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The Classical Limit of QuantumTheory

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THE CLASSICAL LIMIT OF QUANTUM THEORY

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ABSTRACT

This essay treats of the relationship between quantum and classical mechanics. Both physicists and philosophers hold that quantum mechanics reduces to classical mechanics as $\hbar \to 0$, or that classical mechanics is a special case of quantum mechanics in this limit. If one theory reduces to another, certain formal and nonformal conditions must be satisfied. These conditions are formulated and it is shown that the Wigner transformation can serve as a natural reduction function in a reduction which satisfies the formal and non-formal conditions. Finally, it is argued that this reduction does not aid in solving the problem of providing an adequate metaphysical interpretation of quantum theory.

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I. INTRODUCTION

A question of particular interest for the philosophy of physics is in what sense, if any, is classical mechanics a limiting case of quantum mechanics. A more general philosophical question also arises: What does it mean to say that one theory is a limiting case of another? Neither question is easy to answer. The present study attempts to formulate a rigorous answer to the first question in the hope that it will generate some insight into the methodological issues surrounding the second, more general question.

With respect to the relation between classical mechanics and quantum mechanics, physicists assume one of two positions. Most physicists believe that classical mechanics is a limiting case of quantum mechanics in the sense that classical mechanics can be derived from quantum mechanics in the limit $\hbar \to 0$. For example, Gottfried writes: ". . . from a satisfactory quantum theory one must be able to deduce classical mechanics and electrodynamics by taking an appropriate limit" (Gottfried 1966:6). Call this the orthodox view. A second, minority view is that classical mechanics is not a limiting case of quantum theory. Bohr maintains that classical theory is logically prior to quantum theory and that while a formal analogy obtains between the two theories, there is no stronger relation between them (Bohr 1949). David Bohm also subscribes to the view that quantum theory logically presupposes classical theory. Furthermore, Bohm believes that the concepts of classical theory are generally valid and that the concepts of quantum theory are special cases of their classical analogues. In contrast to the orthodox position, the minority stance is that it is not possible to "deduce classical concepts as limiting cases of quantum concepts" (Bohm 1951:625).

There are difficulties with holding either the orthodox or the minority view. The orthodox view appears to imply that there is a well-defined, mathematical limit as $\hbar \to 0$ whereby, given quantum theory, classical physics can be recovered. However, arguments in support of the orthodox view do not justify the existence of such a limit. The arguments in the literature are usually based on formal analogies between the two theories or on a correspondence between the central concepts of quantum theory and those of classical theory. For example, quantum equations of motion can be written in terms of the commutator bracket of two dynamical variables and classical equations can be written in terms of the Poisson bracket of dynamical variables. Such formal analogies are instructive, but the existence of such analogies does not imply the existence of a conceptual relation between the theories, which would allow one to claim that classical mechanics is a limiting case or special case of quantum mechanics. There are far-reaching analogies between hydrodynamics and the theory of heat, yet one cannot maintain that hydrodynamics can be derived from the theory of heat. Hence, a proponent of the orthodox view is obliged to show that the relation between classical and quantum theory goes beyond that of formal analogy.

Difficulties with the minority view are of a different kind. The minority view has undersirable metaphysical and methodological consequences. On the minority view, classical mechanics is the correct universal theory with quantum mechanics appearing as an appendage which serves to generate predictions for micro-phenomena. However, quantum theory does appear to offer a complete account of the structure of matter in terms of its atomic and subatomic constituents. Quantum theory seems to be our best candidate for a universal theory of matter. Hence, one would expect that classical mechanics, valid for macroscopic objects, should be accounted for in terms of quantum theory, the best theory of micro-phenomena. The minority view denies the possibility of so accounting for the success of classical theory. This denial violates our metaphysical intuitions because we do believe that adequate explanations of macro-phenomena can be given by examining the behavior of their micro-constituents.

A rigorous answer to the first question raised above would serve to settle the issue between the orthodox and the minority views.

Analogous positions can be delineated with respect to the more general methodological question. The issue of whether one theory is a limiting case of another arises in the context of intertheoretic reduction. An approximation or limiting case is a kind of reductive relation that might hold between two theories.

Here it is necessary to clarify a terminological problem. Both physicists and philosophers believe that under certain circumstances one theory reduces to another, but they use the term differently. In philosophical parlance, the less fundamental theory is said to reduce to the more fundamental theory. A philosopher would say that thermodynamics reduces to statistical mechanics. For the physicist, reduction is based on the idea of reducing the more fundamental theory to the less fundamental theory by applying some operation to the former theory. The physicist would say that statistical mechanics reduces to thermodynamics, if by applying some operation to the statistical mechanical formalism, the equations of thermodynamics could be obtained. In this study, the physicists' usage will be adopted. It will be said that the primary theory reduces to the secondary theory, or that the new theory reduces to the old theory.

One can also distinguish two general positions on intertheoretic reduction. The orthodox view holds that such reductions are central to scientific progress. A new scientific theory for a class of phenomena, as a general rule, subsumes the prior theory for that domain. One should be able to deduce the old theory from the new theory (cf. Nagel 1961:Ch. 11). The minority position is that science progresses by means of "scientific revolutions" (Kuhn 1962). On this account, science develops via conceptual revolutions, wherein one scientific paradigm gives way to another. In many instances, this change of paradigm, or change or world view, is so severe that the successive theories are logically incommensurate.

Consistently maintaining either of these positions also proves to be problematic. The notion of scientific revolution is supposed to explain how science progresses and develops. A new theory emerges when, faced with theoretical anomalies, the scientific community begins to view the data differently. The paradigm shifts, a new theory comes forth, and the anomalies are resolved. On this account, an explanation is forthcoming as to how a new theory might emerge, but the explanation in terms of a change of paradigm isolates the new theory, logically and conceptually, from its precursor. One is hard pressed, in this circumstance, to explain how scientific knowledge develops. Apparently we do not progressively learn more about the physical world, rather from time to time we are prone to view things differently and offer new explanations for our observations which need bear no conceptual relation to our previous system of beliefs.

The orthodox view of progress by reduction also has a shortcoming, a shortcoming which is exemplified nicely in the purported relation of classical mechanics to quantum mechanics as a limiting case. The advantage of the orthodox view is that it does attempt to provide some account of how scientific knowledge accumulates. A new theory for a given range of phenomena subsumes, and in some case corrects, the previous theory. The belief that the old theory can be derived from the new one reflects our confidence in our ability to accumulate knowledge about the world.

The difficulty with the thesis of scientific development by reduction is that it seldom is the case that the old theory can be derived exactly from the new one. Related to the thesis of development by reduction is the doctrine of scientific realism. One tenet of scientific realism is that well-confirmed theories are (in some sense) approximately true. The primary problem with the orthodox view is in clarifying this notion of approximate truth. For a successful reduction, the reduced theory must be approximately true in a sense strong enough to allow for a logical derivation of the old theory from the new one.

Proponents of the orthodox view recognize that the notion of an approximate derivational reduction, a reduction where the primary theory yields an approximation to the secondary theory, requires considerable clarification (Sklar 1967:111, Schaffner 1967:136). The revolutionaries attempt to refute or discredit the orthodox view by attacking the notions of approximate derivation and approximate truth. They argue that if only an approximation to the old theory can be derived from the new one, then there can be no logical relation of reduction between the reduced and the reducing theories, as in most cases the approximation to the old theory is logically incompatible with the old theory (Feyerabend 1962: 46-8, Kuhn 1962:Sec. IX).

Thus, the questions posed at the outset are closely related. Philosophical positions on the issue of development by reduction parallel the views of physicists on the nature of the relation between quantum and classical mechanics. Quantum mechanics is a well-confirmed theory and one of our fundamental theories of physical phenomena. Classical mechanics is a well-confirmed theory within its range of applicability. If quantum theory, as a universal theory, does not subsume classical mechanics in some sense, then one would have to abandon the doctrine of scientific realism or admit that, contrary to prevailing belief, one of the theories is not well-confirmed. The orthodox physicist and the orthodox philosopher feel compelled to maintain the connection between the theories on grounds that the minority views have undesirable metaphysical and methodological consequences. In order to maintain the orthodox positions, the philosopher must give a convincing explication of approximate truth and the physicist must give a rigorous characterization of the classical limit of quantum theory.

The complications that plague the orthodox philosophical position can be traced to two related sources, the formal framework within which philosophers attempt to explicate reduction and the belief that the reductive relationship between theories must be a relationship of strict logical derivability.

The philosophical literature on reduction is characterized by two kinds of discussion. In one type, reduction is discussed by means of examples from theories. Here typical questions are of the form: Is temperature definable as mean kinetic energy? Is a gene a muton, a cistron, or a recon? (Schaffner 1967:142). The basic preoccupation is with whether one term, say from genetics, can be defined in terms of notions from molecular biology. This ordinary language approach can offer no insight into the problems of approximate reduction because there are no interesting cases of reduction where ordinary language is the theoretical vernacular. Even if there were such cases, no precise notion of limit or approximation would be forthcoming. Concepts are either definable in terms of others or they are not, "approximately definable" is a nonsensical notion. (Although it offers no solution to the problems raised here, Teller (1971) does make some progress toward explicating the relationship between ordinary language and scientific language. The scientific concepts refine the concepts expressed in ordinary language. For example, quantum mechanics refines our prescientific concept of position and refines it in a manner different from the refinement suggested by classical physics.)

The second type of discussion attempts to introduce some formal rigor by relying on the predicate calculus and model theory. These formal discussions are not applicable in any straightforward manner to particular cases of purported approximate reduction. For example, the following are among the necessary and sufficient conditions Schaffner puts on a successful reduction: (i) There is a correspondence, ϕ, between the primitive terms of the primary theory and the primitive terms of the secondary theory; (ii) every n-place primitive predicate of the secondary theory is effectively associated with an open sentence in n free variables of the primary theory such that the open sentence $T(x_n)$ is true if and only if $F(\phi(x_n))$ is true (Schaffner 1967:144).

This kind of formal rigor is not very helpful. It is just not the case that the theories of central interest in this debate are formulated, or even can be readily formulated, in the predicate calculus. What are the primitive terms and predicates of quantum theory, Newtonian mechanics, or General Relativity? In these cases, where approximate reductions and limiting cases are of crucial importance, one is given no indication as to how proposed reduction patterns are to be applied.

Sneed (1971) offers the most exhaustive attempt at a highly formalized approach to reduction. However, on Sneed's account all that is required for intertheoretic equivalence or intertheoretic reduction is the existence of an appropriate mapping from the intended models of one theory to the models of the other theory (Sneed 1971:Ch. VII). Sneed's approach again assumes that the theories in question can be readily formulated in the language of the predicate calculus. A more serious fault is that Sneed's conditions on reduction contain no apparent requirement that the reduction function preserve any structure between the theories. If one is discussing a limiting case reduction between two theories with such explicit mathematical structures as the quantum and classical theories, one would expect, or one would at least like to show, that as the limit is approached, the structure of quantum theory approaches the structure of classical theory.

One difficulty, then, in presenting and defending a notion of approximate reduction is that the conceptual apparatus which the philosopher brings to bear is not adequate for the task. This difficulty can be overcome by noting that the interesting cases of approximations and limits of theories arise most often with respect to theories that have well-defined mathematical structures. In most of these cases, the structures of the theories are such that one can employ the methods of mathematical analysis. As a mathematical theory, analysis provides a proven and natural idiom for discussing limits and approximations.

The second difficulty a proponent of the orthodox position must deal with in explicating the notion of an approximate reduction is the belief that the secondary theory must be logically derivable from the primary theory. This belief derives from the traditional logical empiricist view that in a reduction the primary theory must explain the secondary theory and that in any explanation the explanandum must be logically derivable from the explanans. Here the orthodox philosopher must recognize the cogency of the minority view criticism alluded to above. In most cases, the primary theory and the secondary theory are logically incompatible; so, the traditional, orthodox view could not possibly succeed.

This criticism forces the orthodox philosopher to abandon the narrow logical empiricist construal of explanation which over-emphasizes the formal component of an explanation. The philosopher must also consider the epistemological, or pragmatic, component of an explanation. For discussions of reduction, the broadened outlook amounts to recognizing that there is both a formal and a nonformal aspect to a successful reduction. The <u>formal</u> aspect is concerned with the relation between the languages of the primary and secondary theory. Where the theories involved are capable of mathematical formulation, this becomes a concern with the relation between the mathematical structures of the theories. As a species of explanation, a satisfactory reduction must also satisfy certain pragmatic or epistemological requirements which can be called the <u>nonformal</u> conditions on an adequate reduction. The orthodox tenet of requiring some conceptual continuity between successive theories can be maintained by requiring that an adequate reduction consists of defining some formal relation between the languages of the two theories such that this relation, or reduction function, generates a plausible account of the apparent success and the limitations of the secondary theory in terms of the primary theory.

In this study, it will be shown that there is a definite sense in which classical mechanics is a strict limiting case of quantum theory. In the process of establishing an affirmative answer to the physical question, it will also be shown that the relation between the two theories is an example of an approximation or a limiting case reduction.

Chapter II is concerned with the formal aspect of the reduction. The reduction function must be a structure preserving mapping between the two formalisms. A discussion of abstract mechanics motivates a decision as to which structures of the formalisms must be preserved, yielding the formal conditions on the reduction. A transformation due to Wigner (1932) is shown to be a natural choice for a reduction function. Using methods of mathematical analysis, three propositions are derived. On the basis of these propositions it can be claimed that the Wigner transformation satisfies the formal conditions.

The nonformal conditions are discussed in Chapter III. It is shown, with the aid of a reduction scheme due to Glymour (1970), that the Wigner transformation allows the formulation of an account of the apparent success and the limitations of the classical theory. It is concluded that the Wigner transformation provides an adequate reduction of quantum mechanics to classical mechanics and that classical mechanics is a limiting case of quantum mechanics as $\hbar \to 0$. Finally, it is argued that although the Wigner transformation leads to an adequate reduction, it does not aid in solving the main interpretative problem of quantum theory.

The discussion in this study is confined to the case of a non-relativistic system with one degree of freedom. This simplification is justified on two grounds. First, most of the interesting theoretical problems already arise in this simple case. The results below can easily be extended to systems of more degrees of freedom. Second, many classical problems can be simplified to the one dimensional case. Where this is not possible, classical mechanics continues to be a lively area of mathematical research.

II. FORMAL CONDITIONS ON THE REDUCTION

The formal conditions on an adequate reduction are most easily determined by viewing a scientific theory as a collection of sentences formulated in a formalized, mathematical language. If quantum mechanics reduces to classical mechanics in the limit as $\hbar$ goes to zero, then some relation between the mathematical structures of these two theories must be demonstrated in this limit. In particular it must be shown that the appropriate mathematical structures of the classical theory can be derived from quantum theory as $\hbar \to 0$.

According to the philosophical paradigm of reduction, this derivation should be achieved by means of a reduction function which maps the reducing theory into the reduced theory in such a way that the essential mathematical relations within the theories are preserved under the mapping. This suggests that the reduction function must be some kind of homomorphism from the reducing theory to the reduced theory. The kind of homomorphism required depends on the particular mathematical structures involved. Thus, in order to formulate an acceptable reduction function, it is necessary to specify explicitly the mathematical structures of the theories involved in the reduction and to show that the proposed reduction function is a homomorphism of these structures.

The essential mathematical structures involved become obvious when quantum and classical mechanics are treated as two different mathematical formalizations of an abstract concept of a mechanical system. A mechanical system consists of a system of particles, or mass points, the behavior of which is described by a law of motion. To describe a mechanical system two types of entities, dynamical variables and states, are posited, and two rules, a kinematical law and a dynamical law, are given. The dynamical variables are a set of properties of the system which are assumed to be pertinent to any description of the dynamical behavior of the system. A state of a system is simply the situation or disposition of the system's constituent particles at an instant of time. The kinematical rule relates the dynamical variables and states at an instant of time. At each instant, every dynamical variable has a particular real value. The kinematical law states a rule whereby the states of the system map the dynamical variables onto the real numbers. The value this rule gives for a state and a dynamical variable at a time is called the expectation value of that variable for that state at that time. This kinematical rule is also such that given expectation values for all dynamical variables of the system a unique state is determined. The dynamical law is a rule telling how the states of the system change over time. This abstract notion of a mechanical system is formalized when suitable mathematical entities are chosen to represent the dynamical variables and states, and the kinematical and dynamical laws are represented by functional relationships between these entities. Abstract mechanics is the study of the mathematical entities and structures which can be employed in the formalization of a mechanical theory (cf. Sudarshan 1962, Prosser 1966).

