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Growing a Greater Understanding of Multiplication through Lesson Study: Mathematics Teacher Educators' Professional Development

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Abstract: *Research cites the need for developing teachers' mathematical knowledge for teaching (MKT) as well as for developing mathematics teacher educators' (MTEs) mathematical knowledge for teaching teachers (MKTT). Using the framework of lesson study: formulating goals and researching, planning, implementing and observing, and reflecting (Lewis & Hurd, 2011), a group of MTEs designed and analyzed a lesson on multiplication for prospective elementary teachers. A qualitative analysis of MTE journal reflections and prospective teacher work showed a greater understanding of MTEs' MKTT related to multiplication after completion of the lesson study. The authors recommend MTEs conduct lesson studies for other mathematics topics to further understand what MKTT MTEs need to develop to best support prospective teachers' MKT.*

Keywords: mathematics teacher educator, lesson study, elementary teacher education, mathematical knowledge for teaching teachers.

Introduction

Mathematical knowledge for teaching (MKT) is comprised of different types of knowledge used by mathematics teachers in the work of teaching mathematics to children (Ball & Bass, 2000). Teachers' MKT greatly influences what and how teachers teach, and how students learn mathematics (Hill, Rowan & Ball, 2005). MKT has been studied in great detail and much literature exists on ways to develop and assess prospective teachers'(PTs') MKT (e.g.

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Charalambous, 2010; Dick, 2017; Patterson, Parrott & Belnap, 2020, Rowland & Ruthven, 2011; Wilson, Sztajn, Edgington, & Confrey, 2013). In doctoral programs, mathematics teacher educators (MTEs) are exposed to the types of knowledge they must develop for their PTs. In contrast, the field of mathematics teacher education has only recently begun to research the types of mathematical knowledge needed for teaching teachers (MKTT) (Masinglia, Olanoff & Kimani, 2017; Superfine & Li, 2014; Zopf, 2010). These researchers have not fully defined what comprises MKTT, but agree that MKTT differs from MKT in that MTEs "begin with teachers' compressed mathematical knowledge and attempt to decompress it for the work of teaching children" (Zopf, 2010, p. 198). MTEs must therefore have knowledge about how PTs' mathematics learning influences and informs their future teaching practice.

In 2014, Superfine & Li stated, "the field of teacher education lacks an evidence-based understanding of the knowledge MTEs need to carry out their work" (p. 305). However, Masingila et al. (2017) recently discussed how MTEs engaged in a community of practice (Wenger, 1998) to develop their MKTT. In this paper, five second-year MTEs established a community of practice and used lesson study to develop their MKTT. For the lesson study, we developed and implemented a multiplication-focused lesson. Throughout this paper, we discuss the lesson study process and share MTEs' developing MKTT about teaching multiplication to PTs.

Lesson Study

Lesson study is practice-based professional development where teachers work together to research and plan a lesson, implement and observe the lesson, and afterwards reflect on children's learning (Lewis & Tsuchida, 1998; Lewis, Perry, & Murata, 2006; Hart, Alston, & Murata, 2011). The aim of lesson study is to strengthen collaboration among teachers, deepen teachers' content knowledge and knowledge of children's understanding, and encourage reflection. The lesson study process involves four phases: (1) formulate goals and conduct research, (2) plan, (3) implement and observe, and (4) reflect (Lewis & Hurd, 2011). Though lesson study began in Japan, it has been adopted internationally as a means of professional development for K-12 teachers. Within mathematics teacher education, lesson studies have been used to develop PTs' MKT (e.g. Alvine, Judson, Schien, & Yoshida, 2007; Amador & Weiland, 2015; Appova & Arbaugh, 2018). However, lesson study as a tool for developing MTEs' MKTT has not been as widely adopted. One exception is Cooper et al. (2011), who to developed a lesson to develop PTs' understanding of teaching multi-digit multiplication.

This paper differs from the Cooper et al. (2011) study in two ways: 1) when we began, we were in our second year of tenure-track positions at different U.S. universities, and 2) we were located in five U.S. states and used technology to complete the lesson study. Google Hangouts, Docs, and Sheets as well as Zoom were used to meet, engage in the lesson study process, and collaborate on a regular basis. Although not the traditional set up for lesson study, these technological modifications enabled us to research, plan, implement, and reflect together (Soto, Gupta, Dick, & Appelgate, 2019).

