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Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS)

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Euler's Square Root Laws for Negative Numbers

Dave Ruch

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1 Introduction

We learn in elementary algebra that the square root product law

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \tag{1}$$

is valid for any positive real numbers a, b. For example, $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$. An important question for the study of complex variables is this: will this product law be valid when a and b are complex numbers? The great Leonard Euler discussed some aspects of this question in his 1770 book *Elements of Algebra*, which was written as a textbook [Euler, 1770]. However, some of his statements drew criticism [Martinez, 2007], as we shall see in the next section.

2 Euler's Introduction to Imaginary Numbers

In the following passage with excerpts from Sections 139–148 of Chapter XIII, entitled *Of Impossible or Imaginary Quantities*, Euler meant the quantity a to be a positive number.

> The squares of numbers, negative as well as positive, are always positive.
>
> ...To extract the root of a negative number, a great difficulty arises; since there is no assignable number, the square of which would be a negative quantity. Suppose, for example, that we wished to extract the root of -4; we here require such as number as, when multiplied by itself, would produce -4; now, this number is neither $+2$ nor -2, because the square both of -2 and of -2 is $+4$, and not -4.
>
> We must therefore conclude, that the square root of a negative number cannot be either a positive number or a negative number, since the squares of negative numbers also take the sign *plus*: consequently, the root in question must belong to an entirely distinct species of numbers; as it cannot be ranked either among positive, or negative numbers.
>
> ...

*Department of Mathematical and Computer Sciences, Metropolitan State University of Denver, Denver, CO; ruch@msudenver.edu

We are led to the idea of numbers, which from their nature are impossible; and therefore they are usually called *imaginary quantities*, because they exist merely in the imagination.[1]

All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, etc. are consequently impossible, or imaginary numbers, since they represent roots of negative quantities.

...

The square of $\sqrt{-3}$, for example, or the product of $\sqrt{-3}$ and $\sqrt{-3}$, must be -3; and in general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$ or by taking the square of $\sqrt{-a}$, we obtain $-a$.

...

The root of a times -1, or $\sqrt{-a}$, is equal to $\sqrt{a}$ multiplied by $\sqrt{-1}$; but $\sqrt{a}$ is a possible or real number, consequently the whole impossibility of an imaginary number may be always reduced to $\sqrt{-1}$; for this reason, $\sqrt{-4}$ is equal to $\sqrt{4}$ multiplied by $\sqrt{-1}$.

...

Moreover, as $\sqrt{a}$ multiplied by$\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$.

Some mathematicians have argued that Euler made a mistake here: $\sqrt{-2} \cdot \sqrt{-3} \neq \sqrt{6}$. To see why, carry out the following task.

Task 1 Which statement in Euler's passages above allows us to write $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$?

(a) Write a general defining equation for $\sqrt{-a}$ for positive a based on the excerpt above. Indicate which excerpt statement you are using.

(b) Use part (a) to first rewrite $\sqrt{-2}$ and $\sqrt{-3}$ in terms of $\sqrt{-1}$. Then compute $\sqrt{-2} \cdot \sqrt{-3}$ with these facts to show that

$$\sqrt{-2} \cdot \sqrt{-3} = -\sqrt{6}. \tag{2}$$

(c) Explain why all this leads to the apparently nonsensical statement: $-\sqrt{6} = \sqrt{6}$.

To decide whether Euler really made a mistake here, we should read some more on his notion of square roots[2]. In what follows from Section 150 of his *Elements of Algebra*, remember that Euler used a to indicate a positive real number.

The square root of any number has always two values, one positive and one negative; that $\sqrt{4}$, for example, is both $+2$ and -2, and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a. This remark applies also to imaginary numbers; the square root of $-a$ is both $+\sqrt{-a}$ and $-\sqrt{-a}$.

[1] Descartes was the first to use the term *imaginary*, in a 1637 discussion of roots of cubic equations.

