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Rigorous Bounds on Strong Interaction Coupling Constants

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RIGOROUS BOUNDS ON STRONG INTERACTION COUPLING CONSTANTS

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PREFACE

I am deeply grateful to Professor N.N. Khuri for his constant help and guidance throughout my stay at Rockefeller. It is also a pleasure to thank Professors A. Pais, M.A.B. Bég, and H.R. Pagels for their many contributions to my education. Finally, I want to thank my wife Alice for her patience and encouragement.

SUMMARY

We investigate the problem of deriving bounds on strong interaction scattering amplitudes from the results of axiomatic field theory. The bounds on the π-π scattering amplitude at points within its analyticity domain which have been obtained by Łukaszuk and Martin are especially interesting because they contain no free parameters, and they impose absolute limits on the size of the renormalized coupling constant for π-π scattering.

We improve the rigorous upper bound derived by Łukaszuk and Martin for the π^o-π^o scattering amplitude at the symmetric point. Our principle new tool is the "parametric dispersion relation" of Auberson and Khuri.

Also, for a ϕ^4 type field theory with a scalar bound state which is not too tightly bound, we generalize the methods developed by Martin for π-π scattering to establish upper and lower bounds on the renormalized coupling constant and an upper bound on the physical coupling constant to the bound state. These new bounds are functions of the mass of the bound state. Numerical examples of the bound on the physical coupling constant are given for several bound state masses.

Finally, we discuss the relevance of our results to constructive field theory. We point out that, while our bounds do not apply to a ϕ^4 field theory in 1 space +1 time dimensions, they do limit the values of the renormalized coupling constant for which one could construct a ϕ^4 field theory in 2 space +1 time (and, of course, 3 space +1 time) dimensions.

CONTENTS

CHAPTER I

Introduction

During the decade of the 1950's particle physicists began to realize that, contrary to their previous expectations, the problem of exploiting a realistic quantum field theory for the strong interactions was beyond their mathematical capabilities and that such a theory might not even be self-consistent. The difficulty was that the perturbation expansion approach, which provided nearly all of the results of quantum electrodynamics and which was the only available method for systematically extracting physical consequences from a nontrivial field theory, failed for the strong interactions because the expansion parameter (e.g. the coupling constant) was too large. However, no alternative theory has been proposed which offers a better framework for developing an understanding of the interactions of hadrons. Since the theory of quantum electrodynamics had been, and continues to be, so successful, and since the general principles which are the foundations of any local field theory seem so eminently believable, some physicists began to ask themselves what results could be derived from those principles without getting involved in the intricacies of a particular field theory. Even this modest goal has proved to be difficult to achieve, and the definitive answers have yet to be discovered. Yet the results which have already come out of this program are more than even optimistic physicists would have expected fifteen years ago.

Logically, the first step in this development was to find a mathematically precise formulation of the general physical principles which form the basis of any local field theory. Those principles include: 1) Lorentz invariance; 2) causality; 3) the existence of a unique vacuum state relative to which physical states have positive energy; 4) completeness of the set of physical states; and 5) unitarity. With the addition of certain technical assumptions common to most field theories these principles have been axiomatized in various ways, for example, by Lehmann, Symanzik, and Zimmermann (LSZ)[1] and by Wightman[2].

There are then three relevant questions: 1) Are these axioms self-consistent? If so, 2) what are their consequences, and 3) are those consequences consistent with experiment?

If the axioms are self-consistent, then the best way to prove it would be to actually construct and solve a nontrivial example of a field theory satisfying them. This, of course, is a difficult problem. Nonetheless, it has been possible to demonstrate that in 1 space +1 time dimensions there is such a theory, namely a ϕ^4 field theory, at least for small enough values of the coupling constant.[3] All of the features of this theory have not yet been worked out, but its existence does provide some encouragement. For theories in 3 + 1 dimensions the question is still open. In the absence of evidence to the contrary we assume that the axioms are self-consistent and proceed to the description of some of their consequences.

Most of the physical results which have been obtained from axiomatic field theory take the form of bounds and inequalities which limit the behavior of the scattering amplitude. In the derivation of these bounds the analyticity properties of the scattering amplitude as a function of energy and momentum transfer play a central role. We will therefore begin by enumerating some of the analyticity properties which are rigorous consequences of local field theory.

To do this we must first define our notation. We will limit our discussion to elastic two-body scattering:

$$A + B \rightarrow A + B \ . \tag{1-1}$$

The complications which arise for particles with spin will not be mentioned. Let p_A, p_A', p_B, and p_B' be respectively the ingoing and outgoing four-momenta of particles A and B. Then $s=(p_A+p_B)^2$ is the square of the center of mass energy for the reaction (1-1). Similarly, $t=(p_A-p_A')^2$ and $u=(p_A-p_B')^2$ are the squares of the center of mass energies for the reactions:

$$A + \bar{A} \to B + \bar{B} , \tag{1-2}$$

and

$$A + \bar{B} \to A + \bar{B} . \tag{1-3}$$

The Mandelstam variables s, t, and u satisfy the constraint:

$$s + t + u = 2(M_A^2 + M_B^2). \tag{1-4}$$

They are related to the s-channel center of mass momentum and scattering angle by:

$$\begin{aligned} s &= (\sqrt{M_A^2 + k^2} + \sqrt{M_B^2 + k^2})^2 , \\ t &= -2k^2 (1 - \cos\theta_s) , \\ u &= -2k^2 (1 + \cos\theta_s) . \end{aligned} \tag{1-5}$$

In 1955 Goldberger[4] proposed that the forward pion-nucleon scattering amplitude satisfies a dispersion relation in s. Subsequently, the π-N dispersion relation was proved from the LSZ formulation of the principles of quantum field theory by Bogoliubov and collaborators;[5] an independent proof for the forward dispersion relation was given by Symanzik.[6] The proof has been extended to many other processes of physical interest,[7] including for example $\pi\pi \to \pi\pi$ and $K\pi \to K\pi$; but for some of the most important reactions, especially $NN \to NN$ and $KN \to KN$, there are unphysical thresholds which have so far prevented the proof from being carried out.

We remind the reader that in order for a scattering amplitude to satisfy a dispersion relation it must have no singularities in some simple domain in the complex energy variable, and it must be polynomially bounded in the limit $|s| \to \infty$. What has been proved is the following: a) For fixed physical t, $-t_0 < t \leq 0$, the scattering amplitude $F_{AB\to AB}$ is the boundary value $\lim_{\varepsilon\downarrow 0} F(s+i\varepsilon,t)$ of a function analytic in the complex s-plane with real cuts $s \geq s_0$ and $s \leq 2(M_A^2 + M_B^2) - t - u_0$. Along the left hand cut $\lim_{\varepsilon\downarrow 0} F(s-i\varepsilon,t) = F_{\bar{A}\bar{B}\to\bar{A}\bar{B}}$. The discontinuities of F across the right and left hand cuts are the absorptive parts in the s and u channels, respectively, and s_0 and u_0 are the physical tresholds for the processes (1-1) and (1-3). b) The polynomial boundedness of F has been proved in the LSZ formalism and also from the Wightman axioms by Hepp.[8] Therefore, for t fixed, $-t_0 < t \leq 0$, we can write the scattering amplitude as:

$$F(s,t,u) = \frac{s^N}{\pi} \int_{s_0}^{\infty} \frac{ds' \, A_s(s',t)}{s'^N(s'-s)} + \frac{u^N}{\pi} \int_{u_0}^{\infty} \frac{du' \, A_u(u',t)}{u'^N(u'-u)} \tag{1-6}$$

$$+ \text{ Polynomial in s and u.}$$

Here $A_s(s',t)$ and $A_u(u',t)$ are the absorptive parts in the s and u channels, respectively.

The interpretation of the boundary value of F(s,t,u) on the cuts which we have given above runs into trouble near threshold for fixed negative t. For $s \to s_0$, $\cos\theta_s = 1 + \frac{t}{2k^2}$ becomes large and negative, so it is no longer in the physical region. This difficulty was overcome by Lehmann[9] who showed that for fixed physical s the scattering amplitude is an analytic function of $\cos\theta_s$ inside an ellipse with foci $\cos\theta = \pm 1$ and semi-major axis:

$$\cos\theta_0 = \left(1 + \frac{(M_A'^2 - M_A^2)(M_B'^2 - M_B^2)}{k^2(s - (M_A' - M_B')^2)}\right)^{1/2}. \tag{1-7}$$

Here M_A' and M_B' are the lowest mass states for which $(A'|j_A(0)|0) \neq 0$ and $(B'|j_B'(0)|0) \neq 0$; j_A and j_B are the source currents for the colliding particles. In addition Lehmann showed that the absorptive part $A_s(s,t) = \mathrm{Im}\, F(s,t)$ is analytic in a larger confocal ellipse with semi-major axis

$$\cos\theta_A = (2\cos^2\theta_0 - 1). \qquad (1\text{-}8)$$

It follows that the partial wave expansions for the amplitude and its absorptive part converge within the small and large Lehmann ellipses (1-7) and (1-8), respectively:

$$F(s,t) = \frac{\sqrt{s}}{k} \sum_{\ell=0}^{\infty} (2\ell + 1)\, f_\ell(s)\, P_\ell(\cos\theta), \qquad (1\text{-}9a)$$

$$A(s,t) = \frac{\sqrt{s}}{k} \sum_{\ell=0}^{\infty} (2\ell + 1)\, a_\ell(s)\, P_\ell(\cos\theta), \qquad (1\text{-}9b)$$

where $a_\ell(s) = \mathrm{Im}\, f_\ell(s)$. Therefore, for $\cos\theta_s$ outside the physical range $-1 \leq \cos\theta_s \leq 1$ it is still possible to give a meaning to $F(s,t)$ and $A(s,t)$ via (1-9) within their ellipses of convergence.

Both Lehmann ellipses shrink to the real line $-1 \leq \cos\theta_s \leq 1$ for $s \to \infty$. This is not sufficient for many of the purposes we will encounter. However, Martin[10] proved that for physical s, $F(s,t)$ is regular inside a circle $|t|<R$, where R is independent of s. Now the unitarity constraint,

$$a_\ell(s) \geq |f_\ell(s)|^2, \qquad (1\text{-}10)$$

requires that the absorptive part of the partial wave amplitudes be positive. It follows from the theory of expansion in Legendre polynomials that the ellipse of convergence of (1-9b) must include the right extremity $t = R$ of Martin's circle. The absorptive part $A(s,t)$ is therefore analytic in an ellipse in the t-plane with foci 0 and $-4k^2$ and right extremity R. This is Lehmann-Martin ellipse. Sommer[11] has calculated R, for cases in which $\Delta = (M_A' - M_A) = (M_B' - M_B)$, as:

$$R = \Delta^2 = (M_A' - M_A)^2 . \qquad (1\text{-}11)$$

For $\pi\pi$ and $K\pi$ scattering this gives $R = 4\mu^2$, where μ is the pion mass. Sommer's technique gives a lower value of R for πN scattering, but Bessis and Glaser[12] have succeeded in proving $R = 4\mu^2$ for that case as well. In most cases one gets the physically expected value of R. Martin's enlargement of the Lehmann ellipse makes it possible to extend the range of t for which F(s,t) satisfies a dispersion relation. For $\pi\pi$ scattering Martin[13] has proved the dispersion relation for

$$-28\mu^2 < t < 4\mu^2 \tag{1-12}$$

The analyticity results we have mentioned so far say nothing about the analyticity properties of F(s,t) for s and t both complex. It was first shown by Mandelstam[14] for $\pi\pi$ scattering, where there are dispersion relations in all three channels, that the scattering amplitude is simultaneously analytic in both s and t. He obtained the domain $|st|<256\mu^4$ inside which the only singularities are the physical cuts s, t, and $u \geq 4\mu^2$. Mandelstam's argument was later generalized by Lehmann[15] to processes, such as πN scattering, with only a fixed-t dispersion relation. Finally, for $\pi\pi$ scattering Martin[13] has considerably extended Mandelstam's domain. We mention here only that Martin's domain contains domains of the form:

$$D \supset \{s,t \mid |s-a_1| \cdot |t-a_2| < A_1\} \cup \{s,t \mid |t-a_2| \cdot |u-a_3| < A_2\} \cup \{s,t \mid |u-a_3| \cdot |s-a_1| < A_3\}, \tag{1-13}$$

minus the physical cuts. The constants A_i are known functions of the pion mass and the points a_i. The real sections of these domains contain almost the entire boundary of the Mandelstam double spectral functions.

This concludes our brief discussion of the analyticity results which have been rigorously proved from axiomatic field theory. We now proceed to the enumeration of bounds on the scattering amplitude which follow from these analyticity properties combined with unitarity.

The first example of a bound on the asymptotic behavior of a scattering amplitude at high energy was derived by Froissart.[16]

Starting with the postulate of the Mandelstam representation he proved for the total cross-section the upper bound:

$$\sigma_{tot}(s) \underset{s\to\infty}{<} \text{const}\ \log^2 s\ , \qquad (1\text{-}14a)$$

which is equivalent to the bound on the forward amplitude:

$$|F(s,0)| < \text{const}\ s\ \log^2 s. \qquad (1\text{-}14b)$$

Later Martin[17] proved the Froissart bound from the axioms of field theory; the crucial input for which his proof had to wait was the analyticity of F(s,t) within the Lehmann-Martin ellipse. Upper bounds on the non-forward amplitude have been proved as well. Martin[18] obtained:

$$|F(s,t)| \underset{s\to\infty}{<} \text{const}\ s\ \log^{3/2} s\ , \qquad (1\text{-}15)$$

for fixed physical $t < 0$.

The Froissart bound obviously requires that the dispersion relation for real physical $t \leq 0$ has at most two subtractions. Jin and Martin[19] were able to show that this result also holds in the region $|t| < R$. They also proved[20] that the forward amplitude, which in addition to satisfying a twice-subtracted dispersion relation has a discontinuity of definite sign on each cut, cannot decrease faster than $1/s^2$ for $|s| \to \infty$:

$$|F(s,0)| \underset{s\to\infty}{>} \text{const}/s^2\ . \qquad (1\text{-}16)$$

For asymptotic energies the forward amplitude is thus bounded above and below.

Many other bounds have since been derived. For example, from unitarity one gets an inequality relating the elastic and total cross sections:[21]

$$\sigma_{el}(s) \gtrsim \text{const}\ \frac{\sigma_{tot}{}^2(s)}{\log^2 s}\ . \qquad (1\text{-}17)$$

Unitarity also provides a lower bound on the width of the diffraction peak:[22]

$$\Delta \equiv \frac{|F(s,0)|}{\left|\frac{dF(s,t)}{dt}\right|_{t=0}} \geq \text{const} \ \frac{\sigma_{tot}^{2}(s)}{\log^{4} s} . \qquad (1\text{-}18)$$

To conclude our discussion of asymptotic bounds we would like to mention an important relation which can be deduced from the axioms of field theory only if one makes some additional asusmptions. This is the Pomeranchuk theorem[23] which states that the total cross-sections for scattering of particles and antiparticles on the same target become equal asymptotically. The additional assumptions necessary for the proof have been progressively weakened over the years, and Martin[24] was finally able to reduce them to a single assumption on the growth of the real parts of the forward amplitudes $F_{\pm}(s,0)$ for particle and antiparticle scattering. However, there exists a mathematical counter-example which is consistent with axiomatic field theory (AFT) and which violates Martin's condition. Therefore the Pomeranchuk theorem cannot be proved from the axioms of field theory alone, and additional input is needed to understand why the condition on the real parts should be true.

None of the results derived form local field theory have ever been shown to be in contradiction with experiment. However, the weakness of all of the bounds we have mentioned up to this point is that they contain arbitrary constants relating to the energy at which asymptotic considerations can be expected to become valid. As pointed out by Martin, they provide no restrictions on the scattering amplitude at finite energies.