Background and Group Formation

Our research group formed at Service, Teaching and Research (STaR), a fellowship funded by the National Science Foundation and the U.S. Association of Mathematics Teacher Educators to support early-career MTEs. The group consisted of three MTEs housed in education departments and two in mathematics departments, yet all taught courses for elementary PTs focused on children's thinking about mathematics content. To assist PTs in analyzing children's work, we chose the professional noticing framework as conceptualized by Jacobs, Lamb and

Philipp (2010). The framework consists of three components: "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (Jacobs et al., 2010, p.169). Through previous work studying professional noticing, MTE Dick (2017) found the third component to be the most difficult for PTs and shared her interest in studying this phenomena during our meeting at STaR. Because much of the research on PT noticing has PTs consider work samples of individual children, our research group decided to plan and implement a lesson to develop PTs' skill of making whole class instructional decisions through analysis of multiple children's work samples. We had all experienced challenges with PTs' thinking about multiplication as simply memorizing multiplication facts, thus we chose a multiplication-based case-study with six individual children's work samples entitled, "The Case of Mr. Harris and the Band Concert" (NCTM, 2014; Appendix A) in which none of the children employed a memorization strategy. We hoped the children's work samples would encourage the PTs to think differently about multiplication.

The semester following STaR, three MTEs piloted the lesson. While PTs were successful at attending and interpreting individual children's multiplication strategies, their decisions for the next instructional steps for both individual children and the whole class followed similar patterns. These included PTs providing vague next step suggestions that gravitated towards traditional teaching, such as using times tables and flash cards, and desiring written equations (Gupta, Soto, Dick, Broderick, & Appelgate, 2018). Because we believed the lesson had room for improvements, we chose to complete a lesson study the following semester. This paper discusses the MKTT we developed regarding multiplication as a result of engaging in the lesson study.

The Lesson Study Process

Formulate Goals and Conduct Research

In this phase teachers work together to formulate lesson goals and conduct research on the teaching of the chosen topic. Our main lesson goal remained to support PTs to interpret samples of children's written multiplication work and make sound whole class next-step instructional decisions based on the children's mathematical thinking. For research, we considered literature on the concept of multiplication including multiplication conventions, problem types and solution strategies. While much of the literature discussed below was collected prior to the planning and implementation phase, there were times in later phases where we needed to return to the literature. As early-career MTEs we were not aware of all of the issues that would arise, but as our understanding of PTs' thinking and learning about multiplication grew, we used literature to further grow our MKTT to better assist PTs' developing MKT.

Multiplication Conventions and Problem Types. Multiplication has contextual-based meanings for the multiplicative situation under consideration. For U.S. children, the Common Core State Standards for Mathematics, CCSSM, (NGACBP, 2010) sets the convention that AxB is the total in A groups of B objects, thus A represents the multiplier and B the multiplicand. This interpretation serves as an early definition and structure for multiplication problem types involving equal-sized groups, measurement and multiplicative comparison. According to Carpenter, Fennema, Franke, Levi and Empson (2015) these problem types are asymmetric, meaning "the numbers in them are related to specific referents, and the referents are not interchangeable" because in context, B groups of A does not make sense (p. 67). For example, an equal groups problem for 5 bags of cookies with 7 cookies in each bag (five groups of seven), "It is not obvious to most young children that they could also solve the problem by making 7 groups with 5 objects in each group, or that they could count by fives" (p. 67). This example fits the A groups of B convention and would be solved as 5 x 7. This convention with A as the multiplier and B the multiplicand is just that, a cultural convention. In other countries, such as China and Taiwan, the convention is that the multiplier always appears as the second factor.

However, what if the cookies were on a baking sheet? It would not be obvious whether one should consider the cookies as arranged in 5 rows of 7 or 7 rows of 5. There is not an apparent choice for multiplier or multiplicand. This multiplication problem is considered symmetric because "the two factors do not have distinctly different roles and are not attached to a specific referent" (Carpenter et al., 2015, p. 68). Symmetric multiplication problem types include area and array problems, and combination problems. In the U.S., the CCSSM calls for third-grade children to work with asymmetric equal group problems as well as symmetric array problems. Children are expected to solve these problems "using drawings and equations with a symbol for the unknown number to represent the problem" (NGACBP, 2010, p.23). As children proceed through the grades, they are introduced to the additional multiplication problem types mentioned above.

Multiplication Solution Strategies. When children are first exposed to multiplication, they directly model the story situation, meaning they represent each of the quantities and follow the action in the problem. As their understanding develops, they move to more efficient strategies based on counting (skip counting and repeatedly adding) and eventually the use of derived facts (Carpenter et al., 2015). Derived fact strategies employ different mathematical concepts including place value, properties of operations, understandings of decomposition and compensation, and using multiplication facts children know to solve for facts they do not know, etc. Kling and Bay-Williams (2015) discuss the need for children to halve and double, and decompose factors as they work towards fluency and use of derived facts. They also discuss the importance of providing children with "frequent opportunities to explore, apply, and discuss multiplication strategies and properties" (p. 555). Wallace and Gurganus (2005) further emphasize the need for teachers to "make sure the understanding of the properties is firm" (p. 32). As children spend time developing flexibility using derived facts, they reach the mastery level which is best described as fluency.

