rangle f_j,\ \ k=0,\dots,d, \tag{V.2}$$

where $A_{ij}^{(k)}$, $i,j=1,\dots,n$ are entries of $A^{(k)}$ for $k=0,\dots,d$ and for $v,w\in L^2(\mathcal{M},\varphi)$, $\langle\cdot,v\rangle w$ denotes the 1-rank operator mapping any $u\in L^2(\mathcal{M},\varphi)$ to $\langle u,v\rangle w$. Then $T_1,\dots,T_d$ satisfy

$$\sum_{k=1}^{d}[T_k,X_k]=0.$$

PROOF. First, we write $A(X)f=0$ in entries, namely,

$$\sum_{j=1}^{n}A_{ij}^{(0)}f_j+\sum_{k=1}^{d}\sum_{j=1}^{n}A_{ij}^{(k)}X_kf_j=0,\ \ \forall i=1,\dots,n.$$

Then for each vector e_i, $i=1,\dots,n$, we build an equation of finite rank operators on $L^2(\mathcal{M},\varphi)$ from the above, that is,

$$\sum_{j=1}^{n}A_{ij}^{(0)}\langle\cdot,e_i\rangle f_j+\sum_{k=1}^{d}\sum_{j=1}^{n}A_{ij}^{(k)}\langle\cdot,e_i\rangle X_kf_j=0,\ \ \forall i=1,\dots,n.$$

Summing above equalities over the index i, we have

$$\sum_{i,j=1}^{n}A_{ij}^{(0)}\langle\cdot,e_i\rangle f_j+\sum_{k=1}^{d}\sum_{i,j=1}^{n}A_{ij}^{(k)}\langle\cdot,e_i\rangle X_kf_j=0,$$

or equivalently,

$$T_0+\sum_{k=1}^{d}X_kT_k=0. \tag{V.3}$$

Similarly, from $(A(X))^*e = 0$, i.e.,

$$\sum_{j=1}^{n} \overline{A_{ji}^{(0)}} e_j + \sum_{k=1}^{d} \sum_{j=1}^{n} \overline{A_{ji}^{(k)}} X_k^* e_j = 0, \ \forall i = 1, \ldots, n,$$

we have

$$\sum_{j=1}^{n} \left\langle \cdot, \overline{A_{ji}^{(0)}} e_j \right\rangle f_i + \sum_{k=1}^{d} \sum_{j=1}^{n} \left\langle \cdot, \overline{A_{ji}^{(k)}} X_k^* e_j \right\rangle f_i = 0, \ \forall i = 1, \ldots, n,$$

which yields

$$\sum_{i,j=1}^{n} A_{ji}^{(0)} \langle \cdot, e_j \rangle f_i + \sum_{k=1}^{d} \sum_{i,j=1}^{n} A_{ji}^{(k)} \langle X_k \cdot, e_j \rangle f_i = 0.$$

Actually, that is

$$T_0 + \sum_{k=1}^{d} T_k X_k = 0. \tag{V.4}$$

Therefore, combining (V.3) and (V.4), we conclude that $\sum_{k=1}^{d} [T_k, X_k] = 0$. □

LEMMA V.1.4. *Suppose that $X = (X_1, \ldots, X_d)$ is a tuple of random variables in $\mathcal{M}$ and $(T_1, \ldots, T_d)$ is a tuple of finite rank operators on $L^2(\mathcal{M}, \varphi)$ satisfying*

$$\sum_{k=1}^{d} [T_k, X_k] = 0. \tag{V.5}$$

Let $f = (f_1, \ldots, f_n)$ be an orthonormal family which spans the space of the sum of subspaces $\operatorname{im} T_k + \operatorname{im} T_k^*$, $k = 1, \ldots, d$. We write each T_k as

$$T_k = \sum_{i,j=1}^{n} A_{ij}^{(k)} \langle \cdot, f_i \rangle f_j, \ k = 1, \ldots, d. \tag{V.6}$$

Then the linear matrix

$$A := A^{(0)} - A^{(1)} x_1 + \cdots + A^{(d)} x_d, \tag{V.7}$$

where $A^{(k)} := (A_{ij}^{(k)})_{i,j=1}^{n} \in M_n(\mathbb{C})$, $k = 1, \ldots, d$,

$$B^{(k)} := (\langle X_k f_i, f_j \rangle)_{i,j=1}^{n}, \ k = 1 \ldots, d, \quad \text{and} \quad A^{(0)} := \sum_{k=1}^{d} B^{(k)} A^{(k)},$$

satisfies

$$A(X) f = 0 \quad \text{and} \quad A(X)^* f = 0.$$

PROOF. Substituting the form of T_k, $k = 1, \ldots, d$ in (V.6) into the relation (V.5), we have

$$\sum_{k=1}^{d} \sum_{i,j=1}^{n} A_{ij}^{(k)} \langle X_k \cdot, f_i \rangle f_j = \sum_{k=1}^{d} \sum_{i,j=1}^{n} A_{ij}^{(k)} \langle \cdot, f_i \rangle X_k f_j.$$

Then applying the operators on both sides of the above equation to vectors f_p, $p = 1, \ldots, n$, we obtain

$$\begin{aligned}
\sum_{k=1}^{d} \sum_{i,j=1}^{n} A_{ij}^{(k)} \langle X_k f_p, f_i \rangle f_j &= \sum_{k=1}^{d} \sum_{i,j=1}^{n} A_{ij}^{(k)} \langle f_p, f_i \rangle X_k f_j \\
\sum_{k=1}^{d} \sum_{i,j=1}^{n} B_{pi}^{(k)} A_{ij}^{(k)} f_j &= \sum_{k=1}^{d} \sum_{j}^{n} A_{pj}^{(k)} X_k f_j \\
\sum_{j=1}^{n} A_{pj}^{(0)} f_j &= \sum_{k=1}^{d} \sum_{j}^{n} A_{pj}^{(k)} X_k f_j \qquad \text{(V.8)}
\end{aligned}$$

for $p = 1, \ldots, n$. That is, $A(X)f = 0$, as desired.

Moreover, by taking inner products of both sides of (V.8) with vectors f_q, $q = 1, \cdots, n$, we obtain

$$\begin{aligned}
\sum_{j=1}^{n} A_{pj}^{(0)} \langle f_j, f_q \rangle &= \sum_{k=1}^{d} \sum_{j}^{n} A_{pj}^{(k)} \langle X_k f_j, f_q \rangle \\
A_{pq}^{(0)} &= \sum_{k=1}^{d} \sum_{j}^{n} A_{pj}^{(k)} B_{jq}^{(k)} \qquad \text{(V.9)}
\end{aligned}$$

for $p, q = 1, \ldots, n$. That actually says $\sum_{k=1}^{d} B^{(k)} A^{(k)} = A^{(0)} = \sum_{k=1}^{d} A^{(k)} B^{(k)}$.

Finally, we want to verify the remaining part, i.e., $(A(X))^* f = 0$. For that purpose, we consider

$$\sum_{k=1}^{d} X_k^* T_k^* = \sum_{k=1}^{d} T_k^* X_k^*,$$

which comes from taking conjugation of (V.5). We replace T_k^* in the above equation by

$$T_k^* = \sum_{i,j=1}^{n} \overline{A_{ij}^{(k)}} \langle \cdot, f_j \rangle f_i,$$

for $k = 1, \ldots, d$, then we have

$$\sum_{k=1}^{d} \sum_{i,j=1}^{n} \overline{A_{ij}^{(k)}} \langle \cdot, f_j \rangle X_k^* f_i = \sum_{k=1}^{d} \sum_{i,j=1}^{n} \overline{A_{ij}^{(k)}} \langle X_k^* \cdot, f_j \rangle f_i.$$

Furthermore, for $l = 1, \ldots, n$ the evaluation of both sides of the above equation at f_l yields

$$\sum_{k=1}^{d} \sum_{i,j=1}^{n} \overline{A_{ij}^{(k)}} \langle f_l, f_j \rangle X_k^* f_i = \sum_{k=1}^{d} \sum_{i,j=1}^{n} \overline{A_{ij}^{(k)}} \langle X_k^* f_l, f_j \rangle f_i$$
$$\sum_{k=1}^{d} \sum_{i=1}^{n} \overline{A_{il}^{(k)}} X_k^* f_i = \sum_{k=1}^{d} \sum_{i,j=1}^{n} \overline{A_{ij}^{(k)} B_{jl}^{(k)}} f_i$$
$$\sum_{k=1}^{d} \sum_{i=1}^{n} \overline{A_{il}^{(k)}} X_k^* f_i = \sum_{i=1}^{n} \overline{A_{il}^{(0)}} f_i,$$

where (V.9) is used to arrive at the third line from the second one. The above equation is exactly $(A(X))^* f = 0$, as desired. ☐

V.1.2. Proof of Theorem V.1.1.

Proof. The proof is organized as follows: first we will show the equivalence between (i) and (ii), then the equivalence between (ii) and (iii).

Now we begin to prove (i)$\Longrightarrow$(ii). Let $A \in M_n(\mathbb{C}\langle x_1, \ldots, x_d\rangle)$ be a non-zero linear matrix with vectors $f \in \ker(A(X))$, $e \in \ker(A(X))^*$ such that both e and f have linearly independent components. In order to give a contradiction, we are going to prove $A = 0$. By Lemma V.1.3, finite rank operators T_k, $k = 1, \cdots, d$ defined as in (V.2) satisfy the relation $\sum_{k=1}^{d} [T_k, X_k] = 0$. Hence, according to Item (i), i.e., $\Delta(X) = d$, we conclude that $T_k = 0$ for each $k = 1, \ldots, d$. In particular, we have

$$T_k(e_p) = \sum_{i,j=1}^{n} A_{ij}^{(k)} \langle e_p, e_i \rangle f_j = 0, \;\; p = 1, \ldots, n,$$

and furthermore,

$$\langle T_k(e_p), f_q \rangle = \sum_{i,j=1}^{n} \langle e_p, e_i \rangle A_{ij}^{(k)} \langle f_j, f_q \rangle = 0, \;\; p, q = 1, \ldots, n,$$

for each $k = 1, \ldots, d$. Denoting by $E := (\langle e_p, e_i \rangle)_{p,i=1}^{n}$ and $F := (\langle f_j, f_q \rangle)_{j,q=1}^{n}$ the Gram matrices of e and f, we write the above equation as

$$EA^{(k)}F = 0,$$

for each $k = 1, \ldots, d$. Since both e and f consist of linearly independent components, E and F are invertible. Therefore, $A^{(k)} = 0$, $k = 1, \ldots, d$. It reduces $A(X)f = 0$ to $A^{(0)} f = 0$. However, as f is a linearly independent family, we must have $A^{(0)} = 0$. This completes the proof for the part (i)$\Longrightarrow$(ii).