The basic assumption of abstract mechanics, and one that is most difficult to motivate, is that the mathematical structure of the dynamical variables can be derived from a free associative algebra over the complex numbers. Under this assumption, for any two dynamical variables their formal sum and formal product is also a dynamical variable. The dynamical variables form a free algebra because it is initially assumed that every distinct string of symbols represents a distinct dynamical variable. For the present purposes, two dynamical variables, position (q) and momentum (p), are of primary importance because all other dynamical variables we will be concerned with are functions of position and momentum. A second assumption made about the dynamical variables is that only those elements of the free algebra that are self-conjugate, $A=A^*$, are admitted as physically significant where * is defined by

(i) $(A+aB)^* = A^*+\bar{a}B^*$, $a \in \mathbb{C}$

(ii) $(AB)^* = B^*A^*$

(iii) $q^*=q$, $p^*=p$

Self-conjugacy guarantees that when a physically significant dynamical variable is measured the result of the measurement is always a real number. The elements of the free algebra that are self-conjugate form the set of <u>observables</u>.

Additional structure is imposed on the algebra by requiring that some dynamical variables are equal to others. In classical mechanics it is assumed that $pq=qp$, while in quantum mechanics it is assumed that

$pq-qp=\hbar/i$. This assumption is motivated by physical considerations. If the system under study obeys the laws of classical mechanics, then according to classical theory observations can be made on the system in such a way that disturbances on the system are negligible. The order in which observations are made on the system, in particular fundamental observations of position and momentum, is irrelevant. Quantum mechanically, however, an observation has a non-negligible effect on the observed system. Within the quantum formalism this fact is reflected by the canonical commutation relation on the fundamental dynamical variables, $pq-qp=\hbar/i$ (See Geroch n.d.:114). To incorporate these physical facts into abstract mechanics, the algebra of observables $\mathcal{A}$ is taken to be the quotient algebra of the free algebra by an appropriate ideal. Classically, the appropriate ideal is that generated by the element $pq-qp$ of the free algebra. In the quantum mechanical case, it is the ideal generated by the element $pq-qp-\hbar/i$ of the free algebra. This last assumption has the effect of partitioning the free algebra into equivalence classes, each such class representing a distinct dynamical variable.

The next task in the development of abstract mechanics is to specify the set of admissable states, or the state space of the system. The set of admissable states must map the elements of the observable algebra $\mathcal{A}$ into the real numbers. Linear functionals over an algebra map elements of that algebra into the field of the algebra. The complex numbers form the field of $\mathcal{A}$, so not all linear functionals on $\mathcal{A}$ can represent states. Any element A of $\mathcal{A}$ is called <u>strictly positive</u> if $A=BB^*$. A linear combination of strictly positive elements of $\mathcal{A}$ with real non-negative coefficients is called a <u>positive</u> element of the algebra $\mathcal{A}$. A <u>positive linear functional</u> over $\mathcal{A}$ is a linear functional which maps positive elements of $\mathcal{A}$ into positive numbers, that is F is positive if $F(AA^*) \geqslant 0$. The linear functional F is normalized if $F(\mathbb{1})=1$, where $\mathbb{1}$ is the identity element of $\mathcal{A}$. Thus, the set of states is the set of positive normalized linear functionals over $\mathcal{A}$.

This specification of the dynamical variables and the states leads to a natural formulation of the kinematical rule which generates expectation values. The expectation value of A in state f is the value of the linear functional f(A). Linear functionals on $\mathcal{A}$ form a space which is called the dual space of $\mathcal{A}$. Formally an inner product is a mapping which assigns to each pair (f,A) a scalar from the field of $\mathcal{A}$. So the kinematical rule for calculating expectation values is given by taking the inner product of the state with the dynamical variable. This completes the kinematical structure of a mechanical theory.

The characterization of the dynamical law requires the introduction of another mathematical structure, a Lie algebra. The elements of the Lie algebra $\mathcal{L}$ are the dynamical variables. The product in the Lie algebra is a nonassociative product [A,B] such that

$$[A,B] = -[B,A]$$

$$\left[A,[B,C]\right] + \left[B,[C,A]\right] + \left[C,[A,B]\right] = 0$$

making $\mathcal{L}$ a non-associative algebra. The algebras $\mathcal{A}$ and $\mathcal{L}$ are related by the requirement that the product in $\mathcal{A}$ and the product in $\mathcal{L}$ satisfy

$$[A\,B,C] = A\,[B,C] + [A,C]\,B.$$

When this condition is satisfied, one says that the Lie product is a derivation in a linear associative algebra with the product A B. If a particular associative algebra is chosen as $\mathcal{A}$, it is said that $\mathcal{A}$ provides a realization of $\mathcal{L}$ by derivations.

To specify the dynamical law, a particular element, H, of $\mathcal{A}$, called the Hamiltonian for the system, is designated. The dynamical law is given by

$$\frac{d}{dt} F = -[F, H].$$

In this abstract scheme the dynamical law can be viewed as defining the dynamical operator d/dt. The dynamical operator d/dt operating on an element of the algebra F is equal to the negative of the Lie product of F with the Hamiltonian element.

The formalization of the concept of a mechanical system can be summarized as follows: A theory of mechanics consists of a linear associative algebra $\mathcal{A}$ of dynamical variables which provides a realization of the Lie algebra $\mathcal{L}$ by derivations. States are normalized, positive linear functionals over $\mathcal{A}$. As particular types of mechanical theories, classical mechanics and quantum mechanics share this abstract structure. They differ in being different representations of this abstract structure.

In classical mechanics the basic dynamical variables are position and momentum, q and p. These are associated with the points on a two-dimensional Euclidean space called the classical phase plane. The elements of the associative algebra $\mathcal{R}_c$ are all analytic functions of p and q.

Classical point mechanics treats of the situation in which ideal, perfect data is available. Under such circumstances a state is specified by a point on the phase plane; that is, by an exact value of q and an exact value of p. Statistical mechanics allows for the possibility that such an exact state description might be unattainable in practice. Where ideal precision is unattainable, a state is not represented by a point on the phase plane but rather in terms of an area of the plane, reflecting the inaccuracy of the specification. The earlier definition of a state in abstract mechanics as any normalized positive function in the state space is intended as a general definition of a state which covers the broader notion of state used in statistical mechanics.

Where ideal data are not attainable, one cannot ask whether a particle is at point (q,p) at t; one can only ask with what probability a particle can be found within a given area of the phase plane at t. This requires that states be given a statistical characterization. Classically, a state is then represented by a probability density on the phase plane, $\hat{\rho}$, called a <u>Liouville density</u>. The states of the system are statistical states characterized by the set of Liouville density functions $\hat{\rho}(q,p)$. Such a function of q and p prescribes the joint probability that the position and momentum of a particle lie within any specified pair of ranges. Consistent with the conditions placed on admissable states, the Liouville density $\hat{\rho}(q,p)$ is normalized

$$\int \hat{\rho}(q,p)dpqp = 1$$

and for any function of the canonical coordinates, $\hat{A}(q,p)$, the expectation value is given by

$$\langle \hat{A}(q,p) \rangle = \int \hat{A}(q,p)\hat{\rho}(q,p)dqdp$$

which is an inner product of the functions $\hat{A}(q,p)$ and $\hat{\rho}(q,p)$ on the phase plane. Note that from the normalization condition it follows that the physical dimension of $\hat{\rho}(q,p)$ is $[(qp)^{-1}]$.

The algebra $\mathcal{A}_c$ serves as a realization of the Lie algebra $\mathcal{L}_c$ by derivations as required, where the Lie product is the familiar Poisson bracket; for $\hat{A}(q,p)$ and $\hat{B}(q,p)$

$$\{\hat{A},\hat{B}\} = \frac{\partial \hat{A}}{\partial q}\,\frac{\partial \hat{B}}{\partial p} - \frac{\partial \hat{A}}{\partial p}\,\frac{\partial \hat{B}}{\partial q}\,.$$

The Poisson bracket, as the Lie product for classical statistical mechanics, gives the dynamical law

$$\frac{d}{dt}\,\hat{\rho}(q,p) = -\{\hat{\rho}(q,p)\ ,\ \hat{H}(q,p)\}$$

where $\hat{\rho}$ is a Liouville density and $H \varepsilon \mathcal{A}_c$ is the Hamiltonian of the system. In this form the dynamical law is called the Liouville equation.

In quantum mechanics position and momentum are also basic dynamical variables, but position and momentum are represented by noncommuting Hermitian operators Q and P. The elements of the algebra $\mathcal{A}_Q$ are all "analytic functions" of these two noncommuting operators considered as formal power series.

Quantum mechanical states are usually represented by wave functions $\Psi(q)$, which are elements of a Hilbert space. When the arguments of the wave functions are position coordinates, q's, one is said to be in the <u>coordinate representation</u>. A quantum mechanical system can also be described by wave functions that take momentum values as arguments, $\phi(p)$. This is called the <u>momentum representation</u>. The two representations are equivalent and are related by a Fourier transformation

$$\phi(p) = \frac{1}{2\pi\hbar} \int dq\ \psi(q)\ e^{-ipq/\hbar}.$$

The observables are then Hermitian operators on the Hilbert space. To strengthen the analogy with classical mechanics, a quantum mechanical state here will be represented by a von Neumann density operator or, in von Neumann's words a "statistical operator" (von Neumann 1955:315). Von Neumann showed that each quantum state represented by a wave function can be associated with a density "matrix" defined (in the coordinate representation) by

$$\rho(q,q') = \psi(q)\psi^*(q').$$

This can be thought of as an infinite dimensional matrix where q labels the rows and q' labels the columns. The defining properties of the density matrix are

(i)	$\rho(q,q') = \rho^*(q',q)$	(self-adjoint)
(ii)	$\mathrm{Tr}\rho = \int \rho(q,q)\,dq = 1$	(normalizable)
(iii)	$\mathrm{Tr}\rho\ A^2 \geqslant 0$, any Hermitian A	(positive definite)

Note that these are exactly the formal properties that an admissable state function must satisfy. The diagonal elements of ρ give the probability density for finding a system at coordinate q. Also note that from (ii) it follows that the physical dimension of the von Neumann density matrix $\rho(q,q')$ is (q^{-1}).

The density matrix also provides a ready characterization of quantum mechanical pure states and mixed states or mixtures. A pure state is a state of maximum specificity and is represented by a single vector in the Hilbert space. A state which is represented with the help of at least two states is said to be a mixture. A pure state cannot be represented as a mixture of two others. When states are described by density matrices one has that

$$\mathrm{Tr}\rho^2 = \int dq dq' \rho(q,q')\rho(q',q) \begin{cases} = 1, & \rho \text{ a pure state} \\ < 1, & \rho \text{ a mixed state.} \end{cases}$$

The density matrix, or the density operator, must map vectors of the Hilbert space linearly into vectors of the Hilbert space. The density operator can be expressed as a linear integral operator

$$\rho(\Psi(q)) = \int \rho(q,q')\Psi(q')dq' = \Psi'(q)$$

where $\rho(q,q')$ is called the kernel of the operator ρ. By the above definition of the von Neumann density, ρ is a positive definite, symmetric kernel; hence, by Mercer's theorem (Courant and Hilbert 1937:138) the density matrix can always be expanded as

$$\rho(q,q') = \sum_n \lambda_n F_n(q) F_n^*(q')$$

where

$$\int |F_n(q)|^2 dq = 1$$

$$\int F_n(q) F_m^*(q) dq = 0, \quad m \neq n$$

that is, the density operator can always be expanded in terms of its eigenfunctions. A pure state corresponds to the situation where only one of the λ_n is non-zero and takes the value unity.

If states are represented by density matrices, the prescription for calculating expectation values becomes

$$\langle A \rangle = \int A(q,q')\rho(q',q)dqdq' = \mathrm{Tr}A\rho.$$

By definition the inner product of two operators A and B is the trace of the product A^*B. This again is consistent with the role of an inner product in the kinematical structure of a mechanical theory, as observed in the above presentation of abstract mechanics.

The algebra $\mathcal{A}_Q$ provides a realization by derivations of $\mathcal{L}_Q$ where the Lie product is given by

$$[A,B] = i\hbar\,[AB - BA].$$

The dynamical law is expressed in terms of the Lie product operation. Where ρ is a von Neumann density and H the Hamiltonian operator, the time development of ρ is given by

$$\frac{d}{dt}\rho = -(i\hbar)^{-1}\,[\rho,H].$$

This equation is due to von Neumann and is frequently called the quantal Liouville equation on the basis of its similarity to the classical Liouville equation.

Approaching the reduction of quantum mechanics to classical mechanics by way of abstract mechanics results in an explicit statement of what an adequate reduction function must achieve. Formally, a successful reduction of quantum mechanics to classical mechanics as $\hbar \to 0$ requires that classical kinematics and dynamics be derivable from quantum theory in this limit. $\mathcal{A}_Q$ and $\mathcal{L}_Q$ are the mathematical structures representing quantum kinematics and dynamics. An adequate reduction function should map $\mathcal{A}_Q$ to $\mathcal{A}_c$ and $\mathcal{L}_Q$ to $\mathcal{L}_c$ homomorphically as $\hbar \to 0$. Such a mapping allows for the recovery of classical kinematics and dynamics from quantum theory in the desired limit. Specifically, the mapping from the elements of $\mathcal{A}_Q$ to $\mathcal{A}_c$ should map the trace of A^*B to an inner product of A(q,p) and B(q,p) on the phase plane, the product in $\mathcal{A}_Q$ to the product in $\mathcal{A}_c$ as $\hbar \to 0$, and $\partial\rho/\partial t$ to $\partial\hat{\rho}/\partial t + O(\hbar)$ where as $\hbar \to 0$ the error term $O(\hbar)$ also goes to zero.

Classical statistical mechanics represents states of a system by Liouville density functions on the classical phase plane. In von Neumann's formulation of quantum theory, states are represented by von Neumann density operators. Eugene Wigner, in "On the Quantum Correction for Thermodynamical Equilibrium" (Wigner 1932), derived a transformation function, the Wigner transformation, which maps von Neumann density operators to density functions on the classical phase plane. This suggests that the Wigner transformation might serve as a reduction function from quantum mechanics to classical statistical mechanics.

Wigner's paper appeared at a time when attempts were being made to interpret quantum theory as a theory of classical probabilistic or stochastic processes. These attempts were based on formal analogies between quantum equations and classical transport or diffusion equations.

The similarity between the Schrödinger equation and a classical diffusion equation was noted by Schrödinger (Schrödinger 1931, 1932). However, in Schrödinger's mind the disanalogies far outweighed the analogies and he neither endorsed nor suggested a stochastic interpretation of quantum theory. In the same vein, Furth showed that just as there is a stochastic analog to the Schrödinger equation there is also a stochastic analog to the Heisenberg uncertainty relations (Furth 1933).

Wigner's 1932 paper encouraged the search for classical interpretations of the quantum theory. His result suggested that the relation between quantum and classical theory might be stronger than mere formal analogy. Wigner observed that the relative probability of momentum and position for a classical statistical density function is given by

$$\rho(q,p)\,dq\,dp = e^{(-1/kT)H}\,dq\,dp$$

where k is Boltzmann's constant, T the absolute temperature, and $H = p^2/2m+V$ the classical Hamiltonian. For a quantum mechanical system, the expectation value of a physical quantity is given by von Neumann's prescription

$$\mathrm{Tr}(A\ e^{(-1/kT)H})$$

where A is the operator representing the quantity, H the quantum Hamiltonian, and $e^{(-1/kT)H}$ the von Neumann density operator representing the state in question. Explicit calculations using the von Neumann density proved to be cumbersome. Wigner's insight was that for a wave function $\psi(q)$, and hence for its associated von Neumann density, a density function ρ on the phase plane could be constructed by

$$\rho(q,p) = \int \psi(q-\tau/2)\ e^{(i/\hbar)\tau p}\ \psi^*(q+\tau/2)\,d\tau.$$

The resulting density $\rho(q,p)$, called the Wigner density, is always real but is not everywhere positive. It has the following interesting properties: (i) When integrated with respect to p it yields the correct quantum mechanical expectation values for position; (ii) when integrated with respect to q it yields the correct quantum mechanical expectation values for momentum; (iii) by applying classical techniques it yields the correct quantum mechanical expectation values of any function of position or any function of momentum for a given state; (iv) it similarly yields the correct expectation values for a sum of a function of position and a function of momentum.