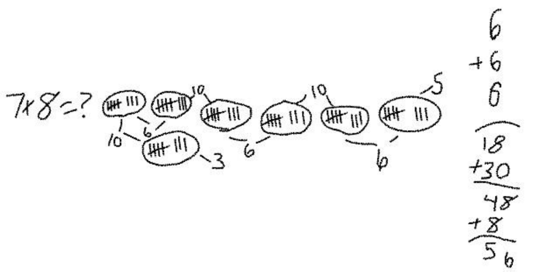
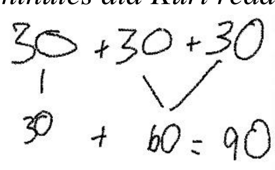
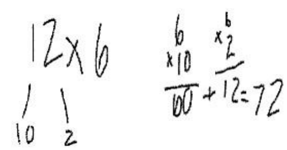
To illustrate these levels, Table 1 contains authentic classwork samples from Ivy, one of the author's daughters, when she was in third grade. These examples highlight the importance of understanding how children view and develop their knowledge of different story problem types. The literature is clear that children first need exposure to story problems and not memorization of multiplication facts. Fuson explains,

> ...seeing problem situations only after learning the mathematical operations keeps students from linking those operations with aspects of the problem situations. This isolation limits the meaningfulness of the operations and the ability of children to use the operations in a variety of situations. (2003, p. 300)

In sample 1, Ivy solved the symmetric area problem using a direct modeling strategy. Ivy drew out 7 groups of 8 with the 8 represented as a group of five tally marks and another three tally marks, directly representing each area unit. To count, she combined her groups of five tallies into 10s and then groups of three tallies into 6s and added them together with the ones that didn't fit neatly into 10s or 6s. In sample 2, Ivy solved the asymmetric grouping problem using a counting strategy. She viewed the problem as repeated addition of 3 groups of 30, she then added 30 + 30 = 60 and then added 30 + 60 = 90. She did not directly model the three groups of thirty, but instead abstractly represented 30 with numerals. In sample 3, Ivy solved the symmetric area problem using derived facts. She decomposed the 12 into 10 and 2 employing the distributive property and showed her understanding of the commutative property when she wrote 6 x 10 instead of 10 x 6. These examples show how children may move back and forth between different strategy levels as they are exposed to different problem types, number choices, and as they work towards developing fluency with multiplication facts.

Table 1

Solution Strategies for Different Multiplication Problems from Ivy's Schoolwork

Sample 1: Direct Modeling	Sample 2: Counting/Addition	Sample 3: Derived Fact
The toddler section of pool measures 7 feet by 8 feet. What is the area of the toddler section? 	*Kurt read from his book three times each day. He read for 30 minutes each time. For how many total minutes did Kurt read?* 	*The dimensions of the front cover of a textbook are 12 inches by 6 inches. What is the area of the front cover?* 

Lesson study's emphasis on formulating goals and delving into research forced us, as MTEs, to think deeply about how children learn multiplication and how PTs think and learn about teaching multiplication. Our discussions about how PTs perceive conventions for multiplication, multiplication interpretations within different story contexts, and children's multiplication solutions strategies developed our MKTT regarding the development of PTs' MKT related to multiplication.

Planning

Once the topic of multiplication was researched and possible materials that would engage PTs in the goal of the lesson were gathered, planning of the lesson began. Figure 1 provides an overview of the main activities that comprised the lesson.

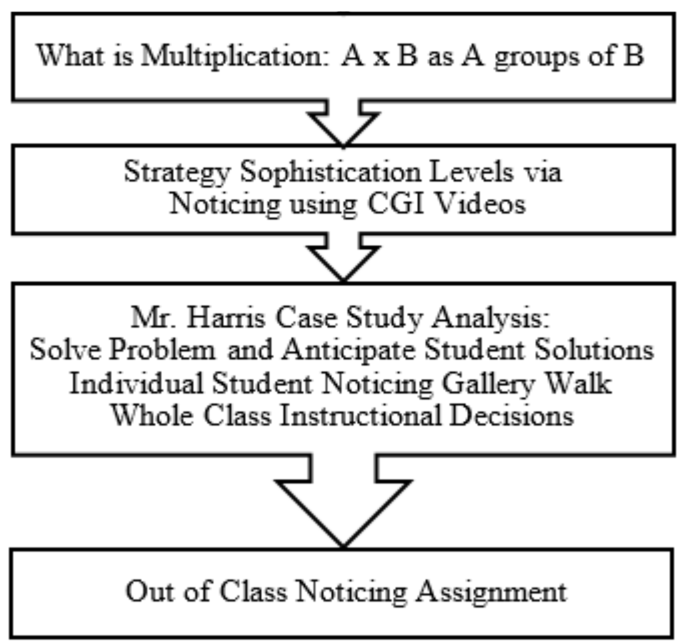


Figure 1. Lesson outline.