Next, we want to show (ii)$\Longrightarrow$(i). In order to prove $\Delta(X) = d$, let $(T_1, \ldots, T_d)$ be a tuple of finite rank operators on $L^2(\mathcal{M}, \varphi)$ satisfying $\sum_{k=1}^{d} [T_k, X_k] = 0$. Then by Lemma V.1.4, the linear matrix A defined as in (V.7) satisfies $A(X)f = 0$ and $(A(X))^* f = 0$, where f is an orthonormal family spanning the space of the sum of $\operatorname{im} T_k + \operatorname{im} T_k^*$, $k = 1, \ldots, d$. Hence we have $A = 0$ according to Item (ii) because the components of f are linearly independent. This enforces immediately that $T_k = 0$, $k = 1\dots, d$ from (V.6). Therefore, we conclude that $\Delta(X) = d$.

Now, we want to prove (ii)$\Longrightarrow$(iii). We proceed by induction on the matrix size n. First, we prove the result by contradiction when $n = 1$. Suppose $f \in L^2(\mathcal{M}, \varphi)$ is a non-zero vector such $A(X)f = 0$. Then there also exists a non-zero vector $e \in L^2(\mathcal{M}, \varphi)$ such that $(A(X))^*e = 0$ by Lemma I.5.8. Therefore, $A = 0 \in \mathbb{C}$ according to Item (ii). This is a contradiction since A is full, i.e., $A \neq 0$.

In the following, we assume that the result holds for the size $n-1$ and we want to prove it for the size n. Let A be a linear full matrix over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ such that $A(X)f = 0$ for some vector $f \neq 0$ in $(L^2(\mathcal{M}, \varphi))^n$. Then the components of f have to be linearly independent by the following reasoning. If these components are not linearly independent, then we can find an invertible matrix $U \in M_n(\mathbb{C})$ such that

$$Uf = \begin{pmatrix} f' \\ 0 \end{pmatrix},$$

where $f' \neq 0$ in $(L^2(\mathcal{M}, \varphi))^{n-1}$. Putting $AU^{-1} = \begin{pmatrix} B & b \end{pmatrix}$ in the corresponding block structure, we have $B(X)f' = 0$. Moreover, B is a full matrix as AU^{-1} is full. Then we can further choose $n-1$ rows of B to form a full matrix $B' \in M_{n-1}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ by Theoerem III.2.3. Hence, we have $B'(X)f' = 0$, which yields $f' = 0$ by the induction hypothesis. This is a contradiction since $f' \neq 0$. Thus the components of f have to be linearly independent.

For $(A(X))^*$, by Lemma I.5.8, there also exists a vector $e \neq 0$ in $(L^2(\mathcal{M}, \varphi))^n$ such that

$$(A(X))^*e = A^*(X^*)e = 0.$$

If the components of e are not linearly independent, then by a similar argument as the case $A(X)f = 0$, we can construct a linear full matrix $B' \in M_{n-1}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ out of A^* such that $B'(X^*)$ has a non-trivial kernel. Hence $(B'(X^*))^* = B'^*(X)$ has a non-trivial kernel. However, since B'^* is a linear full matrix of dimension $n-1$, the kernel of $B'^*(X)$ is trivial by the induction hypothesis. This yields a contradiction. Thus we see that e also has linearly independent components.

Therefore, $A = 0$ follows from Item (ii) as we have seen that the vectors e and f have independent components. So we arrive at a contradiction with the fullness of A. This complete our induction argument.

Finally, we want to show (iii)$\Longrightarrow$(ii). Suppose that $n \geq 1$, $A \in M_n(\mathbb{C}\langle x_1 \dots, x_d\rangle)$ is a linear matrix such that $f \in \ker(A(X))$ and $e \in \ker((A(X))^*)$ and both have linearly independent components. Our goal is to prove $A = 0$, so we assume $\rho(A) > 0$ in order to obtain some contradiction. First, we note that A is not full, otherwise $\ker(A(X)) = \{0\}$ according to Item (iii). This is a contradiction since $f \in \ker(A(X))$ has linearly independent components. Therefore, we may additionally assume that $\rho(A) < n$. By Theorem III.3.3, there are invertible matrices U and V in $M_n(\mathbb{C})$ such that

$$UAV = \begin{pmatrix} B & \mathbf{0} \\ C_1 & C_2 \end{pmatrix}$$

where $B \in M_{n-s,\rho(A)-s}(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ has inner rank $\rho(B) = \rho(A) - s$.

If $s < \rho(A)$, i.e., the block B does not disappear in the above form of UAV, then let us write

$$V^{-1}f = \begin{pmatrix} f' \\ f'' \end{pmatrix},$$

where $f' \in (L^2(\mathcal{M}, \varphi))^{\rho(A)-s}$. Clearly we have $B(X)f' = 0$ and we consider this equation rather than $A(X)f = 0$. By consulting Theorem III.2.3, we can choose $\rho(A) - s$ rows of B to form a linear full matrix $B' \in M_{\rho(A)-s}(\mathbb{C}\langle x_1, \ldots, x_d\rangle)$. It then follows that $B'(X)f' = 0$. Thus $f' = 0$ according to Item (iii). However, this is impossible since f has linearly independent components and $V \in M_n(\mathbb{C})$ is invertible.

So it remains to deal with the case that $s = \rho(A)$. In this case, Theorem III.3.3 actually says that

$$UAV = \begin{pmatrix} \mathbf{0} \\ C_2 \end{pmatrix},$$

where $C_2 \in M_{\rho(A),n}(\mathbb{C}\langle x_1, \ldots, x_d\rangle)$. From $(A(X))^*e = 0$, we see that

$$(UAV)^*(X^*)(U^*)^{-1}e = V^*A^*(X^*)U^*(U^*)^{-1}e = V^*(A(X))^*e = 0.$$

That is,

$$\begin{pmatrix} \mathbf{0} & C_2^*(X^*) \end{pmatrix} (U^*)^{-1}e = 0.$$

Similarly, we can build out of C_2^* a linear full matrix which has non-trivial kernel. This gives a contradiction as desired.

Therefore, we rule out the possibility that $\rho(A) \neq 0$ when there are vectors e and f that have linearly independent components. This completes the proof for the last part of our theorem.

□

V.2. Realization of the free field

In this section, we will extend the list of equivalent properties in Theorem V.1.1. Actually, we will give another list of equivalent properties for a tuple of random variables. Of course these two lists have a common item. The reason that we separate these two lists is that they rely on very different ideas. We have seen that Theorem V.1.1 relies on our two constructive lemmas to associate finite rank operators to linear matrices with vectors in kernels. However, we will not need to deal with these notions in this section any more. Instead, a more algebraic idea will dominate the proof of the coming theorem. In other words, we have delved into questions on the vectors in $L^2(\mathcal{M}, \varphi)$ in the last section, but from now on we will instead consider objects in $L^0(\mathcal{M}, \varphi)$. Recall that $L^0(\mathcal{M}, \varphi)$ is the $*$-algebra of all unbounded random variables, see Section I.5. The first important fact is that we can evaluate rational functions on this algebra $L^0(\mathcal{M}, \varphi)$.

As we want to evaluate rational functions on $L^0(\mathcal{M}, \varphi)$, we have to show that $L^0(\mathcal{M}, \varphi)$ is stably finite. (Note that we always require that φ is a trace to consider $L^0(\mathcal{M}, \varphi)$.) This fact is probably well-known to experts. We refer the interested reader to [**HMS18, MSY19**] for a proof. We simply record it as the following remark.

Remark V.2.1. $L^0(\mathcal{M}, \varphi)$ is stably finite and thus the evaluation of rational functions as in Definition IV.4.2 is well-defined on $L^0(\mathcal{M}, \varphi)$.

Another important fact about $L^0(\mathcal{M},\varphi)$ is that we can rephrase the triviality of kernels by the invertibility in $L^0(\mathcal{M},\varphi)$ (see Lemma I.5.9). With this rephrasing, we will see that many algebraic tools become available. Moreover, the invertibility is also characterized by the maximality of the analytic rank function given in Definition I.5.10. Recall that Lemma I.5.11 says that any $A \in M_n(L^0(\mathcal{M},\varphi))$ is invertible if and only if $\operatorname{rank}(A) = n$. Then Item (iii) in Theorem V.1.1 can be rephrased as follows:

For any $n \in \mathbb{N}^+$ and linear matrix $A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, if $\rho(A) = n$, then $\operatorname{rank}(A(X)) = n$.

Compare this to the strong Atiyah property (Definition V.0.1) which requires that for any matrix $A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$ we have $\operatorname{rank}(A(X)) \in \mathbb{N}$. One may ask whether we could further have $\operatorname{rank}(A(X)) = \rho(A)$, which in particular implies that $\operatorname{rank}(A(X)) \in \mathbb{N}$. We will show that this equality is actually an equivalent property in our list.

Now we have all the ingredients to state our theorem. Note that the first item is exactly Item (iii) in Theorem V.1.1 (with the triviality of kernels rephrased by the invertibility).

THEOREM V.2.2. *Let $(\mathcal{M},\varphi)$ be a tracial W^*-probability space and $L^0(\mathcal{M},\varphi)$ the $*$-algebra of unbounded random variables. For a given tuple $X = (X_1,\dots,X_d)$ in $\mathcal{M}^n$, we consider the evaluation homomorphism $\operatorname{ev}_X : \mathbb{C}\langle x_1,\dots,x_d\rangle \to L^0(\mathcal{M},\varphi)$. Then the following properties for X are equivalent.*

(i) *For any $n \in \mathbb{N}^+$ and linear full matrix $A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $A(X)$ is invertible in $M_n(L^0(\mathcal{M},\varphi))$.*
(ii) *For any $n \in \mathbb{N}^+$ and full matrix $A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $A(X)$ is invertible in $M_n(L^0(\mathcal{M},\varphi))$.*
(iii) *For any $n \in \mathbb{N}^+$ and matrix $A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$, $\operatorname{rank}(A(X)) = \rho(A)$.*
(iv) *For any $r \in \mathbb{C}\langle\!\!\langle x_1,\dots,x_n\rangle\!\!\rangle$, we have $X \in \operatorname{dom}_{L^0(\mathcal{M},\varphi)}(r)$. Moreover, the evaluation map Ev_X as introduced in Definition IV.4.2 becomes an injective homomorphism $\operatorname{Ev}_X : \mathbb{C}\langle\!\!\langle x_1,\dots,x_n\rangle\!\!\rangle \to L^0(\mathcal{M},\varphi)$ that extends the evaluation homomorphism $\operatorname{ev}_X : \mathbb{C}\langle x_1,\dots,x_d\rangle \to L^0(\mathcal{M},\varphi)$.*

Furthermore, if these equivalent properties are satisfied for a tuple X, then the inner rank stays invariant under the evaluation homomorphism, that is,

(V.10)
$$\rho(A) = \rho_{\mathcal{M}}(A(X)) = \rho_{L^0(\mathcal{M},\varphi)}(A(X)) \qquad \text{for all } n \in \mathbb{N}^+,\ A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle),$$

where $\rho_{\mathcal{M}}$ (respectively $\rho_{L^0(\mathcal{M},\varphi)}$) denotes the inner rank taken over the algebra $\mathcal{M}$ (respectively $L^0(\mathcal{M},\varphi)$).

PROOF. It's easy to see that (ii) $\Longrightarrow$ (i). With Lemma I.5.11 in mind, (iii) $\Longrightarrow$ (ii) follows immediately. We will show (iv) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iv) in the following.