[2] See [Martinez, 2007] for a full and fascinating discussion of this issue.

This passage suggests that we have to be careful when using and interpreting the term "square root" and the radical symbol "$\sqrt{\ }$". In particular, when interpreting Euler's statement that "we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$", we should consider whether to use $+\sqrt{-2}$ or $-\sqrt{-2}$ for $\sqrt{-2}$, and similarly for the values of $\sqrt{-3}$ and $\sqrt{6}$. In this way, we can try to to make sense of the square root product law for negative numbers.

$\boxed{\textbf{Task 2}}$ If we replace each $\sqrt{x}$ by $+\sqrt{x}$ or $-\sqrt{x}$ in $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{6}$, explain why there are eight possible ways to interpret this equality. Write out all eight ways.

We also note that mathematicians agree that

$$+\sqrt{-a} = \left(+\sqrt{a}\right)\left(+\sqrt{-1}\right) \text{ and } -\sqrt{-a} = \left(+\sqrt{a}\right)\left(-\sqrt{-1}\right) \text{ and } b\left(-\sqrt{-1}\right) = (-b)\left(+\sqrt{-1}\right) \tag{3}$$

are valid rules for each positive a and real number b.

$\boxed{\textbf{Task 3}}$ Carry out this task **<u>without</u>** using the square root product law for negative numbers. You will need to use the rules in (3) as well as the square root product law for positive numbers (1).

(a) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{-2}, \sqrt{-3}, \sqrt{6}\right\}$ for which $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{6}$ is valid.

(b) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{-2}, \sqrt{-3}, \sqrt{6}\right\}$ for which $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{6}$ is **not** valid.

$\boxed{\textbf{Task 4}}$ Explain in your own words why the square root product law $\sqrt{-a} \cdot \sqrt{-b} = \sqrt{ab}$ is valid if we interpret it properly with Euler's multiple root meaning of square roots.

$\boxed{\textbf{Task 5}}$ Now consider the statement $\sqrt{5} \cdot \sqrt{-7} = \sqrt{-35}$ with Euler's multiple root meaning of square roots. Carry out this task **<u>without</u>** using the square root product law for negative numbers. You will need to use the rules in (3) as well as the square root product law for positive numbers (1).

(a) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{5}, \sqrt{-7}, \sqrt{-35}\right\}$ for which $\sqrt{5} \cdot \sqrt{-7} = \sqrt{-35}$ is valid.

(b) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{5}, \sqrt{-7}, \sqrt{-35}\right\}$ for which $\sqrt{5} \cdot \sqrt{-7} = \sqrt{-35}$ is **not** valid.

Now recall from your precalculus course that for the *function* $f(x) = \sqrt{x}$ we assign only the positive square root to each positive number x. This is often called the *principal* square root of x. If we allow only *one* value for the square root symbol $\sqrt{x}$ for each number x, then the square root product law (1) is not always valid, as we have seen in Tasks 1 and 3 above.

Needless to say, these different interpretations of the square root can become confusing, especially if we want to view $\sqrt{x}$ as a singe-valued function of x. It is important to consider the context when using these symbols and terms. When we want to consider all roots of a value x, we can say we want all *numerical* values of $\sqrt{x}$ and we can also use the common term *multivalued function* for

$f(x) = \sqrt{x}$ when we want to consider *all* square roots for each value of x. This idea of a multivalued function is important and will arise again in a course on complex variables[3]. An amazing example came from Euler himself, who showed that the natural logarithm $\ln(-1)$ has infinitely many values, all of which are imaginary numbers! See [Klyve, 2018] for more on the mathematical history and controversy over logarithms of negative numbers.

Task 6 This task illustrates these new terms.

(a) Find both numerical values of $\sqrt{-25}$ and of $\left(-\frac{1}{64}\right)^{1/2}$ in terms of $\sqrt{-1}$.

(b) Define the multivalued function $f(z) = \sqrt{z}$ for any real number z. Find all values of $f(-36)$ and $f\left(\frac{1}{49}\right)$.