The lesson began with PTs considering the following questions: What is multiplication? How might you explain to a child what 3x5 means? After PTs shared thoughts in pairs and with the whole class, the class viewed videos (see Appendix B for problems posed and video information) of individual children solving multiplication story problems to introduce the Cognitive Guided Instruction (Carpenter et al., 2015) trajectory of children's solution strategies: direct modeling → counting strategies → derived facts. After watching each video, PTs professionally noticed the child's mathematical thinking. Specifically, the PTs were asked to 1) attend: discuss what the child did to solve the problem; 2) interpret: discuss the child's strategy type and what was learned about the child's thinking; and finally, 3) decide: discuss what they would do next if they were the child's teacher.

Once PTs watched the videos and professionally noticed for individual children, the "Mr. Harris and the Band Concert" case study (NCTM, 2014) was introduced by having PTs solve the case study problem using at least two different strategies (Appendix A). They read the initial portion of the case study highlighting Mr. Harris' goals for the lesson and how he launched the problem to his third grade children. In groups, PTs analyzed the six individual children's work samples from the case study (Appendix A). Each group professionally noticed one individual child's work. Specifically, the PTs were asked to 1) attend: what the child did to solve the problem,2) interpret: identify the child's strategy type and what is learned about the child's thinking or what is still unknown, and finally 3) decide: choose a question to ask to better understand the child's mathematical thinking and develop a next task to pose to the child. The PTs created a poster with their noticing analysis and then engaged in a gallery walk reading and adding to their peer's posters. Through the gallery walk, the PTs analyzed all six children's work samples individually. As a culminating in-class activity the PTs reviewed and discussed Mr. Harris' instructional goal and determined a new instructional goal and task for the next day based on how Mr. Harris' goal was met.

As a final assessment, PTs worked in small groups to analyze a different set of children's multiplication work samples (Appendix C). They noticed each individual child's work sample and then decided on a goal and task for the whole class based on the individual work samples.

These were scored with a rubric created to assess the PTs' professional noticing (Broderick, Gupta, Appelgate, Dick, & Soto, 2017).

Implementation and Observation

The implementation of our lesson began with MTE-1 and 2 teaching and videotaping the lesson. We then independently watched these two video-taped lessons and took notes on what was viewed. After this analysis, we met virtually using Google Hangout to debrief. This involved the instructors sharing their reflections on teaching the lesson, including challenges, successes and changes they recommended for next cycle, as well as the viewers sharing their perspectives on the implementation of the lesson. Based on the discussion, the lesson was revised for the next cycle (see Figure 2).

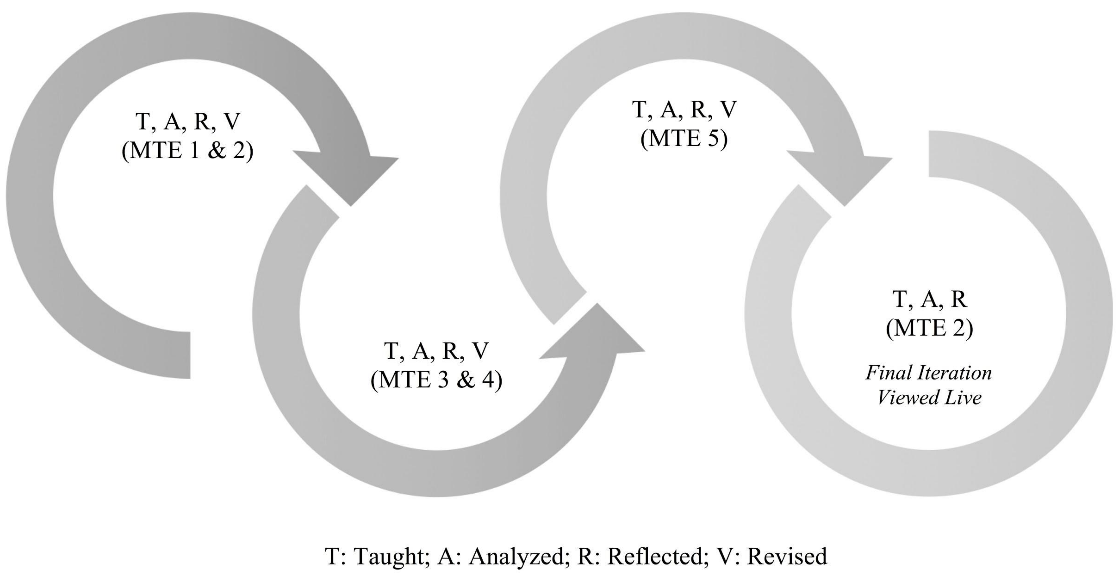


Figure 2. Diagram indicating lesson study cycles that occurred within six months.