The appearance of the Wigner transformation encouraged the belief that quantum theory could be interpreted as a classical probabilistic theory. The Wigner transformation maps quantum mechanical states to density functions on the classical phase plane whereupon in many cases classical methods yield the correct quantum mechanical expectation values. Wigner himself recognized the major obstacle to following through on such a program: The Wigner transformation is not everywhere positive. Probabilities must be non-negative; hence, Wigner believed that his density function could not be consistently interpreted as a simultaneous probability. Wigner viewed his transformation function as a discovery of practical importance that facilitated calculations. He felt that even though the Wigner density can assume negative values, this "must not hinder the use of it in calculations as an auxiliary function which obeys many relations we would expect from such a probability" (Wigner 1932:751).

Other attempts were made at defining an appropriate joint distribution function which would satisfy the conditions on a probability density. Most notable of these attempts were the papers of Groenewold (1946) and Moyal (1949). Moyal concluded that the theoretical difficulties with any such joint distribution are such that it could not be employed to generate an interpretation of quantum mechanics as a classical statistical theory. However, such functions could be used to solve quantum mechanical problems by the methods of classical probability theory. More recently Cohn has shown that no such transformation function can preserve the desired functional relationships between observables (Cohn 1966). These findings have relegated the Wigner transformation to the realm of practical problem solving and have discouraged the belief that one could formulate quantum theory as a classical probability theory. (In Chapter III, these theoretical deficiencies of the Wigner transformation will be related to the proofs that a hidden variable interpretation of quantum mechanics cannot be given.)

The theoretical deficiencies of the Wigner transformation suggest that it is impossible to use the transformation to interpret quantum mechanics as a classical statistical theory. However, the deficiencies of the Wigner transformation are not such as to preclude its use as a reduction function to show that in some sense, specifically in some appropriate limit, quantum mechanics reduces to classical statistical mechanics. The Wigner transformation depends on the value of $\hbar$; hence, it does make sense to inquire as to what becomes of the image of a von Neumann density on the classical phase plane as $\hbar \to 0$. It is also non-singular, as will be shown below, so the transformation and its inverse provide a means to go back and forth between the quantum and classical formalisms.

Although the Wigner transformation appears to satisfy several requirements on a reduction function, nothing has been said about the theoretical basis of this transformation. The transformation is not merely a formal trick that happens to yield the desired result. The next task will be to show that the Wigner transformation, W, and its inverse, W^{-1}, can be derived uniquely given the structure of classical and quantum mechanics.

Before embarking on the derivation of the Wigner transformation, some insight can be gained into the nature of the task by examining the structure of the transformation. The Wigner transformation takes a von Neumann density

$$\rho(q,q') = \sum_{\ell} c_{\ell}\psi(q)\psi^{*}(q') = \sum_{\ell} c_{\ell}\rho_{\ell}(q,q')$$

where $c_{\ell} \gtrless 0$ all ℓ, and $\sum_{\ell} c_{\ell}=1$, into a Wigner density, a function of p and q, by

$$\hat{\rho}(q,p) = \frac{1}{2\pi}\int da e^{ipa}\, \rho(q-a\hbar/2,\ q+a\hbar/2).$$

The action of the Wigner transformation on a von Neumann density can be described by saying that it consists of a linear substitution of variables, $q-a\hbar/2$ for q and $q+a\hbar/2$ for q', and a Fourier transformation of the resulting von Neumann density.

One fact about the quantum and classical theories will be used in the derivation of the Wigner transformation. The derivation will exploit the fact that both classical mechanics and quantum mechanics are invariant under the transformations of the Galilean group. The requirement of Galilean invariance for both theories dictates that this invariance must be preserved under the reduction. Hence, an additional requirement on W and W^{-1} is that they must be invariant under the transformations of the Galilean group, which is to say that W and W^{-1} must commute with the action of the Galilean group. If the transformation can be derived from quantum and classical theory under this assumption, then it can be justifiably claimed that the Wigner transformation provides a natural reduction function between the quantum and classical theories.

The requirement that W and W^{-1} be invariant under the Galilean group leads naturally to the employment of various notions from representation theory in the derivation of the Wigner transformation. A representation of an abstract topological group on a vector space H is a homomorphism

$$\Pi: \quad G \to GL(H)$$

of G into a group of continuous linear automorphisms of the space H such that for every element v of H the map of G into GL(H) given by

$$x \longmapsto \Pi(x)v$$

is continuous. A given abstract group can have different representations on different spaces. For example,

$$\Pi: \quad G \to GL(H)$$
$$\Pi': \quad G \to GL(H')$$

Representations of groups on a vector space form an algebraic category where the morphism of the category is called an intertwining operator or a coupling operator. An intertwining operator between representations is a continuous linear map, A: $H \to H'$, between the spaces such that for every element x of the group G the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "A"] \arrow[d, "\Pi(x)"'] & H' \arrow[d, "\Pi'(x)"] \\ H \arrow[r, "A"'] & H' \end{tikzcd}$$

In other words, the intertwining operator A commutes with the action of the group on H and H'.

The assumption of Galilean invariance for both quantum and classical theory requires, as mentioned above, that W and W^{-1} commute with the action of the Galilean group. Hence, W and W^{-1} are intertwining operators between the Galilean group as represented on the classical phase plane and the Galilean group as represented on the space of von Neumann densities.

The Galilean group, which consists of all translations of position and concurrent changes in velocity (or in momentum) is an abelian group, as a total change in position and momentum is indifferent to the order in which incremental changes are imposed. Any element of the Galilean group evidently may be expressed as a product of two elements, drawn from the two subgroups which give translations in position alone and shifts in momentum alone. An element of the former subgroup, which translates position by an amount q, we will call Q_q, a member of the latter subgroup which shifts momentum by an amount p we will call P_p.

The subgroup elements have an evident natural action upon functions $\rho(q',p')$ over the classical phase plane, namely

$$Q_q\rho(q',p') = \rho(q'-q,p') \qquad (1)$$

$$P_p\rho(q',p') = \rho(q',p'-p). \qquad (2)$$

The set of functions on the phase plane comprise a "function space," that is, a very big vector space, whereby equations (1) and (2) define actions of Q_q and P_p as linear operators upon a vector space. That is to say (1) and (2) define a representation of the Galilean group. Now a representation (with complex scalars) of an <u>abelian</u> group may be resolved into irreducible representations which are <u>one dimensional</u>, with each group element represented by a complex number on the unit circle. We now resolve our representation, of Galilean actions upon functions of the phase plane, in this way. We seek a function $\hat{\rho}_{ks}(q',p')$ upon the phase plane which has the two properties that

$$Q_q\hat{\rho}_{ks} = e^{ikq}\,\hat{\rho}_{ks} \qquad (3)$$

$$P_p\hat{\rho}_{ks} = e^{isp}\,\hat{\rho}_{ks}. \qquad (4)$$

Evidently the function sought is

$$\rho_{ks}(q',p') = e^{-ikq'} e^{-isp'} \tag{5}$$

as substitution of (5) into (1) and (2) shows at once. Moreover, a general normalized function on the phase plane can be resolved as a superposition into 1-dimensional representation functions of the form (5), as

$$\rho(q',p') = \int dkdse^{-ikq'} e^{-isp'} c(k,s) \tag{6}$$

simply asserts that normalized functions on the phase plane may be expressed as Fourier integrals. The appropriate weighting coefficient c(k,s) for any given Liouville density $\rho(q,p)$ is given by the familiar Fourier integral inversion formula

$$c(k,s) = \frac{1}{(2\pi)^2} \int dqdp\ e^{ikq} e^{isp} \rho(q,p). \tag{7}$$

We will now express the way in which Galilean transformations act upon the quantum mechanical von Neumann density operators. As these density operators correspond to integral kernels, which are members of a function space, we will again obtain a representation of the Galilean group as a set of linear operators. This representation may be resolved into irreducible one dimensional representations, which prove to have a one-to-one relationship with counterparts which we have already found on the classical phase plane, at equation (5). Finally we may define a linear transformation which carries each irreducible subspace in the space of von Neumann densities to its irreducible counterpart on the classical phase plane; that transformation is the Wigner transformation.

The Galilean subgroup of position transformations perform shifts in the origin of the position coordinate; the member Q_q shifts the origin by an amount q. This subgroup thus has a natural action on functions $\rho(q',q'')$ in the space of von Neumann densities:

$$Q_q\rho(q',q'') = \rho(q'-q,\ q''-q). \tag{8}$$

The action of P_p on von Neumann densities is less immediate. We note first its natural action upon wave-functions $\psi(q')$: If for the set of momenta p', $\psi_p(q')$ describes the set of quantum-mechanical pure states, each of which corresponds to momentum p', then P_p by its definition gives *

$$P_p\psi_{p'}(q') = \psi_{p'-p}(q'). \tag{9}$$

The momentum pure states are by definition the eigenfunctions of translation

$$\psi_{p'}(q') = e^{-ip'q'/\hbar} \tag{10}$$

whence by substitution in (9), and back to (10),

$$P_p\psi_{p'}(q') = e^{-i(p'-p)q'/\hbar} = e^{ipq'/\hbar}\,\psi_{p'}(q'). \tag{11}$$

We note the coefficient exp $(i\,p\,q'/\hbar)$ is <u>independent</u> of the momentum pure state's momentum value p'. As <u>any</u> pure state $\psi(q')$ may be built up from a superposition of momentum states, we have in general

$$P_p\psi(q') = e^{ipq'/\hbar}\,\psi(q'). \tag{12}$$

Now a general von Neumann density $\rho(q',q'')$ may be constructed from a superposition of densities for pure states, of the form

$$\rho_\psi(q',q'') = \psi(q')\psi(q'')^* \tag{13}$$

and by (12) the action of P_p on such a density operator is

$$P_p\rho_\psi(q',q'') = (P_p\psi(q'))(P_p\psi(q''))^* = e^{ipq'/\hbar}\psi(q')e^{-ipq''/\hbar}\psi(q'')^* \tag{14}$$

As the coefficient $\exp(ipq'/\hbar)\cdot\exp(-ipq''/\hbar)$ does not depend on ψ, by the superposition property we have in general

$$P_p\rho(q',q'') = e^{ip(q'-q'')/\hbar}\rho(q',q''). \tag{15}$$

Equations (8) and (15) are the Galilean transformation counterparts, for von Neumann density functions, of the Galilean transformation equations on the classical phase plane, (1) and (2). These Galilean transformation equations are both less symmetric and less familiar than their classical counterparts (1) and (2), whence the job of finding the one-dimensional irreducible representation functions is not quite trivial. However, the abelian nature of the Galilean group implies that such functions $\rho_{ks}(q',q'')$ exist, and satisfy

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*The minus sign appears as a matter of convention: We index the group element in accordance with its action on the function rather than the coordinate system. This is the same convention which we have already adopted in the classical case.

$$Q_q \rho_{ks} = e^{ikq} \rho_{ks} \tag{16}$$

$$P_p \rho_{ks} = e^{isp} \rho_{ks} \tag{17}$$

in analogy to equations (3) and (4) for the classical case.

If we eliminate $Q_q \rho_{ks}$ from equations (8) and (16), and likewise eliminate $P_p \rho_{ks}$ from equations (15) and (17), we obtain

$$\rho_{ks}\,(q'-q,q''-q) = e^{ikq} \rho_{ks}(q',q'') \tag{18}$$

$$e^{ip(q'-q'')/\hbar} \rho_{ks}(q',q'') = e^{isp} \rho_{ks}(q',q''). \tag{19}$$

These two equations, which express both the action of the Galilean group on quantum-mechanical density operators, and the defining property of irreducible one-dimensional representations, must determine the form of the representation functions $\rho_{ks}(q',q'')$. We now undertake that determination, starting with equation (18).

In form, (18) is a functional equation. For convenience let

$$\rho_{ks} = \exp r \tag{20}$$

and take the logarithm of equation (18); the result is

$$r(q'-q,\ q''-q) - r(q',\ q'') = i\ k\ q. \tag{21}$$

This is a linear, inhomogeneous functional equation for the unknown function r. It shares a generic property of linear inhomogeneous equations: Its general solution is of the form

$$r = r_p + r_H \tag{22}$$

where r_p is <u>any particular</u> solution to equation (21) and r_H is the <u>most general</u> solution to the homogeneous equation

$$r_H(q'-q,\ q''-q) - r(q',\ q'') = 0. \tag{23}$$

We observe that (21) has the immediate particular solution

$$r_p(q',q'') = -ik\,\frac{q'+q''}{2}. \tag{24}$$

(We have chosen to maintain q' and q" on a formally symmetric footing so far.) As the homogeneous equation (23) must hold for all free choices of q, q', q", it must hold if in particular we choose q = q", whence (23) becomes

$$r_H(q',q'') = r(0,q'-q''). \tag{25}$$

Thus $r_H(q',q'')$ must be a function only of the difference q'-q". We observe that an arbitrary function of that difference, $r_H(q'-q'')$, evidently solves the homogeneous equation (23), whence the most general solution to equation (21) is

$$r(q',q'') = -ik\frac{q'+q''}{2} + r_H(q'-q'') \tag{26}$$

where $r_H(q)$ is arbitrary. If we let exp r_H=F, then (20) gives back the corresponding solution to the original functional equation (18):

$$\rho_{ks}(q',q'') = e^{-ik(\frac{q'+q''}{2})}F(q'-q'') \tag{27}$$

where F(u) is an arbitrary function. We observe that the general solution (27) to the functional equation (18) could have been derived without resort to logarithmic transformation: That transformation simply enabled us to appeal to classical results which apply to linear equations, which saved our having to prove the "exponentials" of those results for ourselves.

We have not yet used the "momentum shift" equation (19), whose demand upon ρ_{ks} now can, at most, specify the form of F(q'-q"). Substitute the solution (27) into (19):

$$e^{ip(q'-q'')/\hbar}F(q'-q'') = e^{isp}F(q'-q'') \tag{28}$$

where we have divided both sides by the common non-zero factor exp(-ik(q'+q")/2). To simplify (28) let q'-q" = u, whence

$$e^{\frac{ip}{\hbar}(u-\hbar s)} F(u) = F(u) \tag{29}$$

Thus either

$$F(u) = 0 \tag{30}$$

$$\text{or} \quad u - \hbar s = 0, \tag{31}$$

so F(u) must be of the form

$$F(u) = A\,\delta(u - \hbar s) \tag{32}$$

where A is a yet undetermined constant. Thus finally, from both equations (18) and (19)

$$\rho_{ks}(q',q'') = e^{-ik(\frac{q'+q''}{2})} A\delta(q'-q''-\hbar s) \quad (33)$$

which follows most directly from (32) and (27). The constant A we will regard as fixed, at a value which we will choose below for our convenience.

The functions $\rho_{ks}(q',q'')$ given by (33) are the 1-dimensional representation functions for the action of the Galilean group upon the space of von Neumann densities, as substitution into equations (17), (18) will demonstrate (with equation (18) the vanishing-property of the δfunction must be used). These representation functions also form a complete basis for the space of von Neumann densities, in the usual sense that an arbitrary density may be expanded in the form

$$\rho(q',q'') = \int dkds\rho_{ks}(q'q'')c(k,s). \quad (34)$$

The proof, which we only sketch, is the following: The integral on s may be performed at once, as it only involves the δfunction; the remaining integral is a Fourier transformation from the variable k to the variable $(q'+q'')/2$, and completeness follows from the non-singular nature of the Fourier transform.