Using a different approach to the problem, Martin[25] showed that the axioms of field theory impose quantitative non-asymptotic restrictions on the strength of the strong interactions, at least for the case of $\pi\pi$ scattering. From the requirements of analyticity (as discussed earlier), unitarity, and crossing symmetry he proved that within its analyticity domain, including the symmetric point $s = t = u = 4\mu^2/3$, the $\pi^o\pi^o$ scattering amplitude is bounded above and below as a function of the pion mass alone. From these bounds follows a lower bound on the π^o-π^o s-wave scattering length. Finally, Martin and others[26] have found an interesting group of restrictions on the π-π partial wave amplitudes below threshold $(0 < s < 4)$. For the π^o-π^o s-wave amplitude one has the best results. Among them are:

$$\frac{df_0(s)}{ds} > 0 \qquad 1.697 \leq s < 4,$$

$$\frac{df_0(s)}{ds} < 0 \qquad 0 < s \leq 1.127\ , \qquad (1\text{-}19)$$

$$\frac{d^2 f_0(s)}{ds} > 0 \qquad 1.7 \geq s > 0\ .$$

The minimum of $f_0(s)$, the s-wave amplitude, must occur in the interval $1.127 < s < 1.697$.

This brings us to the subject of this dissertation. Martin's upper and lower bounds on the π^o-π^o scattering amplitude within its analyticity domain are important for two reasons: First, they contain no arbitrary constants and represent quantitative restrictions on the size of the amplitude at finite energies. With the exception of the lower bound on the s-wave scattering scattering length they cannot be compared directly with experiment. However, they do impose restrictions on models for π-π scattering, and therefore one would like to have the best bounds which can be obtained with the analyticity, unitarity, and crossing symmetry from AFT. Using a refinement of Martin's original method, Łukaszuk and Martin[27] succeeded in improving Martin's numerical results, but their bounds are still not the best which can be obtained. We have used a "parametric dispersion relation" for π^o-π^o scattering, which has recently been derived from the results of axiomatic field theory by Auberson and Khuri[28] and which is fully and explicitly crossing symmetric, to derive an improved upper bound on the amplitude at the symmetric point.

In addition to their physical interest for π-π scattering, these bounds are of theoretical interest because they limit the range of renormalized coupling constants for which one could construct a consistent field theory. In a ϕ^4 field theory, for example, the renormalized coupling constant can be directly related to the amplitude at the symmetric point.

In the real world there are no bound states in the π-π system, and their absence has been explicitly incorporated in the analyticity assumptions used to derive the bounds on the π^o-π^o amplitude. However, in more general field theories there may be bound states and one would like to know if there exist bounds on coupling constants when bound states are present. We have generalized the methods developed for π^o-π^o scattering by Martin to the case of scattering of identical neutral pseudoscalar bosons of mass μ which couple to a bound state of mass m with physical coupling constant g^2. For masses in the range $4/3 < \frac{m^2}{\mu^2} < 4$ we have proved a rigorous upper bound on g^2 which depends only on the masses m and μ. We have also proved that if one is willing to accept a larger analyticity domain which has not been proved from the axioms of field theory, but which is true in perturbation theory, then this bound can be extended to $1 \leq \frac{m^2}{\mu^2} < 4$. The latter result also covers the scattering of identical neutral scalar particles with a bound state pole at the particle mass, for example in a ϕ^3 field theory. Finally, we have demonstrated that for $2 < \frac{m^2}{\mu^2} < 4$ one can still obtain upper and lower bounds on the conventionally defined (in terms of the amplitude at the symmetric point) renormalized coupling constant. For $4/3 < \frac{m^2}{\mu^2} \leq 2$ one no longer has the lower bound, but it is possible to choose a different definition of the renormalized coupling constant for which both lower and upper bounds can be calculated. Again, this result can be extended to all masses in the range $1 \leq \frac{m^2}{\mu^2} < 4$ if one is willing to accept some analyticity from perturbation theory.

The organization of this thesis is the following: In Chapter II we give a thorough review of the methods and results of Martin and of Łukaszuk and Martin. Chapter III contains a brief discussion of the Auberson-Khuri representation. In Chapter IV we discuss our improvement of the LM upper bound on the π^o-π^o amplitude at the symmetric point; we also describe our attempts to improve their other numerical results. We generalize Martin's methods to theories with bound states in Chapter V, and we conclude by discussing the relevance of our results to constructive field theory in Chapter VI. In the Appendix we show how the Poisson-Jensen formula can be used to prove a group of inequalities which are used extensively throughout the text.

CHAPTER II

The Bounds of Martin and Łukaszuk

A. Introduction

Martin[25] was the first person to find rigorous bounds with no free parameters on the π^o-π^o scattering amplitude at points within its analyticity domain. He took as his starting point the following results from axiomatic field theory:

1) Analyticity: The π^o-π^o scattering amplitude $F(s,t,u)$ is an analytic function of the Mandelstam variables s, t, and u, with $s + t + u = 4$. Note that we always work in a system of units in which the pion mass is unity. For fixed physical $s \geq 4$, the amplitude is analytic in t within the Lehmann-Martin ellipse, and its partial wave expansion converges in that ellipse:

$$F(s,t,u)= \frac{\sqrt{s}}{k} \sum_{\ell\ \mathrm{even}} (2\ell+1) f_\ell(s) P_\ell(x=1+ \frac{2t}{s-4}). \qquad (2\text{-}1)$$

For fixed real t within the Lehmann-Martin ellipse, $F(s,t,u)$ is analytic in the s-plane with real cuts from $s=4$ to ∞ and from $s = -t$ to $-\infty$; F satisfies a twice subtracted dispersion relation in s. There are no bound states in the π-π system. The amplitude is regular within the Mandelstam triangle where s, t, and u are real and below threshold: $s < 4$, $t < 4$, $u < 4$. Within this triangle is the small Mandelstam triangle $s > 0$, $t > 0$, $u > 0$. Both triangles are displayed in the usual triangular coordinate system in Figure I.

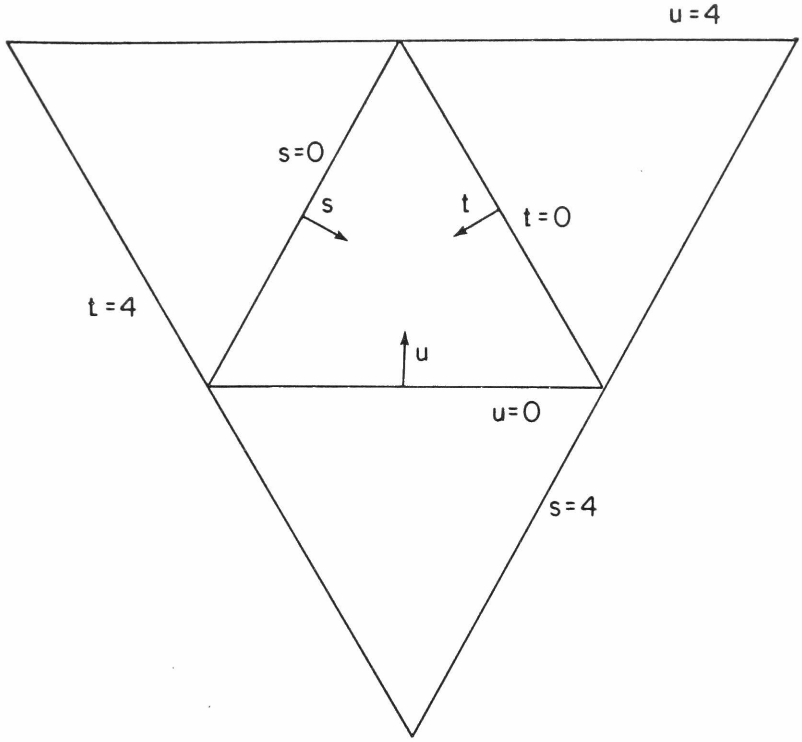

Figure 1

The Mandelstam Triangle. Inside this triangle s, t, and u are real and below threshold: $s < 4$, $t < 4$, $u < 4$. It contains the small Mandelstam triangle $s > 0$, $t > 0$, $u > 0$.

2) Unitarity: Let $A(s,t)$ and $a_\ell(s)$ be the absorptive parts of the full scattering amplitude and the ℓ^{th} partial wave amplitude, respectively:

$$A(s,t)=\mathrm{Im}F(s,t)=\frac{\sqrt{s}}{k}\sum_{\ell\ \mathrm{even}}(2\ell+1)a_\ell(s)P_\ell(x). \qquad (2\text{-}2)$$

Our normalization is defined so that the unitarity constraint takes the simple form:

$$1 \geq a_\ell(s) \geq |f_\ell(s)|^2 > 0. \qquad (2\text{-}3)$$

It follows that, for $t \geq 0$,

$$A(s,t)>0. \tag{2-4}$$

3) Crossing Symmetry: The π^o-π^o scattering amplitude is a completely symmetric function of s,t, and u.

From these three statements Martin demonstrated, using a method first introduced by Meiman[29], that with the Mandelstam triangle, for example at the symmetric point, the π^o-π^o ampltiude is bounded above and below as a function of the pion mass alone. Martin's original method was subsequently refined by Łukaszuk[30], and with this more sophisticated method Łukaszuk and Martin [27] improved the numerical results of Martin.

At this point I would like to mention the significance of these results. The size of the scattering amplitude within the Mandelstam triangle is a measure of the strength of the π^o-π^o interaction. The scattering amplitude takes its minimum value within the Mandelstam triangle at the symmetric point[20]

$$\alpha_o \equiv F(4/3,4/3,4/3). \tag{2-5}$$

The conventional definition, due to Chew and Mandelstam,[31] of the renormalized coupling constant λ for π^o-π^o scattering is:

$$\lambda = -\alpha_o/6. \tag{2-6}$$

The bounds on the scattering amplitude are therefore equivalent to bounds on the renromalized coupling constant. These bounds, which follow from analyticity, unitarity, and crossing symmetry, are not restricted to $\pi^o-\pi^o$ scattering and they apply to any local field theory which satisfies these requirements, i.e. a ϕ^4 field theory with no bound state. In addition, the bounds of Łukaszuk and Martin impose a lower bound on the $\pi^o-\pi^o$ s-wave scattering length. As pointed out earlier, these bounds contain no free parameters, and therefore, unlike the asymptotic bounds, they represent a real constraint on the amplitude at finite energies.

We will divide the rest of this chapter into three sections. First, in Section B we will describe Martin's original method for obtaining bounds on F(s,t,u); since only a very sketchy outline of this procedure is available in the literature, we have explained the technique in complete detail. Then in Section C we will explain Łukaszuk's[30] modification of Martin's[25] method as used by Łukaszuk and Martin[27]. The paper of LM is somewhat misleading in that the procedure used to calculate their bounds does not fully coincide with the method described by them; we present the method actually employed to obtain numerical results. Finally, in Section D we summarize the numerical results. We have independently checked the calculations of LM and found one small error; we correct their mistake and also provide an improved calculation.

B. Martin's Original Method

To begin, Martin wrote a twice-subtracted fixed-t dispersion relation for F(s,t,u) with t real and positive:

$$F(s_1,t,u_1)-F(s_o,t,u_o)$$

$$= \frac{1}{\pi}\int_4^\infty ds'\rho(s';\, s_1,\, s_o,\, t)A(s',t), \qquad (2\text{-}7a)$$

where

$$\rho(s';\ s_1,\ s_o,\ t) \equiv \frac{(s_1-s_o)(s_1-u_o)(2s'-4+t)}{(s'-s_1)(s'-u_1)(s'-s_o)(s'-u_o)}\ . \tag{2-7b}$$

He noticed that for $s_1 > s_o$, $s_1 > u_o$, $u_1 \leq 0$, and both points (s_1,t,u_1) and (s_o,t,u_o) within the Mandelstam triangle, the difference (2-7a) is positive and unitarity allows one to derive a non-linear inequality of the form:

$$F(s_1,t,u_1)-F(s_o,t,u_o) > C|F(s_1,u_1,t)|^N. \tag{2-8}$$

with $N \geq 2$. C is a known constant dependent on N.

Combined with $t\to u$ crossing symmetry this inequality gives an upper bound on $F(s_o,t,u_o)$. Furthermore, if the point (s_o,t,u_o) as well as (s_1,t,u_1) is outside the inner Mandelstam triangle, then one can also obtain a second inequality:

$$F(s_1,t,u_1)-F(s_o,t,u_o) > C_o|F(s_o,u_o,t)|^N. \tag{2-9}$$

Equations (2-8) and (2-9) and $t\to u$ crossing then produce absolute bounds on both $F(s_o,t,u_o)$ and $F(s_1,t,u_1)$. Finally, having obtained bounds on the amplitude at these points it is possible to obtain a lower bound on the amplitude at any point within the inner Mandelstam triangle at a momentum transfer $t' \lesssim t$.

We will now describe in detail how one arrives at the inequalities (2-8) and (2-9). The method for obtaining lower bounds at points within the inner Mandelstam triangle is explained in Section D.

To obtain Equation (2-8) it is best to limit our attention to the case $u_1=0$; then the point (s_o,t,u_o) lies within the inner Mandelstam triangle. The reason for this choice of points will be mentioned later. The first step is to minimize the righthand side of the dispersion relation (2-7) by finding an inequality of the form:

$$A(s,t) \geq g_N(s,t,0)\,|F(s,0)|^N. \tag{2-10}$$

for $s \geq 4$ and $N \geq 2$. From unitarity and the partial wave expansions (2-1) and (2-2) we have:

$$A(s,t) \geq \frac{\sqrt{s}}{k} \sum_{\ell \text{ even}} (2\ell+1)\,|f_\ell(s)|^2 P_\ell(x)$$

$$\equiv \bar{A}(s,t), \tag{2-11a}$$

$$|F(s,0)| \leq \frac{\sqrt{s}}{k} \sum_{\ell \text{ even}} (2\ell+1)\,|f_\ell(s)|$$

$$\equiv \bar{F}(s). \tag{2-11b}$$

By the Lagrange multiplier technique and unitarity it follows that the set $\{f_\ell(s)\}$ minimizing $\bar{A}(s,t)$ for a given $\bar{F}(s)$ has the form:

$$\begin{aligned} f_\ell(s)&=1 \quad , \qquad \ell = 0,2,\ \dots\ ,\ 2L\ , \\ f_\ell(s) &= \frac{c(s)}{P_\ell(x)}\ , \quad \ell = 2L+2,\ \dots\ . \end{aligned} \tag{2-12}$$

The Lagrange multiplier $c(s)$ is a function of s and satisfies the inequalities:

$$1 \leq P_{2L}(x) < c(s) \leq P_{2L+2}(x)\ . \tag{2-13}$$

To obtain the inequality (2-10) we then maximize $\frac{|\bar{F}(s)|^N}{\bar{A}(s,t)}$ as a function of c(s). The result is:

$$g_N(s,t,0) = \frac{N}{2c(s)}\left\{ \frac{\sqrt{s}}{k} \left[\sum_{0}^{2L} (2\ell+1) + c(s) \sum_{2L+2}^{\infty} (2\ell+1)/P_\ell(x) \right]\right\}^{N-1}, \tag{2-14}$$

with c(s) and $L \equiv L(s)$ fixed by the equations:

$$\bar{A}(s,t) = \frac{\sqrt{s}}{k} \left[\sum_{0}^{2L} (2\ell+1)P_\ell(x) + c^2(s) \sum_{2L+2}^{\infty} (2\ell+1)/P_\ell(x) \right], \tag{2-15a}$$

$$\bar{F}(s) = \frac{\sqrt{s}}{k} \left[\sum_{0}^{2L} (2\ell+1) + c(s) \sum_{2L+2}^{\infty} (2\ell+1)/P_\ell(x) \right], \tag{2-15b}$$

$$\bar{A}(s,t) = \frac{2c(s)\bar{F}(s)}{N} , \tag{2-15c}$$

and the inequalities (2-13). Inserting Equation (2-10) into the dispersion relation (2-7) we have the inequality:

$$F(s_1,t,0) - F(s_o,t,u_o) \geq \frac{1}{\pi} \int_4^\infty ds' \frac{\rho(s';s_1,s_o,t)\,|F(s',0)|^N}{g_N(s',t,0)}. \tag{2-16}$$