First, (iv) $\Longrightarrow$ (iii). Suppose that $A \in M_n(\mathbb{C}\langle x_1,\dots,x_d\rangle)$ has inner rank $\rho(A) = r$. By Proposition III.1.8, there exist two invertible matrices U and V over the free field $\mathbb{C}(\langle\!\!\langle x_1,\dots,x_n\rangle\!\!\rangle)$ such that

$$UAV = \begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

According to Item (iv), the extended evaluation $\operatorname{Ev}_X : \mathbb{C}\langle\!\!\langle x_1,\dots,x_n\rangle\!\!\rangle \to L^0(\mathcal{M},\varphi)$, as a homomorphism, implies that $U(X), V(X)$ are invertible. With the help of Lemma I.5.12, we see that

$$\operatorname{rank}(A(X)) = \operatorname{rank}(U(X)A(X)V(X)) = \operatorname{rank}\begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = r.$$

Now, (i) $\Longrightarrow$ (iv). First, recall that by Definition IV.4.2 a rational function r in $\mathbb{C}\,\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle$ satisfies $X \in \operatorname{dom}_{L^0(\mathcal{M},\varphi)}(r)$ if there is a linear representation $\rho = (u, A, v)$ of r such that $X \in \operatorname{dom}_{L^0(\mathcal{M},\varphi)}(\rho)$, i.e., $A(X)$ is invertible. In fact, each linear representation of r (whose existence is guaranteed by Theorem IV.3.2) satisfies this condition due to Item (i) as A is full. Therefore, according to Definition IV.4.2 and Theorem IV.4.1, the evaluation $\operatorname{Ev}_X(r)$ is well-defined for each $r \in \mathbb{C}\,\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle$. So it induces a map $\operatorname{Ev}_X : \mathbb{C}\,\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle \to L^0(\mathcal{M}, \varphi)$. Moreover, we can infer from the proof of Theorem IV.4.1 that this evaluation of rational functions respects the arithmetic operations on rational functions. Hence the evaluation $\operatorname{Ev}_X : \mathbb{C}\,\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle \to L^0(\mathcal{M}, \varphi)$ is a homomorphism which agrees with ev_X on $\mathbb{C}\langle x_1, \ldots, x_d \rangle$. Finally, Ev_X has to be injective as a homomorphism defined on a skew field.

Finally, we want to show that the inner rank stays invariant under the evaluation homomorphism if these equivalent properties are satisfied for a tuple X. Actually, the desired equality (V.10) follows from Item (iii) combined with the inequality (V.11) in the following remark. $\square$

Remark V.2.3. Let X be a tuple of random variables in a tracial W^*-probability space $(\mathcal{M}, \varphi)$. For any matrix A over $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ we have

$$\operatorname{rank}(A(X)) \leq \rho_{L^0(\mathcal{M},\varphi)}(A(X)) \leq \rho_{\mathcal{M}}(A(X)) \leq \rho(A). \tag{V.11}$$

The latter two inequalities follow from Item (v) in Remark III.1.3. We only need to show the first one, i.e., $\operatorname{rank}(A(X)) \leq \rho_{L^0(\mathcal{M},\varphi)}(A(X))$.

Let A be a matrix in $M_n(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$. Suppose $A(X) = PQ$ is a rank factorization of $A(X)$ over $L^0(\mathcal{M}, \varphi)$, where $P \in M_{n,r}(L^0(\mathcal{M}, \varphi))$, $Q \in M_{r,n}(L^0(\mathcal{M}, \varphi))$ and r stands for the inner rank $\rho_{L^0(\mathcal{M},\varphi)}(A(X))$. Then we rewrite this rank factorization as $A(X) = \hat{P}\hat{Q}$ with the square matrices $\hat{P}, \hat{Q} \in M_n(L^0(\mathcal{M}, \varphi))$ that are defined by

$$\hat{P} := \begin{pmatrix} P & \mathbf{0} \end{pmatrix} \qquad \text{and} \qquad \hat{Q} := \begin{pmatrix} Q \\ \mathbf{0} \end{pmatrix},$$

where the zero blocks are chosen to be of appropriate sizes. We see that $(\operatorname{Tr}_n \circ \varphi^{(n)})(p_{\ker(\hat{P})}) \geq n - r$ due to the zero block structure of $\hat{P}$. So we have $\operatorname{rank}(\hat{P}) \leq r$ (recall Equation (I.2)). Finally it follows that

$$\operatorname{rank}(A(X)) \leq \operatorname{rank}(\hat{P}) \leq r,$$

since $\operatorname{im}(A(X)) \subseteq \operatorname{im}(\hat{P})$.

Remark V.2.4. Recall that we have to use the $*$-structure on $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ determined by $1^* = 1$ and $x_k^* = x_k$ for $k = 1, \ldots, d$ (see Item (ii) of Remark V.1.2). It can be extended uniquely to a $*$-structure on the free field $\mathbb{C}\,\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle$ with the help of linear representations. However, $\operatorname{Ev}_X : \mathbb{C}\langle x_1, \ldots, x_d \rangle \to L^0(\mathcal{M}, \varphi)$ will not be a $*$-homomorphism in general. But if the tuple X consists of self-adjoint random variables, then Ev_X is automatically a $*$-homomorphism.

Remark V.2.5. Let us address the commutative counterpart of Theorem V.2.2 in this remark. Suppose that X is a tuple of commuting operators. Clearly, X do not satisfy any of the equivalent properties in Theorem V.2.2. But statements and the proof of Theorem V.2.2 can be adapted to this commutative setting. That is, instead of $\mathbb{C}\langle x_1, \dots, x_d\rangle$ we should evaluate the algebra $\mathbb{C}[x_1, \dots, x_d]$ of commutative polynomials at X. Correspondingly, we should use the field $\mathbb{C}(x_1, \dots, x_d)$ of commutative rational functions to diagonalize matrices over $\mathbb{C}[x_1, \dots, x_d]$. Moreover, recall that a rational function can be written as the quotient of two polynomials in the commutative case. We can see that the well-definedness of the evaluation of rational functions at X is reduced to the invertibility of the evaluation of polynomials at X. Therefore, we have the following analogue of Theorem V.2.2 for commuting operators.

For a given tuple $X = (X_1, \dots, X_d)$ of commuting operators in a tracial W^*-probability space $(\mathcal{M}, \varphi)$, the following statements are equivalent.

(i) For any non-zero polynomial $p \in \mathbb{C}[x_1, \dots, x_d]$ we have that $p(X)$ is invertible in the $*$-algebra $L^0(\mathcal{M}, \varphi)$.
(ii) For any $n \in \mathbb{N}$ and $A \in M_n(\mathbb{C}[x_1, \dots, x_d])$ we have: $\operatorname{rank}(A(X)) = \rho(A)$, where $\rho(A)$ denotes the inner rank of A over $\mathbb{C}(x_1, \dots, x_d)$.
(iii) The evaluation homomorphism $\mathrm{ev}_X : \mathbb{C}[x_1, \dots, x_d] \to L^0(\mathcal{M}, \varphi)$ can be extended to a homomorphism defined on $\mathbb{C}(x_1, \dots, x_d)$.

There are two significant differences between the commutative and the noncommutative case. First, in order to generate the field $\mathbb{C}(x_1, \dots, x_d)$ by a tuple X of commuting operators, it is enough to ask all non-zero polynomials to be invertible after the evaluation at X. Whereas, in the noncommutative case, in Item (i) and (ii) of Theorem V.2.2, we ask all full matrices over noncommutative polynomials to be invertible after the evaluation. Secondly, the inner rank ρ in Item (ii) of the above list cannot be taken over polynomials as in Item (iii) of Theorem V.2.2. It has to be taken over $\mathbb{C}(x_1, \dots, x_d)$ that enables the diagonalization. Recall that if $d > 2$, $\mathbb{C}[x_1, \dots, x_d]$ is not a Sylvester domain (see Remark III.2.5). So the inner rank over $\mathbb{C}[x_1, \dots, x_d]$ may not equal the inner rank over $\mathbb{C}(x_1, \dots, x_d)$ (see Theorem IV.2.5).

A natural question inspired by Remark V.2.5 is whether those equivalent properties in Theorem V.2.2 are equivalent to the invertibility of evaluations of all non-zero (noncommutative) polynomials. This question has a negative answer, provided by Example V.3.2. In this example, we will see a tuple X which has no polynomial relations but has a rational relation. Moreover, X has the strong Atiyah property so that $p(X)$ is invertible for any non-zero polynomial (since $\operatorname{rank}(p(X)) \in \{0, 1\}$).

Let us conclude our excursion on the realization of the free field by the following remark on the rational closure and division closure of X in $L^0(\mathcal{M}, \varphi)$. For their definitions, see Section IV.5.

Remark V.2.6. Let X be a tuple of random variables satisfying the equivalent properties in Theorem V.2.2 and V.1.1. The Item (iv) in Theorem V.2.2 tells us that the image $\mathrm{Ev}(\mathbb{C}\langle\!\langle x_1, \dots, x_d\rangle\!\rangle)$ is a division ring containing $\mathbb{C}\langle X_1, \dots, X_d\rangle$ and sitting inside $L^0(\mathcal{M}, \varphi)$. Remark IV.5.3 then claims that the rational closure $L^0(\mathcal{M}, \varphi)_X$ deduced by X is exactly the division ring $\mathrm{Ev}(\mathbb{C}\langle\!\langle x_1, \dots, x_d\rangle\!\rangle)$. Moreover, in this case, the division closure $\mathbb{C}\langle\!\langle X_1, \dots, X_d\rangle\!\rangle$ actually equals $L^0(\mathcal{M}, \varphi)_X$, by Proposition IV.5.7, and thus also equals $\mathrm{Ev}(\mathbb{C}\langle\!\!\langle x_1,\dots,x_d\rangle\!\!\rangle)$. Hence we have

$$\mathbb{C}\langle\!\!\langle x_1,\dots,x_d\rangle\!\!\rangle \cong \mathbb{C}\langle\!\!\langle X_1,\dots,X_d\rangle\!\!\rangle$$

where the isomorphism is given by the evaluation homomorphism Ev. So we see that X realizes the free field in $L^0(\mathcal{M},\varphi)$. It was first noticed by Linnell that the division closure $\mathbb{C}\langle\!\!\langle X_1,\dots,X_d\rangle\!\!\rangle$ can realize the free field. In [**Lin93**], he showed this for the case of free groups. In Chapter VI, with the help of tools from free probability, we will answer the question when X can maximize $\Delta(X)$ and thus realize the free field. In particular, we will see that freely independent random variables with non-atomic distributions maximize Δ. This in particular recovers the result of Linnell for the case of free groups.

V.3. Strong Atiyah property

In this section, we will investigate the strong Atiyah property (see Definition V.0.1) with the similar ideas as in the last section.

One may notice that in (iii) of Theorem V.2.2 the equality $\mathrm{rank}(A(X)) = \rho(A)$ says that in particular that $\mathrm{rank}(A(X))$ is an integer, since $\rho(A)$ is an integer by definition. Therefore, for a tuple X of random variables, each of these equivalent properties in Theorem V.2.2 and V.1.1 implies that X satisfies the strong Atiyah property. In other words, each of these properties is a special case of the strong Atiyah property.