The Wigner transformation now may be defined by its action on the irreducible representation functions:

$$W(Ae^{-ik(\frac{q'+q''}{2})}\delta(q'-q''-\hbar s)) = e^{-ikq}e^{-isp}, \quad (35)$$

or

$$W^{-1}(e^{-ikq}e^{-isp}) = Ae^{-ik(\frac{q'+q''}{2})}\delta(q'-q''-\hbar s), \quad (36)$$

where q, p are the coordinate and momentum of the classical phase plane. We note that the Wigner transformation is parametric in Planck's constant $\hbar$, but this dependence appears at only one place, where it relates the scale of the coordinate q to that of the reciprocal momentum s.

All that remains is to derive expedient expressions for the ways in which W and W^{-1} carry a general member of the one function space to a member of the other. The easier choice is to first calculate how W^{-1}

moves a function on the phase plane to the space of von Neumann densities. Using equation (36), let W^{-1} act upon $\hat{\rho}(q,p)$ as given by equation (6):

$$W^{-1}\hat{\rho}(q,p)=\rho(q',q'')=A\int dkds\ e^{-ik\left(\frac{q'+q''}{2}\right)}\delta(q'-q''-\hbar s)\,c(k,s) \tag{37}$$

As $ds = (1/\hbar)\ d(\hbar s)$, the integral on s is immediate and gives

$$\rho(q',q'')=(1/\hbar)A\int dke^{-ik\left(\frac{q'+q''}{2}\right)}c\left(k,\frac{q'-q''}{\hbar}\right). \tag{38}$$

Now evaluation of $c\left(k, \frac{q'-q''}{\hbar}\right)$ in terms of $\hat{\rho}(q,p)$, from equation (7) gives

$$\rho(q',q'')=(1/\hbar)\frac{A}{(2\pi)^2}\int dpdqdk\ e^{ik\left(q-\frac{q'+q''}{2}\right)}e^{i\left(\frac{q'-q''}{\hbar}\right)p}\hat{\rho}(q,p). \tag{39}$$

The integral on k is

$$\frac{1}{2\pi}\int dke^{ik\left(q-\frac{q'+q''}{2}\right)} = \delta\left(q-\frac{q'+q''}{2}\right) \tag{40}$$

so that in (39) the integral on q simply replaces $\hat{\rho}(q,p)$ by $\hat{\rho}\left(\frac{q'+q''}{2},p\right)$ and the final result is

$$\rho(q',q'') = \frac{A}{2\pi\hbar}\int dpe^{i\left(\frac{q'-q''}{\hbar}\right)p}\hat{\rho}\left(\frac{q'+q''}{2},p\right). \tag{41}$$

We may now evaluate the constant A by requiring that the Wigner transformation preserve the property of normalization, namely:

$$\int dq'\rho(q',q') = \int dqdp\hat{\rho}(q,p) \tag{42}$$

Set $q''=q'$ in (41) and integrate on q'; we see that (42) is immediately satisfied if we let

$$A = 2\pi\hbar \tag{43}$$

Equation (41) for W^{-1} is in the form of a Fourier transform on the variable $(q'-q'')$, with the other independent variable $(q'+q'')/2$ simply playing the role of a fixed parameter.

The inverse transformation, for W, is easily evaluated. In (41), let $q'-q''=x$ and $(q'+q'')/2=q$. Solving these relations simultaneously for q' and q'' yields

$$q' = q + x/2 \qquad (44)$$
$$q'' = q - x/2$$

Substituting these values of q' and q" into (41) gives

$$r(x,q) \equiv \rho(q + \frac{x}{2}, q - \frac{x}{2}) = \frac{A}{2\pi\hbar}\int dpe^{ixp/\hbar}\,\hat{\rho}(q,p) \qquad (45)$$

or

$$\frac{1}{A}\,r(x,q) = \frac{1}{2\pi}\int dxe^{ixp/\hbar}\,\hat{\rho}(q,p). \qquad (46)$$

Thus, we have expressed the von Neumann density as a Fourier transform of a Liouville density, where p is the Fourier variable and q is a parameter. But then by the Fourier integral theorem we have that

$$\hat{\rho}(q,p) = \int dxe^{-ixp/\hbar}\frac{1}{A}r(x,q) = \frac{1}{A}\int dxe^{-ixp/\hbar}\,\rho(q + \frac{x}{2}, q - \frac{x}{2}). \qquad (47)$$

If in (47), we let $x = -a\hbar$, then

$$\hat{\rho}(q,p) = \frac{1}{A}\int_{-\infty}^{\infty} dae^{-ixp/\hbar}\,\rho(q + \frac{x}{2}, q - \frac{x}{2}) = \frac{\hbar}{2\pi\hbar}\int_{\infty}^{-\infty} dae^{iap}\,\rho(q - \frac{a\hbar}{2}, q + \frac{a\hbar}{2}) \qquad (48)$$

or

$$\hat{\rho}(q,p) = \frac{1}{2\pi}\int dae^{iap}\,\rho(q - \frac{a\hbar}{2}, q + \frac{a\hbar}{2}) \qquad (49)$$

which is exactly the Wigner transformation as it was quoted above.

W and W^{-1} provide one-to-one mappings between the space of classical density functions and the space of von Neumann densities. By linearity this result extends to elements of $\mathcal{A}_Q$ and $\mathcal{A}_c$. The $\hat{\rho}_{ks}(q,p)$ form a complete basis for the classical space. By elementary Fourier analysis, any function $\hat{A}(q,p)$ on the classical space can be expressed as

$$\hat{A}(q,p) = \iint_{ks} \alpha_{sk}\hat{\rho}_{ks}(q,p)\; dkds$$

where $\alpha_{sk} = \int \hat{\rho}^*_{ks}(q,p)\hat{A}(q,p)dqdp$. This is a statement of completeness. W and W^{-1}, as intertwining operators, take one dimensional representations to one dimensional representations; hence, the completeness is inherited by the images in the space of von Neumann densities, and the $\rho_{ks}(q',q'')$ form a complete basis for the Hilbert space on which the von Neumann densities are defined. Because of this relation every operator A can be expressed as

$$A = \iint\limits_{ks} \alpha_{sk}\rho_{ks}(q',q'')\,dk\,ds$$

where $\alpha_{sk}=\mathrm{Tr}(\rho_{ks}A)$.

This derivation of the Wigner transformation from the assumption of Galilean invariance, justifies the Wigner transformation as a natural choice for a reduction function between quantum and classical mechanics. Even though it is a natural choice for a reduction function, if it is to be an adequate reduction function, it must satisfy the formal conditions on such a function. These formal conditions, based on the discussion of abstract mechanics, are that the function map $\mathcal{A}_Q$ into $\mathcal{A}_c$ and $\mathcal{L}_Q$ into $\mathcal{L}_c$ homomorphically as $\hbar \to 0$. Several propositions concerning W and W^{-1} will be established showing that the Wigner transformation fulfills these formal conditions and hence that classical mechanics is a <u>bona fide</u> limiting case of quantum mechanics. The following operations will be investigated: (i) Products of von Neumann densities as $\hbar \to 0$; (ii) commutators of von Neumann densities as $\hbar \to 0$; (iii) anti-commutators of von Neumann densities as $\hbar \to 0$.

The general strategy of the proofs is to begin with two Liouville densities defined on the classical phase plane. Apply the inverse Wigner transformation to carry the Liouville densities to the space of von Neumann densities. Compose the resulting von Neumann densities in the appropriate (quantum mechanical) manner. Apply the Wigner transformation to the result and show that as $\hbar \to 0$ the image of the Wigner transformation is the result of the analogous composition of the original classical densities in the classical phase plane. The reason for starting with classical densities and carrying them to the space of the von Neumann densities, composing, and returning to the classical space is merely the need to be explicit as to what stays fixed when limits are being taken. The arbitrary choice is that the classical density functions stay fixed.

Let ρ, σ denote von Neumann densities, $\hat{\rho}$, $\hat{\sigma}$ denote functions on the classical phase plane, $\tilde{\rho}$ denote the Fourier transform of $\hat{\rho}[(q'+q'')/2, p]$ with respect to its second argument, and $\tilde{\rho}_k$ denote the first derivative of $\tilde{\rho}$ with respect to its $k^{\underline{th}}$ argument. The basic result to be established is the following:

Proposition 1. $\rho\sigma(q',q'') = W^{-1}\hat{\rho}\hat{\sigma}(q,p)+O(\hbar)$.

Proof. By definition of the inverse Wigner transformation

$$W^{-1}\hat{\rho}(q,p)=\rho(q',q'')=\int dp\, e^{(i/\hbar)p(q'-q'')}\hat{\rho}[\frac{q'+q''}{2},p]=\tilde{\rho}[\frac{q'+q''}{2},\frac{q'-q''}{\hbar}] \tag{50}$$

$$W^{-1}\hat{\sigma}(q,p)=\int dp\; e^{(i/\hbar)p(q'-q'')}\hat{\sigma}[\frac{q'+q''}{2},\; p]=\tilde{\sigma}[\frac{q'+q''}{2},\; \frac{q'-q''}{\hbar}]. \tag{51}$$

The product of von Neumann densities is given by

$$\rho\sigma(q',q'') = \int dy\; \rho(q',y)\sigma(y,q'').$$

Let $q'' = y$ in (50) and $q' = y$ in (51). Then

$$\rho\sigma(q',q'') = \int dy\tilde{\rho}[\frac{q'+y}{2},\; \frac{q'-y}{\hbar}]\quad \tilde{\sigma}[\frac{y+q''}{2},\; \frac{y-q''}{\hbar}]. \tag{52}$$

Add and subtract $q''/2$ from the first argument of $\tilde{\rho}$ and add and subtract $q'/2$ from the first argument of σ in (52)

$$\rho\sigma(q',q'')=\int dy\tilde{\rho}[\frac{q'+q''}{2}+\frac{(-q''+y)}{2},\; \frac{q'-y}{\hbar}]\tilde{\sigma}[\frac{q'+q''}{2}+\frac{(-q'+y)}{2},\; \frac{y-q''}{\hbar}]$$

$$=\int dy\tilde{\rho}[\frac{q'+q''}{2}+\frac{\hbar}{2}\frac{(-q''+y)}{\hbar},\frac{q'-y}{\hbar}]\tilde{\sigma}[\frac{q'+q''}{2}+\frac{\hbar}{2}\frac{(-q'+y)}{\hbar},\frac{y-q''}{\hbar}] \tag{53}$$

Expand the integral in (53) as a power series in $\hbar$, obtaining

$$\rho\sigma(q',q'') = \int dy\Big[\tilde{\rho}[\frac{q'+q''}{2},\; \frac{q'-y}{\hbar}] + \frac{\hbar}{2}\frac{y-q''}{\hbar}\tilde{\rho}_1[\frac{q'+q''}{2},\; \frac{q'-y}{\hbar}]\ \mathrm{X}$$

$$\mathrm{X}\ \tilde{\sigma}[\frac{q'+q''}{2},\; \frac{y-q''}{\hbar}] + \frac{\hbar}{2}\frac{y-q'}{\hbar}\tilde{\sigma}_1[\frac{q'+q''}{2},\; \frac{y-q''}{\hbar}]\Big] + O(\hbar^2). \tag{54}$$

Carrying out the multiplication and recombining terms yields

$$\rho\sigma(q',q'') = \int dy\tilde{\rho}[\frac{q'+q''}{2},\; \frac{q'-y}{\hbar}]\;\tilde{\sigma}\;[\frac{q'+q''}{2},\; \frac{y-q''}{\hbar}] +$$

$$+\frac{\hbar}{2}\int dy\;\Big[\tilde{\rho}_1[\frac{q'+q''}{2},\; \frac{q'-y}{\hbar}]\cdot\frac{y-q''}{\hbar}\tilde{\sigma}[\frac{q'+q''}{2},\; \frac{y-q''}{\hbar}] -$$

$$-\frac{q'-y}{\hbar}\tilde{\rho}[\frac{q'+q''}{2},\; \frac{q'-y}{\hbar}]\;\tilde{\sigma}_1\;[\frac{q'+q''}{2},\; \frac{y-q''}{\hbar}]\Big] + O(\hbar^2). \tag{55}$$

In both integrals $(q'+q'')/2$ appears simply as a parameter. For functions $\ell(x)$, $k(x)$ on the real line, their convolution is defined as

$$\ell * k(x) = \int dy\; \ell(x-y)k(y) = \int dy\; k(x-y)\ell(y).$$

The same rule of convolution holds for translations $\ell(x-z)$, $k(x-z)$

$$\ell * k(x-z) = \int dy\, \ell(x-y)k(x-z) = \int dw\, \ell[(x-z)-w]k(w).$$

The first integral in (55) is a convolution. A fundamental result of Fourier analysis is the Convolution Theorem: The Fourier transform of the product of two functions equals the convolution of the Fourier transform. Accordingly,

$$\int dy\, \tilde{\rho}[\frac{q'+q''}{2}, \frac{q'-y}{\hbar}]\, \tilde{\sigma}[\frac{q'+q''}{2}, \frac{y-q''}{\hbar}] = \widetilde{\rho\cdot\sigma}[\frac{q'+q''}{2}, \frac{q'-q''}{\hbar}] = W^{-1}\hat{\rho}\hat{\sigma}(q,p) \qquad (56)$$

Another basic relation in Fourier analysis is that if $\tilde{f}(\xi)$ is the Fourier transform of $\hat{f}(x)$, then $\xi\tilde{f}(\xi)$ is the Fourier transform of $i d/dx\, \hat{f}(x)$. This relation and the Convolution Theorem allow one to express the second integral in (55) as

$$\frac{\hbar}{2}\int dy \Big[\tilde{\rho}_1[\frac{q'+q''}{2}, \frac{q'-y}{\hbar}] \cdot \frac{y-q''}{\hbar}\, \tilde{\sigma}[\frac{q'+q''}{2}, \frac{y-q''}{\hbar}] -$$

$$- \frac{q'-y}{\hbar}\, \tilde{\rho}[\frac{q'+q''}{2}, \frac{q'-y}{\hbar}]\, \tilde{\sigma}_1[\frac{q'+q''}{2}, \frac{y-q''}{\hbar}]\Big]$$

$$= \frac{\hbar}{2}\Big[\widetilde{\rho_1\cdot i\sigma_2}[\frac{q'+q''}{2}, \frac{q'-q''}{\hbar}] - \widetilde{i\rho_2\cdot\sigma_1}[\frac{q'+q''}{2}, \frac{q'-q''}{\hbar}]\Big]$$

$$= \frac{i\hbar}{2} W^{-1}[\hat{\rho}_1\cdot\hat{\sigma}_2(q,p) - \hat{\rho}_2\cdot\hat{\sigma}_1(q,p)]$$

$$= \frac{i\hbar}{2} W^{-1}[\frac{\partial}{\partial q}\hat{\rho}(q,p)\frac{\partial}{\partial p}\hat{\sigma}(q,p) - \frac{\partial}{\partial p}\hat{\rho}(q,p)\frac{\partial}{\partial q}\hat{\sigma}(q,p)]. \qquad (57)$$

So combining (56) and (57)

$$\rho\sigma(q',q'') = W^{-1}\hat{\rho}\hat{\sigma}(q,p) + \frac{i\hbar}{2} W^{-1}\{\hat{\rho}(q,p), \hat{\sigma}(q,p)\} + O(\hbar^2). \qquad (58)$$

Therefore, as $\hbar \to 0$ only the first term in this expression remains and

$$\rho\sigma(q',q'') = W^{-1}\hat{\rho}\hat{\sigma}(q,p). \qquad (59)$$

In the limit as $\hbar \to 0$ under the action of the Wigner transformation the noncommutative operator product in the space of von Neumann densities becomes the commutative product defined in the classical space.

The results for commutators and anticommutators are simple corollaries of Proposition 1.

<u>Proposition 2</u>. $-\frac{i}{\hbar}[\rho\sigma-\sigma\rho] = W^{-1}\{\varrho,\varsigma\} + O(\hbar^2)$.