The next step is to transform the s-plane with real cuts $s \geq 4$ and $s \leq 0$ onto the unit disk in the y-plane. As an intermediate step, transform the twice-cut s-plane onto the once-cut z-plane with real cut $z \geq 1$:

$$z = \left(\frac{s-2}{2} \right)^2 . \tag{2-17}$$

Then transform the cut z-plane onto the unit disk $|y| \lesssim 1$:

$$y = \frac{i - \sqrt{\frac{z-1}{1-c}}}{i + \sqrt{\frac{z-1}{1-c}}} \quad , \tag{2-18a}$$

with

$$c \equiv z(s_1). \tag{2-18b}$$

This sequence of transformations maps the point $s = s_1$ onto the point $y = 0$. The integration range $s \gtrsim 4$ is mapped onto the unit semi-circle in the upper-half y-plane: $y = e^{i\phi}$, $0 \leq \phi \leq \pi$. The amplitude $F(s(y),0)$ is analytic in the interior of the unit circle. With this change of variables Equation (2-16) becomes:

$$F(s_1,t,0) - F(s_o,t,u_o) >$$

$$\geq \frac{1}{\pi} \int_0^{\pi} d\phi \; W(\phi) \left| F(s'(e^{i\phi}),0) \right|^N, \tag{2-19a}$$

where

$$W(\phi) \equiv \frac{\rho(s'(e^{i\phi}); s_1, s_o, t) J(s', \phi, s_1)}{g_N(s'(e^{i\phi}),t,0)} \quad , \tag{2-19b}$$

and $J(s',\phi,s_1)$ is the Jacobian of the transformation from s' to ϕ. The reality property $F(s^*,0) = F^*(s,0)$ and the theorem on arithmetic and geometric means allow us to get from (2-19) the inequality:

$$F(s_1,t,0) - F(s_o,t,u_o)$$

$$\geq C \exp\{\frac{1}{\pi} \int_0^{\pi} d\phi \ln \left| F(s'(e^{i\phi}),0) \right|^N \} \tag{2-20a}$$

$$= C \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \ln \left| F(s'(e^{i\phi}),0) \right|^N \},$$

where

$$C = \exp\ [\frac{1}{\pi} \int_0^{\pi} d\phi \ln W(\phi)]. \tag{2-20b}$$

Finally, $t \leftrightarrow u$ crossing symmetry and Poisson' inequality reguire:

$$\begin{aligned} &F(s_1,t,0) - F(s_o,t,u_o) \\ &\quad > C|F(s_1,0,t)|^N \\ &\quad = C|F(s_1,t,0)|^N. \end{aligned} \tag{2-21}$$

For a detailed discussion of the Poisson and related inequalities we refer the reader to the Appendix. It follows immediately from (2-21) that $F(s_o,t,u_o)$ has the upper bound:

$$F(s_o,t,u_o) < (1-\frac{1}{N})\ (\frac{1}{NC})^{\frac{1}{N-1}} \tag{2-22}$$

which is the result we are looking for.

Next we will show how one obtains Equations (2-8) and (2-9), and from them absolute bounds on $F(s_1,t,u_1)$ and $F(s_o,t,u_o)$, for both points (s_1,t,u_1) and (s_o,t,u_o) outside or on the border of the inner Mandelstam triangle. It will be simplest, as well as desirable (as will be explained later), to limit ourselves to the case $u_o = 0$. Then starting from the dispersion relation (2-7), the identical considerations which led to (2-21) now require:

$$\begin{aligned} &F(s_1,t,u_1) - F(s_o,t,0) \\ &\quad > C_o|F(s_o,t,0)|^N. \end{aligned} \tag{2-23a}$$

where

$$C_o = \exp\ \{\frac{1}{\pi} \int_0^{\pi} d\phi \ln\ [\frac{\rho(s'(e^{i\phi});s_1,s_o,t)J(s',\phi,s_o)}{g_N(s'(e^{i\phi}),t,0)}]\}\ , \tag{2-23b}$$

The only difference between this and Equation (2-21) is that here the point $(s_o,0,4-s_o)$, rather than $(s_1,0,4-s_1)$, has been mapped onto the center of the unit circle in the y-plane; that is, (2-18b) must be replaced by

$$c = z(s_o). \qquad \text{(2-18b')}$$

To derive absolute bounds on $F(s_1,t,u_1)$ and $F(s_o,t,u_o)$ we will need in addition to Equation (2-23) an inequality of the same form, but with $|F(s_o,t,0)|$ replaced by $|F(s_1,t,u_1)|$ on the right hand side. The first step in obtaining that inequality is to find an inequality of the form

$$A(s,t) \geq g_N(s,t,u_1)\,|F(s,u_1)|^N \qquad \text{(2-24)}$$

to replace (2-10). Let us write a partial wave expansion for the non-forward amplitude $F(s,u_1)$:

$$F(s,u_1) = \frac{\sqrt{s}}{k} \sum_{\ell \text{ even}} (2\ell+1) f_\ell(s) P_\ell\left(y=1+\frac{2u_1}{s-4}\right) ; \qquad \text{(2-25)}$$

and define:

$$I(s,u_1) \equiv \mathrm{Im}F(s,u_1) = \frac{\sqrt{s}}{k} \sum_{\ell \text{ even}} (2\ell+1) a_\ell(s) P_\ell(y), \qquad \text{(2-26a)}$$

$$R(s,u_1) \equiv \frac{\sqrt{s}}{k} \sum_{\ell \text{ even}} (2\ell+1) \sqrt{a_\ell(s)(1-a_\ell(s))}\, |P_\ell(y)|. \qquad \text{(2-26b)}$$

It follows from the unitarity constraint (2-3) that

$$R(s,u_1) \geq |\mathrm{Re}F(s,u_1)| , \qquad \text{(2-27a)}$$

and hence,

$$|\tilde{F}(s,u_1)|^2 \equiv R^2(s,u_1) + I^2(s,u_1) \geq |F(s,u_1)|^2. \qquad \text{(2-27b)}$$

For $4 \leq s < s_1 + t$, the amplitude $F(s,u_1)$ is unphysical (i.e. $u = 4-s-u_1>0$) and the partial wave expansion, with partial wave amplitudes determined by the minimization procedure we are about to describe, will diverge. Therefore we must minimize $A(s,t)$ by zero in the interval $4 \lesssim s < s_1 + t$. For $s \gtrsim s_1 + t$, $F(s,u_1)$ is physical and so we can maximize $|\frac{k}{\sqrt{s}} F(s,u_1)|^2$ for a given $[\frac{k}{\sqrt{s}} A(s,t)]$. From the Lagrange multiplier method we obtain:

$$\frac{1-2a_\ell(s)}{2\sqrt{a_\ell(s)(1-a_\ell(s))}} = \frac{\lambda(s)P_\ell(x)-I(s,u_1)P_\ell(y)}{R(s,u_1)\,|P_\ell(y)|} , \tag{2-28}$$

where $\lambda(s)$ is the undetermined multiplier. To fix $\lambda(s)$ we maximize $\frac{|\widetilde{F}(s,u_1)|^N}{A(s,t)}$ for $N \geq 2$. The extremum occurs at:

$$\lambda(s) = \frac{R^2(s,u_1)+I^2(s,u_1)}{NA(s,t)} . \tag{2-29}$$

With this value for $\lambda(s)$, (2-28) becomes:

$$\frac{1-2a_\ell(s)}{2\sqrt{a_\ell(s)(1-a_\ell(s))}} = \frac{1}{R(s,u_1)\,|P_\ell(y)|} \times$$

$$\times\left\{\frac{[R^2(s,u_1)+I^2(s,u_1)]P_\ell(x)}{NA(s,t)} - I(s,u_1)P_\ell(y)\right\}$$

$$\equiv K_\ell(s)/2. \tag{2-30a}$$

Solving for $a_\ell(s)$ we find:

$$a_\ell(s) = \tfrac{1}{2}\left[1-\frac{K_\ell(s)}{\sqrt{4+K_\ell^2(s)}}\right] . \tag{2-30b}$$

The solution to the set of Equations (2-26), (2-30), and (2-2) determines

$g_N(s,t,u_1)$ by:

$$g_N(s,t,u_1) = \frac{[R^2(s,u_1)+I^2(s,u_1)]}{A(s,t)} . \tag{2-31}$$

Inserting the inequality (2-24) into the dispersion relation (2-7) we get:

$$F(s_1,t,u_1)-F(s_o,t,0)$$

$$\geq \frac{1}{\pi} \int_{s_1+t}^{\infty} ds' \frac{\rho(s';s_1,s_o,t)}{g_N(s',t,u_1)} |F(s',u_1)|^N . \tag{2-32}$$

As before, the next step is to map the twice-cut s-plane, with real cuts $s \geq 4$ and $s \leq s_1 + t - 4 = -u_1$, onto the unit disk in the y-plane. Here, however, the transformation must be slightly different in order to deal with the part of the integration range $4 \leq s < s_1 + t$ where $A(s,t)$ was minimized by zero. The necessary sequence of mappings is:

$$z = \frac{\left(s - \frac{s_1+t}{2}\right)^2}{\left(4 - \frac{s_1+t}{2}\right)^2} , \tag{2-33}$$

and

$$y = \frac{i - \sqrt{\frac{z-a}{a-c}}}{i + \sqrt{\frac{z-a}{a-c}}} , \tag{2-34a}$$

with

$$a \equiv z(s_1+t) = z(0),$$

$$c \equiv z(s_1). \tag{2-34b}$$

Note that after the second mapping, $s = s_1$ is mapped onto $y = 0$. The part of the integration range $4 \leq s < s_1 + t$ is mapped onto the segment

$$\beta \equiv y(s=4) \leq y < 1 ; \tag{2-35}$$

and the remainder of the integration range $s \geq s_1 + t$ is mapped onto the unit semi-circle in the upper-half y-plane: $y = e^{i\phi}$, $0 \leq \phi \leq \pi$. The amplitude $F(s(y), u_1)$ is analytic inside the unit circle $|y|<1$ except for a cut from $y = \beta$ to $y = 1$.

Introducing this change of variables into (2-32), it follows from the theorem on arithmetic and geometric means and the reality property $F(s_1^*, u_1)=F^*(s,u_1)$ that:

$$F(s_1,t,u_1)-F(s_o,t,0)$$
$$\geq C_1 \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \ln|F(s'(e^{i\phi}),u_1)|^N\}, \qquad (2\text{-}36a)$$

where

$$C_1 = \exp\{\frac{1}{\pi}\int_0^{\pi} d\phi \ln[\frac{\rho_1(s'(e^{i\phi});s_1,s_o,t)J_1(s',\phi,s_1,t)}{g_N(s',t,u_1)}]\quad, \qquad (2\text{-}36b)$$

and $J_1(s',\phi,s_1,t)$ is the Jacobian of the transformation from s' to ϕ.

Because $0 < u = 4 - s - u_1 < 4$ in the region $4 \leq s < s_1 + t$, we know from unitarity and the partial wave expansion that the absorptive part of $A(s',u_1)$ (and hence the discontinuity of $\ln F(s',u_1)$) across the cut $\beta \leq y < 1$) is positive in that interval. Therefore we can employ a modified Poisson inequality (as discussed in the Appendix) and $t \leftrightarrow u$ crossing symmetry to obtain:

$$F(s_1,t,u_1)-F(s_o,t,0) > C_1\beta^N|F(s_1,u_1,t)|^N$$
$$= C_1\beta^N|F(s_1,t,u_1)|^N. \qquad (2\text{-}37)$$

This is the second of the pair of inequalities for which we are searching.

The maximal allowed values for $|F(s_1,t,u_1)|$ and $|F(s_o,t,0)|$ will occur when $F(s_1,t,u_1) > 0$, $F(s_o,t,0) < 0$, and the inequalities (2-23) and

(2-37) are saturated. Solving these equations we find:

$$F_o \equiv |F(s_o,t,0)| < \left[\frac{1+\frac{1}{\beta}\left(\frac{C_o}{C_1}\right)^{\frac{1}{N}}}{C_o}\right]^{\frac{1}{N-1}} , \tag{2-38a}$$

$$F_1 \equiv F(s_1,t,u_1)| < \left[\frac{1+\beta\left(\frac{C_1}{C_o}\right)^{\frac{1}{N}}}{C_1\beta}\right]^{\frac{1}{N-1}} . \tag{2-38b}$$

These are the absolute bounds on the amplitude.

Before proceeding to Łukaszuk's more refined method, let us mention the reason we have restricted ourselves to the cases $u_o=0$ or $u_1=0$. To use points within the inner Mandelstam triangle it would be necessary to minimize A(s,t) for a given $|F(s,u_i)|$ with $t > u_i > 0$. With the set of partial wave amplitudes determined by this minimization, the partial wave expansion does not converge. Therefore, this minimization cannot be performed in practice, at least using this technique. On the other hand, for points outside the inner triangle, but within the large Mandelstam triangle, one must minimize A(s,t) for a given $|F(s,u_i)|$ with $t > 0$ and $u_i < 0$. Along part of the integration range $F(s,u_i)$ becomes unphysical (i.e. $u=4-s-u_i > 0$) and, again, it is possible to perform the minimization described above. In this case one can minimize A(s,t) by zero for that part of the integration range where $F(s,u_i)$ is unphysical, but then one loses information. The best results are therefore obtained by considering cases in which one point is on the border of the inner triangle.

C. Łukaszuk's Modification of Martin's Method

There is no reason to believe that the bounds (2-22) and (2-38) are the best which can be obtained starting from a fixed-t dispersion relation. They can in fact be improved by a refinement of Martin's method developed by Łukaszuk[30] and applied to the cases we are interested in by Łukaszuk and Martin.[27]

The relevant observation is that we are really interested in maximizing the right hand side of the dispersion relation (2-7) for given values of $|F(s_1,t,u_1)|$ and $|F(s_o,t,u_o)|$, and then showing that these amplitudes cannot be increased arbitrarily without making the right hand side of (2-7) larger than the left hand side.

To illustrate what we have in mind it is simplest to first consider the upper bound on $F(s_o,t,u_o)$. As in Section B we specialize the dispersion relation to the case $u_1=0$. The method for obtaining an upper bound on $F(s_o,t,u_o)$ consists of two steps: 1) First minimize the absorptive part appearing in the dispersion relation as a function of the magnitude of the full amplitude in the forward direction:

$$A(s,t) \geq A_{min}(|F(s,0)|). \tag{2-39}$$

2) Second, maximize the difference

$$F(s_1,t,0) - \frac{1}{\pi}\int_4^\infty ds'\rho(s';s_1,s_o,t)A_{min}(|F(s',0)|) \geq F(s_o,t,u_o) \; . \tag{2-40}$$

The maximum value of the left hand side of (2-40) is then an upper bound of $F(s_o,t,u_o)$.

Again let us define $\bar{A}(s,t)$ and $\bar{F}(s)$ by (2-11). Then the set of partial wave amplitudes $\{f_\ell(s)\}$ minimizing $\bar{A}(s,t)$ for a given $\bar{F}(s)$ has the form (2-12) and (2-13). Therefore, from the dispersion relation we have the inequality :

$$F(s_1,t,0)-F(s_1,t,u_1) \geq \frac{1}{\pi}\int_4^\infty ds'\rho(s';s_1,s_o,t)\bar{A}_{min}(\bar{F},s,t), \tag{2-41}$$

with $\overline{A}_{min}(\overline{F},s,t)$ given by

$$\overline{A}_{min}(F,s,t)=\frac{\sqrt{s}}{k}\left[\sum_{o}^{2L}(2\ell+1)P_\ell(x)+c^2(s)\sum_{2L+2}^{\infty}(2\ell+1)/P_\ell(x)\right], \tag{2-42a}$$

$$\overline{F}(s)=\frac{\sqrt{s}}{k}\left[\sum_{o}^{2L}(2\ell+1)+c(s)\sum_{2L+2}^{\infty}(2\ell+1)/P_\ell(x)\right], \tag{2-42b}$$

and the subsidiary condition

$$1\leq P_{2L}(x)<c(s)\leq P_{2L+2}(x). \tag{2-42c}$$

It is easy to see that for a given $\overline{F}(s)$ the parameters $c(s)$ and $L\equiv L(s)$ are uniquely determined by (2-42b) and (2-42c). If, on the contrary, there existed another $L'>L$ and c' satisfying these equations, then one could obtain from (2-42b) the equation

$$0=\sum_{2L+2}^{2L'}(2\ell+1)\left(1-\frac{c(s)}{P_\ell(x)}\right)+(c'(s)-c(s))\sum_{2L'+2}^{\infty}(2\ell+1)/P_\ell(x). \tag{2-43}$$

However, for $L'>L$, $c'(s)>c(s)$, and for $\ell\geq 2L+2$, $c(s)\leq P_\ell(x)$, so the right hand side will be greater than zero unless $L'=L$ and $c'(s)=c(s)$, or $L'=L+1$ and $c'(s)=c(s)=P_{2L+2}(x)$. The second possibility is ruled out by the condition (2-42), and in any case it gives the same set of partial waves for $L'=L$ and $L'=L+1$, so $c(s)$ and L are uniquely determined.