So it is natural to ask the question what is the gap between these properties in Theorem V.2.2 and V.1.1 and the strong Atiyah property? Answering this question is our main task of this section. Our answer consists of two parts. First, we will provide examples to demonstrate the difference between them. Secondly, we will show an analogue of Theorem V.2.2 for the strong Atiyah property. Namely, we will derive a list of equivalent properties for the strong Atiyah property based on the ideas used in Theorem V.2.2. Comparing this list with the list in Theorem V.2.2 also provides an answer to our question.

Example V.3.1. Clearly, if a tuple, say (X,Y), has a polynomial relation between its components X and Y, then (X,Y) cannot realize the free field. Namely, if there exists a non-zero polynomial $p\in\mathbb{C}\langle x,y\rangle$ such that $p(X,Y)=0$, then Ev_X cannot be defined on some rational functions, such as p^{-1}.

For example, we can take X and Y as two classical random variables in a probability space $(L^\infty(\Omega,\mathbb{P}),\mathbb{E})$. Then they have a polynomial relation $XY-YX=0$. However, if X and Y are independent random variables with non-atomic distributions, then they satisfy the strong Atiyah property (see [**SS15**, Lemma 2.3]).

A more interesting example[1] is the following one. We will see in this example that a tuple might have a rational relation though it has no polynomial relations. Moreover, we will also see that the rank equality in (iii) in Theorem V.2.2 breaks down for this tuple. However, one can still see that the rank function *rank* takes values in $\mathbb{N}$ due to some algebraic reason.

Example V.3.2. Let X and Y be two freely independent semicircular random variables. By the results in [**SS15**] (or by Theorem V.1.1 and V.2.2), they satisfy the strong Atiyah property. Let

$$A=Y^2,\ B=YXY,\ C=YX^2Y.$$

[1]We thank Ken Dykema and James Pascoe for providing us this example.

Then A, B, C also have the strong Atiyah property as any polynomial in them can be reduced to a polynomial in X and Y. Moreover, they do not satisfy any non-trivial polynomial relation. But they do have a rational relation:

$$BA^{-1}B - C = 0$$

in $L^0(\mathcal{M}, \varphi)$. Definitely they do not satisfy the last property in Theorem V.2.2. Furthermore, we see that a matrix like

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

has inner rank 2 if it is regarded as a matrix of three formal variables. While it has

$$\operatorname{rank}\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \operatorname{rank}\begin{pmatrix} Y^2 & YXY \\ YXY & YX^2Y \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & X \\ X & X^2 \end{pmatrix} = \rho\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} = 1.$$

(We use the equality between rank and ρ for X, Y since (X, Y) satisfy Theorem V.2.2 as we will see in Section VI.1.) Therefore, (A, B, C) violates all the properties in Theorem V.2.2 though it has the strong Atiyah property.

Now here comes the promised list of equivalent properties for the strong Atiyah property.

Theorem V.3.3. *Let $(\mathcal{M}, \varphi)$ be a tracial W^*-probability space and $L^0(\mathcal{M}, \varphi)$ the $*$-algebra of unbounded random variables. For a given tuple $X = (X_1, \ldots, X_d)$ in $\mathcal{M}^n$, we consider the evaluation homomorphism $\mathrm{ev}_X : \mathbb{C}\langle x_1, \ldots, x_d\rangle \to L^0(\mathcal{M}, \varphi)$. For succinctness, we denote by $\mathcal{R}$ the rational closure $L^0(\mathcal{M}, \varphi)_X$ of X in $L^0(\mathcal{M}, \varphi)$. So the inner rank over $\mathcal{R}$ is denoted by $\rho_{\mathcal{R}}$. Then the following properties for X are equivalent.*

(i) *For any $n \in \mathbb{N}^+$ and $A \in M_n(\mathbb{C}\langle x_1, \ldots, x_d\rangle)$, if $A(X)$ is full over $\mathcal{R}$, then $A(X) \in M_n(L^0(\mathcal{M}, \varphi))$ is invertible in $\mathcal{M}_n(L^0(\mathcal{M}, \varphi))$.*

(ii) *For any $n \in \mathbb{N}^+$ and $A \in M_n(\mathbb{C}\langle x_1, \ldots, x_d\rangle)$, $\operatorname{rank}(A(X)) = \rho_{\mathcal{R}}(A(X))$.*

(iii) *The rational closure $\mathcal{R}$ is a division ring.*

(iv) *The division closure $\mathbb{C}\langle\!\langle X_1, \ldots, X_d\rangle\!\rangle$ of X in $L^0(\mathcal{M}, \varphi)$ is a division ring.*

(v) *X has the strong Atiyah property, i.e., for any $n \in \mathbb{N}^+$ and $A \in M_n(\mathbb{C}\langle x_1, \ldots, x_d\rangle)$, $\operatorname{rank}(A(X)) \in \mathbb{N}$.*

Furthermore, if these equivalent properties are satisfied for a tuple X, then the inner rank stays invariant when passing from the rational closure $\mathcal{R}$ to a much larger algebra $L^0(\mathcal{M}, \varphi)$, that is,

$$\rho_{\mathcal{R}}(A(X)) = \rho_{L^0(\mathcal{M},\varphi)}(A(X)) \qquad \text{for all } n \in \mathbb{N}^+,\ A \in M_n(\mathbb{C}\langle x_1, \ldots, x_d\rangle), \tag{V.12}$$

where $\rho_{L^0(\mathcal{M},\varphi)}$ denotes the inner rank taken over $L^0(\mathcal{M}, \varphi)$.

Proof. (ii) $\Longrightarrow$ (i) follows from Lemma I.5.11. The equivalence (iii)$\Longleftrightarrow$(iv) is due to Proposition IV.5.7. In the following we will show that (iii) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii). This gives us a proof for the equivalence of the first four items. The last one, the strong Atiyah property clearly follows from Item (ii). Then we will show (v) $\Longrightarrow$ (iii) to complete the proof for the equivalence of all items. Note that the inequality (V.11) in Remark V.2.3 holds for arbitrary tuples X over $\mathcal{M}$. Moreover, since $\mathcal{R}$ is a subalgebra of $L^0(\mathcal{M}, \varphi)$, we see that

$$\operatorname{rank}(A(X)) \leq \rho_{L^0(\mathcal{M},\varphi)}(A(X)) \leq \rho_{\mathcal{R}}(A(X))$$

for all $n \in \mathbb{N}^+$ and $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$. So the inner rank equality V.12 follows from this inequality combined with the equality in (ii).

Now we begin to show that (iii) $\Longrightarrow$ (ii). Let A be a matrix in $M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ for some $n \in \mathbb{N}^+$. We assume that the evaluation $A(X)$ has inner rank r over $\mathcal{R}$. By Proposition III.1.8, there exist two invertible matrices U and V over the division ring $\mathcal{R}$ such that

$$UA(X)V = \begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then we have

$$\operatorname{rank}(A(X)) = \operatorname{rank}(UA(X)V) = \operatorname{rank} \begin{pmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = r$$

with the help of Lemma I.5.12.

Next, we want to show that (i) $\Longrightarrow$ (iii). Let $r \in \mathcal{R}$ be a non-zero element. Then there exist $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$, $u \in M_{1,n}(\mathbb{C})$, $v \in M_{n,1}(\mathbb{C})$ such that $A(X)$ is invertible in $M_n(L^0(\mathcal{M}, \varphi))$ and $r = u(A(X))^{-1}v$. By consulting Schur's Lemma (Lemma IV.5.4), it suffices to show that

$$B := \begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix}$$

is invertible in $M_{n+1}(L^0(\mathcal{M}, \varphi))$ in order to show $r = u(A(X))^{-1}v$ is invertible in $L^0(\mathcal{M}, \varphi)$. Therefore, according to Item (i), it boils down to showing that the matrix A is full over $\mathcal{R}$. For that purpose, we apply Proposition III.1.6 ($\mathcal{R}$ is stably finite because it is a subalgebra of the stably finite algebra $L^0(\mathcal{M}, \varphi)$) to B. Then we see that $A(X)$ is full over $\mathcal{R}$ due to $-u(A(X))^{-1}v = -r \neq 0$. So we have seen that any non-zero $r \in \mathcal{R}$ is invertible in $L^0(\mathcal{M}, \varphi)$. Recalling that the rational closure $\mathcal{R}$ is division closed, we conclude $\mathcal{R}$ is a division ring.

Finally, we want to prove (v) $\Longrightarrow$ (iii). Let $r \in \mathcal{R}$ be a non-zero element. Then there exist $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$, $u \in M_{1,n}(\mathbb{C})$, $v \in M_{n,1}(\mathbb{C})$ such that $A(X)$ is invertible in $M_n(L^0(\mathcal{M}, \varphi))$ and $r = u(A(X))^{-1}v$. We consider the following factorization

$$\begin{pmatrix} -r & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix} = \begin{pmatrix} 1 & -u(A(X))^{-1} \\ \mathbf{0} & (A(X))^{-1} \end{pmatrix} \begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ -(A(X))^{-1}v & \mathbf{1}_n \end{pmatrix}.$$

By Lemma I.5.12 we see that

$$\operatorname{rank} \begin{pmatrix} -r & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix}$$

since they only differ by invertible matrices over $L^0(\mathcal{M}, \varphi)$. According to Item (v), we know that

$$\operatorname{rank} \begin{pmatrix} 0 & u \\ v & A(X) \end{pmatrix} \in \mathbb{N}.$$

Combining this with the fact that

$$\operatorname{rank} \begin{pmatrix} -r & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix} = \operatorname{rank} r + n,$$

we obtain $\operatorname{rank}(r) \in \{0, 1\}$. Then, as $r \neq 0$, we see that $\operatorname{rank}(r) = 1$. Thus r is invertible according to Lemma I.5.11. Hence $\mathcal{R}$ is a division ring as $\mathcal{R}$ is division closed. $\square$

Remark V.3.4. The equivalence between the strong Atiyah property and (iii) or (iv) is known for the group case in the context of the strong Atiyah conjecture. For example, see [**Lüc02**, Lemma 10.39].

CHAPTER VI

Atoms and zero divisors

In Section I.3 we have introduced the basics on atoms and zero divisors for non-commutative random variables. We have seen that atoms can be defined for a probability measure on $\mathbb{C}$ (see Definition I.3.1) and thus for the analytic distribution (see Definition I.2.4) of a normal random variable. Moreover, zero divisors (see Lemma I.3.7) and the invertiblilty (see Lemma I.5.9) provide an algebraic way to detect atoms for random variables. We will continue the investigation on atoms and zero divisors in context of free probability in this chapter. There are several results in this direction which were established before. For example, in [**CS16, MSW17**], it was shown that for any non-constant polynomial $p \in \mathbb{C}\langle x_1, \ldots, x_d\rangle$ its evaluation $p(X)$ cannot be a zero divisor for a tuple X of self-adjoint random variables that has maximal non-microstates free entropy dimension δ^*. This is a generalization of a previous result in [**SS15**] that showed the absence of atoms for non-constant polynomials in freely independent random variables with non-atomic analytic distributions.