<u>Proof</u>. Suppressing the arguments of the density functions

$$-\frac{i}{\hbar}[\rho\sigma-\sigma\rho] = -\frac{i}{\hbar}W^{-1}\varrho\varsigma - \frac{i}{\hbar}\left[\frac{i\hbar}{2}\right]W^{-1}\{\varrho,\varsigma\} +$$

$$+\frac{i}{\hbar}W^{-1}\varsigma\varrho + \frac{i}{\hbar}\left[\frac{i\hbar}{2}\right]W^{-1}\{\varsigma,\varrho\} + O(\hbar^2)$$

$$= -\frac{i}{\hbar}W^{-1}(\varrho\varsigma-\varsigma\varrho) + \frac{1}{2}W^{-1}\{\varrho,\varsigma\} - \frac{1}{2}W^{-1}\{\varsigma,\varrho\} + O(\hbar^2)$$

$$= \frac{1}{2}W^{-1}\{\varrho,\varsigma\} + \frac{1}{2}W^{-1}\{\varrho,\varsigma\} + O(\hbar^2) = W^{-1}\{\varrho,\varsigma\} + O(\hbar^2).$$

So in the limit as $\hbar \to 0$ under the action of the Wigner transformation the Lie product in the space of von Neumann densities becomes the Lie product in the classical space.

<u>Proposition 3</u>. $(\varrho\varsigma+\varsigma\varrho) = 2W^{-1}\varrho\varsigma + \frac{\hbar^2}{2}\left[\frac{\partial^2\varrho}{\partial p\partial q}\frac{\partial^2\varsigma}{\partial p\partial q}\right] + O(\hbar^3)$.

<u>Proof</u>.

$$\rho\sigma+\sigma\rho = W^{-1}(\varrho\varsigma+\varsigma\varrho) + \frac{i\hbar}{2}W^{-1}\{\varrho,\varsigma\} + \frac{i\hbar}{2}W^{-1}\{\varsigma,\varrho\} + O(\hbar^2) = 2W^{-1}\varrho\varsigma + O(\hbar^2).$$

If the power series expansion of $\rho\sigma(q',q'')$ in (54) is carried out one more term, the Convolution Theorem and the result on the relation between Fourier transforms and their derivatives can again be applied, allowing one to calculate $O(\hbar^2)$ as

$$O(\hbar^2) = \frac{\hbar^2}{2}W^{-1}\left[\frac{\partial^2\varrho}{\partial p\partial q}\frac{\partial^2\varsigma}{\partial p\partial q}\right].$$

Then

$$\rho\sigma+\sigma\rho = 2W^{-1}\varrho\varsigma + \frac{\hbar^2}{2}W^{-1}\left[\frac{\partial^2\varrho}{\partial p\partial q}\frac{\partial^2\varsigma}{\partial p\partial q}\right] + O(\hbar^3).$$

Every operator in $\mathcal{A}_Q$ and <u>a fortiori</u> every element of $\mathcal{L}_Q$ can be expanded in terms of the functions $\rho_{ks}(q',q'')$. Propositions 1 and 2 show that the Wigner transformation carries the operator product of $\mathcal{A}_Q$ to the commuting product of $\mathcal{A}_c$ and the Lie product of $\mathcal{L}_Q$ (commutator bracket) to the Lie product of $\mathcal{L}_c$ (Poisson bracket). In the limit as $\hbar \to 0$, the Wigner transformation carries the quantum mechanical operation to its classical counterpart homomorphically. The Wigner transformation is then not only a natural choice for a reduction function; it is also a reduction function which satisfies the formal requirements that have been placed on such a function. Now it must be determined whether the proposed reduction of quantum mechanics to classical mechanics by means of the Wigner transformation meets the nonformal conditions on a reduction.

III. NONFORMAL CONDITIONS ON THE REDUCTION

The results of the previous chapter show that the formal requirements for a successful reduction of quantum mechanics to classical mechanics are satisfied: A reduction function, the Wigner transformation, maps $\mathcal{R}_Q \to \mathcal{R}_c$ and $\mathcal{L}_Q \to \mathcal{L}_c$ in such a way that as $\hbar \to 0$ the classical kinematical and dynamical laws are recovered from the quantum structure. Yet several nonformal requirements must be satisfied if this reduction is to be entirely adequate.

These nonformal requirements derive directly from the philosophical problem outlined in Chapter 1. The philosopher's interest in reduction is a response to a common phenomenon in the development of science. As a science develops theories are proposed and tested; some are accepted as well-confirmed. In many instances, these well-confirmed theories are found to be deficient in some respect and are replaced by new, well-confirmed theories. A philosophical account of intertheoretic reduction is an attempt to give a rational account of the process whereby one well-confirmed theory is superceded by another. Given the assumption that science advances, the general theme of these rational reconstructions is that the old theory must be explained or accounted for by the new theory. Thus, intertheoretic reduction becomes intertheoretic explanation in which the secondary science is explained by the primary science.

The formal and nonformal requirements for an adequate reduction mirror the formal and nonformal conditions for an adequate explanation. According to the accepted philosophical paradigm for explanation, the explanandum must be derivable from a set of premises, the explanans. In intertheoretical explanation the reduction function is intended to mediate this derivation. Of course, the construction of such a derivation does not guarantee that any explanation at all has been given. For the present case, the Wigner transformation *per se* is a function that maps a noncommutative operator algebra to a commutative algebra of functions on the phase plane. This in itself tells us nothing about the relation between quantum and classical mechanics as physical theories. Such a formal relation is at best a necessary, and certainly not a sufficient, condition for an adequate explanation of classical mechanics by quantum mechanics.

In fact, the formal relation does not treat the most interesting facet of reduction, namely of how a well-confirmed theory can be rationally replaced by another well-confirmed theory that is incompatible with it. This facet of reduction is addressed by the nonformal requirements. Explanations are stories or accounts of a certain kind. Usually an explanation gives an account of something poorly understood in terms of something that is more clearly understood or an account of some particular thing in terms of a more general framework. Thus, there are pragmatic and epistemic features present in an explanation. These features must also be present in any adequate intertheoretical explanation. The nonformal requirements which are of epistemic and pragmatic character, ensure that the inductive support of the secondary theory is passed on to the primary theory, thus showing how one well-confirmed theory can be replaced, consistently, by another well-confirmed theory.

It is the requirement that the primary theory inherit the inductive support of its predecessor that forces attention to the epistemic and pragmatic aspects of reduction. These aspects, or the nonformal conditions on a reduction, must pay special attention to the particular interpretations of the mathematical formalisms of the theories in question. Certain of the mathematical entities in each formal structure are interpreted as standing for physical properties of a system. Laws are statements relating the entities so interpreted. Experiment and observation provide inductive support for the laws and hence indirectly for the physical interpretation of the mathematical formalism. If the nonformal conditions on reduction are to be met, some account is required of how evidence for the primary theory is related to that for the secondary theory.

The difficulty of specifying these conditions exactly is exacerbated by the fact that in many cases the primary and secondary theories are logically incompatible. The logical incompatibility of the primary and secondary theories was the crux of Feyerabend's critique of reduction given in Chapter I. To circumvent Feyerabend's criticism, it is necessary to state the conditions under which an approximate reduction obtains. Formally, the Wigner transformation serves this purpose well, as the Wigner transformation allows for the recovery of the classical theory. This is achieved by taking the usual interpretation of the quantum formalism and mapping it onto the classical phase plane. The Wigner transformation generates a model for quantum theory in terms of the classical quantities; that is, under that transformation the quantum laws are approximations to classical laws such that in the limit as $\hbar \to 0$ the approximations become the classical laws. This relation in the limit is not a relation of logical derivability but it is as strong a formal relation as one would expect to obtain between logically incompatible theories.

Inductive support for a theory is generated when observation gives evidence that the theory is true. If the theories are logically incompatible, both of them cannot be true, and we were mistaken in believing that one of the theories was well-confirmed. This complication leads to a natural formulation of inter-theoretical explanation given by Glymour (1970:341):

> Intertheoretical explanation is an exercise in the presentation of counterfactuals. One does not explain one theory from another by showing why the first is true; a theory is explained by showing under what conditions it would be true, and by contrasting those conditions with the conditions that actually obtain.

This is a natural formulation of the reductive relation in that the emergence of the new theory and the deficiencies of the old theory reveal that the old theory is wrong, whereas the reduction tells us in addition why we were wrong and how it was possible that the old theory could have become accepted. This relation goes far beyond a mere formal relation and does capture the pragmatic and epistemic aspects of intertheoretic explanation.

For the case of quantum and classical mechanics the formal condition as met by the Wigner transformation is that quantum theory under its usual interpretation be modeled in the usual interpretation of classical theory in such a way that as $\hbar \to 0$ this model of quantum theory becomes a formulation of the classical theory. Such a reduction function leads to an adequate reduction if the following two nonformal conditions are also fulfilled:

(i) Explanatory condition. The reduction must generate an account of how and where the secondary theory was successful and deficient, where this account is told in terms of the primary theory;

(ii) Unity condition. The reduction must unify physical theory.

Fulfillment of the first condition guarantees that the reducing theory accounts for the phenomena that the reduced theory accounted for and that it accounts for these phenomena in such a way that the inductive support of the reduced theory is inherited by the reducing theory. It is this condition, too, that contains the counterfactual aspect of the reduction. The proposed account contains statements of the form: "Classical statistical mechanics would be true if it were the case that ..." where the ellipses are replaced by some statement that is contrary to fact about the structure of the physical world correctly described by quantum theory.

As a simple example illustrating this condition consider the relation between the van der Waals gas law and the ideal gas law. The ideal gas law may be derived from the assumption that gas molecules are point masses and that there are no forces of intermolecular attraction present. To derive his law, van der Waals assumes that molecules are solid spheres with weak intermolecular forces present. The two laws are logically incompatible, as are assumptions on which they are based. The ideal gas law would be true if molecules had no volume and if there were no intermolecular forces. The ideal law works well in many cases because for extremely dilute gases, molecular diameters and intermolecular forces are insignificant. It is possible here to compare and contrast the conditions placed on the molecules by the two laws and give an account in terms of the properties of the van der Waals molecules of why the ideal law is generally incorrect and why it worked as well as it did.

The unity condition has not been motivated as well as the explanatory condition. The explanatory condition is concerned with the inductive support of theories, the observational support for the theories, and the inheritance of this inductive support in reduction. Strong inductive grounding is one reason why theories are accepted. A second subsidiary feature that might lead one to accept one theory over another is the degree of systematization that the theory imposes on the phenomena in question. The degree of systematization, or the theoretical simplicity or elegance, of a theory is not a feature that is easily characterized. For the present purposes it can be said that the degree of systematization is related to the number of *ad hoc* assumptions the theory requires. Fewer *ad hoc* assumptions indicate a greater degree of systematization. Another desideratum that a reduction should fulfill is that some of the assumptions of the secondary theory be derivable from the laws of the primary theory. Fulfillment of this desideratum unifies physical theory in that it shows that the primary theory has greater theoretical economy, allowing the science to proceed on fewer *ad hoc* assumptions. This condition is not usually considered in reductions of the kind treated here.

In our particular case, an investigation of whether the non-formal requirements for a successful reduction are satisfied requires, as we have seen, that certain relationships obtain between the specific interpretations of the abstract algebras on which the Wigner transformation acts. The process by which these interpretations are formulated presents another interesting area of philosophical research. It is not easy to specify how abstract mathematical entities are endowed with empirical significance. As stated in the earlier discussion of abstract mechanics, it is extremely difficult to motivate the assumptions leading to the rigorous mathematization of physical theories. For the present purposes, it will be assumed that this interpretative problem of the abstract algebras has been solved by means of Weyl's group theoretic approach (see Stein 1972:Sec. XI). It will be assumed that $\mathcal{A}_Q$ is an operator algebra generated by the operators P and Q which represent position and momentum. Likewise it will be assumed that $\mathcal{A}_c$ is an algebra of functions generated by p and q where these canonical coordinates represent position and momentum in the classical theory.

Interestingly, we do not view the problem of interpreting the classical formalism as presenting the same kind of difficulties as interpreting the quantum formalism. As far as interpreting an abstract formalism is concerned, it is difficult to motivate the interpretation of either theory. However, these conceptual problems of interpretation are not as pressing in the case of classical theory. The interpretation we have adopted in the classical case is psychologically pleasing and epistemologically accessible in that the interpretation is clear, vivid, and easily visualized. One might say that we have an excellent model of classical theory in terms of functions on the classical phase plane. The dynamical variables and states of the theory are defined in terms of this plane. For the physically interesting functions of position and momentum, one can picture these functions on the phase plane. This modelling process gives us the feeling that we have a clear understanding of what the notions of classical theory mean. This model puts us epistemologically at ease with the classical theory, in a way that we are not at ease with quantum theory. Classical theory is epistemologically accessible in a way that quantum theory is not.

The epistemological accessibility of the classical theory can be readily exploited to show that the first nonformal condition on a reduction, the explanatory condition, is satisfied in this case. The Wigner transformation maps quantum theory into classical theory. But classical mechanics is formulated and visualized as a theory of the classical phase plane; so, under W quantum mechanics can be formulated as a theory of the phase plane. The proposed reduction function, when this is viewed as operating on the interpreted observable algebras $\mathcal{A}_Q$ and $\mathcal{A}_C$, allows us to compare and contrast the two theories on common ground as different theories of the phase plane. Formal requirements on the quantum mechanical pure states impose a definite structure on their phase-plane images, which leads to an explicit criterion whereby it can be determined when classical mechanics can serve as an adequate theory of the phase plane and when quantum mechanics must be used. The criterion, although based on a formal requirement, is rendered empirically significant by the connection between pure states and measurement. Pure states are the states of maximum specificity allowed by a theory given ideal measurements. On the assumption that quantum mechanics is the true universal theory, it is the true theory of the phase plane, and classical mechanics is at best an approximation to it. The proposed criterion can be used to give an account, in terms of quantum mechanics, of how "almost true" classical mechanics is, of why classical mechanics appears to be true in certain cases, and of why classical mechanics fails as a universal theory. This is just the kind of account required by the first nonformal condition on an adequate reduction.

The explanatory condition is the condition which a reduction must satisfy if we are to be assured that some plausible account can be given of how the primary theory might inherit inductive support from the secondary theory. Because of the connection between pure states and measurements, an investigation of what happens to pure states under the action of the Wigner transformation and its inverse should generate an account of the inductive and confirmatory relations between the two theories.

Pure states are the states of maximal specificity. Classically, a pure state is specified by giving precise numerical values for the position and the momentum of a particle. These numerical values are determined by measurement. A classical pure state is given by a classical maximal measurement, the accurate determination of position and momentum. According to classical theory it is possible in principle to make such exact measurements. In classical statistical mechanics, the pure states defined by maximal measurements are the states of optimal knowledge from which all other density functions can be constructed by taking convex superpositions of the pure states. The pure states are represented on the phase plane by delta functions.

Quantum mechanical pure states are also states of maximal specificity from which all other states can be constructed by taking convex superpositions. A maximal measurement in quantum mechanics is an experiment designed to uniquely specify the wave function for a system. The maximal measurements of quantum mechanics are motivated and given operational meaning in expositions of the theory by recourse to filtration experiments where a measurement on a system is construed as a filter that selects for a value of a certain dynamical quantity. Weyl gives a criterion for the determination of a pure state where the experimental conditions S represent such filtration processes:

> We say that the conditions S' effect a greater homogeneity than the conditions S if (1) every quantity which has a sharp reproducible value under S has the same definite value under S' and if (2) there exists a quantity which is strictly determined under S' but not under S. The desired criterion is obviously this: _The conditions S guarantee a pure state if it is impossible to produce a further increase in homogeneity_ (Weyl 1950:78).

Classically, maximal homogeneity is achieved when all quantities of a system have a definite value. According to quantum mechanics some quantities, such as position and momentum, are incompatible in that they cannot both have precise simultaneous values. It is exactly this situation that Weyl's criterion takes into account. Thus, a quantum mechanical pure state must be specified by data somewhat less complete than exact values for position and momentum. Yet this is still a pure state determined by a maximal measurement and the conditions for a maximal measurement are the conditions of optimal precision for the theory. Quantum mechanical pure states are represented in the space of von Neumann densities by projection operators, $\rho(x,x') = E_n(x,x') = \psi(x)\psi^*(x')$.