The next step is to determine the Lagrangian multiplier $c(s)$ in such a way as to maximize the difference

$$F(s_1,t,0)-\frac{1}{\pi}\int_4^{\infty}ds'\rho(s';s_1,s_o,t)\overline{A}_{min}(\overline{F},s',t). \tag{2-44}$$

With that goal in mind, we will conformally transform the twice-cut s-plane onto the unit disk in the y-plane by the sequence of transformations (2-17) and (2-18). Equation (2-41) can then be written as:

$$F(s_1,t,0)-F(s_o,t,u_o)$$

$$\geq \frac{1}{\pi}\int_0^{\pi} d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(e^{i\phi}),t), \tag{2-45a}$$

where

$$W(\phi)=\rho(s'(e^{i\phi});s_1,s_o,t)J(s',\phi,s_1), \tag{2-45b}$$

and $J(s',\phi,s_1)$ is the Jacobian of the transformation from s' to ϕ.

Now since $F(y) \equiv F(s(y),0)$ is analytic in $|y| < 1$, crossing symmetry and Poisson's inequality tell us that

$$|F(s_1,t,0)| = |F(s_1,0,t)| \equiv |F(y=0)|$$

$$\lesssim \exp[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \ln|F(y=e^{i\phi})|]. \tag{2-46}$$

It then follows from the reality property $F^*(y)=F(y^*)$ and the inequality $\bar{F}(s) \gtrsim |F(s,0)|$ that:

$$|F(s_1,t,0)| \lesssim \exp[\frac{1}{\pi}\int_0^{\pi} d\phi \ln\bar{F}(y=e^{i\phi})] \equiv F_o. \tag{2-47}$$

The Lagrange multiplier c(s) is now fixed by minimizing the integral

$$\frac{1}{\pi}\int_0^{\pi} d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t). \tag{2-48}$$

for a given F_o. This imposes the condition:

$$W(\phi)\bar{F}(e^{i\phi})\frac{\delta\bar{A}_{min}(\bar{F},s(\phi),t)}{\delta\bar{F}(e^{i\phi})} = \gamma \tag{2-49}$$

The new Lagrange multiplier γ is independent of s. From Equations (2-42) we have:

$$\frac{\delta\bar{A}_{min}(\bar{F},s,t)}{\delta\bar{F}(s)} = c(s) > 0, \tag{2-50a}$$

and

$$\frac{\delta^2\bar{A}_{min}(F,s,t)}{\delta\bar{F}(s)^2} = \frac{2\ k/\sqrt{s}}{\sum\limits_{2L+2}^{\infty}(2\ell+1)/P_\ell(x)} > 0\ . \tag{2-50b}$$

At the changeover points $L\to L+1$, $\bar{A}_{min}$ and $\frac{\delta\bar{A}_{min}}{\delta F}$ are continuous, so these points present no difficulty. Therefore, $\bar{A}_{min}$ and $\frac{\delta\bar{A}_{min}}{\delta F}$ are increasing functions of $\bar{F}$, and hence Equation (2-49) has a unique solution as a function of $\bar{F}(s)$.

Now define

$$I_{min}(\gamma) \equiv \frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F}(\gamma),s'(\phi),t), \tag{2-51a}$$

$$F_o(\gamma)=\exp[\frac{1}{\pi}\int_0^\pi d\phi \ln\bar{F}(e^{i\phi},\gamma)], \tag{2-51b}$$

with $\bar{F}(y,\gamma)$ determined by the condition (2-49). Then:

$$\begin{aligned}\Delta(\gamma) &\equiv F_o(\gamma)-I_{min}(\gamma)\\ &\geq F(s_1,t,0)-\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t)\\ &\gtrsim F(s_o,t,u_o).\end{aligned} \tag{2-52}$$

The final step is to determine the Lagrange multiplier so as to maximize $\Delta(\gamma)$. Taking the functional derivative of $\Delta(\gamma)$ with respect to $\bar{F}(y,\gamma)$ we find:

$$\begin{aligned}\frac{\delta\Delta(\gamma)}{\delta\bar{F}(y,\gamma)} &= \frac{1}{\pi}\left[\frac{F_o(\gamma)}{\bar{F}(y,\gamma)} - W(\phi)\frac{\delta\bar{A}_{min}(\bar{F}(y,\gamma),s'(\phi),t)}{\delta\bar{F}(y,\gamma)}\right]\\ &= \frac{[F_o(\gamma)-\gamma]}{\pi\bar{F}(y,\gamma)}\ ,\end{aligned} \tag{2-53a}$$

and

$$\frac{\delta^2\Delta(\gamma)}{\delta\bar{F}(y,\gamma)^2} = -\frac{1}{\pi}\{(1-\frac{1}{\pi})\frac{F_o(\gamma)}{\bar{F}(y,\gamma)^2}$$

$$+W(\phi)\frac{\delta^2\bar{A}_{min}(\bar{F}(y,\gamma),s'(\phi),t)}{\delta\bar{F}(y,\gamma)^2}\}<0 \ . \tag{2-53b}$$

Clearly, the extremum of $\Delta(\gamma)$ occurs at $F_o(\gamma_m)=\gamma_m$ and is a maximum. Therefore we have found the new maximum for $F(s_o,t,u_o)$,

$$\Delta(\gamma_m)\gtrsim F(s_o,t,u_o) \ , \tag{2-54}$$

which is an improvement over (2-22).

The same technique can be used to improve bounds (2-38). However, despite some misleading statements to the contrary, LM never carried out the improved method for the non-forward case. That is, they used essentially the same method we have just described to improve on the inequality (2-23), while they retained the inequality (2-37).

To obtain an improved substitute for (2-23) we specialize to the case $u_o=0$ and then follow the steps which led to (2-45) to find:

$$F(s_1,t,u_1)-F(s_o,t,0)\gtrsim\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t), \tag{2-55a}$$

where

$$W(\phi)=\rho(s'(e^{i\phi});s_1,s_o,t)J(s',\phi,s_o) \ , \tag{2-55b}$$

and $\bar{A}_{min}$ is determined as a function of $\bar{F}$ by (2-41) and (2-42). Equations (2-55) differ from (2-45) in that here the point $(s_o,0,4-s_o)$, rather than $(s_1,0,4-s_1)$, has been mapped onto the center of the unit circle in the y-plane; i.e. as in Section B, (2-18b) must be replaced by (2-18b'). Poisson's inequality, crossing symmetry, and the reality of $F(s,0)$ now require:

$$|F(s_o,t,0)|=|F(s_o,0,t)|\equiv F(y=0)\lesssim\exp[\frac{1}{\pi}\int_0^\pi d\phi\ln|F(e^{i\phi})|]$$

$$\lesssim\exp[\frac{1}{\pi}\int_0^\pi d\phi\ln\bar{F}(e^{i\phi})]\equiv F_1. \tag{2-56}$$

Next we minimize the right hand side of (2-55a) for a given F_1 to fix the Lagrange multiplier c(s):

$$W(\phi)\bar{F}(e^{i\phi}) \frac{\delta\bar{A}_{min}(\bar{F},s(\phi),t)}{\delta\bar{F}(e^{i\phi})} = \gamma \ . \tag{2-57}$$

The determination of the Lagrange multiplier γ is somewhat different in this case. Here we are looking for the largest values of $|F(s_1,t,u_1)|$ and $|F(s_o,t,0)|$ such that the inequalities (2-37) and (2-55) can be satisfied subject to the conditions (2-56) and (2-57). That these equations cannot be satisfied for arbitrarily large values of $|F(s_1,t,u_1)|$ and $|F(s_o,t,0)|$ can be seen most easily by adding (2-37) and (2-55) to obtain:

$$\begin{aligned}
&[2|F(s_1,t,u_1)|-C_1\beta^N|F(s_1,t,u_1)|^N] \\
&\quad +[2\exp(\frac{1}{\pi}\int_0^\pi d\phi\ln\bar{F}(e^{i\phi}))-\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t)] \\
&\geq [2F(s_1,t,u_1)-C_1\beta^N|F(s_1,t,u_1)|^N] \\
&\quad +[2|F(s_o,t,0)|-\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t)] \\
&\geq 0 \ .
\end{aligned} \tag{2-58}$$

The first bracketed term in (2-58) will obviously become negative for sufficiently large $|F(s_1,t,u_1)|$. To see that the second bracketed term must also become negative for sufficiently large $|F(s_o,t,0)|$, we use the theorem on arithmetic and geometric means to bound it by:

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\pi d\phi[2\bar{F}(e^{i\phi})-W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t)] \\
&= \frac{1}{\pi}\int_0^\pi d\phi\bar{F}(e^{i\phi})[2-W(\phi)\frac{\bar{A}_{min}(\bar{F},s'(\phi),t)}{\bar{F}(e^{i\phi})}] \\
&\geq [2\exp(\frac{1}{\pi}\int_0^\pi d\phi\ln\bar{F}(e^{i\phi})-\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F},s'(\phi),t)] \\
&\geq 0
\end{aligned} \tag{2-59}$$

Now increasing $|F(s_0,t,0)|$ increases $\bar{F}(e^{i\phi})$ for all ϕ. Also, $\frac{\bar{A}_{min}(\bar{F},s,t)}{\bar{F}}$ is easily seen to be an increasing function of $\bar{F}$:

$$\frac{\delta}{\delta\bar{F}}\left[\frac{\bar{A}_{min}(\bar{F},s'(\phi),t)}{\bar{F}}\right]$$

$$=\frac{\sqrt{s}/k}{\bar{F}^2}\left[\frac{\left(\frac{k}{\sqrt{s}}\right)^2\bar{F}^2}{\sum_{2L+2}^{\infty}(2\ell+1)/P_\ell(x)}-\sum_0^{2L}(2\ell+1)P_\ell(x)\right.$$

$$\left.-\frac{\left(\sum_0^{2L}(2\ell+1)\right)^2}{\sum_{2L+2}^{\infty}(2\ell+1)/P_\ell(x)}\right]>0\ . \qquad (2\text{-}60)$$

So for $|F(s_0,t,0)|$ sufficiently large, (2-59) will be violated and the second bracketed term in (2-58) will also be nagative.

The set of equations (2-55), (2-56), and (2-57) can therefore be used to replace (and improve) the inequality (2-23), and combined with (2-37) they impose absolute bounds on the amplitudes $|F(s_1,t,u_1)|$ and $|F(s_0,t,0)|$.

D. Numerical Results

We will now enumerate the numerical results obtained by Martin[25] and by Łukaszuk and Martin.[27] By specializing (2-52) and (2-54) to $s_1 = 8/3$, $t = 4/3$, $s_0 = 4/3$, LM found

$$\alpha_0 \equiv F(4/3,\ 4/3,\ 4/3) < 16, \qquad (2\text{-}61)$$

which is an improvement of nearly 40% over Martin's previous result, obtained using (2-22):

$$\alpha_0 < 25\ . \qquad (2\text{-}62)$$

Similarly, with $t = 2$, $s_1 = 3$, $s_0 = 2$, LM found from (2-37), (2-55), (2-56), and (2-57):

$$|F(3,2,-1)| < 150,$$

$$|F(2,2,0)| < 37. \qquad (2\text{-}63)$$

Martin's previous results, calculated from (2-38), were:

$$|F(3,2,-1)| < 150,$$

$$|F(2,2,0)| < 50. \qquad (2\text{-}64)$$

While Martin claimed to get these results for N=5 as the optimal value for the exponent N in (2-23) and (2-24), we found that they are actually produced with N=10, and that this value of N is the optimal one. We assume that Martin meant to say that he optimized $\frac{(|F|^2)^N}{A}$, rather than $\frac{|F|^N}{A}$, to get inequalities (2-10) and (2-24).

Martin also showed that it is possible to obtain, from these bounds, lower bounds on the amplitude within the inner Mandelstam triangle, in particular at the symmetric point. His numerical result, which is also quoted by LM, is:

$$\alpha_0 > -100 \, . \qquad (2\text{-}65)$$

This is not correct. After we pointed out the error to Martin, he constructed the following counter-example:[32] The function $F(s,t,u) = A+B(s^2+t^2+u^2)$ is a limit of a twice-subtracted, crossing symmetric amplitude with the correct positivity properties. When A and B are adjusted to fit $F(3,2,-1)=150$ and $F(2,2,0)=-37$, one finds $\alpha_0 = F(4/3,4/3,4/3) = -120$.

We will now show that the calculation described by LM gives

$$\alpha_0 > -130 \, , \qquad (2\text{-}66)$$

and that this result can be improved to:

$$\alpha_0 > -122 \ . \tag{2-67}$$

Take

$$\alpha_0 = F(2,2,0) - [F(2,2,0) - F(4/3,2,2/3)] - [F(2,4/3,2/3) - \alpha_0], \tag{2-68}$$

and write the dispersion relations:

$$F(3,2,-1) - F(2,2,0) = \frac{1}{\pi}\int_4^\infty ds'\rho(s')A(s',2) \ ,$$
$$F(2,2,0) - F(4/3,2,2/3) = \frac{1}{\pi}\int_4^\infty ds'\rho_1(s')A(s',2),$$
$$F(2,4/3,2/3) - \alpha_0 = \frac{1}{\pi}\int_4^\infty ds'\rho_2(s')A(s',4/3). \tag{2-69}$$

Then we have immediately:

$$\alpha_0 > F(2,2,0) - [F(3,2,-1) - F(2,2,0)] \times \underset{s'\geq 4}{\mathrm{Max}} \frac{\rho_1(s')}{\rho(s')} \left[1 + \underset{s'\geq 4}{\mathrm{Max}} \frac{\rho_2(s')}{\rho_1(s')}\right] , \tag{2-70}$$

which is the equation quoted by LM. It is easy to see that this can be improved to:

$$\alpha_0 > F(2,2,0) - [F(3,2,-1) - F(2,2,0)] \times \underset{s'\geq 4}{\mathrm{Max}} \left[\frac{\rho_1(s') + \rho_2(s')}{\rho(s')}\right] . \tag{2-71}$$

With the bounds (2-63), Equations (2-70) and (2-71) give the results (2-66) and (2-67), respectively.

Martin[32] has suggested that it may be possible to obtain the result $\alpha_o > -100$ by first calculating absolute bounds on F(8/3,4/3,0) and F(10/3,4/3,-2/3) and then using these absolute bounds to get a lower bound on α_o by the method described above. We have tried this, and it gives a worse result than (2-67). We have also tried using bounds at a variety of other points to calculate a better lower bound on α_o, but with no success.