Following the idea of [**MSW17**], it was shown in [**MSY18**] that the absence of zero divisors actually holds for all non-constant rational functions in random variables with maximal free entropy dimension δ^*. This extension was proven with the help of Theorem V.2.2 as well as an adaptation of the non-commutative derivative idea in [**MSW17**] to the matricial level. Another possible extension is to further weaken the assumption on a d-tuple X of self-adjoint random variables. A candidate was suggested by the inequality

$$\delta^*(X) \leq \Delta(X) \leq d$$

that was proven in [**CS05**]. This inequality tells us that in particular the maximality of Δ follows from the maximality of δ^*. So it is fesible that the absence of zero divisors for non-constant polynomials or rational functions in X can be implied from the maximality of Δ. For the polynomial case, such a result was also conjectured as a corollary of [**CS16**, Conjecture 5]. As we have already seen in Chapter V, the absence of zero divisors for non-constant rational functions in random variables is actually equivalent to the maximality of Δ according to Theorem V.1.1 and Theorem V.2.2.

In conclusion, the equivalence of Item (i) in Theorem V.1.1 and Item (iv) in Theorem V.2.2 is a vast generalization of the results from [**CS16, MSW17**] on the absence of zero divisors for non-commutative random variables. In particular, as we will see in Section VI.1, this implies that all non-constant rational functions in freely independent semicircular random variables (defined in Example I.4.6) are not zero divisors. Moreover, the result holds for freely independent Haar unitary random variables (defined in Example I.4.3). But since these random variables are not self-adjoint, we will introduce a notion, called dual system, to see that they have maximal Δ.

In Section VI.2, we will consider the zero divisors for the matrices over polynomials, or equivalently, polynomials with matrix coefficients. It can be regarded as a kind of generalization of results in [**CS16, MSW17**] but one cannot expect the absence of zero divisors any more. Results in this direction can also be found in [**SS15**]. Comparing to [**SS15**, Theorem 1.1] which addresses the allowed size for atoms of matrices in freely independent random variables with information on the size of atoms of each variable, we can draw more precise information on atoms or zero divisors for matrices with the help of Item (iii) in Theorem V.2.2. Actually, this information can be extracted by a pure algebraic calculation on central eigenvalues and inner ranks once we know the random variables in question satisfy Item (iii) in Theorem V.2.2. Moreover, the equality in Item (iii) in Theorem V.2.2 also allows us to prove results the other way around. For example, Proposition III.4.2 claims that every $n \times n$ matrix over $\mathbb{C}\langle x_1, \dots x_d\rangle$ has at most n central eigenvalues. We will see that this purely algebraic result can be derived from its counterpart over von Neumann algebras.

VI.1. Absence of rational relations and zero divisors

In this section, we will introduce two criteria, provided by free probabitilty theory, for the maximality of Δ. The first one is the maximality of free entropy dimension δ^*. We will see that it in particular applies to a tuple of freely independent semicircular random variables. The second one is the existence of the a dual system, which applies to a tuple of freely independent Haar unitary random variables.

VI.1.1. Non-microstates free entropy dimension. In order to define the non-microstates free entropy dimension δ^*, we need several notions as follows.

Definition VI.1.1. Let $\mathbb{C}\langle x_1, \dots, x_d\rangle$ be the algebra of non-commutative polynomials. We endow the tensor product $\mathbb{C}\langle x_1, \dots, x_d\rangle \otimes \mathbb{C}\langle x_1, \dots, x_d\rangle$ with a $\mathbb{C}\langle x_1, \dots, x_d\rangle$-bimodule structure with the left and right multiplications given by

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2) \quad \text{for all } p_1, p_2, q_1, q_2 \in \mathbb{C}\langle x_1, \dots, x_d\rangle.$$

Then we define the *partial non-commutative derivatives* ∂_i ($i = 1, \dots, d$) as linear mappings

$$\partial_i : \mathbb{C}\langle x_1, \dots, x_d\rangle \to \mathbb{C}\langle x_1, \dots, x_d\rangle \otimes \mathbb{C}\langle x_1, \dots, x_d\rangle$$

determined by

$$\partial_i 1 = 0, \quad \partial_i x_j = \delta_{ij} 1 \otimes 1 \quad \text{for } j = 1, \dots, d$$

and by the Leibniz rule

$$\partial_i(pq) = (\partial_i p) \cdot q + p \cdot (\partial_i q) \quad \text{for all } p, q \in \mathbb{C}\langle x_1, \dots, x_d\rangle.$$

Definition VI.1.2. Let $X = (X_1, \dots, X_d)$ be a tuple of self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M}, \varphi)$. We say $\xi_1, \dots, \xi_d \in L^2(\mathcal{M}, \varphi)$ satisfy the *conjugate relations* for $X_1, \dots, X_d$ if

$$\varphi\big(\xi_j P(X_1, \dots, X_d)\big) = (\varphi \otimes \varphi)\big((\partial_j P)(X_1, \dots, X_d)\big)$$

holds for all $p \in \mathbb{C}\langle x_1, \dots, x_d\rangle$ and $j = 1, \dots, d$. If additionally $\xi_1, \dots, \xi_d$ belong to $L^2(X_1, \dots, X_d, \varphi)$, then we call $(\xi_1, \dots, \xi_d)$ a *conjugate system* for $(X_1, \dots, X_d)$.

A conjugate system, in case of its existence, is automatically unique. We refer the interested reader to [**Voi98**] for more discussion on conjugate variables.

Let us remark here, as an example of the existence of conjugate systems, that its existence for freely independent semicircular random variables can be deduced from the *Schwinger-Dyson equation* (see [**Gui09**, Chapter 8]). Namely, the unique solution to

$$\varphi\big(X_j P(X_1,\dots,X_d)\big) = (\varphi\otimes\varphi)\big((\partial_j P)(X_1,\dots,X_d)\big)$$

is actually given by the freely independent semicircular random variables. In other words, a tuple $S=(S_1,\dots,S_d)$ of freely independent semicircular random variables has S as its own conjugate system.

Definition VI.1.3. The *(non-microstates) free Fisher information* of $X=(X_1,\dots,X_d)$ is defined by

$$\Phi^*(X_1,\dots,X_d) := \sum_{i=1}^{d} \|\xi_j\|_2^2$$

if a conjugate system $(\xi_1,\dots,\xi_d)$ for $(X_1,\dots,X_d)$ exists and $\Phi^*(X_1,\dots,X_d):=\infty$ if there is no conjugate system for $(X_1,\dots,X_d)$.

Similar to the fact that classical Fisher information is additive with respect to independent random variables, the free Fisher information was also proven to be additive with respect to freely independent random variables.

Remark VI.1.4. Let $X=(X_1,\dots,X_d)$ and $Y=(Y_1,\dots,Y_s)$ be two tuples of self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M},\varphi)$.

(i) We have

$$\Phi^*(X_1,\dots,X_d,Y_1,\dots,Y_s) \geq \Phi^*(X_1,\dots,X_d)+\Phi^*(Y_1,\dots,Y_s).$$

(ii) Suppose that $\Phi^*(X_1,\dots,X_d)+\Phi^*(Y_1,\dots,Y_s)<\infty$. Then the tuple X is freely independent from Y if and only if

$$\Phi^*(X_1,\dots,X_d,Y_1,\dots,Y_s) = \Phi^*(X_1,\dots,X_d)+\Phi^*(Y_1,\dots,Y_s).$$

Now we can define a free entropy via the free Fisher information for a tuple of self-adjoint random variables.

Definition VI.1.5. (See [**Voi98**]) Let $X=(X_1,\dots,X_d)$ be a tuple of self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M},\varphi)$. Its *non-microstates free entropy* is defined by

$$\chi^*(X_1,\dots,X_d) := \frac{1}{2}\int_0^\infty \Big(\frac{d}{1+t} - \Phi^*(X_1+\sqrt{t}S_1,\dots,X_d+\sqrt{t}S_d)\Big)\,dt + \frac{d}{2}\log(2\pi e),$$

where $S_1,\dots,S_d$ are freely independent semi-circular random variables that are freely independent from $X_1,\dots,X_d$.

Definition VI.1.6. Let $X=(X_1,\dots,X_d)$ be a tuple of self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M},\varphi)$. Its *non-microstates free entropy dimension* is defined by

$$\delta^*(X) := d - \liminf_{\varepsilon\searrow 0} \frac{\chi^*(X_1+\sqrt{\varepsilon}S_1,\dots,X_d+\sqrt{\varepsilon}S_d)}{\log(\sqrt{\varepsilon})}.$$

In the case $d = 1$, we know that

$$\delta^*(X) = 1 - \sum_{t\in\mathbb{R}} \mu_X(\{t\})^2, \tag{VI.1}$$

where μ_X is the analytic distribution of X. In particular, $\delta^*(X) = 1$ if and only if μ_X has no atoms.

A variant of $\delta^*(X)$ defined by

$$\delta^\star(X) := d - \liminf_{\varepsilon\searrow 0} \varepsilon\Phi^*(X_1 + \sqrt{\varepsilon}S_1, \ldots, X_d + \sqrt{\varepsilon}S_d),$$

was introduced in [**CS05**] and was shown to satisfy

$$\delta^*(X) \le \delta^\star(X).$$

Moreover, the inequality was also extended to the quantity $\Delta(X)$ (see (V.1) for its definition) as follows:

$$\delta^*(X) \le \delta^\star(X) \le \Delta(X) \le d.$$

Actually, we have a long chain of implications as follows:

$$\begin{aligned} \Phi^*(X_1, \ldots, X_d) < \infty &\implies \chi^*(X_1, \ldots, X_d) > -\infty \\ &\implies \delta^*(X) = d \\ &\implies \delta^\star(X) = d \\ &\implies \Delta(X) = d, \end{aligned}$$

where the weakest assumption on the tuple X is $\Delta(X) = d$.

In particular, since $\Phi^*(S_1, \ldots, S_d) = d\|S_1\|_2^2 = d < \infty$ for a tuple $S = (S_1, \ldots, S_d)$ of freely independent semicircular random variables, we conclude that $\Delta(S) = d$.

VI.1.2. Dual systems. Dual systems were introduced by Voiculescu in [**Voi98**], and appeared also in [**CS05**] as an important technical tool for getting the maximality of the free entropy dimension.

Definition VI.1.7. Let $X = (X_1, \ldots, X_d)$ be a tuple of random variables in a tracial W^*-probability space $(\mathcal{M}, \varphi)$. A *dual system* to $(X_1, \ldots, X_d)$ is a tuple $(D_1, \ldots, D_d)$ of operators in $B(L^2(\mathcal{M}, \varphi))$ such that

$$[X_i, D_j] = \delta_{ij} P_\Omega, \tag{VI.2}$$

where $P_\Omega \in B(L^2(\mathcal{M}, \varphi))$ is the orthogonal projection onto the subspace $\mathbb{C}1$ in $L^2(\mathcal{M}, \varphi)$.