The pure states of each theory are specified by the maximal measurements under each theory. Quantum mechanical pure states are not as precise as the pure states hypothesized by classical theory; each theory makes a different claim as to what is a theoretically most precise observation. Under W both theories are theories of the phase plane and maximal measurements can be compared as they relate to the phase plane.

If quantum mechanics is the correct, universal theory, then the maximal measurements allowed by quantum mechanics represent the theoretically most precise determination of the state of a system. One thing that quantum theory tells us, then, is that the precision assumed by classical theory is illusory; there can be no classical pure states. The images of the quantum mechanical pure states under the Wigner transformation represent optimal knowledge of the system, as expressed on the phase plane. Thus, quantum mechanics is the correct "fine-grain" theory of the phase plane. If classical mechanics is to be applicable at all, it must be as an approximation to this correct "fine-grain" theory. Classical mechanics can serve as an adequate "coarse-grain" theory of the phase plane. It works reasonably well if we do not look too closely at the phase plane with extremely precise measuring instruments.

In order to investigate the relation between quantum and classical pure states,* it will first be shown that the delta function representing any classical pure state can be recovered, as $\hbar \to 0$, from an appropriate wave function, or quantum mechanical pure state. Any wave function $\delta(q)$ which has an expectation value of position and momentum can be expressed as

$$\psi(q) = e^{ip_c q/\hbar} f\left[\frac{q-q_c}{b\sqrt{\hbar}}\right]$$

where q_c and p_c are particular values of position and momentum, and where the position dependence of the wave function has been scaled by a factor proportional to $1/\sqrt{\hbar}$. As particular values of position and momentum, q_c and p_c are the coordinates of a point on the classical phase plane. The pure state corresponding to the point is represented by the delta function $\delta(x-x_c, p-p_c)$.

Consider the normalized wave function

$$\psi(q) = \frac{1}{(b\sqrt{\hbar})^{1/2}} [\tfrac{2}{\pi}]^{1/4} e^{ip_c q/\hbar} e^{-[(q-q_c)/b\sqrt{\hbar}]^2} .$$

The von Neumann density corresponding to this wave function is given by $\rho(q,q') = \psi(q)\psi^*(q') =$

$$\frac{1}{b\sqrt{\hbar}} [\tfrac{2}{\pi}]^{1/2} e^{ip_c(q-q')/\hbar} e^{-[(q-q_c)/b\sqrt{\hbar}]^2} e^{-[(q'-q_c)/b\sqrt{\hbar}]^2} .$$

The Wigner transformation applied to this density gives

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*A particularly elegant relationship obtains between pure states of the quantum-mechanical harmonic oscillator and functions on the classical phase plane. See Appendix.

$$W\rho(q,q') = \frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2}\int da\, e^{ipa}\,\rho\left[q - \frac{a\hbar}{2},\, q + \frac{a\hbar}{2}\right]$$

$$= \frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2}\int da\, e^{ipa}\, e^{-ip_c a}\, e^{-(2/b^2\hbar)(q-q_c)^2}\, e^{-a^2\hbar/2b^2}$$

$$= \frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2} e^{-2(q-q_c)^2/b^2\hbar}\int da\, e^{-\hbar a^2/2b^2}\, e^{i(p-p_c)a} .$$

This integral is evaluated by completing the square:

$$\frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2} e^{-2(q-q_c)^2/b^2\hbar}\int da\, e^{-\hbar a^2/2b^2}\, e^{i(p-p_c)a}$$

$$= \frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2} e^{-2(q-q_c)^2/b^2\hbar}\, e^{-b^2(p-p_c)^2/2\hbar}\ \mathrm{X}$$

$$\int da\, e^{-[(\sqrt{\hbar}a/\sqrt{2}b + ib(p-p_c)/\sqrt{2}\sqrt{\hbar}]^2}$$

$$= \frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2} e^{-2(q-q_c)^2/b^2\hbar}\, e^{-b^2(p-p_c)^2/2\hbar}\left[\frac{2b}{\sqrt{\hbar}}\frac{1}{\sqrt{2}}\right]\int dy\, e^{-y^2/2}$$

$$= \frac{1}{2\pi}\frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2}\frac{2b}{\sqrt{\hbar}}\frac{1}{\sqrt{2}}\cdot\sqrt{2\pi}\, e^{-2(q-q_c)^2/b^2\hbar}\, e^{-b^2(p-p_c)^2/2\hbar}$$

$$= \frac{\sqrt{2}}{\pi}\frac{1}{\hbar}\, e^{-2(q-q_c)^2/b^2\hbar}\, e^{-b^2(p-p_c)^2/2\hbar} .$$

What happens to this function on the classical phase plane as $\hbar \to 0$? As $\hbar \to 0$, the negative exponentials drop off more rapidly and will have an appreciable value only very near the point (q_c, p_c) on the phase plane. But this is to say that as $\hbar \to 0$ the image of the quantum mechanical pure state tends to a delta function $\delta(q-q_c,\, p-p_c)$ on the classical phase plane. Delta functions represent classical pure states; hence, as $\hbar \to 0$, the Wigner transformation allows one to recover classical pure states. The classical pure states are literally embedded in the set of quantum mechanical pure states as $\hbar \to 0$.

Investigating what happens to the formalism of quantum theory as $\hbar \to 0$ was the topic of the previous chapter. As shown there, this is a strictly mathematical problem which considers how the mathematical entities of quantum theory behave in this limit. It was found that in this limit the classical algebras could be recovered from the quantum algebras. Here it has been shown that in this limit the classical pure states can be recovered from the quantum pure states. Although this is a formal result, nevertheless, it does tell us something about the relationship between the two theories. This result can be interpreted as showing that a relation exists between two physically possible worlds, the world as described by quantum mechanics and another physically possible world as described by classical mechanics. These worlds differ in that $\hbar$ has a non-zero value in the quantum world and is equal to zero in the classical world. The reduction shows that by making the counterfactual assumption that $\hbar \to 0$, the classical world is accessible from the quantum world. The accessability of the classical world from the quantum world in this limit shows that classical mechanics is literally embedded in quantum theory as $\hbar \to 0$. If a theory is explained by showing under what conditions it would be true and by contrasting these conditions with the conditions that obtain in the actual world, then part of that explanation is given by literally letting $\hbar$ go to zero. Classical theory is explained here by saying that it <u>would</u> be true if $\hbar$ <u>were</u> equal to zero; but it is false because $\hbar$, as a universal physical constant, has an exact, experimentally determinable non-zero value. This "explanation" of classical theory by quantum mechanics is of some significance because it does show that an intimate conceptual connection exists between the two theories. However, letting $\hbar \to 0$, while enlightening vis-a-vis relations between physically possible worlds, says nothing as to the applicability of classical mechanics to the actual world where quantum theory is the true theory. Something must be said of this applicability if an explanation that satisfies the first nonformal condition on reduction is to be forthcoming.

To be assured that the explanatory condition is satisfied, one must consider what it means to say that quantum theory is the correct fine-grain theory of the phase plane and state a condition under which classical theory, the coarse-grain theory, would be applicable. Suppose one has a measuring instrument on the phase plane and suppose that one attempts to make simultaneous measurements of position and momentum with this device. The results of a series of measurements on similarly prepared systems would be represented by a Liouville density on the phase plane that covers an area of the plane. As measurements are made more and more precisely, the area covered becomes smaller and smaller. According to classical theory, there is no limit as to how small this area might become. Indeed, on the classical theory it can shrink to a point, in which case one has a classical pure state.

Quantum theory implies that there are no classical pure states and *a fortiori* implies that there is a limit to how small the area of the phase plane under the Liouville density can become: The area under the Liouville density can be no smaller than $\hbar$. Hence, a Liouville density which is the image of a quantum mechanical pure state must at least cover an area of the phase plane of order $\hbar$. Quantum mechanics, as the correct fine-grain theory of the phase plane, tells us that we can only look at areas of the phase plane larger than $\hbar$. One would expect classical mechanics, as a coarse-grain theory of the phase plane, to provide a good approximation for the situations in which an area considerably larger than $\hbar$ is scrutinized, that is, in those situations in which we are far from a quantum mechanical pure state.

These reflections on what fine-grain and coarse-grain theories of the phase plane provide lead to a criterion whereby one can judge where classical mechanics might be applicable. It was shown above that the image under W of the pure state

$$\rho(q,q') = \frac{1}{b\sqrt{\hbar}}\left[\frac{2}{\pi}\right]^{1/2} e^{ip_c(q-q')/\hbar}\, e^{-[(q-q_c)/b\sqrt{\hbar}]^2}\, e^{-[(q'-q_c)/b\sqrt{\hbar}]^2}$$

was

$$\hat{\rho}(q,p) = \frac{\sqrt{2}}{\pi}\frac{1}{\hbar}\, e^{-2(q-q_c)^2/b^2\hbar}\, e^{-b^2(p-p_c)^2/2\hbar} \quad .$$

The function $\hat{\rho}$ has its maximum value at (q_c,p_c) and

$$\hat{\rho}(q_c,p_c) = \frac{\sqrt{2}}{\pi}\frac{1}{\hbar} .$$

Its maximum value is of order $1/\hbar$. This is true for the classical image of any quantum mechanical pure state. As a coarse-grain theory, classical mechanics is applicable in situations far from a quantum mechanical pure state. So one can say that classical mechanics is applicable in those situations in which the maximum value of $\hat{\rho}(q,p)$ is much less than $1/\hbar$, or

$$\text{Max}[\hat{\rho}(q,p)] << 1/\hbar.$$

This criterion states the limit of the validity of the classical approximation. It states a condition for the applicability of the fine and coarse-grain theories of the phase plane.

The criterion reveals another relationship between quantum and classical theory other than the conceptual relationship as $\hbar \to 0$. By dint of the connection between pure states and measurement the metaphor of fine and coarse-grain can be extended to the measuring instruments themselves. This makes it possible to say something about the use of classical mechanics in <u>this</u> world.

A fine-grain theory requires measurements of high resolution, a coarse-grain theory employs low resolution measurements. Hence, if the resolving power of the measuring devices used in a given situation is relatively low, only imprecise measurements are possible. In such a situation the coarse-grain theory, classical mechanics, would appear to be correct. It would appear to be correct in that low resolution observation of the phase plane would yield no data that would be inconsistent with classical theory. At a time when only low resolution observation was possible, the success of classical mechanics might encourage the belief that a state could be given an ideal representation by a point on the phase plane. Even after the advent of high resolution instruments, there would still be situations in which low resolution measurements would be adequate for the practical specification of a state (e.g., macroscopic systems). In other words, a quantum mechanical state would be indistinguishable from a phase point where the precision of the measurements is significantly less than the precision allowed by the maximal measurements of quantum theory.

On the basis of these considerations an account or explanation of classical theory is forthcoming that satisfies the first nonformal condition on reduction. If quantum mechanics is a true universal theory, then classical mechanics is an approximation that would hold only (i) if $\hbar$ were equal to zero; or (ii) if relatively imprecise measurements were used to specify the states of the system. Classical mechanics fails as a universal theory because Planck's constant is not equal to zero, because precise measurements are possible, and because there are no classical pure states. The precision assumed by classical theory is unrealistic and unattainable.

Condition (ii) above is the facet of the explanation generated by the reduction which ensures the fulfillment of the explanatory condition. A reliance on low resolution observations explains the apparent success of classical theory, why it appears to work for macroscopic systems and why a false theory appeared to work for so many cases. This relation between the theories shows how the classical picture is incorporated into the quantum picture and shows how confirmatory relations of evidential support are established between the theories. It is also (ii) that explains the success and justifies the use of classical approximation techniques.

The intertheoretic explanation given of classical mechanics by quantum mechanics differs somewhat from the typical examples of limiting case reductions. First of all, it has been shown that one can consider a limiting case by literally letting $\hbar \to 0$ [(i) above] or limiting cases where the $\hbar$ dependence is negligible [(ii) above]. It is typically the latter case that is discussed in the literature. As shown, such discussions validate the use of classical mechanics as an approximation under certain experimental conditions or for certain systems. This is the usual type of explanation given in limiting case reductions. For example, in the explanation of the ideal gas law by the van der Waals law, the van der Waals law states the following relation among volume, pressure, and temperature.

$$(V - b)(P + \frac{a}{V^2}) = RT$$

where R is the universal gas constant, _b_ a constant characteristic of the substance depending on the molecular diameter and the number of molecules present, and _a_ another constant characteristic of the system. Where _a_ and _b_ are small, that is as _a_, _b_ $\to 0$, the van der Waals law becomes

$$PV = RT$$

the ideal gas law. Because _a_ and _b_ are system dependent, this limiting case argument explains the apparent success of the ideal law and justifies its use as an approximation for certain systems. Similarly, Newtonian mechanics explains Galileo's law of falling bodies. According to Newtonian theory

$$ma = F = \frac{GmM}{R^2} \frac{1}{(1 + h/R)^2}$$

where G is the gravitational constant, m the mass of the body, M the mass of the Earth, R the radius of the Earth, and h the height of the body above the surface of the Earth. As h/R goes to zero

$$a = \frac{GM}{R^2}$$

or the acceleration of a body falling near the surface of the Earth is constant, which is Galileo's law. This limiting case argument justifies Galileo's law as an approximation valid for bodies very close to the Earth's surface. In both of these paradigmatic cases, an explanation is given of why the secondary theory might appear to be successful in some cases and not in others.

For the quantum mechanical case, in addition to this usual type of explanation, an explanation was also forthcoming where one let $\hbar \to 0$ and where this limit could be approached in a mathematically rigorous manner. This formal result generates an explanation of classical theory in terms of a relation between physically possible worlds, emphasizing the conceptual connection between the two theories. The present result suggests that two types of intertheoretic explanation might be possible in limiting case reductions, one where the limit is taken with respect to a system specific parameter of function and one where the limit is taken with respect to a universal physical constant. In the former case empirical connections are established between the theories; in the latter case conceptual connections are established between the theories.

It should be mentioned here that a given limiting case reduction might not always fall entirely under one or the other of these two types of intertheoretic explanation. The example of Galileo's law and the quantum mechanical cases under (ii) above are clearly limits with respect to system specific parameters or functions. The van der Waals case, where the constants $\underline{a}$, $\underline{b} \to 0$, is usually interpreted as establishing empirical connections between the two laws, justifying the applicability of the ideal law under certain conditions. However, the constants $\underline{a}$ and $\underline{b}$, even though they refer to system specific properties of molecular diameter and intermolecular attraction, do characterize actual properties of gas molecules. One could as well interpret $\underline{a}$, $\underline{b}$, $\to 0$ as envisioning a possible world where the molecules were ideal mass points with no forces of intermolecular attraction. Under such an interpretation, an explanation emphasizing the conceptual connections between the two gas laws might be forthcoming.

The explanation given to classical mechanics in the present reduction also differs from the typical examples in being extremely abstract. Quantum theory and classical theory are compared and contrasted as theories of the phase plane, where this is a mathematical construct invented to facilitate the description of mechanical systems. On the formulation of intertheoretic explanation adopted, the explanation consists of contrasting conditions that do obtain with the conditions under which the secondary theory would be true, where these conditions can be cited as causes for the success of the primary theory and the failure of the secondary theory. The abstractness of the proffered explanation of classical mechanics might be criticized as being no "causal" explanation at all.

Explanations of the ideal gas law or of phenomenological thermodynamics are more concrete, or at least more readily visualizable. For the gas laws one imagines that $\underline{a}$, $\underline{b} \to 0$. These constants are dependent upon molecular diameters and the density of the gas. By considering limits on these constants, one is contrasting the van der Waals assumption of molecules as hard spheres subject to forces of intermolecular attraction with the ideal law assumption of molecules as point masses subject to no intermolecular forces. There is a visualizable picture at hand in terms of which it is possible to see exactly what the ideal law left out and to envisage situations in which that oversight is insignificant, namely, for dilute gases. Similarly, statistical mechanics as a theory of the behavior of large numbers of particles explains the gross thermodynamical properties of matter by showing how systems of particles produce the gross effects. The formal aspect of this reduction leads to an account of how collisions of the particles on a container wall can be associated with the pressure of the gas in the container. In both of these cases, there is available an acceptable visualizable model for both the primary and secondary theory which facilitates the understanding of what causal conditions must be varied or ignored to formulate an adequate explanation.