In summary, the best bounds obtained by the methods of LM for the amplitude at the symmetric point are:

$$-122 < \alpha_o < 16. \tag{2-72}$$

Recalling that the Chew Mandelstam coupling constant is defined as

$$\lambda = -\alpha_o/6, \tag{2-73}$$

we see that these correspond to the bounds on the coupling constant:

$$20.33 > \lambda > -2.67 \ . \tag{2-74}$$

Finally, we would like to point out that the bounds quoted above can be used to produce a lower bound on the π^o-π^o s-wave scattering length a_o. Recalling that

$$a_o = \frac{F(4,0,0)}{2}, \tag{2-75}$$

and that the dispersion relation coupled with the positivity of the absorptive part of the forward amplitude requires

$$F(4,0,0) > F(2,0,2) \ , \tag{2-76}$$

it is obvious from (2-63) that the scattering length has the lower bound:

$$a_o > -18.5. \tag{2-77}$$

Using a different method, Bonnier and Vinh Mau[33] were able to show that the bounds (2-63) and (2-65) require

$$a_o > -4.0 \ , \tag{2-78}$$

and this estimate was subsequently improved by Martin[7] to

$$a_o > -3.5 \ . \tag{2-79}$$

We will not describe the method of Bonnier and Vinh Mau here because it would take us too far afield.

CHAPTER III

The Auberson-Khuri Representation

Recently Auberson and Khuri[28] derived a "parametric dispersion relation" for π-π scattering which exhibits the three-fold symmetry of the π-π scattering amplitude in a completely explicit way, and which is a rigorous consequence of local field theory. The fixed-t dispersion relation is a special case of the more general class of parametric dispersion relations of which the Auberson-Khuri (A-K) representation is also an example. The advantage of having these more general representations is that for any given problem one can select the representation best suited to it. As we shall see in Chapter IV, for the purpose of deriving an upper bound on the π^o-π^o scattering amplitude at the symmetric point, the A-K representation is superior to the fixed-t dispersion relation.

We now give a brief derivation of the A-K representation for π^o-π^o scattering. For more details, and for the more general case of scattering of charged pions, we refer the reader to the paper of Auberson and Khuri.[28]

The π^o-π^o scattering amplitude F(s,t,u) is a completely symmetric function of the Mandelstam variables s, t, and u(s+t+u=4). Auberson and Khuri pointed out that for $0 \leq a < 4$, the cubics

$$(s-a)(t-a)(u-a) = (4/3-a)^3 \tag{3-1}$$

lie within the analyticity domain D of the amplitude which Martin[13,7] has proved from axiomatic field theory. For the purpose of writing a dispersion relation they parameterized these cubics by the rational mapping:

$$s_k = a+(4/3-a)\,\frac{(z-z_k)^3}{(z^3-1)}\quad ,\ k = 1,\ 2,\ 3\quad , \tag{3-2}$$

where $s_1=s$, $s_2=t$, $s_3=u$ and z_k are the cube roots of unity. For all complex z the amplitude $F(s(z), t(z), u(z))$ can then be considered as an analytic function of z and a:

$$\bar{F}(z,a) = F(s(z), t(z), u(z)) . \tag{3-3}$$

The only singularities of $\bar{F}(z,a)$ in the z plane are the image V(a) of the physical cuts $s_k \geq 4$, k=1,2,3. A-K proved that the set of admissable values of the parameter a can be extended to include $-28.19 < a < 4$. For our purposes the range $4/9 < a < 4$ will be more than sufficient, and we will not mention the complications which arise for smaller values of a.

From (3-2) we can calculate the image V(a) in the z plane of the three physical cuts as:

$$V(a) = V_1(a) \cup V_2(a) \cup V_3(a) , \tag{3-4a}$$

where

$$V_1(a) = \begin{cases} z \mid & |z|=1,\ \frac{2\pi}{3} \leq |\arg z| \leq \phi_o(a) \text{ if } \frac{4}{9} < a < 4/3 \quad \text{(Case I)} \\ z \mid & |z|=1,\ \phi_o(a) \leq |\arg z| \leq \frac{2\pi}{3} \text{ if } \frac{4}{3} < a < 4 \quad \text{(Case II)} \end{cases} \tag{3-4b}$$

and

$$V_2(a) = e^{2i\pi/3}\, V_1(a) ,$$

$$V_3(a) = e^{4i\pi/3}\, V_1(a) . \tag{3-4c}$$

The function $\phi_o(a)$ is given by

$$\phi_o(a) = \tan^{-1}\left\{\frac{\sqrt{(4-a)(a-4/9)}}{a-20/9}\right\},\ 0 < \phi_o \leq \pi . \tag{3-5}$$

In both cases I and II the image of the cuts lies on the unit circle in the z-plane but does not fully cover the circle. Therefore one can analytically continue from $z < 1$ to $z > 1$. In the case a=4/3 the cubic degenerates into three straight lines and one would obtain a fixed-t dispersion relation. We do not need to consider this case. In Figure 2 we display, for examples of both cases I and II, the image V(a) of the physical cuts in the z-plane, and the corresponding contours in the Mandelstam plane.

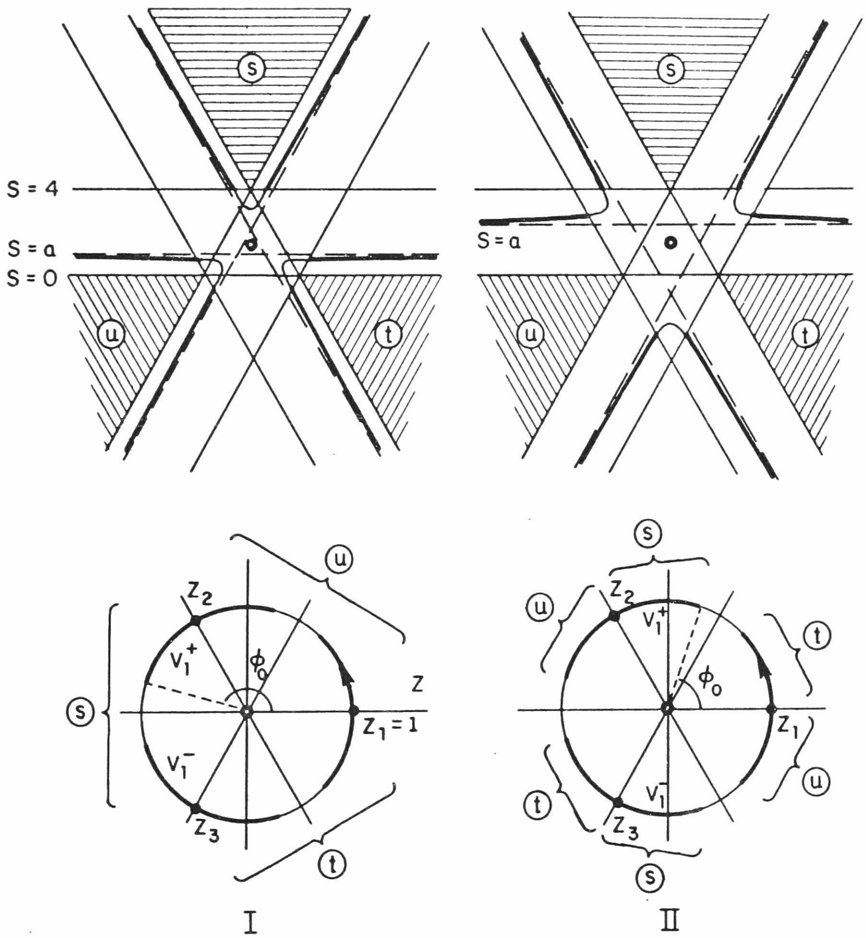

Figure 2

Integration paths for the Auberson-Khuri representation, Equations (3-7) and (3-23), are displayed in the Mandelstam plane and in the z-plane. The two cases correspond to values of the parameter a in the intervals: $4/9 < a < 4/3$ (I) and $4/3 < a < 4$ (II).

Along the cuts V(a) the amplitude is either physical or obtainable from a convergent physical partial wave expansion. This is guaranteed by the choice of cubics (3-1) which requires that along each physical cut the corresponding values of momentum transfer are always within the Lehmann-Martin ellipse for that channel.

To write a dispersion relation in z it is necessary to relate the discontinuity of $\bar{F}(z,a)$ across V(a) to the absorptive parts of the amplitude F(s,t,u) in the s,t, and u channels:

$$A_k(s_k,s_{k+1}) = \lim_{\varepsilon \downarrow 0} \frac{1}{2i} [F(s_k+i\varepsilon, s_{k+1}) - F(s_k-i\varepsilon, s_{k+1})], \qquad (3\text{-}6)$$

for $s_k \geq 4$ and $s_4 = s_1$. A_k is the continuation via the appropriate partial wave expansion of the physical absorptive part in the k channel.

From (3-2) the reality condition $F^*(s,t) = F(s^*,t^*)$ becomes:

$$\bar{F}^*(z,a) = \bar{F}(\frac{1}{z^*},a). \qquad (3\text{-}7)$$

We define the discontinuity of $\bar{F}(z,a)$ across V(a) by:

$$\bar{A}(z,a) = \lim_{\varepsilon \downarrow 0} \frac{1}{2i} [\bar{F}((1+\varepsilon)z,a) - \bar{F}((1-\varepsilon)z,a)],$$

$$|z| = 1, \qquad (3\text{-}8a)$$

where

$$\bar{A}(z,a) = \bar{A}^*(z,a) \ . \qquad (3\text{-}8b)$$

The sign of $\text{Im}(s_k)$ on each side of V(a) is determined by:

$$\text{Im } s = 3\varepsilon(4/3-a) \frac{2 \sin\phi}{(1+2\cos\phi)^2}, \quad z = (1+\varepsilon)e^{i\phi}. \qquad (3\text{-}9)$$

To relate $\bar{A}(z,a)$ to $A_k(s_k,s_{k+1})$ we define $V_k^{\pm}$ by:

$$V_k(a) = V_k^+(a) \cup V_k^-(a) \ , \qquad (3\text{-}10a)$$

with

$$\mathrm{Im}\,\frac{z}{z_k}\;\begin{cases} \geq 0 & \text{on } V_k^+(a) \\ \leq 0 & \text{on } V_k^-(a)\ . \end{cases} \tag{3-10b}$$

Then

$$\bar{A}(z,a)=\pm A_k(s,t)\begin{cases} z\varepsilon V_k^+(a<4/3) \text{ or } V_k^-(a>4/3) \\ z\varepsilon V_k^-(a<4/3) \text{ or } V_k^+(a>4/3) \end{cases} \tag{3-11}$$

We must also settle the question of subtractions. Jin and Martin[19] have shown that:

$$F(s,t)=o(s^2) \text{ for } |s|\to\infty\ ,\ t \text{ fixed,}$$
$$(s,t)\ \varepsilon\, D. \tag{3-12}$$

Therefore, using the fact that for $z\to z_k$, $s_k\to a$ and $s_j\to \mathrm{const}\frac{(a-4/3)}{(z-z_k)}$, $j\neq k$, we see that:

$$F(z,a)\to o\left(\frac{1}{(z-z_k)^2}\right)\ , \tag{3-13}$$

for $z\to z_k$ with a fixed. By writing a contour integral for the function $(z^3-1)\bar{F}(z,a)$ we can eliminate the contributions from the singularities at $z=z_k$. This is equivalent to the introduction of two subtractions into the usual fixed-t dispersion relation. Here the subtraction constants are related to the coefficients in the Taylor expansion

$$\bar{F}(z,a)=\sum_{n=0}^{\infty} f_n(a)z^n, \tag{3-14}$$

which is convergent for $|z|<1$. From the full crossing symmetry of the $\pi^o-\pi^o$ amplitude we know that $\bar{F}(z,a)$ is a function only of z^3, so the only powers of z contributing in (3-14) are those for which n is an integral multiple of three. Noting that

$$f_o=\bar{F}(0,a)=F(4/3.4/3,4/3)\equiv\alpha_0, \tag{3-15a}$$

we see that f_o is real and independent of a. Also, from (3-7) we have:

$$\bar{F}(\infty,a)=\bar{F}^*(0,a)=f_o. \tag{3-15b}$$

Writing Cauchy formulas for two circles C_i and C_e, interior and exterior to V(a), we get for $z<1$:

$$\frac{1}{2\pi i}\oint_{C_i} dz' \frac{z'^3-1}{z'^3(z'-z)}\bar{F}(z',a)=\frac{z^3-1}{z^3}\bar{F}(z,a)+\frac{f_o}{z^3}, \tag{3-16a}$$

$$\frac{1}{2\pi i}\oint_{C_i} dz' \frac{z'^3-1}{z'^3(z'-z)}\bar{F}(z',a)=f_o. \tag{3-16b}$$

Subtracting and letting C_e and C_i approach V(a) we obtain the dispersion relation on the cubic (3-1):

$$\bar{F}(z,a)=f_o+\frac{z^3}{1-z^3}\frac{1}{\pi}\int_{V(a)} dz' \frac{z'^3-1}{z'^3(z'-z)}\bar{A}(z',a)\ . \tag{3-17}$$

Since the discontinuities across the three physical cuts are identical,

$$A_s(s,t)=A_t(s,t)=A_u(s,t), \tag{3-18a}$$

it follows that:

$$\bar{A}(z,a)=\bar{A}(e^{2i\pi/3}z,a)=\bar{A}(e^{4i\pi/3}z,a). \tag{3-18b}$$

Thus we can reduce (3-17) to an integral over only one segment of V(a):

$$\bar{F}(z,a)=f_o+\frac{3z^3}{1-z^3}\int_{V_1^+(a)}\frac{dz'}{z'}(z'^3-1)\bar{A}(z',a)\left[\frac{1}{z'^3-z^3}-\frac{1}{1-z^3z'^3}\right]. \tag{3-19}$$

In order to rewrite the parametric dispersion relation (3-19) in terms of the usual Mandelstam variables it is convenient to introduce the variables:

$$\bar{s}_k = s_k - 4/3 \quad , \quad k = 1,2,3,$$

$$\bar{a} = a - 4/3 \; . \tag{3-20}$$

Simple algebra gives us the relation:

$$\bar{a} = \frac{\bar{s}\,\bar{t}\,\bar{u}}{\bar{s}\bar{t}+\bar{t}\bar{u}+\bar{u}\bar{s}} \tag{3-21}$$

On V_1^+ we have, from (3-11):

$$\bar{A}(z,a) = \pm A(s,t_+(s,a)) \text{ for } a \lessgtr 4/3, \tag{3-22a}$$

where

$$t_+(s,a) = 4/3 + \frac{\bar{s}}{2}\left\{\sqrt{\frac{\bar{s}+3\bar{a}}{\bar{s}-\bar{a}}} - 1\right\} . \tag{3-22b}$$

Transforming from (z,a) to s,t,u) we get:

$$F(s,t,u) = \alpha_o + \frac{1}{\pi}\int_{8/3}^{\infty} d\bar{s}'\rho(\bar{s}';\bar{s},\bar{t},\bar{u})A(\bar{s}',t_+(\bar{s}';\bar{s},\bar{t},\bar{u})), \tag{3-23}$$

where

$$\rho(\bar{s}';\bar{s},\bar{t},\bar{u}) = \frac{1}{\bar{s}'}\left\{\frac{\bar{s}}{\bar{s}'-\bar{s}} + \frac{\bar{t}}{\bar{s}'-\bar{t}} + \frac{\bar{u}}{\bar{s}'-\bar{u}}\right\} , \tag{3-23b}$$

and

$$t_+(\bar{s}';\bar{s},\bar{t},\bar{u}) = 4/3 + \frac{\bar{s}'}{2}\left\{\sqrt{\frac{\bar{s}'+3\bar{a}}{\bar{s}'-\bar{a}}} - 1\right\} , \tag{3-23c}$$

with

$$\bar{a} = \frac{\bar{s}\,\bar{t}\,\bar{u}}{\bar{s}\bar{t}+\bar{t}\bar{u}+\bar{u}\bar{s}} \; . \tag{3-23d}$$

This is the Auberson-Khuri representation for the π^0-π^0 scattering amplitude. The representation is fully and explicitly crossing symmetric, and there is only one subtraction constant, $\alpha_0=F(4/3,4/3,4/3)$. Our discussion has been limited to points on the cubics (3-1) with $4/9<a<4$, which will be enough for our applications. However, Auberson and Khuri have proved that (3-23) can be analytically continued to all values of (s,t,u) such that $t_+(\bar{s}';\bar{s},\bar{t},\bar{u})$ lies within the analyticity domain of $A(s',t_+)$ (which contains the Lehmann-Martin ellipse) for all $\bar{s}'\geq 8/3$.