Proposition VI.1.8. *Let $(X_1, \ldots, X_d)$ be a tuple of operators in a tracial W^*-probability space $(\mathcal{M}, \varphi)$. If $X_1, \ldots, X_d$ has a dual system, then $\Delta(X_1, \ldots, X_d) = d$.*

We refer the interested reader to [**CS05**, Theorem 4.11] and [**MSY19**, Proposition 6.2] for a proof. Now we want to apply it to a tuple of freely independent Haar unitary random variables. Namely, we need to construct a tuple of bounded operator in $B(l^2(\mathbb{F}_d))$ that are satisfying (VI.2). These operator actually are actually given by the left transductions on $\mathbb{F}_d$. Let $g_1, \ldots, g_d$ be the generators of the free group $\mathbb{F}_d$. For each $i = 1, \ldots, d$, the *left transduction* of g_i is map $L_i : \mathbb{F}_d \to \mathbb{F}_d$ defined by

$$L_i w := \begin{cases} wg_i^{-1}, & \text{if } w \text{ is a reduced word ending with } g_i, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following result, see [**MSY19**, Proposition 6.3] for a proof.

PROPOSITION VI.1.9. *Let $U_1 = \lambda(g_1), \ldots, U_d = \lambda(g_d)$ be as defined in Example I.4.3. For each $i = 1, \ldots, d$ let $D_i \in B(l^2(\mathbb{F}_d))$ be the bounded operator extended from the left transduction L_i on $\mathbb{F}_d$. Then $(D_1, \ldots, D_d)$ is a dual system to $(U_1, \ldots, U_d)$.*

VI.1.3. Absence of rational relations and zero divisors. Finally, let us spell out the consequences of the maximality of Δ on atoms and zero divisors. These consequences are in particular satisfied for freely independent semicircular or Haar unitary random variables.

COROLLARY VI.1.10. *Let $X = (X_1, \ldots, X_d)$ be a tuple of random variables in a tracial W^*-probability space $(\mathcal{M}, \varphi)$. If $\Delta(X) = d$, then for any non-zero rational function $r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, $r(X)$ is well-defined as an unbounded random variable in $L^0(\mathcal{M}, \varphi)$. Moreover, $r(X)$ is invertible in $L^0(\mathcal{M}, \varphi)$.*

Note that for a non-zero rational function r, we have $r(X) \neq 0$ since $r^{-1}(X)$ is well-defined and $r^{-1}(X)r(X) = (r^{-1}r)(X) = 1$. So X cannot satisfy any rational relation. Moreover, for each non-constant rational function r such that $r(X)$ is normal, its analytic distribution $\mu_{r(X)}$ has no atoms. This is because for any $\lambda \in \mathbb{C}$, $\lambda - r \neq 0$ and thus $\lambda - r(X)$ is invertible according to Corollary VI.1.10.

VI.2. Zero divisor for matrices with polynomial entries

In this section we will investigate the zero divisors for square matrices in random variables with maximal Δ. To be more precise, we will study the point spectrum (see Definition I.5.4) of $A(X)$, i.e., the set of all $\lambda \in \mathbb{C}$ for which $\ker(\lambda - A(X)) \neq \{0\}$, where A is a square matrix over $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ and X is a d-tuple of random variables maximizing Δ. If $A(X)$ is normal, this will allow us to locate the position and to specify the size of atoms of the analytic distribution $\mu_{A(X)}$ by purely algebraic quantities associated to A.

Let $(\mathcal{M}, \varphi)$ be a tracial W^*-probability space. We denote the point spectrum of each $A \in M_n(L^0(\mathcal{M}, \varphi))$ by

$$\sigma_p(A) := \big\{ \lambda \in \mathbb{C} \,\big|\, \operatorname{rank}(\lambda - A) < N \big\}.$$

Then for a tuple X with maximal Δ, we have the following corollary that says that the point spectrum of $A(X)$ agrees with the set $\sigma^{\text{full}}(A)$ of central eigenvalues (see Definition III.4.1).

COROLLARY VI.2.1. *Suppose that $X = (X_1, \ldots, X_d)$ is a tuple of elements in a tracial W^*-probability space $(\mathcal{M}, \varphi)$ that satisfies $\Delta(X) = d$. Then for any $A \in M_n(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$,*

$$\sigma^{\text{full}}(A) = \sigma_p(A(X)).$$

Moreover, we have that

$$\rho(\lambda - A) = \operatorname{rank}(\lambda - A(X)) \quad \textit{for all } \lambda \in \mathbb{C}, \tag{VI.3}$$

where ρ stands for the inner rank over $\mathbb{C}\langle x_1, \dots, x_d\rangle$.

Note that in general, for a matrix $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ and any tuple X of random variables in $\mathcal{M}$, we have

$$\sigma^{\text{full}}(A) \subseteq \sigma^{\text{full}}_{L^0(\mathcal{M},\varphi)}(A(X)) \subseteq \sigma_p(A(X))$$

where $\sigma^{\text{full}}_{L^0(\mathcal{M},\varphi)}(A(X))$ is the set of all central eigenvalues over $L^0(\mathcal{M},\varphi)$ (see Definition III.4.1). These inclusions follow from the inequalities $\operatorname{rank}(A(X)) \leq \rho_{L^0(\mathcal{M},\varphi)}(A(X)) \leq \rho(P)$; see (V.11). So Corollary VI.2.1 provides a criterion for the equality of all these spectra and rank functions.

Moreover, if $A(X)$ is normal, then the conclusion on the atoms of the analytic distribution $\mu_{A(X)}$ can be drawn as follows.

Corollary VI.2.2. *Let $X = (X_1, \dots, X_d)$ be a tuple of elements in a tracial W^*-probability space $(\mathcal{M}, \varphi)$ such that $\Delta(X) = d$. Let $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$ be given such that the random variable $A(X)$ is normal. Then the analytic distribution $\mu_{A(X)}$ has atoms precisely at the points in $\sigma^{\text{full}}(A)$. In fact, we have that*

$$\mu_{A(X)}(\{\lambda\}) = \frac{1}{n}\big(n - \rho(\lambda - A)\big) \quad \textit{for each } \lambda \in \mathbb{C}.$$

Remark VI.2.3. In the situation of Corollary VI.2.2, if $A(X)$ is self-adjoint, we can extract some information about the non-microstates free entropy dimension $\delta^*(A(X))$. Recall that for one self-adjoint random variable Y its free entropy dimension $\delta^*(Y)$ is determined by the sizes of all atoms of μ_Y (see (VI.1)). Therefore, the formula (VI.3) given in Corollary VI.2.1 allows to express $\delta^*(A(X))$ in terms of purely algebraic quantities associated to A. Namely, we have that

$$\delta^*(A(X)) = 1 - \sum_{\lambda\in\sigma_p(A(X))} \mu_{A(X)}(\{\lambda\})^2 = 1 - \frac{1}{n^2} \sum_{\lambda\in\sigma^{\text{full}}(A)} \big(n - \rho(\lambda - A)\big)^2.$$

Note that $\rho(\lambda - A) \in \mathbb{N}$, so that

$$\delta^*(A(X)) \in \Big\{\frac{k}{n^2} \;\Big|\; k \in \mathbb{N} \cap [0, n^2]\Big\}.$$

Recall that in Remark V.2.6, we have seen that every tuple $X = (X_1, \dots, X_d)$ of random variables satisfying $\Delta(X) = d$ provides an analytic model for the algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ as well as the free field $\mathbb{C}\,\langle\!\!\langle x_1, \dots, x_d\rangle\!\!\rangle$. So Corollary VI.2.1 have strengthened this correspondence to the matrix level through identifying spectra and rank functions. This connection is clearly bidirectional though so far we have only seen that information on the analytic side can be drawn from the algebraic side. So in the following we will give an example to show the other possibility.

To be more precise, we want to present an analytic counterpart of Proposition III.4.2. Recall that Proposition III.4.2 says that the cardinality of $\sigma^{\text{full}}(A)$ is at most n for a matrix $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$. It is clearly an algebraic fact, but it can be implied by its analytic counterpart. Namely, we will show that the cardinality of $\sigma_p(A(X))$ is at most n for each $n \times n$ matrix A over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ and a given tuple X with maximal $\Delta(X)$. Then Proposition III.4.2 follows from Corollary VI.2.1. We put this analytic counterpart as the following proposition.

PROPOSITION VI.2.4. *Let $X = (X_1, \dots, X_d)$ be a tuple of elements in a tracial W^*-probability space $(\mathcal{M}, \varphi)$ that satisfies $\Delta(X) = d$. The for any $A \in M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$, $\sigma_p(A(X))$ has at most n elements.*

In order to prove this we need the following lemma.

LEMMA VI.2.5. *Let A be a random variable in a tracial W^*-probability space $(\mathcal{M}, \varphi)$. Assume that we have distinct eigenvalues $\lambda_1, \cdots, \lambda_k$ of A on $L^2(\mathcal{M}, \varphi)$. Denote by $p_i \in \mathcal{M}$ the orthogonal projection onto the eigenspace $\ker(\lambda_i - A)$ of λ_i. Then we have*

$$\sum_{i=1}^{k} \varphi(p_i) = \varphi(\bigvee_{i=1}^{k} p_i).$$

This result is probably well-known for experts. See, for example, [**CDSZ17**, Lemma 2.2.3] which implies the case $k = 2$. Then this lemma can be proven by an induction argument; see [**MSY19**, Lemma 5.18] for such a proof.

PROOF OF PROPOSITION VI.2.4. For each $\lambda \in \sigma_p(A(X))$, we write $p_\lambda := p_{\ker(\lambda - A(X))}$. Then we have

$$\mathrm{Tr}_n \circ \varphi^{(n)}(p_\lambda) = n - \mathrm{rank}(\lambda - A(X)) = n - \rho(\lambda - A)$$

by combining (I.2) with Corollary VI.2.1. Since for each $\lambda \in \sigma^{\mathrm{full}}(A)$, $\rho(A)$ is an integer strictly less than n, we have

$$\mathrm{Tr}_n \circ \varphi^{(n)}(p_\lambda) \geq 1.$$

Now we consider the W^*-probability space $(M_n(\mathcal{M}), \mathrm{tr}_n \circ \varphi^{(n)})$. We have $\mathrm{tr}_n \circ \varphi^{(n)}(p_\lambda) \geq \frac{1}{n}$ for any $\lambda \in \sigma_p(A(X))$. Suppose that we have distinct $\lambda_1, \dots, \lambda_k \in \sigma_p(A(X))$ for a positive integer k, then, by Lemma VI.2.5,

$$1 \geq \varphi(\bigvee_{i=1}^{k} p_{\lambda_i}) = \sum_{i=1}^{k} \varphi(p_{\lambda_i}) \geq \frac{k}{b},$$

and thus $k \leq n$, as desired. □

REMARK VI.2.6. Note that for the case that $A(X)$ is normal, the fact that the cardinality of $\sigma_p(A(X))$ is at most n can be proved easily without the help of Lemma VI.2.5. This is because the analytic distribution of $\mu_{A(X)}$ is a probability measure. So the inequality

$$\sum_{i=1}^{k} \varphi(p_{\lambda_i}) = \sum_{i=1}^{k} \mu_{A(X)}(\{\lambda_i\}) = \mu_{A(X)}(\{\lambda_1, \dots, \lambda_d\}) \leq 1$$

follows immediately.

Let us end this section by the following remark which an immediate consequence of Proposition III.4.3 and Corollary VI.2.1.