The explanation of classical mechanics by quantum mechanics might be criticized as being too abstract or may not seem to be as compelling as the explanation in the above examples. These qualms stem from the fact that there is no visualizable model available for quantum theory in its usual formulation. However, such criticisms are misguided. The Wigner transformation yields a phase plane version of quantum theory. As long as the problem of the interpretation of the classical formalism is deemed solved by appeal to the phase plane model, the present reduction does employ as adequate an interpretation of the quantum formalism as is possible at present. This reduction fulfills the requirements for a significant reduction by making full use of the developed formalisms of each theory to generate a picture of the two theories. That this picture must be formulated within an abstract, mathematical space is no impediment to the adequacy of the reduction, given the role of that space in the interpretation of classical theory and the epistemological accessability of that interpretation.

The second nonformal condition on an adequate reduction, the unity condition, states that the reduction must result in the unification of physical theory. Fulfillment of this condition guarantees that the primary theory is superior to the secondary theory in its range of explanatory power and in its theoretical economy. This superiority is manifested (i) by the primary theory being able to account for a significant portion of the phenomena, and in ideal cases all of the phenomena, explained by the secondary theory and (ii) by the primary theory being able to justify some of independent primitive assumptions of the secondary theory. (It is necessary to distinguish the ideal case from the general case because there are instances where one wants to claim that a reduction does unify physical theory, but where the primary theory cannot, in its current formulation, give a completely adequate account of all the phenomena explained by the secondary theory. Quantum mechanics is one such case. Quantum mechanics reduces to classical mechanics and, as will be shown, unifies physical theory; yet, there is at present no adequate quantum theory of gravity). Where the primary and secondary theories treat of apparently distinct phenomena, this particular aspect of reduction is most noticeable and significant.

Again, the reduction of statistical mechanics to thermodynamics affords a prime example of how reduction can unify physical theory. Thermodynamics was a well-confirmed theory, describing relations between certain gross properties of systems such as temperature, pressure, and entropy. Another extremely successful theory emerged, the atomic theory of matter, which implied that all matter is composed of small particles which obey certain physical laws. The conceptual problem was that these two successful theories appeared to be independent of each other, where one would expect that if the atomic theory of matter is correct, the gross properties of material systems should be explicable in terms of their constituent particles. The successful reduction did show that the thermodynamical properties of matter could be explained in terms of the constituent particles. Where previously there were two sets of laws explaining two independent sets of phenomena, the reduction shows that one set of laws will suffice and that the previously independent sets of phenomena are indeed related. That greater theoretical economy should also be expected from the reduction is made a condition on its success by Khinchin:

> ...statistical mechanics considers every kind of matter as a certain mechanical system and tries to deduce the general physical (in particular, thermodynamical) laws governing the behavior of this matter from the most general properties of mechanical systems, and _eo ipso_ to eliminate from the corresponding parts of physics any theoretically unjustified postulation of their fundamental laws (Khinchin 1949:7).

Thus, in this reduction both the second law of thermodynamics and the ideal gas law can be derived from statistical mechanics, whereas in classical thermodynamics these laws are stated as independent primitive assumptions. Statistical mechanics unifies physical theory because it not only accounts for the data of classical thermodynamics, it also explains and justifies primitive assumptions of that theory (cf. Friedman 1974).

The reduction of statistical mechanics to thermodynamics is an example of a heterogeneous reduction--a reduction wherein the domains of the primary and secondary theory are distinct. And it is with this type of reduction that the unification of physical theory is most dramatic. Where the primary and secondary theories are two different theories of the same set of phenomena, a homogeneous reduction, reduction is often viewed as only a formal problem of relating the structures of the two theories. The case of the reduction of quantum mechanics to classical mechanics is clearly a case of homogeneous reduction, if for no other reason than that both theories were believed to be universally applicable. This being the prevailing belief, there is no possibility of viewing the theories as treating of distinct, independent domains. Even so, significant unification of physical theory results from this reduction.

The reduction of quantum mechanics to classical mechanics unifies physical theory by explaining or justifying some of the independent primitive assumptions of classical theory. In particular, the reduction explains or justifies the laws of conservation of linear momentum, angular momentum, and energy.

Newton's great achievement was the discovery of his dynamical law. This law, as the second law of motion, is the principle postulate of Newtonian mechanics. However, given the second law and the assumption of Galilean invariance the laws of conservation of linear and angular momentum, which are equivalent to Newton's third law, cannot be deduced; and given these laws the conservation of energy cannot be derived. The classical conservation laws can be derived using the second law and the additional assumption that the forces involved can be derived from a potential depending only on the distances between the particles. Mach's derivation of these laws requires two additional assumptions: That the force on any particle can be resolved into a sum of forces each due to another particle and that such forces depend only on the positions, and not on the velocities of the interacting pair (Mach 1893:376). There are then two alternatives open to us in Newtonian mechanics. Either the second law and the conservation laws must be assumed as primitive postulates of a somewhat arbitrary kind, or additional special assumptions must be made which allow the derivation of the conservation laws from Newton's second law.

Newton did advance an argument for the third law from the first law (cf. Home 1968:43). He gave a reductio ad absurdam for the third law based on the following thought experiment: Imagine an obstacle placed between two attracting bodies. If one body experiences a stronger attraction than the other, the pressure of this body on the obstacle will overcome the pressure of the other body on the obstacle causing the system to accelerate to infinity. But such a circumstance is impossible by the first law. This argument, however, would also appear to make use of Mach's first assumption that the force on any particle can be resolved into a sum of forces each due to another particle. In this case, the force on the obstacle is resolved into a sum of forces due to the two attracting bodies.

For quantum mechanics the situation is much simpler. The conservation laws follow directly from quantum kinematics and the assumption of Galilean invariance without appeal to any dynamical law.

The one assumption required for the derivations is von Neumann's postulate that states are represented by vectors in an abstract Hilbert space and that physical quantities are represented by self-adjoint operators on these vectors. Suppose that in the Schrödinger picture the time development of the expectation value of an operator A is calculated

$$i\hbar \frac{d}{dt} (\psi, A\psi) = (\psi, Ai\hbar \frac{\partial\psi}{\partial t}) - (i\hbar \frac{\partial\psi}{\partial t}, A\psi) + (\psi, i\hbar \frac{\partial A}{\partial t} \psi)$$

$$= (\psi, AH\psi) - (\psi, HA\psi) + i\hbar (\psi, \frac{\partial A}{\partial t} \psi)$$

where H is the Hamiltonian operator for the system. Where A does not depend explicitly on time,

$$i\hbar \frac{d}{dt} (\psi, A\psi) = (\psi, AH\text{-}HA\ \psi).$$

If A commutes with the Hamiltonian, AH = HA, then the time development of the expectation value of A is zero, and A is a quantum mechanical <u>constant of the motion</u> or a <u>conserved quantity</u>. If A is a generator of the Galilean group, say the total momentum operator P, then by the assumption of invariance

$$PH\psi = HP\psi$$

for all states ψ. P commutes with H and is a conserved quantity, which is to say that linear momentum is conserved. Analogous arguments establish the conservation of angular momentum, and for energy the result is immediate, as A = H and $H^2\text{-}H^2 = 0$.

Quantum mechanically, a quantity is conserved if its operator commutes with the Hamiltonian operator, that is if its commutator with H is zero. According to Proposition 2 of the preceding chapter in the limit as $\hbar \to 0$ the image of the commutator equals the Poisson bracket; so

$$0 = W \frac{i}{\hbar} (AH - HA) = \{\hat{A}, \hat{H}\} .$$

But if the Poisson bracket of a classical quantity and the classical Hamiltonian is zero, then, according to classical theory, that quantity is conserved; hence, under the Wigner transformation in the limit $\hbar \to 0$, the quantum mechanical conservation laws yield the conservation laws for the corresponding classical quantities.

This is not a new result, indeed it is widely accepted that "the simplest proof of the conservation laws in classical theory is based on the remark that classical theory is a limiting case of quantum theory" (Wigner 1967:20). But this simple proof of the classical conservation laws requires considering the behavior of the quantum equations as $\hbar \to 0$. To the extent that this limiting relation is imprecise, the proofs are imprecise. Exhibiting the limiting relation in a rigorous manner by means of the Wigner transformation removes this imprecision. Furthermore, the Wigner transformation is the reduction function for the reduction of quantum mechanics to classical mechanics, so these proofs show that the reduction unifies physical theory. By means of the reduction, the classical conservation laws need not be assumed as postulates nor need they rely on additional ad hoc assumptions for their derivation. Quantum mechanics explains and justifies these primitive assumptions of classical theory and consequently the reduction satisfies the second nonformal condition on reductions.

Although the reduction satisfies all of the proposed conditions on an adequate reduction and it can be claimed that quantum mechanics does explain classical mechanics, there is another facet to the relation between the theories which has not yet been addressed. A compelling motive for the sustained research, by both philosophers and physicists, into the foundations of quantum mechanics is the presence of formidable problems of theoretical interpretation. The preceding discussions of the relation of quantum theory to classical theory assumed that the quantum theory is suitably interpreted by associating elements of the algebra $\mathcal{A}_Q$ with the states and observables of mechanical systems. The mathematical formalism when interpreted in this way is extremely successful over a wide range of phenomena. In the discussion of the first nonformal condition, the maximal measurements of the two theories were easily compared and contrasted. The fulfillment of the first condition was defended from various criticisms on the ground that quantum mechanics is as adequately interpreted as is classical mechanics in terms of the phase plane model. From this point of view, quantum theory is as adequately interpreted as any other theory of mathematical physics. However, this interpretation of the quantum formalism which allows for its successful experimental application solves the interpretative problem of quantum mechanics only in what Howard Stein calls the epistemological sense (Stein 1972:369). What remains problematic is an adequate metaphysical or ontological interpretation of the theory. The difficulty is that certain conceptual questions arise out of the formalism that admit no easy answer. Lacking such answers, it is extremely difficult to say what is really going on in the world as it is described by quantum theory. Can the reduction of quantum mechanics to classical mechanics aid in answering these questions?

It is a testimony to the clarity and precision of the quantum formalism that these conceptual problems can be given a rigorous formulation. The root of the metaphysical problem of interpretation is that what appears to be the case when the theory is applied to specific atomic systems is demonstrably inconsistent with the formalism of the theory. In particular, every time a dynamical quantity of an elementary particle is measured a reasonably precise value is ascertained for that quantity. Thus, measurements would lead one to believe that for any atomic system each dynamical quantity has, at all times, a precise value. But although the evidence suggests the general hypothesis that all dynamical quantities have precise simultaneous values, the existence of precise simultaneous values is inconsistent with the formalism of quantum mechanics. The inconsistency of what is observed with the formal theory follows from a corollary of Gleason's theorem: If the dimensionality of the state space is greater than two, the additivity requirement for expectation values of commuting operators cannot be met by dispersion free states (cf. Bell 1966). This corollary treats of the case of an infinite number of operators. The proof of Kochen and Specker (1967) establishes a similar result for the case of a finite number of operators, and it is this proof that will be discussed here.

Proofs such as that of Kochen and Specker are quite sophisticated mathematically and are the subject of considerable current research in their own right; but for the purpose of investigating the relation between reduction and the metaphysical problem of interpretation, it suffices to consider simply the structure and basic assumptions of these proofs. Results like the corollary to Gleason's Theorem are usually invoked to show that it is impossible to construct a hidden variable interpretation of quantum theory. In a hidden variable interpretation, a phase space of hidden states, having the same formal structure as the phase space of classical statistical mechanics, is posited. The quantum theory is then interpreted by defining a mapping, w, which takes the quantum states to the phase space of the hidden states. The mapping w must satisfy two conditions. First of all it is required that the mapping be such that the expectation value of the quantum observable A in the state ρ be given by phase-space averaging of the image of ρ and A

$$\mathrm{Tr}(\rho A) = \int_{\Omega} \hat{\rho}(\omega)\hat{A}(\omega)\,d\omega. \qquad (1)$$

This is to say that in the hidden variable theory expectation values are to be calculated by the classical prescription. Kochen and Specker show that this requirement can be trivially satisfied, but satisfied in such a way that observables become independent random variables over the phase space of hidden states. However, the observables of a theory are not all independent of one another; indeed, some observables are functions of others. This being the case, observables have an algebraic structure. Kochen and Specker make it a second condition on w that it preserve this structure. This second condition on w can be formulated

$$A \overset{W}{\to} \hat{A} \Rightarrow f(A) \overset{W}{\to} f(\hat{A}) \qquad (2)$$

where w maps quantum operators into real valued functions on the space of hidden states and where f is any function of A.

In terms of conditions (1) and (2), Kochen and Specker formulate a precise necessary condition for the existence of hidden variables: If a hidden variable interpretation of quantum mechanics is possible then there must be an embedding, w, of the noncommutative algebra of quantum mechanics into a commutative algebra. Kochen and Specker formulate this condition in terms of the partial algebra of quantum observables. However, the notion of a partial algebra is introduced solely for the purpose of making condition (2) more tractable (cf. Kochen and Specker 1967:64). In the present context this complication can be avoided, as the introduction of partial algebras is not crucial to the strategy or general assumptions of the proof. Kochen and Specker proceed to prove that no such embedding w can exist. The proof consists of showing that (2) is inconsistent with another general constraint on the structure of quantum theory, the additivity requirement:

$$A \overset{W}{\to} \hat{A} \text{ and } B \overset{W}{\to} \hat{B} \Rightarrow A + B \overset{W}{\to} \hat{A} + \hat{B}. \qquad (3)$$

This condition must be satisfied if it is to be maintained that for all quantum states the expectation value of a sum of operators is the sum of the expectation values. Quite simply then, the strategy of the proof is to show that the existence of the algebra homomorphism required by (2) is inconsistent with the additivity requirement (3).

Note that this is a very strong result. It establishes that there can be no homomorphism from the noncommutative algebra of quantum observables, $\mathcal{A}_Q$, into any commutative algebra whatsoever; a fortiori there can be no homomorphism from $\mathcal{A}_Q$ to $\mathcal{A}_c$. Hidden variable interpretations of quantum theory, the existence of dispersion free measures (a probability measure is dispersion free if and only if it assigns 0 or 1 to each indempotent of $\mathcal{A}_Q$), and truth value assignments to quantum mechanical propositions are likewise all shown to be impossible as they require the existence of a homomorphism from $\mathcal{A}_Q$ to a commutative algebra.

A further consequence of the Kochen-Specker result is that precise simultaneous values of all dynamical observables cannot exist. For suppose that every observable has a precise value at a given time. Then there must exist a real valued function w which maps each operator A into the value that it has at that time. If it is required for any function f(A) that

$$w[f(A)] = f[w(A)]$$

then w must be a homomorphism from the noncommutative algebra of quantum observables $\mathcal{A}_Q$ into the commutative algebra of real numbers. By the Kochen-Specker result no such w can exist. Therefore, precise simultaneous values of all observables cannot exist.

In this way the Kochen-Specker result throws the conceptual difficulties of a metaphysically adequate interpretation of quantum theory into sharp relief. On our best interpretation of that theory, whenever the theory is used and measurements are made, it appears as if all dynamical quantities have precise simultaneous values; yet the existence of such values is inconsistent with the quantum formalism.

The Wigner transformation maps quantum mechanics onto the classical phase plane. It might be thought that the reduction achieved by means of this transformation could aid in resolving the conceptual problems surrounding the quantum theory.