CHAPTER IV

New Upper bound on F(4/3,4/3,4/3)

In this chapter we will derive an improved upper bound on the π^o-π^o scattering amplitude at the symmetric point. We remind the reader that in order to obtain the bound $\alpha_o<16$, Łukaszuk and Martin started from the fixed-t dispersion relation:

$$F(8/3,4/3,0)-\alpha_o = \frac{1}{\pi}\int_4^\infty ds'\rho(s')A(s',4/3). \tag{4-1}$$

They minimized the absorptive part $A(s',4/3)$ in the dispersion integral for a given forward amplitude $|F(s',0)|$ using the Lagrange multiplier method. After bounding $|F(8/3,4/3,0)|$ as a function of an integral over the forward amplitude using Poisson's inequality, they determined the Lagrange multiplier by requiring that the difference

$$F(8/3,4/3,0)-\frac{1}{\pi}\int_4^\infty ds'\rho(s')A_{min}(F(s',0))\geq\alpha_o \tag{4-2}$$

be a maximum. This maximum was their upper bound on α_o.

The dispersion relation is explicitly crossing symmetric in s and u. In addition, LM employed the relation $F(8/3,4/3,0)=F(8/3,0,4/3)$which follows from $t\leftrightarrow u$ crossing symmetry. There is, of course, much more information contained in the full crossing symmetry of the π^o-π^o amplitude. In particular, the fixed-t dispersion relation is somewhat restrictive in that it allows direct comparison only between points at the same momentum transfer t. The Auberson-Khuri representation, which contains in an explicit way the full crossing symmetry of the π^o-π^o scattering amplitude, and which has only one subtraction constant, namely α_o, will allow us to directly compare any point within the Mandelstam triangle with the symmetric point. By applying the method of LM to the A-K representation and selecting the optimal point for comparison with the symmetric point, we will derive an improved upper bound on α_o. This chapter is mainly devoted to that derivation.

Examples of cubics

$$(s-a)\ (t-a)\ (u-a) = (4/3-a)^3 \tag{4-3}$$

passing through the Mandelstam triangle are displayed in Figure 2. In addition to providing a representation for the difference between the amplitude at any point within the Mandelstam triangle (all points within that triangle lie on the cubics (4-3)) and the amplitude at the symmetric point (which is common to all the cubics (4-1)), the A-K representation permits direct comparison between any two points on the same cubic, i.e. for the same value of a. This makes possible the calculation of absolute bounds on the amplitude at pairs of points (s_1,t_1,u_1) and (s_o,t_o,u_o). Unfortunately, none of these bounds represent a significant improvement over the bounds LM found on $|F(3,2,-1)|$ and $|F(2,2,0)|$. Since the methods used are the same as in Chapter II, and since there is no real improvement, we only list the results of these calculations.

We will now show that the methods of LM can be applied to the A-K representation:

$$F(s,t,u)=\alpha_o+\frac{1}{\pi}\int_{8/3}^{\infty} d\bar{s}'\rho(\bar{s}';\bar{s},\bar{t},\bar{u})A(\bar{s}',t_+), \tag{4-4}$$

with $\rho(\bar{s}';\bar{s},\bar{t},\bar{u})$ and $t_+=t_+(\bar{s}';\bar{s},\bar{t},\bar{u})$ given by Equations (3-23). Two conditions must be satisfied: The kernal $\rho(\bar{s}';\bar{s},\bar{t},\bar{u},)$ and the momentum transfer $t_+(\bar{s}';\bar{s},\bar{t},\bar{u})$ in the absorptive part $A(\bar{s}',t_+)$ must both be real and positive for all $\bar{s}'\geq 8/3$. Then by unitarity $A(\bar{s}',t_+)$ will be positive for $s'\geq 8/3$, and we can use the unitarity constraint to minimize the absorptive part as a function of the forward amplitude. We will prove that these properties hold for all points (s,t,u) within the Mandelstam triangle.

For proving the positivity of both ρ and t_+, the first important observation is that for all points within the Mandelstam triangle the variable a defined by (3-21) is in the range

$$-8/9 < \bar{a} < 16/9, \tag{4-5a}$$

or equivalently,

$$4/9 < a < 28/9. \tag{4-5b}$$

To see this note that:

$$\bar{a}(s=-4,t=4,u=4)=16/9,$$

$$\bar{a}(s=4,t=0,u=0)=-8/9, \tag{4-6a}$$

and

$$\left.\frac{d\bar{a}}{d\bar{s}}\right|_{\bar{t}\text{ fixed}} = \frac{\bar{t}^3(\bar{s}-\bar{u})}{(\bar{s}^2+\bar{s}\bar{t}+\bar{t}^2)^2}, \tag{4-6b}$$

$$\left.\frac{d\bar{a}}{d\bar{s}}\right|_{\bar{t}=\bar{u}} = -1/3 .$$

The bounds (4-5) on the parameter a follow immediately from Equations (4-6) and confirm the statement made in Chapter III that the range of values $4/9 < a < 4$ for which we exhibited a proof of the A-K representation will be sufficient for the applications we have in mind.

It is now trivial to see that the kernal ρ of the A-K representation, which can be written in the form

$$\rho(\bar{s}';\bar{s},\bar{t},\bar{u}) = \frac{(2\bar{s}'-3\bar{a})}{(\bar{s}^2+\bar{t}^2+\bar{s}\bar{t})\ \bar{s}'(\bar{s}'-\bar{s})\ (\bar{s}'-\bar{t})\ (\bar{s}'-\bar{u})}, \tag{4-7}$$

is real and positive for all points (s,t,u) within the Mandelstam triangle and $\bar{s}' \geq 8.3$.

Next we show that t_+ is real and positive for $\bar{a}$ in the range (4-5), and so for all points within the Mandelstam triangle the absorptive part $A(\bar{s}',t_+)$ is a positive definite function of $\bar{s}' \geq 8/3$. The reality property of t_+ follows immediately from the definition (3-23c) and the observation that for $\bar{a}$ in the specified range,

$$\frac{\bar{s}'+3\bar{a}}{\bar{s}'-\bar{a}} > 0 \; . \qquad (4\text{-}8)$$

To prove positivity is only a little more difficult. Notice that for $\bar{a}$ bounded by (4-5) and $\bar{s}' \geq 8/3$, $t_+(\bar{s}',\bar{a}) \geq t_+(\bar{s}',\bar{a}=-8/9)$. Since $t_+(8/3,-8/9)=0$, all we have to do is show that $\frac{dt_+(\bar{s}',-8/9)}{d\bar{s}'} > 0$ for $\bar{s}' \geq 8/3$. From (3-23c) we have:

$$\frac{dt_+(\bar{s}',\bar{a})}{d\bar{s}'} = \frac{}{2(\bar{s}'-\bar{a})\sqrt{\frac{\bar{s}'+3\bar{a}}{\bar{s}'-\bar{a}}}} \times$$

$$\times[(\bar{s}'^2-3\bar{a}^2)-(\bar{s}'-\bar{a})\sqrt{\bar{s}'^2+2\bar{a}\bar{s}'-3\bar{a}^2}\;]. \qquad (4\text{-}9)$$

The denominator in (4-9) is clearly positive, so it is left to show that the bracketed term is positive. For $\bar{s}'\to\infty$,

$$[(\bar{s}'^2-3\bar{a}^2)-(\bar{s}'-\bar{a})\sqrt{\bar{s}'^2+2\bar{a}\bar{s}'-3\bar{a}^2}\;]$$

$$\longrightarrow -\frac{4\bar{a}^3}{\bar{s}'} > 0, \text{ for } \bar{a}=-8/9. \qquad (4\text{-}10)$$

Also, $\frac{dt_+}{d\bar{s}'}$ has no zero between $\bar{s}'=8/3$ and ∞ because the condition for a zero is $\bar{s}'=3\bar{a}/2$, which is negative for $\bar{a}=-8/9$. Therefore, $\frac{dt_+(\bar{s}',-8/9)}{d\bar{s}'} > 0$, and so $t_+(\bar{s}',\bar{a})$ is positive for all points within the Mandelstam triangle.

Since the kernal $\rho(\bar{s}';\bar{s},\bar{t},\bar{u})$ and the momentum transfer $t_+(\bar{s}';\bar{s},\bar{t},\bar{u})$ are real and postive, the method of Łukaszuk and Martin can be applied directly to the Auberson-Khuri representation to obtain a new upper bound on α_o. Our task is to find the best point (s,t,u) inside the Mandelstam triangle for comparison with the symmetric point. Recall from Chapter II that the optimization procedure used to obtain the LM bounds can only be applied to the amplitude at points outside or on the border of the inner Mandelstam triangle (s>0,t>0,u>0). For points outside the inner triangle there is a loss of information because the minimization procedure fails along part of the integration range. Therefore we know from the start that it is best to consider only points on the border of the inner Mandelstam triangle, and crossing symmetry then allows us to limit our attention to points (see Figure 3):

$$(s_1, t, 0),\ 2 \leq s_1 < 4. \tag{4-11}$$

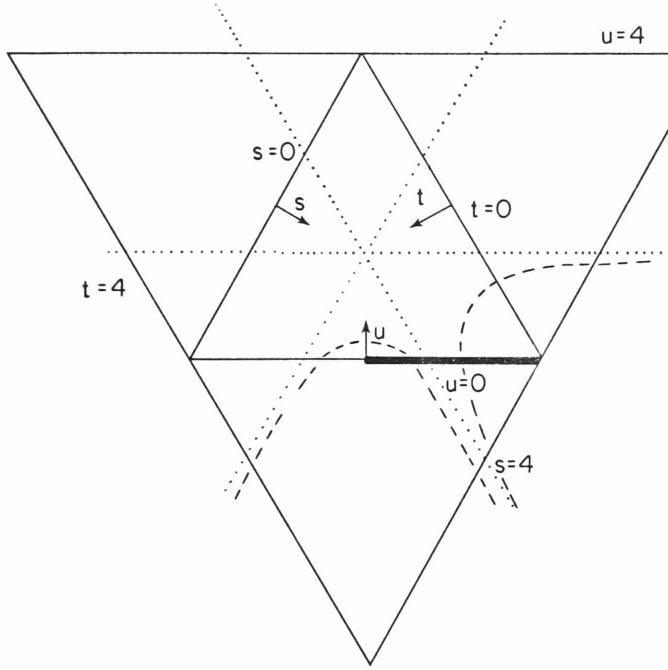

Figure 3

The Mandelstam Triangle. The segment $u_0=0$, $2 \leq s_0 < 4$ is indicated by a thick line. Sections of two cubics which intersect this segment are represented by dashed curves. The symmetric point is the intersection of the dotted lines (s=4/3, t=4/3, u=4/3).

Specializing to this case, the A-K representation becomes:

$$F(s_1, t, 0) = \alpha_0 + \frac{1}{\pi} \int_4^\infty ds' \rho(s', s_1) A(s', t_+), \tag{4-12a}$$

where

$$\rho(s', s_1) = \frac{2s'(s_1^2 - 4s_1 + \frac{16}{3}) + \frac{4}{3} s_1(s_1 - 4)}{s'(s' - s_1)(s' + s_1 - 4)(s' - 4/3)}, \tag{4-12b}$$

and t_+ is given by (3-23c).

To derive a new upper bound on α_o, we now make the replacements:

$$\begin{aligned} &\rho(s';s_1,s_o,t)\to\rho(s',s_1),\\ &F(s_o,t,u_o)\to\alpha_o\ ,\\ &A(s',t)\ A(s',t_+)\ , \end{aligned} \tag{4-13}$$

in Equations (2-40) through (2-54). We remind the reader that the procedure is the following: Minimize the absorptive part $A(s',t_+)$ as a function of the magnitude of the amplitude in the forward direction, $|F(s',0)|$, using the Lagrange multiplier method:

$$F(s_1,t,0)-\alpha_o\geq\frac{1}{\pi}\int_4^\infty ds'\rho(s',s_1)\bar{A}_{min}(\bar{F}(s'),s',t_+). \tag{4-14}$$

Transform the twice-cut s-plane of the forward amplitude onto the unit circle in the y-plane and maximize the magnitude of the amplitude at the center of the circle (chosen to be $F(y=0)=F(s_1,t,0)$) using the Poisson inequality:

$$|F(s_1,t,0)|\leq\exp[\frac{1}{\pi}\int_0^\pi d\phi\ln\bar{F}(y=e^{i\phi})\equiv F_o. \tag{4-15}$$

Fix the s-dependent Lagrange multiplier by minimizing the integral

$$\begin{aligned} &\frac{1}{\pi}\int_4^\infty ds'\rho(s',s_1)\bar{A}_{min}(\bar{F}(s'),s',t_+)\\ &=\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F}(y=e^{i\phi}),s'(\phi),t_+), \end{aligned} \tag{4-16}$$

for a given value of F_o, again using the Lagrange multiplier method. This determines $\bar{F}(s')$ as a function of the new s-independent Lagrange multiplier γ . Finally, maximize the difference:

$$\Delta(\gamma)\equiv F_o(\gamma)-\frac{1}{\pi}\int_0^\pi d\phi W(\phi)\bar{A}_{min}(\bar{F}(y,\gamma),s'(\phi),t_+). \tag{4-17}$$

The extremum is our upper bound on α_o:

$$\Delta(\gamma_m) > \alpha_o, \tag{4-18}$$

To numerically evaluate our upper bound on $\alpha_o = F(4/3,4/3,4/3)$ we performed a computer calculation in which, for s_1 in the range (4-11) and starting from $\gamma=0$ we successively calculated $\Delta(\gamma)$ (using Equations (2-42), (2-46), (2-49), (2-51), and (2-52) with the replacements (4-13)) and incremented γ until the maximum of $\Delta(\gamma)$ was reached. Varying s_1 we found that the upper bound on α_o had a maximum at $s_1=2$, decreased to its minimum value

$$\alpha_o < 11 \tag{4-19}$$

at $s_1=47/12$, and then increased again as s_1 approached 4. Equation (4-19) is an improvement of more than 30% over the result $\alpha_o<16$ obtained by LM. Our bound corresponds to the lower bound

$$\lambda \equiv -\alpha_o/6 > -1.84 \tag{4-20}$$

on the Chew Mandelstam coupling constant λ .

We have also tried to improve the other bounds of LM by using the A-K representation, but with no success. For completeness we list in Table I the absolute bounds we have calculated at various pairs of points (s_1,t_1,u_1) and (s_o,t_o,u_o) using both the fixed-t dispersion relation and the A-K representation. From each pair of absolute bounds in Table I a lower bound on α_o can be calculated by the simple procedure described in Chapter III, Section D; but the bound

$$\alpha_o > -122 \tag{4-21}$$

calculated by us in Chapter II is the best result we have been able to find.

To summarize, our best bounds on the π^o-π^o scattering amplitude at the symmetric points and on the Chew-Mandelstam coupling constant are:

$$\begin{aligned} -122 < \alpha_o &\equiv F(4/3,4/3,4/3) < 11 \\ 20.33 > \lambda &\equiv -\alpha_o/6 > -1.84 \end{aligned} \tag{4-22}$$

CHAPTER V

Bounds on Coupling Constants in the Presence of Bound States

In the previous chapters we have discussed bounds on the π^o-π^o scattering amplitude at points within its analyticity domain, i.e. at the symmetric point, which follow from analyticity, unitarity, and crossing symmetry. There are no bound states in the π-π system, and they have been explicitly excluded. The bounds we have considered are not limited to π^o-π^o scattering, and they apply to any field theory which satisfies these requirements. They are important because in a ϕ^4 field theory, for example, the renormalized coupling constant λ can be defined as

$$\lambda = -F(4/3,4/3,4/3)/6 \ , \tag{5-1}$$

and a bound on the amplitude at the symmetric point is therefore also a bound on the renormalized coupling constant. In Chapter VI we will mention the possible future relevance of these results for the constructive field theorists.

While we know that in the real world there are no bound states in the π-π system, it is theoretically interesting to consider as well theories with bound states. We will show that the methods used to obtain bounds on the π^o-π^o scattering amplitude can be generalized to prove that there exists an upper bound on the coupling to the bound state which is a function of the particle and bound state masses. We will also show that upper and lower bounds on the renormalized coupling constant can still be derived.