The revised lesson was then taught by MTE-3 and 4. This process of teaching, analyzing, reflecting, and revising the lesson was completed in several cycles. The last cycle of implementation was observed in real time with MTE-2's summer class as the other MTEs observed the lesson via Google Hangouts (Soto et al., 2019). While these four cycles were completed within six months, four of the MTEs have continued teaching the lesson and meeting informally to discuss and share results.

Reflecting

During the lesson study, we kept online journals documenting our thoughts and experiences after each group meeting. As is standard in autoethnographic research (Ellis & Bochner, 2000), these journal entries did not follow any particular prompt, rather they were organic. We were free to write anything that came to mind about the lesson study process and agreed to share honest reflections. In addition, after subsequent semesters of teaching of the lesson, we each completed a cumulating written reflection focused specifically on what we learned about teaching PTs to teach multiplication throughout this process.

MTE MKTT about Multiplication

This manuscript shares results from the analysis of PTs' work and MTEs' online journals to identify changes in MTEs' MKTT with respect to teaching PTs about teaching multiplication. Analysis revealed three MKTT issues that emerged throughout the lesson study. Specifically, they showed the need to:

1. Support PTs to understand CCSSM's convention for multiplication as A groups of B.
2. Provide PTs with opportunities to make sense of nuances within and between the addition and multiplication trajectories of children's solution strategies.
3. Ensure PTs understand that the multiplication symbol and naked equations are not necessary as evidence of children's understanding of multiplication.

We share details about how the issues emerged and how they were addressed throughout the lesson study process. Issues 1 and 2 emerged while we were planning and helped in anticipating struggles PTs may encounter while engaging in the lesson. Issue 3 was not anticipated and emerged as we assessed the PTs at the conclusion of the lesson.

Understanding CCSSM's Convention for Multiplication

The first common issue that arose was the need for additional clarification on the CCSSM convention that multiplication, as represented by AxB, means A groups of B. In discussing the lesson plan, MTE-1 struggled to understand the value of the CCSSM convention and wondered how, if she saw it as arbitrary, she would explain it to her PTs.

> I wondered about the value of saying the first number is the [number of] groups and the second number is the number in each group. To me this seems too confining but when I asked MTE-3 about it, I completely understand her reasoning and will most likely share that with my [PTs]. (MTE-1 journal)

MTE-3 shared that the convention of A groups of B provides PTs with common language and a structure to use with story problems. This understanding of the convention of AxB helped MTE-1 come to terms with the perceived rigidity. As anticipated during planning, this issue arose in MTE-2's class,

> We discussed [the definition of multiplication as A groups of B] and some [PTs] were talking about it doesn't matter if you say it is 3 x 5 or 5 x 3- –however some of them argued that it is the same thing, as the answer is the same. (MTE-2 journal)

The PTs were correct. The communicative property does hold and the answer would be the same. However, beyond saying this was the CCSSM's chosen convention, in cycle one of teaching the lesson, we struggled with how to engage the PTs in thinking about the importance of having a convention. When the PTs pushed us with questions as to why it really mattered, we were unsure how to explain the value of convention.

To better respond, we delved further into multiplication research. Using language from Carpenter et al. (2015), in future cycles of teaching the lesson, MTEs named and discussed the difference between symmetric and asymmetric multiplication problem types. In gaining this knowledge, the next MTEs to teach the lesson shared carefully chosen examples of symmetric and asymmetric multiplication story problems. This language and these examples helped the PTs accept the convention.

Navigating the Trajectories of Children's Solution Strategies

Valuing children's thinking and using it to guide instruction was a vital piece of our pedagogical approach. We thus began our investigation into multiplication with the strategy trajectory (Carpenter et al., 2015). Discussing this trajectory using the third grade children's work from the Mr. Harris Band Concert Task (NCTM, 2014; Appendix A) raised important issues in our understanding of the nuances of children's strategies and the meaning of the trajectory levels. After a group discussion while planning the lesson, MTE-2 reflected,

> The talk about solution strategies made me think more and grow as a teacher. It was an awesome discussion... talking about whether skip counting or repeated addition is more sophisticated... the discussion really helped me better understand the student [children's] work and also understand what I want my PTs to know... The fact that we can represent Jasmine work as 20(5+2) was wow! What was interesting was that MTE-1 had a different point of view and hence the discussion generated clearer understandings about the [children's] work and [made me] think deeply... it also left me with the question – what do I want my [PT]s to get to? (MTE-2 journal; see Appendix A)

MTE-2 reflected on her growing MKTT regarding how to support PTs' understanding of how children's strategies represented by Mr. Harris' children's solutions, fit into the trajectory of solution strategies. Believing that PTs would benefit from engaging in these types of discussions, we incorporated time in the lesson for PTs to ponder where and why each of Mr. Harris' children fell on the strategy trajectory.