W is a one-to-one, and hence invertible, mapping of $\mathcal{A}_Q$ into $\mathcal{A}_c$. The Wigner transformation was originally proposed as a means for computing quantum mechanical expectation values by the method of classical statistical mechanics. So at least in some instances W satisfies (1). However, W does not satisfy (2) and hence it is not a homomorphism of a noncommuting algebra of Hermitian operators into a commuting algebra. A simple counter-example suffices to show that (2) is violated. Consider the case of the linear harmonic oscillator and let $\hat{H}$ be its classical Hamiltonian. Then

$$W^{-1}(\hat{H}) = \frac{P^2}{2m} + \frac{1}{2} aQ^2 \tag{4}$$

and

$$W^{-1}(\hat{H}^2) = \frac{P^4}{4m^2} + \frac{a}{2m} Q^2P^2 + \frac{1}{4} a^2Q^4 \tag{5}$$

but

$$(W^{-1}(\hat{H}))^2 = \frac{P^4}{4m^2} + \frac{a}{4m} P^2Q^2 + \frac{a}{4m} Q^2P^2 + \frac{a^2}{4} Q^4 \tag{6}$$

$$= \frac{P^4}{4m^2} + \frac{aP^2Q^2}{2m} + \frac{a^2}{4} Q^4 - \frac{a\hbar}{2im} (PQ+QP)$$

$$= W^{-1}(H^2) = \frac{ia\hbar}{m} QP + \frac{a\hbar^2}{2m}$$

that is, $W^{-1}(\hat{H}^2) \neq [W^{-1}(\hat{H})]^2$; so W, W^{-1} are not homomorphisms and (2) is not satisfied.

According to Proposition 1, the basic result of the reduction, there is a mutual embedding of the commutative algebra of classical mechanics into the noncommutative algebra of quantum mechanics in the limit as $\hbar \to 0$. Also, in this limit (5) equals (6) in the above counter-example. Thus, if $\hbar$ has any non-zero value, W does not satisfy the conditions placed by Kochen and Specker on an embedding from a noncommutative algebra into a commutative algebra. Where $\hbar \neq 0$ and W is used to map $\mathcal{A}_Q$ into $\mathcal{A}_c$ all of the negative results of the Kochen and Specker proof stand, and precise simultaneous values are impossible. Because the conceptual problems of an adequate metaphysical interpretation of quantum theory can be formulated in terms of the existence of precise simultaneous values, the Wigner transformation and the associated reduction of quantum mechanics to classical statistical mechanics do not aid in solving these conceptual problems. As $\hbar \to 0$, W becomes a homomorphic mapping of quantum theory into classical theory. Whereas W does not sanction the reduction of a theory of classical structure to quantum theory, as the hidden variable theorists desire, it does yield a reduction of quantum mechanics to classical mechanics as $\hbar \to 0$.

SUMMARY

At the outset of this essay, two questions were posed: In what sense, if any, is classical mechanics a limiting case of quantum mechanics? What does it mean to say that one theory is a limiting case of another? After reviewing the orthodox and heterodox answers to these questions, it was concluded that the orthodox position would be tenable only if classical mechanics were a bona fide mathematical limit of quantum mechanics as $\hbar \to 0$ and only if the orthodox view of intertheoretic explanation could be broadened to allow intertheoretic explanation where the secondary theory was not strictly derivable from the primary theory. This broadened outlook requires that an adequate reduction satisfy both formal and nonformal conditions.

In Chapter II the formal conditions on an adequate reduction of quantum mechanics to classical mechanics were discussed. A detour through abstract mechanics motivated the claim that, formally, an adequate reduction of quantum mechanics to classical mechanics requires the existence of a structure preserving mapping, or reduction function, from the quantum algebra of observables, $\mathcal{A}_Q$, to the classical algebra of observables, $\mathcal{A}_c$, and from the Lie algebra of quantum theory, $\mathcal{L}_Q$, to the Lie algebra of classical mechanics, $\mathcal{L}_c$, in the limit as $\hbar \to 0$. The Wigner transformation, W, was shown to be a natural reduction function. Three propositions were proved showing that W satisfies the formal conditions on an adequate reduction of quantum theory to classical theory as $\hbar \to 0$.

Nonformal conditions on the reduction were discussed in Chapter III. The pragmatic and epistemic facets of explanation require that a reduction, as a species of explanation, satisfy an explanatory condition and a unity condition in addition to the formal conditions presented in Chapter II. The Wigner transformation as a reduction function generated a reduction of quantum mechanics to classical mechanics which satisfies the explanatory and unity conditions. The explanations of classical theory by quantum theory resulting from this reduction were compared and contrasted with other examples of intertheoretical explanation.

An additional topic was treated in Chapter III. The problem of formulating an adequate metaphysical interpretation of quantum theory was presented and made precise in terms of the Kochen-Specker result. Where $\hbar \neq 0$ and W is used to map $\mathcal{A}_Q$ into $\mathcal{A}_c$, the negative results of Kochen-Specker stand and W provides no obvious solution to the metaphysical problems of interpretation.

On the basis of these arguments, it can be concluded that classical mechanics is a strict limiting case of quantum mechanics as $\hbar \to 0$ and that if one theory is to be a limiting case of another, certain formal and nonformal conditions must be stated and satisfied. These conclusions support the orthodox physical and philosophical positions.

APPENDIX: THE IMAGE OF HARMONIC OSCILLATOR PURE STATES UNDER W

In the position representation, the normalized pure states of the one-dimensional harmonic oscillator are*

$$\psi_n(q) = \frac{1}{\sqrt{2^n n!\,\sqrt{\pi\hbar}}}\, e^{-(1/2\hbar)q^2}\, H_n\left[\frac{q}{\sqrt{\hbar}}\right]$$

where $H_n(q/\sqrt{\hbar})$ is the n^{th} Hermite polynomial. The von Neumann density corresponding to this wave function is

$$\psi_n(q)\psi^*_n(q') = \frac{1}{2^n n!}\,\frac{1}{\sqrt{\pi\hbar}}\, e^{-(1/2\hbar)(q^2+q'^2)}\, H_n\left[\frac{q}{\sqrt{\hbar}}\right] H_n\left[\frac{q'}{\sqrt{\hbar}}\right] .$$

For products of Hermite polynomials the following generating function obtains (Morse and Feshbach 1953:786)

$$\frac{e^{-(x^2+y^2-2xyz/1-z^2)}}{\sqrt{1+z^2}} = e^{-x^2-y^2} \sum_{n=0}^{\infty} \left[\frac{z^n}{2^n n!}\right] H_n(x)H_n(y).$$

Multiplying both sides of this expression by $e^{(x^2+y^2)/2}$ and simplifying the left-hand side yields

$$\frac{e^{\frac{-(x^2+y^2)(1+z^2)+2xy(2z)}{2(1-z^2)}}}{\sqrt{1+z^2}} = e^{(-x^2-y^2)/2} \sum_{n=0}^{\infty} \left[\frac{z^n}{2^n n!}\right] H_n(x)H_n(y). \qquad (1)$$

The Wigner transformation is

$$W\rho(q,q') = \frac{1}{2\pi}\int da\; e^{ipa}\,\left[q - \frac{a\hbar}{2},\; q + \frac{a\hbar}{2}\right] .$$

The Wigner transformation is linear, so it can be applied to (1). Letting $x = (2q-a\hbar)/2\sqrt{\hbar}$ and $y = (2q+a\hbar)/2\sqrt{\hbar}$ in (1), one gets

$$\frac{1}{2\pi}\,\frac{1}{\sqrt{1+z^2}}\, e^{-(q^2/\hbar)(1-z)/(1+z)} \int da\; e^{ipa}\, e^{-(\hbar a^2/4)(1+z)/(1-z)} =$$

$$= \sum_{n=0}^{\infty} \left[\frac{z^n}{2^n n!}\right] \int da\; e^{ipa}\, e^{-q^2/\hbar}\, e^{-\hbar a^2/4}\, H_n\left[\frac{2q-a\hbar}{2\sqrt{\hbar}}\right] H_n\left[\frac{2q+a\hbar}{2\sqrt{\hbar}}\right]$$

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*Note that here q is the q/b of p. 42 and p is the bp of p. 42.

The left hand integral can be evaluated by completing the square:

$$\frac{1}{2\pi}\frac{1}{\sqrt{1+z^2}} e^{-(q^2/\hbar)(1-z)/(1+z)} \int da\, e^{ipa} e^{-(\hbar a^2/4)(1+z)/(1-z)}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1+z^2}} e^{-(q^2/\hbar)(1-z)/(1+z)}$$

$$e^{-(p^2/\hbar)(1-z)/(1+z)} \int da\, e^{-\left[\frac{\sqrt{\hbar}}{2}\sqrt{\frac{1+z}{1-z}}\, a + \frac{ip}{\sqrt{\hbar}}\sqrt{\frac{1-z}{1+z}}\right]^2}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1+z^2}}\frac{2\sqrt{1-z}}{\sqrt{\hbar}\sqrt{1+z}}\frac{1}{\sqrt{2}} e^{-(q^2/\hbar)(1-z)/(1+z)}$$

$$e^{-(p^2/\hbar)(1-z)/(1+z)} \int dy\, e^{-y^2/2}$$

$$= \frac{\sqrt{2\pi}}{2\pi}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{\hbar}}\frac{\sqrt{1-z}}{\sqrt{1+z^2}\sqrt{1+z}} e^{-[(q^2+p^2)/\hbar]\,(1-z)/(1+z)}$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\hbar}}\frac{\sqrt{1-z}}{\sqrt{1+z^2}\sqrt{1+z}} e^{-[(q^2+p^2)/\hbar]\,(1-z)/(1+z)}\ .$$

In this expression, let $t/1-t = (1-z)/(1+z)$; that is, let $z = 1-2t$. On making this substitution and simplifying, the resulting expression one finds that

$$\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\hbar}}\frac{\sqrt{1-z}}{\sqrt{1+z^2}\sqrt{1+z}} e^{-[(q^2+p^2)/\hbar]\,(1-z)/(1+z)}$$

$$= \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{\hbar}}\sqrt{\frac{2t}{t+1}}\,\frac{e^{-[(q^2+p^2)/\hbar]t/1-t}}{1-t}$$

$$= \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{\hbar}}\sqrt{\frac{2t}{t+1}}\sum_{n=0}^{\infty} t^n \frac{L_n[(q^2+p^2)/\hbar]}{n!}$$

Hence, the result of applying the Wigner transformation to the generating function for products of Hermite polynomials is the generating function for the Laguerre polynomials in $(q^2+p^2)/\hbar$:

$$W \sum_{n=0}^{\infty} [\frac{z^n}{2^n n!}] \; e^{-(1/2\hbar)(q^2+q'^2)} \; H_n[\frac{q}{\sqrt{\hbar}}] \; H_n[\frac{q'}{\sqrt{\hbar}}]$$

$$= \sum_{n=0}^{\infty} \frac{(1-2t)^n}{2^n n!} \int da \; e^{ipa} \; e^{-q^2/\hbar} \; e^{-\hbar^2 a^2/4} \; H_n[\frac{2q-a\hbar}{2\sqrt{\hbar}}] \; H_n[\frac{2q+a\hbar}{2\sqrt{\hbar}}]$$

$$= \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\hbar}} \sqrt{\frac{2t}{t+1}} \; \frac{e^{-[(q^2+p^2)/\hbar]t/1-t}}{1-t} = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\hbar}} \sqrt{\frac{2t}{t+1}} \sum_{n=0}^{\infty} t^n \frac{L_n[(q^2+p^2)/\hbar]}{n!}$$

Quantum mechanical pure states for the harmonical oscillator are mapped to Laguerre functions in the action variable on the classical phase plane.

BIBLIOGRAPHY

Bell, J. 1966. On the Problem of Hidden Variables in Quantum Mechanics, Review of Modern Physics 38:447-452.

Bohr, Niels 1949. Discussion with Einstein on Epistemological Problems in Atomic Physics, in Albert Einstein: Philosopher Scientist, P. A. Schilpp, ed., Open Court, La Salle, Illinois, 199-242.

Bohm, David 1951. Quantum Theory, Prentice Hall, New York.

Cohn, L. 1966. Can Quantum Mechanics be Formulated as a Classical Probability Theory, Philosophy of Science 33:317-322.

Courant, R. and Hilbert, D. 1937. Methods of Mathematical Physics, Volume 1, Interscience Publishers, Inc., New York.

Dirac, P. A. M. 1958. Principles of Quantum Mechanics, 4th ed., Oxford University Press, Oxford.

Feyerabend, Paul 1962. Explanation, Reduction, and Empiricism, in Minnesota Studies in the Philosophy of Science, Volume III, H. Feigl and G. Maxwell, eds., University of Minnesota, Minneapolis, 28-97.

Friedman, Michael 1974. Explanation and Scientific Understanding, Journal of Philosophy 71:5-19.

Furth, R. 1933. Über einige Beziehungen zwischen klassischer Statistik und Quanten mechanik, Zeitschrift fur Physik 81:143-162.

Geroch, Robert n.d. Mathematical Physics, (mimeographed course notes), University of Chicago, Chicago, Illinois.

Glymour, Clark 1970. On Some Patterns of Reduction, Philosophy of Science, 37:340-353.

Gottfried, Kurt 1966. Quantum Mechanics, W. A. Benjamin, Reading, Mass.

Groenewald, H. J. 1946. On the Principles of Quantum Mechanics, Physica 12:405-460.

Home, Roderick W. 1968. The Third Law in Newton's Mechanics, British Journal for the History of Science 4:39-51.

Jordan, Thomas F. 1969. Linear Operators for Quantum Mechanics, John Wiley and Sons, Inc., New York.

Khinchin, A. I. 1949. Mathematical Foundation of Statistical Mechanics, Dover Publications, Inc., New York.

Kochen, S. and Specker, E. P. 1967. The Problem of Hidden Variables in Quantum Mechanics, Journal of Mathematics and Mechanics 17: 59-87.

Kuhn, Thomas 1962. The Structure of Scientific Revolutions, University of Chicago, Chicago, Illinois.

Mach, E. 1893. The Science of Mechanics, Open Court, La Salle, Illinois.

Mackey, George 1963. Mathematical Foundations of Quantum Mechanics, W. A. Benjamin, Inc., New York.

Morse, P. and Feshbach, H. 1953. Methods of Theoretical Physics, Part I, McGraw-Hill, New York.

Moyal, J. E. 1949. Quantum Mechanics as a Statistical Theory, Proceedings of the Cambridge Philosophical Society 45:99-124.

Nagel, Ernest 1961. The Structure of Science, Routledge and Kegan Paul, London.

von Neumann, J. 1955. Mathematical Foundations of Quantum Theory, Princeton University, Princeton, New Jersey.

Prosser, Reese 1966. A New Formulation of Particle Mechanics, Memoirs of the American Mathematical Society, No. 61.

Schaffner, Kenneth 1967. Approaches to Reduction, Philosophy of Science, 34:137-147.

Schrödinger, E. 1931. Über die Umkehrung der Naturgesitze, Berliner Sitzungberichte 144-153.

Schrödinger, E. 1932. Sur la théorie relativiste de l'électron et l'interpretation de la mécanique quantique, Annales de l'Institut Henri Poincaré 2:269-310.

Sklar, Lawrence 1967. Types of Inter-theoretic Reduction, British Journal for the Philosophy of Science 18:109-124.

Sneed, Joseph 1971. The Logical Structure of Mathematical Physics, D. Reidel, Dordrecht, Holland.

Stein, Howard 1972. On the Conceptual Structure of Quantum Mechanics, in Paradigms and Paradoxes, R. Colodny, ed., University of Pittsburgh Press, Pittsburgh, 367-438.

Sudarshan, E. C. G. 1962. Structure of Dynamical Theories, Lectures in Theoretical Physics, Brandeis Summer Institute of 1961, Benjamin, Inc., New York, 143-199.

Teller, Paul 1977. On the Interpretation of Continuous Quantities in Quantum Mechanics, (unpublished lecture notes), University of Illinois at Chicago.

Weyl, H. 1950. The Theory of Groups and Quantum Mechanics, Dover Publications, Inc., New York.

Wigner, E. 1932. On the Quantum Correction for Thermodynamic Equilibrium, Physical Review 40:749-759.

Wigner, E. 1967. Symmetries and Reflections, M. I. T. Press, Cambridge, Mass.