To that end we will consider the scattering of neutral pseudoscalar bosons of unit mass μ=1 which couple to a scalar bound state of mass m with physical coupling constant g. We assume, as for π^o-π^o scattering, that the scattering amplitude is fully crossing symmetric and that we have the usual unitarity constraints on the partial wave amplitudes. It then follows from axiomatic field theory that for fixed physical energy

$s \geq 4$ the scattering amplitude is an analytic function[11] of momentum transfer inside the circle $|t|<m^2$. For t real and within this domain the scattering amplitude is analytic in the twice-cut s-plane with real cuts $s \geq 4$ and $s \leq -t$ and poles at $s=m^2$ and $u=4-s-t=m^2$. In perturbation theory one can show that for fixed $s \geq 4$ the amplitude is analytic in $|t| < 4$ except for a pole at $t=m^2$. However, to the best of our knowledge this has not been proved from the axioms of field theory. The rigorously proven domain $|t|<m^2$ will permit us to derive an upper bound on the physical coupling constant g^2, provided $4/3<m^2<4$, and upper and lower bounds on the Chew-Mandelstam coupling constant λ for $2<m^2<4$. To find both upper and lower bounds on the renormalized coupling constant for $4/3<m^2 \leq 2$ it is necessary to adopt instead of (5-1) a different definition of λ. We will also mention how one could extend our methods to the entire range $1 \leq m^2<4$ using the perturbation theory domain $|t|<4$. The scattering of identical neutral scalar bosons via a ϕ^3 interaction, for which there is a pole at $m^2=1$, has already been treated by Martin[7] with a different method using analyticity in $|t|<4$, and we mention later in this section how our more general results compare with his.

We start with the twice-subtracted fixed-t dispersion relation:

$$F(s_1,t,u_1)-F(s_o,t,u_o)=-g^2\left[\frac{1}{s_1-m^2}+\frac{1}{u_1-m^2}-\frac{1}{s_o-m^2}-\frac{1}{u_o-m^2}\right]+\frac{(s_1-s_o)(s_1-u_o)}{\pi}\int_4^\infty ds'\,\frac{(2s'+t-4)A(s',t)}{(s'-s_1)(s'-u_1)(s'-s_o)(s'-u_o)}. \tag{5-2}$$

Both points (s_1,t,u_1) and (s_o,t,u_o) will be taken to be inside the Mandelstam triangle. We will generalize the techniques developed by Martin[25] and reviewed in Chapter II, Section A, to demonstrate that there exists an upper bound on the physical coupling constant, $g^2_{max}(m^2)>g^2$, which depends only on the particle and bound state masses for $4/3<m^2<4$. We have chosen this method for clarity of presentation, and because we are mainly interested in showing that these bounds exist. Our numerical results could, of course, be improved by generalizing the more refined method of Łukaszuk and Martin[27], and by replacing the dispersion relation (5-2) with an Auberson-Khuri representation to make better use of crossing symmetry. Later we will use our generalization of Martin's method to discuss the existence of bounds on the renormalized coupling constant in the presence of bound states.

For definiteness we specialize the dispersion relation (5-2) to $u_o=0$:

$$F(s_1,t,u_1)-F(s_o,t,0)=g^2f(m^2)+ \\ +\frac{1}{\pi}\int_4^\infty ds'\rho(s',s_1,t)A(s',t) \quad , \tag{5-3a}$$

where

$$f(m^2) = \frac{s_1(s_1+t-4)(2m^2+t-4)}{m^2(m^2+t-4)(m^2-s_1)(m^2+s_1+t-4)},$$

$$\rho(s,s_1,t)= \frac{s_1(s_1+t-4)(2s+t-4)}{s(s+t-4)(s-s_1)(s+s_1+t-4)} \quad . \tag{5-3b}$$

We temporarily restrict ourselves to masses in the range $2<m^2<4$. Then for $m^2>s_1\geq t\geq 2$ and $s\geq 4$, $f(m^2)$ and $\rho(s,s_1,t)$ are both positive. From the dispersion relation (5-3) we will obtain two nonlinear inequalities:

$$F(s_1,t,u_1)-F(s_o,t,0)\geq g^2f(m^2)+A|F(s_o,t,0)|^N, \tag{5-4a}$$

and

$$F(s_1,t,u_1)-F(s_o,t,0)\geq g^2f(m^2)+B|F(s_1,t,u_1)|^N. \tag{5-4b}$$

As we shall see, these inequalities lead immediately to an upper bound on g^2.

The first step in getting inequality (5-4a) is to find a lower bound on the right hand side of (5-3a) using the inequality:

$$A(s,t) \geq \frac{|F(s,0)|^N}{g_N(s,t,0)} \quad , \tag{5-5}$$

with $N \gtrsim 2$. We have already calculated $g_N(s,t,0)$ in Chapter II, and it is given by Equations (2-14) and (2-15). Inserting (5-5) into the dispersion relation (5-3) we have:

$$F(s_1,t,u_1)-F(s_o,t_,o) \gtrsim g^2 f(m^2) + \\ + \frac{1}{\pi} \int_4^{\infty} ds' \frac{\rho(s',s_1,t)\,|F(s',0)|^N}{g_N(s',t,0)} \, . \tag{5-6}$$

The second step is to transform the twice-cut s-plane onto the unit disk in the y-plane by the sequence of transformations (2-17) and (2-18). The new feature here is that the poles at $s=m^2$ and $u=4-s=m^2$ are mapped onto a single pole at

$$R_o(m^2) \equiv y(s=m^2)=y(u=m^2) \\ = \frac{1-\sqrt{\dfrac{4m^2-m^4}{s_o t}}}{1+\sqrt{\dfrac{4m^2-m^4}{s_o t}}} \, . \tag{5-7}$$

This pole lies on the real y axis in the interval $0<y<1$. Making the transformation $s' \rightarrow y$ in (5-6), and using the theorem on arithmetic and geometric means and the reality of $F(s,0)$, we get the inequality:

$$F(s_1,t,u_1)-F(s_o,t,0) \gtrsim g^2 f(m^2) + \\ + C_o \exp\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \ln|F(s'(\phi),0)|^N \}, \tag{5-8a}$$

with

$$C_o = \exp\{\frac{1}{\pi} \int_0^{\pi} d\phi \ln \frac{\rho(s'(\phi),s_1,t) J(s',\phi,s_o)}{g_N(s'(\phi),t,0)} \} \, . \tag{5-8b}$$

$J(s',\phi,s_o)$ is the Jacobian of the transformation from s' to $\phi, y=e^{i\phi}$. Finally, we can use the Poisson-Jensen formula, as shown in the Appendix, and crossing symmetry to obtain:

$$F(s_1,t,u_1)-F(s_o,t,0) \geq g^2 f(m^2)+ \\ +C_o R_o^N(m^2)\,|F(s_o,t,0)|^N . \qquad (5\text{-}9)$$

This is the first of the pair of inequalities for which we are looking.

To get the second inequality (5-4b) we will need an inequality of the form

$$A(s,t) \geq \frac{|F(s,u_1)|^N}{g_N(s,t,u_1)} \quad , \quad s \geq s_1+t \quad , \qquad (5\text{-}10)$$

to replace (5-5). The function $g_N(s,t,u_1)$ has also been calculated in Chapter II and is given by Equations (2-31), (2-2), (2-26), and (2-30). As for the case of $\pi^o-\pi^o$ scattering, we must minimize $A(s,t)$ by zero for $4 \leq s < s_1+t$. Inserting the inequality (5-10) into (5-3) we get:

$$F(s_1,t,u_1)-F(s_o,t,0) \geq g^2 f(m^2) \\ +\frac{1}{\pi}\int_{s_1+t}^{\infty} ds' \frac{\rho(s',s_1,t)\,|F(s',u_1)|^N}{g_N(s',t,u_1)} . \qquad (5\text{-}11)$$

The next step is to transform the s-plane, with real cuts $s \geq 4$ and $s \leq s_1+t-4=-u_1$ and poles at $s=m^2$ and $u=s_1+t-s=m^2$, onto the unit disk in the y-plane by the sequence of transformations (2-33) and (2-34). As in Chapter II the point $s=s_1$ is mapped onto the point $y=0$; the part of the integration range $4 \leq s < s_1+t$ is mapped onto the segment

$$\beta \equiv y(s=4) \leq y < 1 \ ; \qquad (5\text{-}12a)$$

and the remainder of the integration range $s \gtrsim s_1+t$ is mapped onto the unit semi-circle in the upper half y-plane: $y=e^{i\phi}$, $0 \lesssim \phi \lesssim \pi$. Here the poles at $s=m^2$ and $u=m^2$ are mapped onto a single pole on the real axis at

$$R_1(m^2) \equiv y(s=m^2)=y(u=m^2)$$
$$= \frac{1-\sqrt{\frac{(s_1+t)m^2-m^4}{s_1t}}}{1+\sqrt{\frac{(s_1+t)m^2-m^4}{s_1t}}} \quad , \tag{5-12b}$$

with

$$0<R_1(m^2)<\beta \; . \tag{5-12c}$$

Introducing this change of variables into (5-11), it follows from the theorem on arithmetic and geometric means and the reality of $F(s,u_1)$ that:

$$F(s_1,t,u_1)-F(s_o,t,0) \gtrsim g^2 f(m^2)$$
$$+C_1 \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \ln|F(s'(\phi),u_1)|^N\right\} \; , \tag{5-13a}$$

where

$$C_1=\exp\left\{\frac{1}{\pi}\int_0^{\pi} d\phi \ln \frac{\rho(s'(\phi),s_1,t)J_1(s',\phi,s_1)}{g_N(s'(\phi),t,u_1)}\right. \quad , \tag{5-13b}$$

and $J_1(s',\phi,s_1)$ is the Jacobian of the transformation from s' to ϕ.

Because $0<u=4-s-u_1<m^2$ in the regions $4 \lesssim s<s_1+t$, we know from unitarity and the partial wave expansion that the absorptive part $A(s,u_1)$ (and hence the discontinuity of $\ln(F(s,u_1))$ across the cut $\beta \lesssim y<1$) is positive in that interval. Therefore we can employ a modified form of the Poisson inequality deduced from the Poisson-Jensen formula in the Appendix, to obtain:

$$F(s_1,t,u_1)-F(s_o,t,u_o) \gtrsim g^2 f(m^2)+$$
$$+C_1 R_1^N(m^2)\beta^N|F(s_1,t,u_1)|^N \; . \tag{5-14}$$

This inequality can be improved by noting that the imaginary part of the amplitude above the pole at $y=R_1(m^2)$ is positive, and so as shown in the Appendix the Poisson-Jensen formula requires:

$$F(s_1,t,u_1)-F(s_o,t,0)\geq g^2 f(m^2)$$
$$+C_1 R_1^N(m^2)\,|F(s_1,t,u_1)|^N. \tag{5-15}$$

This is the second of the inequalities for which we have been searching.

It is easy to see that Equations (5-9) and (5-15) impose an upper bound on g^2. The maximum value of g^2 will occur when both inequalities are saturated, at which point:

$$|F(s_1,t,u_1)|= \left(\frac{C_o}{C_1}\right)^{1/N} \frac{R_o(m^2)}{R_1(m^2)}\,|F(s_o,t,0)|\ . \tag{5-16}$$

Inserting this value for $|F(s_1,t,u_1)|$ into (5-9) we get the inequality:

$$|F(s_o,t,0)|\left[1+\left(\frac{C_o}{C_1}\right)^{\frac{1}{N}} \frac{R_o(m^2)}{R_1(m^2)}\right] -$$
$$-C_o R_o^N(m^2)\,|F(s_o,t,0)|^N$$
$$\geq g^2 f(m^2). \tag{5-17}$$

Maximizing the left hand side of (5-17) as a function of $|F(s_o,t,0)|$ we find:

$$g^2 \leqslant \frac{(1-\frac{1}{N})}{f(m^2)}\ \frac{\left[1+\left(\frac{C_o}{C_1}\right)^{\frac{1}{N}} \frac{R_o(m^2)}{R_1(m^2)}\right]^{\frac{N}{N-1}}}{\left[NC_o R_o^N(m^2)\right]^{\frac{1}{N-1}}}\ . \tag{5-18}$$

All quantities in (5-18) are determined by Equations (5-3b), (5-7), (5-8b), (5-12b), and (5-13b). This is our upper bound on the coupling constant g^2. It is an explicit function of the ratio of the bound state mass m to the particle mass $\mu=1$, and it has the dimensions of mass.

We remind the reader that s_1 must be chosen so that $t \leq s_1 < m^2$ for masses in the range $2 \leq t < m^2 < 4$.

We can now remove the restriction $m^2 > 2$. To do this we note that for $2 \geq m^2 > t \geq 4/3$ and $s_1 > s_o = 4 - t > m^2$, $f(m^2)$ and $\rho(s, s_1, t)$ are both positive again. We can therefore carry out the derivations of (5-9) and (5-14), the only change being that here $R_o(m^2)$ and $R_1(m^2)$ are negative, i.e. $-1 < y(s=m^2) < 0$, so they must be replaced by their absolute values in (5-9) and (5-14). As pointed out in the Appendix, the nagativity of R_1 prevents us from deriving (5-15). The inequalities (5-9) and (5-14) are nonetheless sufficient to impose a bound on g^2, and it is given by (5-18) with the replacements:

$$R_o(m^2) \to |R_o(m^2)| \quad ,$$
$$R_1(m^2) \to \beta |R_1(m^2)| \quad . \tag{5-19}$$

We have now proved using the rigorous analyticity domain $|t| < m^2$ that g^2 is bounded as a function of m^2 for $4 > m^2 > 4/3$.

We would like to make some remarks about the possibility of extending these results to smaller masses $1 \leq m^2 \leq 4/3$. If we are willing to assume that the amplitude is analytic in $|t| < 4$ for fixed $s \geq 4$ except for a pole at $t = m^2$, then we can also write the dispersion relation (5-3) for $m^2 < t$. We point out that $f(m^2)$ is positive for $\frac{4-t}{2} < m^2 < 4 - t \leq t < s_1$, as well as for $2 \leq t \leq s_1 < m^2 < 4$. The derivation of (5-9) and (5-14) can still be carried out for this case. Now the u pole in the physical amplitudes $F(s,0)$ and $F(s,u_1)$ lies to the right of the s pole, and, as for the case $4/3 < m^2 \leq 2$, one can no longer obtain (5-15). The bound on g^2 is now given by (5-18) with the replacement:

$$R_1(m^2) \to \beta R_1(m^2). \tag{5-20}$$

Therefore, if we are willing to accept the analyticity domain $|t| < 4$ from perturbation theory, then we can use the method we have described above to derive bounds on the coupling constant g^2 for any bound state mass in the range $1 \leq m^2 < 4$.