REMARK VI.2.7. Let $X = (X_1, \dots, X_d)$ be a tuple of random variables in a tracial W^*-probability space $(\mathcal{M}, \varphi)$ such that $\Delta(X) = d$. Let $A = A^{(0)} + A^{(1)}x_1 + \dots + A^{(d)}x_d$ be a linear matrix in $M_n(\mathbb{C}\langle x_1, \dots, x_d\rangle)$.

(i) We have $\sigma_p(A(X)) \subseteq \sigma(A^{(0)})$, i.e., $\lambda - A(X)$ is a zero divisor can only happen when λ is an eigenvalue of the constant matrix $A^{(0)}$.

(ii) If the homogeneous part $A^{(1)}x_1 + \cdots + A^{(d)}x_d$ of A is full, then $\sigma_p(A(X)) = \emptyset$.

Moreover, in the case that $A(X)$ is normal we know that the analytic distribution $\mu_{A(X)}$ has atoms precisely at the points in $\sigma^{\text{full}}(A) \subseteq \sigma(A^{(0)})$. However, at the moment the author is not aware of any analytic proof in the spirit of Proposition VI.2.4.

CHAPTER VII

Strong convergence in distribution for rational functions

In Chapter V and VI, the investigation of rational functions was carried out for the non-commutative random variables in W^*-probability spaces. Our investigation of this chapter, based on [**Yin18**], will turn to random matrices that are introduced in Chapter II.

Recall that in Chapter II we learned that some random variables, such as freely independent semicircular and Haar unitary random variables, serve as limits of some independent random matrices such as GUE and respectively Haar unitary random matrices. Note that freely independent semicircular and Haar unitary random variables are random variables with maximal Δ, as shown in Section VI.1. So in particular we know that for every rational function r it has a well-defined evaluation as an unbounded random variable at freely independent semicircular or Haar unitary random variables. Then a natural question arises: does r also have a well-defined evaluation at the corresponding random matrices?

Actually, this question has been raised earlier in the work [**HMS18**] that provides an algorithm for computing analytic distributions for rational functions in non-commutative random variables. Due to the very convincing matching of the histograms of random matrices and the distributions yielded by the algorithm in [**HMS18**], one would expect the convergence in distribution holds for rational functions in these random matrices. However, on the one hand, we will see that this convergence actually cannot be described by moments. This is because the convergence in trace is not stable under taking inverses. On the other hand, the well-definedness of rational functions in random matrices cannot be controlled by the convergence in distribution. We will provide an example to exhibit both issues at the beginning of Section VII.1. But if the convergence is in norm, we will show that it has some stableness as we will see in the remaining part of Section VII.1.

In Section VII.2, we will offer a solution to the well-definedness and convergence problem for rational functions. On the one hand, instead of the convergence in distribution we will consider the strong convergence in distribution. On the other hand, we will limit the evaluations of rational functions in limiting random variables to the bounded ones. It turns out that this allows us to have a nice control on least eigenvalues of random matrices such that the well-definedness of evaluations of rational functions become feasible. Moreover, the well-defined evaluations of random matrices then also strongly converge in distribution to the corresponding rational functions in limiting random variables.

VII.1. The stableness of inverses

First, let us show that the convergence in trace is not stable for taking an inverse by the following example. Let $(X^{(N)})_{N=1}^{\infty}$ be a sequence of scalar-valued Hermitian matrices that converges in distribution to a self-adjoint random variable X, which lies in some faithful tracial C^*-probability space $(\mathcal{A}, \varphi)$. We suppose that X is invertible in $\mathcal{A}$ and $X^{(N)} \in M_N(\mathbb{C})$ is invertible for each $N \in \mathbb{N}^+$. Let us additionally assume that

$$\lim_{N\to\infty} \mathrm{tr}_N((X^{(N)})^{-1}) = \varphi(X^{-1}).$$

Then we set a matrix as follows:

$$Y^{(N+1)} := \begin{pmatrix} \frac{1}{N+1} & \mathbf{0} \\ \mathbf{0} & X^{(N)} \end{pmatrix} \in M_{N+1}(\mathbb{C}).$$

It is clear that $Y^{(N)}$ also converges in distribution to X since

$$\lim_{N\to\infty} \mathrm{tr}_N(p(Y^{(N)})) = \lim_{N\to\infty} \frac{p(\frac{1}{N+1})}{N+1} + \lim_{N\to\infty} \frac{N}{N+1} \mathrm{tr}_N(p(X^{(N)})) = \varphi(p(X))$$

for any polynomial p. Moreover, $Y^{(N)}$ is invertible as $X^{(N)}$ is invertible. However,

$$\lim_{N\to\infty} \mathrm{tr}_N((Y^{(N)})^{-1}) = 1 + \lim_{N\to\infty} \frac{N}{N+1} \mathrm{tr}_N((X^{(N)})^{-1}) = 1 + \varphi(X^{-1}).$$

So we have seen that a slight modification of a convergent sequence $(X^{(N)})_{N=1}^\infty$ can lead to the failure of the convergence $((X^{(N)})^{-1})_{N=1}^\infty$ in trace.

Similar, a modification like

$$Z^{(N+1)} := \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & X^{(N)} \end{pmatrix} \in M_{N+1}(\mathbb{C})$$

will not change the convergence in distribution of $(X^{(N)})_{N=1}^\infty$ but make any matrix in the sequence singular. So the notion of convergence in distribution as in Definition II.2.1 does not fit very well the algebraic operation of taking inverses.

However, the strong convergence in distribution does not have such an issue due to the convergence in norm. We will see this through Lemma VII.1.3 in the following. First, let us reintroduce the strong convergence in distribution given in Definition II.2.4 with a modification on its framework.

Definition VII.1.1. Let $(\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N})$ be a family of C^*-probability spaces with faithful states. For each $N \in \mathbb{N}^+$, $X^{(N)} = (X_1^{(N)}, \ldots, X_d^{(N)})$ denotes a d-tuple of random variables in $(\mathcal{A}_N, \varphi_N)$. Let $X = (X_1, \ldots, X_d)$ denote a d-tuple of random variables in $(\mathcal{A}_0, \varphi_0)$. We say the sequence $(X^{(N)})_{N=1}^\infty$ *strongly converges in distribution* to X if

(i) the sequence $(X^{(N)})_{N=1}^\infty$ converges in distribution to X, i.e.

$$\lim_{N\to\infty} \varphi_N(p(X^{(N)})) = \varphi_0(p(X))$$

for all polynomials $p \in \mathbb{C}\langle x_1, \ldots, x_d, x_1^* \ldots, x_d^*\rangle$;

(ii) we have

$$\lim_{N\to\infty} \|p(X^{(N)})\|_{\mathcal{A}_N} = \|p(X)\|_{\mathcal{A}_0}$$

for all polynomials $p \in \mathbb{C}\langle x_1, \ldots, x_d, x_1^* \ldots, x_d^*\rangle$.

Since the strong convergence in distribution of those random matrices introduced in Section II.2 always holds almost surely, the above definition will simplify our notations in the following discussion. From now on, we will also abuse the notation and write the norm $\|\cdot\|_{\mathcal{A}_N}$ as $\|\cdot\|$ for each $N \in \mathbb{N}$. There is no danger of confusion.

The following lemma shows that the invertibility of a positive element in a C^*-algebra can be detected by the least eigenvalue disguised as a formula in norms. This lemma is standard but we present a proof for the reader's convenience.

LEMMA VII.1.2. *Let a be a positive element in a C^*-algebra $\mathcal{A}$. Then $a \in \mathcal{A}$ is invertible if and only if*

$$\big\| \|a\| - a \big\| < \|a\|.$$

Moreover, in this case, we have

$$\|a^{-1}\| = (\|a\| - \big\|\|a\| - a\big\|)^{-1} \tag{VII.1}$$

PROOF. Let $\sigma(a) \subseteq [0, +\infty)$ be the spectrum of the positive element a. We denote

$$R := \sup_{\lambda \in \sigma(a)} \lambda = \|a\| \quad \text{and} \quad r := \inf_{\lambda \in \sigma(a)} \lambda.$$

Then we have $\sigma(R - a) = R - \sigma(a)$ and

$$\|R - a\| = R - r.$$

Suppose that a is invertible, i.e. $0 \notin \sigma(a)$. We see that $r > 0$ since $\sigma(a)$ is compact. So we conclude that $\|R - a\| < R$. Conversely, if $\|R - a\| < R$ holds, then we see $r > 0$. It follows that $0 \notin \sigma(a)$ and thus a is invertible.

It remains to show (VII.1). We have already seen that $\|R - a\| = R - r$. So $\|a\| - \big\|\|a\| - a\big\| = R - \|R - a\| = r$. Note that $\sigma(a^{-1}) = (\sigma(a))^{-1} := \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$. We deduce that

$$\|a^{-1}\| = \sup_{\lambda \in \sigma(a^{-1})} \lambda = \sup_{\lambda \in \sigma(a)} \lambda^{-1} = r^{-1} = (\|a\| - \big\|\|a\| - a\big\|)^{-1}.$$

□

Now we give a lemma[1] that shows how the convergence of norms can be used to control the least singular values of a sequence of operators such that these operators become invertible eventually, provided that the limiting operator is invertible.

LEMMA VII.1.3. *Let $\mathcal{A}_N$ ($N \in \mathbb{N}$) be a sequence of C^*-algebras. Let $A^{(N)}$ ($N \in \mathbb{N}^+$) and respectively A be elements in $\mathcal{A}_N$ ($N \in \mathbb{N}^+$) and respectively $\mathcal{A}_0$. Suppose that A is invertible in $\mathcal{A}$ and*

$$\lim_{N \to \infty} \|p(A^{(N)})\| = \|p(A)\|$$

for all polynomials $p \in \mathbb{C}\langle x, x^* \rangle$. Then $A^{(N)}$ is invertible in $\mathcal{A}_N$ for N large enough and

$$\lim_{N \to \infty} \|(A^{(N)})^{-1}\| = \|A^{-1}\|.$$

PROOF. First, note that AA^* is positive and invertible. We denote its norm $\|AA^*\|$ by R. Then we have $\|R - AA^*\| < R$ by consulting Lemma VII.1.2. According to our assumption,

$$\|R - AA^*\| = \lim_{N \to \infty} \|R - A^{(N)}(A^{(N)})^*\| \quad \text{and} \quad R = \|AA^*\| = \lim_{N \to \infty} \|A^{(N)}(A^{(N)})^*\|.$$

[1]We thank Guillaume Cébron for the inspiring discussion that led to this lemma.

Now, let us denote the norm $\|A^{(N)}(A^{(N)})^*\|$ by $R^{(N)}$. By the reverse triangle inequality, we have

$$\big|\|R^{(N)} - A^{(N)}(A^{(N)})^*\| - \|R - A^{(N)}(A^{(N)})^*\|\big| \leq |R^{(N)} - R|.$$

It follows that

$$\begin{aligned}\lim_{N\to\infty} \|R^{(N)} - A^{(N)}(A^{(N)})^*\| &= \lim_{N\to\infty} \|R - A^{(N)}(A^{(N)})^*\| \\ &= \|R - AA^*\| \\ &< R \\ &= \lim_{N\to\infty} \|A^{(N)}(A^{(N)})^*\|.\end{aligned}$$

So we see that $\|R^{(N)} - A^{(N)}(A^{(N)})^*\| < \|A^{(N)}(A^{(N)})^*\|$ if N is large enough. Namely, $A^{(N)}(A^{(N)})^*$ is invertible if N is large enough. Then we conclude that $A^{(N)}$ is surjective. Similarly, we can prove that $(A^{(N)})^*A^{(N)}$ is invertible and so $A^{(N)}$ is injective, when N is large enough.