During each cycle of teaching, the issue of differentiating between trajectory levels and appreciating the nuances of children's solution strategies continued to arise. Specifically, we found PTs often viewed children's multiplication strategies that used repeated addition/counting strategies as not "understanding" multiplication. MTE-3, anticipating this issue based on previous cycles, grounded the discussion with PTs in how her daughter solved a real problem.

> When I originally introduced multiplication, I shared a story from my first grader that week. [The problem was] 3 cupcakes on 4 tables. Her answer was that she knew 3 + 3 = 6 and 6 + 4 = 10, so 6 + 3 is one less = 9. Then 9 + 1 = 10 and 10 + 2 = 12. [The PTs] then discussed and decided that she had used a derived fact addition strategy which correlated into a counting strategy as her multiplication level. Focusing on the two different levels (addition and multiplication) was SO key for my [PTs] as they moved into Mr. Harris. (MTE-3 journal)

With this example, MTE-3 led the PTs to deeply examine trajectory nuances, in particular, how the multiplication trajectory related to the addition trajectory, which the PTs had discussed previously. Comparing the solution strategy as it might be interpreted as an addition problem with how it might be interpreted as a multiplication problem helped PTs see interconnections between trajectory levels for addition and multiplication. Focusing on the differences and similarities helped clarify questions PTs had about why children may add when they are solving a multiplication problem. It also raised the important point that for a solution strategy to be categorized as derived facts in the multiplication solution strategy trajectory, it must use known multiplication facts as building blocks to solve the unknown product, which provides evidence of number sense.

Another issue that arose was PTs' thinking that derived facts are "too complicated." For example, after watching video #4 (see Appendix B) in which the child solved 7 x 8 by decomposing the 8 into 3 and 5 and multiplying those by 7, thereby using the distributive property, MTE-4 shared in her post-lesson reflection her PTs' conceptions of the child's solution strategy,

> I had [PTs] that thought that [the child] made things too complicated and even though she got the right answer, it was too many steps. Some were unable to recognize the sophistication in her strategy and their focus was on efficiency and they felt she was not efficient, even though she was thorough. (MTE-4 journal)

The crux of this issue was to determine how to support PTs to see the sophistication of the child's strategy as evidence of number sense. MTE-4 pressed her PTs to describe what the child knew and understood in order for them to attend to the child's rich understanding and use of number relationships and properties of multiplication. The PTs continued to focus on the child's inefficiency until a classmate shared that it was evidence of the distributive property and probably, no matter what number choices the child was given, she would most likely be able to solve any problem. This exchange influenced MTE-4's big take away from her instruction during this cycle.

> ...[PTs] are not realizing all the work that it takes to get to those number facts and how [children] go about building on. Also, I am not sure [PTs] are aware of the power of these properties of operations. I need to do better in highlighting these properties because I think we take them for granted...It wasn't until I really started diving into [children's] work samples and seeing all the interesting ways that children solve problems, that I realized just how important these properties are. (MTE-4 journal)

After reflecting together, we realized the issue is communicating to PTs the importance of number sense. When children display evidence of number sense, they know *why* numbers can be manipulated (decomposed and composed), can generalize, and solve problems with larger quantities. Children with number sense are more flexible with numbers, willing to take risks and be creative when solving problems, and reason about quantities rather than just compute.

We found PTs may not realize or value the strong number sense that is involved when a child solves problems using derived facts. We also found that the PTs often would skip directly to recall rather than seeing the use of derived facts as an important developmental step. Overall, spending time focused on children's solution strategies along different trajectory levels was immensely important for PTs' understanding of how children think about and solve multiplication problems, and also for supporting them to understand the value of using number sense to solve problems.