3 The Square Root Division Law

Euler argued in Section 149 of his *Elements of Algebra* that there is a division square root law similar to the product square root law.

It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{a/b}$, it is evident that $\sqrt{-4}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$ or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided by $\sqrt{-1}$ gives $\sqrt{\frac{+1}{-1}}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$.

From our work, with the product square root law, it is reasonable to view Euler's division square root law with caution. If we use Euler's multiple root meaning of square roots, there are eight possible ways to interpret $\sqrt{a}/\sqrt{b}=\sqrt{a/b}$ (*why?*).

Task 7 Show that

$$\frac{1}{+\sqrt{-1}} = -\sqrt{-1} \tag{4}$$

Task 8 Carry out this task **without** using the square root division law. You may use the rules in (3) and (4).

(a) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{3}, \sqrt{-3}, \sqrt{-1}\right\}$ for which $\sqrt{3}/\sqrt{-3} = \sqrt{-1}$ is valid.

(b) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{3}, \sqrt{-3}, \sqrt{-1}\right\}$ for which $\sqrt{3}/\sqrt{-3} = \sqrt{-1}$ is **not** valid.

[3]Strictly speaking, a multivalued function is a type of mathematical *relation*, not a true function. However, this mild abuse of terminology is common and useful in the study of complex variables.

(c) A typical modern student outside this class would likely interpret the statement $\dfrac{\sqrt{3}}{\sqrt{-3}} = \sqrt{-1}$ as we would interpret $\dfrac{+\sqrt{3}}{+\sqrt{-3}} = +\sqrt{-1}$. According to your answers to parts (a) and (b), is this a valid statement?

Task 9 Carry out this task **without** using the square root division law. You may use the rules in (3) and (4) as well as the square root product law for positive numbers (1).

(a) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{-10}, \sqrt{-5}, \sqrt{2}\right\}$ for which $\dfrac{\sqrt{-10}}{\sqrt{-5}} = \sqrt{2}$ is valid.

(b) Using $+, -$ signs before $\sqrt{\ }$ signs for clarity, find four different sets of values for $\left\{\sqrt{-10}, \sqrt{-5}, \sqrt{2}\right\}$ for which $\dfrac{\sqrt{-10}}{\sqrt{-5}} = \sqrt{2}$ is **not** valid.

In modern terminology, we use the symbols i and $-i$ to denote the two imaginary square roots of -1: $i^2 = -1$ and $(-i)^2 = -1$.

Can we create a single-valued square root function for which the division square root law is always valid? Let's try! For real x, we will define a single-valued principle square root function

$$SQ\left(x\right) = \begin{cases} +\sqrt{x} & \text{if } x \geq 0 \\ +\sqrt{|x|}i & \text{if } x < 0 \end{cases} . \tag{5}$$

In this context, $\sqrt{4} = SQ\left(4\right) = 2$ and $SQ\left(-4\right) = 2i$.

Task 10 Using the single-valued principal square root function definition $SQ\left(x\right)$ from (5), show that $\frac{SQ(3)}{SQ(-3)} = SQ(-1)$ leads to the nonsensical statement $-1 = 1$. What can you conclude from this?

Task 11 Can you find a single-valued square root function that agrees with the ordinary square root of a for $a \geq 0$ and for which the division square root law is always valid with any real numbers a, b?

4 Conclusion

We have introduced the idea of a multivalued function in this project to help make sense of square root laws when imaginary numbers come into play. Square root properties are not the only algebra rules for complex numbers that need careful consideration. Several mathematicians of the eighteenth century became embroiled in an argument over how to define logarithms of negative numbers. Euler eventually put the controversy to rest when he showed in 1747 that the natural logarithm $\ln\left(-1\right)$ has infinitely many values, all of which are imaginary: $\ln\left(-1\right) = i\pi\left(1 + 2n\right)$ for all integers n, an amazing fact! This suggests that we will need to define $\ln z$ as a multivalued function, and will have to be careful when interpreting the logarithm property $\ln(A/B) = \ln(A) - \ln(B)$ when A or B is negative.