Moreover, with the help of (VII.1), we see that

$$\begin{aligned}\|A^{-1}\|^{-2} = \|(AA^*)^{-1}\|^{-1} &= \|(AA^*)\| - \big\|\|(AA^*)\| - (AA^*)\big\| \\ &= \lim_{N\to\infty} (R^{(N)} - \|R^{(N)} - A^{(N)}(A^{(N)})^*\|) \\ &= \lim_{N\to\infty} \|(A^{(N)})^{-1}\|^{-2}.\end{aligned}$$

So the asserted norm convergence of $(A^{(N)})^{-1}$ follows. □

VII.2. Strong convergence in distribution for rational functions

In this section, we will see that the strong convergence in distribution provides an answer to the question on the well-definedness and the convergence of rational functions in random matrices. It can also be regarded as a generalization of the strong convergence results for random matrices. Recall that the linearization trick is one of the main ingredients used in [**HT05, HST06**] to show the strong convergence of GUE random matrices. In Chapter IV we have seen that such a linearization works equally well for non-commutative rational expressions or functions. So we can expect the proof for convergence of norm in [**HT05, HST06**] can apply also to some rational functions in GUE as well as many other random matrix modes. This is indeed true according to the following theorem. But we will see that this can be done without going into detailed estimations on quantities of specified random matrices. The strong convergence in distribution follows automatically for rational functions that have bounded evaluations, provided that the random matrices in question strongly converge in distribution.

THEOREM VII.2.1. *Let $(\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N})$ be a family of C^*-probability spaces with faithful states. Let $X^{(N)}$ $(N \in \mathbb{N}^+)$ and X respectively be d-tuples of random variables in $(\mathcal{A}_N, \varphi_N)$ $(N \in \mathbb{N})$ and $(\mathcal{A}_0, \varphi_0)$ respectively. We assume the following two conditions on $X^{(N)}$ and X.*

(i) *$(X^{(N)})_{N=1}^{\infty}$ strongly converges in distribution to X.*
(ii) *Let $r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_d, x_1^*, \ldots, x_d^*\rangle\!\rangle$ be any rational function such that the evaluation $r(X)$ is well-defined in $\mathcal{A}_0$.*

Then we have the following conclusions.

(i) $r(X^{(N)})$ *is well-defined in* $\mathcal{A}_N$ *for* N *large enough.*
(ii) $\lim_{N\to\infty} \varphi_N(r(X^{(N)})) = \varphi(r(X))$.
(iii) $\lim_{N\to\infty} \|r(X^{(N)})\| = \|r(X)\|$.

We need the following lemma to prove the theorem. This fact is probably well-known as a folklore result in C^*-algebra theory. We refer to [**Mal12**, Proposition 7.3] for a proof.

LEMMA VII.2.2. *Suppose that a sequence* $(X^{(N)})_{N=1}^{\infty}$ *strongly converges in distribution to* X *as in Definition VII.1.1. Then for any* $n \in \mathbb{N}^+$ *and* $n \times n$ *matrix* A *over* $\mathbb{C}\langle x_1, \dots, x_d, x_1^* \dots, x_d^* \rangle$, *we have*

$$\lim_{N\to\infty} \|A(X^{(N)})\| = \|A(X)\|.$$

PROOF OF THEOREM VII.2.1. First, let us prove $r(X^{(N)})$ is well-defined in $\mathcal{A}_N$ for N large enough. Since $r(X)$ is well-defined in $\mathcal{A}_0$, there exist, according to Definition IV.4.2 and Theorem IV.4.1, a full $n \times n$ matrix A over $\mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^* \rangle$ and scalar-valued row and column vectors u and v such that $A(X)$ is invertible as a matrix over $\mathcal{A}_0$ and $r(X) = u(A(X))^{-1}v$. So $r(X^{(N)})$ will be well-defined in $\mathcal{A}_N$ if $A(X^{(N)})$ is invertible in $M_n(\mathcal{A}_N)$. According to Lemma VII.2.2, we know that for all polynomial $p \in \mathbb{C}\langle x, x^* \rangle$, $\|p(A(X))\| = \lim_{N\to\infty} \|p(A(X^{(N)}))\|$. Hence by applying Lemma VII.1.3 to $A(X)$, we see that $A(X^{(N)})$ is invertible when N is large enough and

$$\|(A(X))^{-1}\| = \lim_{N\to\infty} \|(A(X^{(N)}))^{-1}\|. \tag{VII.2}$$

Now, we want to show that $(r(X^{(N)}))_{N=1}^{\infty}$ converges to $r(X)$ in norm. Since we have shown that $r(X^{(N)})$ is eventually well-defined, we may assume that it is well-defined for all $N \in \mathbb{N}^+$ to simplify arguments in the following. To prove the desired norm convergence, we want to find polynomials approximating r uniformly throughout X and $X^{(N)}$ for all $N \in \mathbb{N}^+$. For that purpose, let we consider the following product of C^*-algebras (see [**Bla06**, II.8.1.2])

$$\mathcal{A} = \prod_{N\in\mathbb{N}} \mathcal{A}_N := \big\{ (a^{(N)})_{N=0}^{\infty} \bigm| a^{(N)} \in \mathcal{A}_N, \sup_{N\in\mathbb{N}} \|a^{(N)}\| < \infty \big\}.$$

It is a C^*-algebra with the norm

$$\|a\| = \sup_{N\in\mathbb{N}} \|a^{(N)}\| \quad \text{for all } a = (a^{(N)})_{N=0}^{\infty} \in \mathcal{A}.$$

Let us denote

$$\mathbb{X}_i := (X_i, X_i^{(1)}, X_i^{(2)}, \dots)$$

for each $i = 1, \dots, d$ and $\mathbb{X} = (\mathbb{X}_1, \dots, \mathbb{X}_d)$. Then we have

$$p(\mathbb{X}) = (p(X), p(X^{(1)}), p(X^{(2)}), \dots) \in \mathcal{A}$$

for all polynomial $p \in \mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^* \rangle$, due to the strong convergence in distribution of $(X^{(N)})_{N=1}^{\infty}$. We denote by $\mathcal{B}$ the C^*-subalgebra of $\mathcal{A}$ generated by the tuple $\mathbb{X}$. By the previous paragraph, we know that the sequence

$$r(\mathbb{X}) := (r(X), r(X^{(1)}), r(X^{(2)}), \dots)$$

is well-defined. This sequence also lies in $\mathcal{A}$, i.e. $\sup_{N\in\mathbb{N}^+}\|r(X^{(N)})\| < \infty$. It follows from

$$r(\mathbb{X}) = u(A(\mathbb{X}))^{-1}v \quad \text{and} \quad (A(\mathbb{X}))^{-1} \in M_n(\mathcal{A}),$$

where the latter is due to the norm convergence of $((A(X^{(N)}))^{-1})_{N=1}^{\infty}$ given in (VII.2). Actually, we have $(A(\mathbb{X}))^{-1} \in M_n(\mathcal{B})$ since any C^*-subalgebra is division closed (see [**Bla06**, II.1.6.7]). So we have $r(\mathbb{X}) \in \mathcal{B}$. Then for any $\varepsilon > 0$, there exists some polynomial $p \in \mathbb{C}\langle x_1, \dots, x_d, x_1^*, \dots, x_d^*\rangle$ such that $\|r(\mathbb{X}) - p(\mathbb{X})\| < \varepsilon$. Namely, we have

$$\|r(X) - p(X)\| < \varepsilon \quad \text{and} \quad \|r(X^{(N)}) - p(X^{(N)})\| < \varepsilon, \forall N \in \mathbb{N}^+.$$

Finally, from

$$\begin{aligned}\big|\|r(X)\| - \|r(X^{(N)})\|\big| &\le \big|\|r(X)\| - \|p(X)\|\big| + \big|\|p(X)\| - \|p(X^{(N)})\|\big| \\ &\quad + \big|\|p(X^{(N)})\| - \|r(X^{(N)})\|\big| \\ &\le \|r(X) - p(X)\| + \big|\|p(X)\| - \|p(X^{(N)})\|\big| \\ &\quad + \|p(X^{(N)}) - r(X^{(N)})\| \\ &\le 2\varepsilon + \big|\|p(X)\| - \|p(X^{(N)})\|\big|,\end{aligned}$$

we conclude that

$$\begin{aligned}\limsup_{N\to\infty}\big|\|r(X)\| - \|r(X^{(N)})\|\big| &\le 2\varepsilon + \lim_{N\to\infty}\big|\|p(X)\| - \|p(X^{(N)})\|\big| \\ &\le 2\varepsilon.\end{aligned}$$

Since ε can be chosen arbitrary, we see that $\|r(X)\| = \lim_{N\to\infty}\|r(X^{(N)})\|$.

Finally, we want to show the convergence in trace of $(r(X^{(N)}))_{N=1}^{\infty}$. For $\varepsilon > 0$, let p be a polynomial such that $\|r(\mathbb{X}) - p(\mathbb{X})\| < \varepsilon$ like in the end of last paragraph. Then

$$\begin{aligned}\big|\varphi(r(X)) - \varphi_N(r(X^{(N)}))\big| &\le \big|\varphi(r(X)) - \varphi(p(X))\big| + \big|\varphi(p(X)) - \varphi_N(p(X^{(N)}))\big| \\ &\quad + \big|\varphi_N(p(X^{(N)})) - \varphi_N(r(X^{(N)}))\big| \\ &\le \|r(X) - p(X)\| + \big|\varphi(p(X)) - \varphi_N(p(X^{(N)}))\big| \\ &\quad + \|p(X^{(N)}) - r(X^{(N)})\| \\ &\le 2\varepsilon + \big|\varphi(p(X)) - \varphi_N(p(X^{(N)}))\big|.\end{aligned}$$

So we infer that $\varphi(r(X)) = \lim_{N\to\infty}\varphi_N(r(X^{(N)}))$. □

REMARK VII.2.3. Theorem VII.2.1 actually implies that if $(X^{(N)})_{N=1}^{\infty}$ strongly converge in distribution to X, then for any fixed rational function r such that $r(X)$ is invertible we have $(r(X^{(N)}))_{N=1}^{\infty}$ strongly converge in distribution to $r(X)$. In particular, if $(r(X^{(N)}))_{N=1}^{\infty}$ and $r(X)$ are normal, then the sequence $(\mu_{r(X^{(N)})})_{N=1}^{\infty}$ of analytic distributions converges weakly to the analytic distribution $\mu_{r(X)}$. So our theorem verifies the matching of the histograms of random matrices and analytic distributions for rational functions that satisfy certain assumptions. For example, the rational function and random matrices in [**HMS18**, Example 4.14] satisfy the assumption of Theorem VII.2.1.

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