Symbol and Naked Equations as Evidence of Multiplying

After teaching the lesson, the professional noticing assessment (Appendix C) highlighted the PTs' desire to see written equations with the multiplication symbol as evidence of children's understanding of multiplication. For example, a PT said, "writing an equation would also help me understand how much she really knows about multiplication" (MTE-3's PT work), but did not include any details as to how exactly this information would help. PTs at another university stated that they would "...ask [the child] to multiply completely without grouping. For example, we would have him multiply 2x12 without doing 12+12" (MTE-2's PT work). Another group stated that,

> Students should know times tables and know multiplication... It is important to use multiplication every day because it gets students in the habit of using it, so having a times tables quiz at the end of each day to refresh their memory would help the students improve. (MTE-2's PT work)

These examples highlight PTs' focus on the multiplication symbol and recall as evidence of knowing multiplication rather than developing understanding and flexibility to solve multiplication problems in different ways. When PTs saw a written equation with the multiplication symbol they often concluded the child "understands multiplication." For example, a PT interpreted Olivia's mathematical thinking as, "Olivia understood immediately that this was going to be a multiplication problem by writing the two numbers and showing their relationship to the multiplication symbol" (MTE-4's PT work; Appendix C). PTs often suggested asking children to write a multiplication equation for the problem.

Asking children to connect different representations (i.e. direct model and equation) is beneficial (NCTM, 2014) and is not problematic if the child is ready for abstract representation.

However, PTs' suggestions for writing an equation was often without consideration for the children's understanding of number relationships. As we have improved in our MKTT in this area, it has become easier to push back on this by returning to the definition of multiplication as is evidenced in MTE-4's journal response:

> I did have one [PT] ask if they are skip counting, does that mean that they are multiplying. I posed the question to the class and at first it was a split between yes and no. Then PT 2 did connect to the videos and say, yes, they were multiplying because they were thinking of groups (MTE-4 Journal).

Similarly in her culminating reflection MTE-1 noted,

> [PTs] want to use naked equations ALL the TIME! In my 9am class when someone [shared] naked equations for the video's next-steps discussion I tried to ask questions and I feel like I did a good job using her thinking and that of others to push back on that idea fruitfully. However, now that we [MTEs] can ANTICIPATE what [PTs] will say, what are our questions to push them on this? And how do we [best] respond? (MTE-1 journal)

MTE-1 felt knowing what PTs might say better prepared her for questioning her PTs. However, in each case, MTE-1 and MTE-4 anticipated responses and used questioning to build on the contributions of the PTs in order to support and value the idea of using context without requiring naked equations when children are developing an understanding of multiplication.

Recommendations for MTEs

Throughout the lesson study cycle, we better engaged PTs with the concept of multiplication and developed our MKTT regarding teaching PTs multiplication. From this work, we conclude with two recommendations for MTEs regarding continuing professional development of MKTT.

Recommendation One: Participate in a Lesson Study with other MTE Colleagues as Means of Developing MKTT.

The process of lesson study can serve as a means of professional development for MTEs, particularly when engaging with colleagues to form a community of practice (Wenger, 1998).

Lesson study among university faculty is becoming more prevalent as a means of connecting with and learning from others. Whether within your own university (Druken & Marzocchi, 2017) or using technology to connect with other MTEs across distances (Cooper et al., 2011; Soto et al., 2019), lesson study has the potential to support university faculty in their professional growth, overcoming isolation to develop MKTT. Working with colleagues at other universities from different backgrounds and experience, allowed us to learn new pedagogies and think more deeply about how PTs develop MKT. In any lesson study group, regardless of members' backgrounds, each participant is put into a vulnerable position of opening their classroom. Yet, through the process, we identified areas that could be improved individually and also shared teaching struggles. Through the lesson cycles, we developed our knowledge of PTs' thinking and learning about how to teach multiplication.

For someone new to lesson study, an invaluable resource is the Lewis and Hurd (2011) book that includes a step-by-step guide on how to engage in lesson study. To locate lesson study collaborators, we suggest connecting with special interest groups (SIG) at national conferences such as the U.S. American Educational Research Association, which has a Lesson Study SIG, and conducting working groups at conferences to connect with individuals with similar teaching interests. If new collaborators are located afar, ensuring that everyone has access to video conferencing and collaborative online tools to access documents and track changes in real time, is key. See Soto et al. (2019) for a detailed discussion on the use of technology to support professional learning in a lesson study across geographical distances. To find local collaborators, consider your university's teaching and learning resource center to support university-wide lesson study groups or discuss these possibilities within your own department.

After forming a lesson study group, mathematics education research should be viewed, and reviewed often, as a foundational resource when planning elementary mathematics content lessons as it provides not only research into the mathematical concept, but also background on children's mathematical thinking and potential pedagogical approaches for engaging PTs. It is through reviewing this literature in concert with teaching the lesson to PTs that MKTT is developed.

Recommendation Two: Ground Mathematical Concepts in Children's Mathematical Thinking with the Goal of Promoting the Value of Number Sense.