In fact, Euler's contemporary d'Alembert tried to convince Euler that $\ln(-1) = 0$ before he read Euler's proof that $\ln(-1) = i\pi(1+2n)$ [Klyve, 2018]. Here is d'Alembert's argument[4]:

> All difficulties reduce, it seems to me, to knowing the value of $\ln(-1)$. Now why may we not prove it by the following reasoning? $-1 = 1/-1$, so $\ln(-1) = \ln(1) - \ln(-1)$. Thus $2\ln(-1) = \ln(1) = 0$. Thus $\ln(-1) = 0$.

Task 12 In your own words, explain the flaw in d'Alembert's argument.

Task 13 Based on Euler's fact $\ln(-1) = i\pi(1+2n)$, explain how to properly interpret the logarithm property $\ln(A/B) = \ln(A) - \ln(B)$ when $A = 1, B = -1$.

For a modern definition of a single-valued function f, which allows for only *one* value of $f(z)$, we have defined the function $\sqrt{z}$ or principal square root of z when z is real, in (5) above. But this is not the end of the story! We must decide how to define a single-valued square root function $\sqrt{z}$ for non-real complex numbers such as $3-4i$. In addition, we want to consider logarithms, cube roots and other rational exponents for complex numbers. Moreover, in a course on complex variables we also want to consider derivative and integrals of our complex functions, so we have to investigate where the single-valued functions are differentiable and integrable. These are all topics for further study.

Task 14 Since each negative real number n has two square roots, it seems plausible that a negative number n should have three cube roots c (values for which $c^3 = n$). For example, one cube root of -8 is clearly -2.

(a) Use algebra to verify that $1+i\sqrt{3}$ is a cube root of -8, where $\sqrt{3}$ is the positive (principal) square root of 3.

(b) Use Euler's ideas on square roots to find a third cube root of -8, and verify your claim.

[4] For clarity, we have written ln where d'Alembert actually wrote log.

References

L. Euler. *Elements of Algebra*. 1770. Also published in *Opera Omnia*, translated from the French edition into English by J. Hewlett, Longman/Hurst/Rees/Orme, London, 1822; available as E387, Section 1, at `maa.eulerarchive.org`.

D. Klyve. The logarithm of -1. 2018. Available at TRIUMPHS site `digitalcommons.ursinus.edu/triumphs`.

A. Martinez. Euler's "mistake"? the radical product rule in historical perspective. *MAA Monthly*, 114:273–285, 2007.

Notes to Instructors

This mini-PSP is designed to be used in a course on complex variables or history of mathematics.

PSP Content: Topics and Goals

1. Interpret Euler's square root product and division laws for negative numbers.

2. Introduce the idea of a multi-valued function.

Student Prerequisites

The mini-PSP is written with very few assumptions about student background beyond the basic sophistication developed in an introductory calculus course. For a complex variables course, the PSP is designed to be used very early in the course, ideally at the first or second class meeting.

PSP Design, and Task Commentary

This is roughly a one or two period mini-project. The conclusion section tasks are designed to extend the idea of multivalued functions beyond the square root, and can be omitted.

Suggestions for Classroom Implementation

Advanced reading through the first Euler passage and tackling Task 1 is ideal but not necessary.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50 minute class period)

Students read through the first Euler passage and try Task 1 before class. Students work through Task 8 in class. Tasks 5, 9 - 14 can be done in class or given as homework as time permits. Tasks 12 and 13 on the logarithm property may create some confusion and merit follow-up discussion.

Additional Historical Notes

The Martinez article given in the references is quite interesting and valuable. It provides a very scholarly and rich treatment of the "Euler mistake" issues and controversy.

5 Acknowledgments

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