We have calculated several examples of these bounds on the physical coupling constant. The results are listed in Table II and displayed in Figure 4. In the Figure, the points calculated using the rigorous

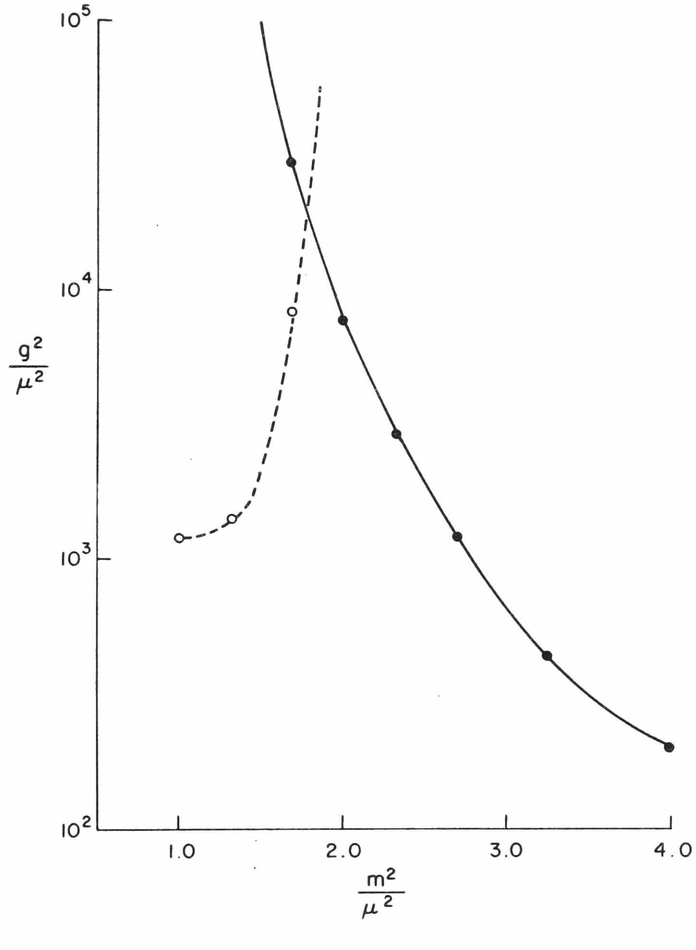

Figure 4

Examples of bounds on physical coupling constants g^2 to bound states of mass m. The points connected by a solid line were calculated using the rigorous domain $|t|<m^2$. Those connected by a dashed line required the use of the perturbation theory domain $|t|<4\mu^2$.

domain $|t|<m^2$ are connected by a solid line, which gives a rough indication of the behavior of our bound on g^2 as a function of m^2. It varies from 200 μ^2 for $\frac{m^2}{\mu^2}\to 4$ to ∞ for $\frac{m^2}{\mu^2}\to 4/3$. The bounds calculated using the perturbation theory domain $|t|<4$ are connected with a dashed line. At $\frac{m^2}{\mu^2}=1$ our result is $g^2<1200\mu^2$; this also applies to the renormalized coupling constant for the scattering of neutral scalar particles via a ϕ^3 interaction, in which case the bound state is just the particle itself. For this coupling constant Martin[7] derived the crude estimate $g^2 < 2\times 10^7\mu^2$, which we have improved by four orders of magnitude.

We want to emphasize that these are not the best bounds which can be obtained. Using the method we have described, they could be improved by varying s_1 and t to find the optimal bound for any given m^2. Then they could be further improved by generalizing the methods of Łukaszuk and Martin and by replacing the dispersion relation with an Auberson-Khuri representation. Our expectation is that the bounds we have calculated explicitly give a good order of magnitude estimate of the best bounds which could be obtained by the techniques we have mentioned. The possibility of improving these methods will be touched on in Chapter VI.

Finally, we want to discuss bounds on the renormalized coupling constant λ in the presence of bound states. As for the bound on the physical coupling constant g^2, the simplest case is for masses in the range $2<m^2<4$. For this case the rigorous analyticity domain $|t|<m^2$ is sufficient. Now for $2\leq t\leq s_1<m^2$ we have inequalities (5-9) and (5-15) which require, since $g^2f(m^2)$ is positive:

$$\begin{aligned} &F(s_1,t,u_1)-F(s_o,t,0)>C_oR_o^N(m^2)\,|F(s_o,t,0)|^N, \\ &F(s_1,t,u_1)-F(s_o,t,0)>C_1R_1^N(m^2)\,|F(s_1,t,u_1)|^N. \end{aligned} \tag{5-21}$$

These inequalities have the same form as (2-23) and (2-37), and they impose absolute bounds on the magnitudes of $F(s_1,t,u_1)$ and $F(s_o,t,0)$.

In particular, choosing $t=2<s_1<m^2$ we get bounds on $|F(2,2,0)|$ and $|F(s_1,2,2-s_1)|$. By the same method as was used in Chapter II, Section D, for π^o-π^o scattering, we can get from this pair of bounds a lower bound on α_o which will, of course, depend on the mass of the bound state. To get an upper bound on α_o we need only note that, as for π^o-π^o scattering, $F(2,2,0)-F(4/3,2,2/3))$ and $(F(2,4/3,2/3)-\alpha_o)$ are both positive, so the upper bound on $F(2,2,0)$ is also an upper bound on α_o. For $m^2>8/3$ we could get an upper bound on α_o more directly by considering $(F(8/3,4/3,0)-\alpha_o)$ as was done by Martin for π^o-π^o. The important point is that for $4>m^2>2$ we can obtain upper and lower bounds on α_o as a function of the bound state mass m^2, and that these correspond respectively to lower and upper bounds on the Chew-Mandelstam coupling constant λ.

For $4/3<m^2\lesssim 2$ we can still get a lower bound on α_o, but not an upper bound. For the lower bound we could, for example, calculate bounds on $|F(s_1,4/3,8/3-s_1)|$ and $|F(8/3,4/3,0)|$, with $8/3<s_1$, using Equations (5-21) with the substitutions (5-19). Then we could immediately get a lower bound on α_o:

$$\alpha_o>-|F(8/3,4/3,0)|-K[|F(s_1,4/3,8/3-s_1)|+|F(8/3,4/3,0)|]. \tag{5-22}$$

where K is determined from the dispersion relations for $(F(8/3,4/3,0)--\alpha_o)$ and $(F(s_1,4/3,8/3-s_1)-F(8/3,4/3,0))$ by the method described in Chapter II. However, we see no way of getting an upper bound on α_o, even using the larger perturbation theory analyticity domain $|t|<4$. The reason is that for $4/3<m^2\lesssim 2$ there is no way to show for any point (s,t,u) outside or on the border of the inner Mandelstam triangle that $F(s,t,u)>\alpha_o$.

This difficulty is not as serious as it may seem, because the definition of the renormalized coupling constant is to some extant arbitrary. For a given bound state mass we could, for example, equally well define the renormalized coupling constant as

$$\lambda' = -F(s,t,0) , \tag{5-23}$$

with $4\geq t\geq 4/3$. For suitable choice of t we could then calculate upper and lower bounds on λ' for any bound state mass in the range $4/3<m^2<4$.

For $1\leq m^2\leq 4/3$ we can only calculate bounds on the renormalized coupling constant if we are willing to assume the perturbation theory analyticity domain $|t|<4$. Even then we can calculate only a lower bound on α_o. The situation is essentially the same as for $4/3<m^2\leq 2$. Again, by defining the renormalized coupling constant by (5-23) with t chosen appropriately for the mass m, we can get upper and lower bounds on λ' for any mass m. We must point out, however, that there is no single definition of the renormalized coupling constant which will allow us to derive both upper and lower bounds on the coupling constant for any bound state mass $1\leq m^2<4$.

CHAPTER VI

Discussion and Conclusions

While our new upper bound on the π^o-π^o scattering amplitude at the symmetric point represents a significant improvement over the bound obtained by Łukaszuk and Martin, there is no reason to believe that it is the best bound which could be obtained from analyticity, unitarity, and crossing symmetry. In particular, we have utilized only a very weak form of unitarity. It is reasonable to expect that this entire class of bounds for π-π scattering would benefit from an improved use of unitarity. The virtue of the method of LM is that, after one has selected a representation for the amplitude and employed unitarity to minimize $A(s,t)$ for a given $|F(s,0)|$, one knows at each succeeding step that the prescribed procedure is the optimal one. It would be desirable to have a method in which, from the start, one knows that one has the best optimization at every step.

As we have already mentioned, our bounds on coupling constants in theories with bound states are definitely not the best which we could obtain. Nonetheless, they are better than one might have expected from Martin's result[7] for a ϕ^3 interaction. These bounds also might be greatly improved by a better use of unitarity.

We want to emphasize that the upper and lower bounds on the Chew-Mandelstam coupling constant which were derived by LM and improved by us are rigorous consequences of axiomatic field theory; they are true for the renormalized coupling constant in any ϕ^4 type field theory with no bound state. For a ϕ^4 field theory with a bound state we have shown in Chapter V that it is still possible to calculate rigorous upper and lower bounds on the renormalized coupling constant, defined à la Chew and Mandelstam, provided $4>m^2>2$. The upper bound can also be derived for $4/3<m^2\leqslant 2$, but the lower bound no longer exists in that case. However, we have shown that for a ϕ^4 type theory with a bound state of mass $4/3<m^2<4$ it is always possible to choose a definition for the renormalized coupling constant such that the magnitude of that coupling constant cannot be arbitrarily large. For a bound state which is more tightly bound (i.e. for $1 \lesssim m^2 \lesssim 4/3$) we can obtain this result only if we are willing to use the analyticity domain found in perturbation theory.

Our discussion so far has been restricted to field theories in 3 space + 1 time dimensions. The bounds on coupling constants require the use of analyticity of the scattering amplitude in two independent variables. In 1 + 1 dimensional theories there is only one independent variable, and so there are no bounds of the type we are considering.

For theories in 2 + 1 dimensions there are again two independent variables. We have checked that for this case the scattering amplitude has the same analyticity domain in s and t as it does for 3 + 1 dimensions, and that therefore the bounds on the renormalized coupling constant which we have obtained hold in 2 + 1 dimensions. In an actual field theory these bounds should, of course, be even more restrictive. This will further complicate the already difficult task of constructing a ϕ^4 field theory in 2 + 1 dimensions.[3] When such a theory has been constructed, it will be interesting to see how the bounds on the coupling constant arise.

APPENDIX

The Poisson-Jensen Formula

We will discuss here several inequalities which follow from the Poisson-Jensen formula[34] and a simple generalization of it. Let $f(y)$ be a function analytic inside the unit circle $|y|<1$ except for poles at the points $p_1, p_2, \dots, p_n$, $1>|p_i|>0$, and let $f(y)$ have zeros at the points $q_1, q_2, \dots, q_m$, $1>|q_i|>0$. Then the Poisson-Jensen formula for $f(y)$ is:

$$\ln|f(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \ln|f(e^{i\phi})| - \sum_{i=1}^{n} \ln|p_i| + \sum_{j=1}^{m} \ln|q_j| \;. \qquad \text{(A-1)}$$

We also wish to consider functions $F(y)$ which have the same properties as $f(y)$ above except that $F(y)$ has as an additional singularity a cut along the real y-axis for $1>y\geqslant\beta>0$. In that case we can generalize the Poisson-Jensen formula (A-1) by including the integral around the cut as well as the integral around the circle:

$$\ln|F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi \ln|F(e^{i\phi})| - \sum_{i=1}^{n} \ln|p_i| + \sum_{j=1}^{m} \ln|q_j| + \frac{1}{2\pi}\int_{\beta}^{1} dx \, \frac{\mathrm{Im}[\ln F(x+i\varepsilon) - \ln F(x-i\varepsilon)]}{x} \;. \qquad \text{(A-2)}$$

provided $F(y)$ is not zero on a finite segment of the cut. In the cases of interest to us, $F(y)$ satisfies the reality property $F(y^*)=F^*(y)$ and the phase $\psi(y)$ of $F(y)$ above the cut $(\beta,1)$ is between 0 and π. Also, we will need to consider functions with at most one real pole at $y=R$. Therefore we can simplify (A-2) to:

$$\ln|F(0)| = \frac{1}{\pi}\int_0^\pi \ln|F(e^{i\phi})|d\phi + \frac{1}{\pi}\int_\beta^1 \frac{dx\psi(x)}{x}$$

$$-\ln|R| + \sum_{j=1}^{m} \ln|q_j|$$

$$\leq \frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})| - \ln\beta$$

$$-\ln|R| + \sum_{j=1}^{m} \ln|q_j| \ . \tag{A-3}$$

The only zeros of importance to us will be those on the real axis for the case $1>\beta>|q_j|>R>0$. There can at most a finite number of such zeros, and it will be enough to retain only one of them if there are any. All other zeros can be ignored by remembering that $|q_j|<1$. Therefore we can further simplify (A-3) to:

$$\ln|F(0)| \leq \frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})| + \ln\frac{Q}{\beta|R|} \ , \tag{A-4}$$

where Q is the location of the (possible) zero. Exponentiating both sides of (A-4) we find:

$$|F(0)| \leq \frac{Q}{\beta|R|} \exp\{\frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})|\} \ . \tag{A-5}$$

If the pole, cut, or zero are absent, then the corresponding factor R, β, or Q should be replaced by unity in (A-2) through (A-5). For example, if there is no pole, in which case we ignore the possible zero, then (A-5) reduces to the modified Poisson inequality discussed by Drell, Finn, and Hearn,[35] and by Łukaszuk and Martin:[27]

$$|F(0)| \leq \frac{1}{\beta} \exp\{\frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})|\}. \tag{A-6}$$

If in addition there is no cut, then we get the standard Poisson inequality:

$$|F(0)| \leq \exp\{\frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})|\} \ . \tag{A-7}$$

The most interesting situation occurs when there is a cut $(\beta,1)$ and a pole at $y=R$, $0<R<\beta$. Then if there is a zero at $y=Q$, $R<Q<\beta$, we will have from (A-5):

$$|F(0)|<\frac{1}{R}\exp\{\frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})|\}. \tag{A-8}$$

On the other hand, if there is no zero, but the imaginary part of $F(y)$ above the pole is positive, then $F(y)$ is negative in the gap $R<y<\beta$ and $\mathrm{Im}(\ln F(x+i\varepsilon))=\pi$. Therefore the phase of $F(x+i\varepsilon)$ is continuous and between 0 and π along the entire interval $R<x<1$, so we can just as well treat that interval as a single cut in $\ln F(y)$ with positive definite discontinuity. Then we again get the inequality (A-8).

In general we do not know if there exists a zero on the interval $R<y<\beta$. However, as long as we know that the imaginary part of $F(y)$ above the pole at $y=R>0$ is positive, then we can get the inequality (A-8). If we do not have positivity above the pole, or if $R<0$, then we must settle for the weaker inequality:

$$|F(0)|\leq\frac{1}{\beta|R|}\exp\{\frac{1}{\pi}\int_0^\pi d\phi \ln|F(e^{i\phi})|\}. \tag{A-9}$$

Table I

Absolute bounds on the π^o-π^o scattering amplitude calculated at points within the Mandelstam triangle using the method of Łukaszuk and Martin. The representation used to calculate each pair of bounds from $(F(s_1,t_1)-F(s_o,t_o))$ is indicated.

s_1	t_1	s_o	t_o	$\lvert F(s_1,t_1)\rvert_{max}$	$\lvert F(s_o,t_o)\rvert_{max}$	Representation Used
3.2	1.8	2.0	2.0	179	39	A-K
3.0	1.805	2.0	2.0	122	39	A-K
2.8	1.814	2.0	2.0	92	40	A-K
2.75	2.75	2.0	2.0	283	37	A-K
2.5	2.5	2.0	2.0	136	37	A-K
3.2	2.0	2.0	2.0	222	37	DR
3.0	2.0	2.0	2.0	150	37	DR
2.8	2.0	2.0	2.0	111	37	DR
2.5	2.0	2.0	2.0	69	40	DR
2.5	2.5	1.5	2.5	134	36	DR
11/3	4/3	8/3	4/3	388	44	DR
10/3	4/3	8/3	4/3	135	42	DR
7/3	8/3	4/3	8/3	134	36	DR

Table II

Examples of bounds on physical coupling constants to bound states. The parameters used to calculate the bounds for each bound state mass m^2 and the analyticity required (AFT or PT) are also tabulated. All results were calculated using N=10.

m^2/μ^2	g^2_{max}/μ^2	Analyticity	s_1	t	s_o
5/3	$3.\mathrm{x}10^4$	AFT	10/3	4/3	8/3
2.	$7.7\mathrm{x}10^3$	AFT	10/3	4/3	8/3
7/3	$2.9\mathrm{x}10^3$	AFT	10/3	4/3	8/3
2.7	$1.2\mathrm{x}10^3$	AFT	2.5	2.	2.
3.25	$4.3\mathrm{x}10^2$	AFT	3.	2.	2.
4.	$2.\mathrm{x}10^2$	AFT	10/3	4/3	8/3
1.	$1.2\mathrm{x}10^3$	PT	7/3	8/3	4/3
4/3	$1.4\mathrm{x}10^3$	PT	2.5	2.5	1.5
5/3	$8.4\mathrm{x}10^3$	PT	3.	2.	2.

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