Many PTs learned elementary mathematics content in a traditional manner and enter mathematics teacher content courses believing they have adequate mastery of the content to teach it (Thanheiser, 2018). Browning et al. (2016) posited that exposing PTs to children's solutions "may provide them with a sense of urgency for them to truly understand the mathematics they will teach" (p. 47) and Castro Superfine, Prasad, Welder, Olanoff, and Eubanks-Turner (2020) found it is not until PTs observe children thinking differently about mathematics that they realize the need to grapple deeply with previously-learned mathematical ideas. PTs can experience children thinking differently and develop greater understanding of the mathematics through direction interaction with young people (e.g. interviews) or indirectly (e.g. looking at student work) (Appova &Taylor, 2020). Philipp (2008) concurs and further explain it is through observing children solving problems and analyzing children's work samples that PTs begin to "look at mathematics through the lens of children's mathematical thinking... [and thus] come to care about mathematics, not as mathematicians, but as teachers" (p. 10). Thus, MTEs should design learning experiences for PTs that expose them to opportunities to engage with children's work samples.

During our lesson study, we found children's work samples served as basis for motivating and developing PTs MKT regarding multiplication and the importance of building on childrens' number sense (Jones et al., 1998). As we interacted with our PTs as they engaged with children's work samples, we developed our MKTT. For example, videos of children solving multiplication problems using derived facts brought to our attention how PTs struggled to recognize or value the depth of knowledge needed for children to use derived facts and become flexible mathematical thinkers. As MTEs who had developed MKTT through the lesson study, we were able to press PTs, guide them to return to children's thinking, and focus on the value of children's use of number sense to build understanding of multiplication.

The following are suggestions to ground mathematical concepts in children's mathematical thinking and number sense into one's course. First, ensure there is space in the course to include children's work samples, either through videos, written work samples, or providing opportunities for PTs to interview children. Second, when using children's work, it is important to press on the mathematical ideas within children's strategy development, particularly derived facts. It is important within children's work to provide a range of solution strategies (both correct and incorrect) across grade levels and children with varying backgrounds (e.g. race, ethnicity, socioeconomic status, geographic areas) so PTs are exposed to the full spectrum of children's thinking and understand that all children are capable of learning and doing mathematics. See Appendix B of Max and Welder (2020) for a list of sources of children's mathematical thinking that could be used with PTs.

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Appendix A: Mr. Harris and the Band Concert (NCTM, 2014)

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Exploring Representations for Multiplication
The Case of Mr. Harris and the Band Concert Task

Mr. Harris wanted his third-grade students to understand the structure of multiplication and decided to develop a task that would allow students to explore multiplication as equal groups through a familiar context—the upcoming spring band concert. He thought that the Band Concert Task (shown below) would prompt students to make or draw arrays and provide an opportunity to build conceptual understanding toward fluency in multiplying one-digit whole numbers by multiples of 10 using strategies based on place value and properties of operations—all key aspects of the standards for third grade students. He felt that the task aligned well with his math goals for the lesson and supported progress along math learning progressions, had multiple entry points, would provide opportunities for mathematical discourse, and it would challenge his students. As students worked on the task he would be looking for evidence that his students could identify the number of equal groups and the size of each group within visual or physical representations, such as collections or arrays, and connect these representations to multiplication equations.

> The third-grade class is responsible for setting up the chairs for the spring band concert. In preparation, the class needs to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area?

Mr. Harris began the lesson by asking students to consider how they might represent the problem. "Before you begin working on the task, think about a representation you might want to use and why, and then turn and share your ideas with a partner." The class held a short conversation sharing their suggestions, such as using cubes or drawing a picture. Then the students began working individually on the task.

As Mr. Harris made his way around the classroom, he noticed many students drawing pictures. Some students struggled to organize the information, particularly those who tried to represent each individual chair. He prompted these students to pause and review their work by asking, "So, tell me about your picture. How does it show the setup of the chairs for the band concert?" Other students used symbolic approaches, such as repeated addition or partial products, and a few students chose to use cubes or grid paper. He made note of the various approaches so he could decide which students he wanted to present their work, and in which order, later during the whole class discussion.

In planning for the lesson, Mr. Harris prepared key questions that he could use to press students to consider critical features of their representations related to the structure of multiplication. As the students worked, he often asked: "How does your drawing show the seven rows?" "How does your drawing show that there are 20 chairs in each row?" "Why are you adding all those twenties?" "How many twenties are you adding and why?"

He also noticed a few students changed representations as they worked. Dominic started to draw tally marks, but switched to using a table. When Mr. Harris asked her why, she explained she got tired of making all those marks. Similarly, Jamal started to build an array with cubes, but then switched to drawing an array. Their initial attempts were valuable, if not essential, in helping each of these students make sense of the situation.

Before holding a whole class discussion, Mr. Harris asked the students to find a classmate who had used a different representation and directed them to take turns explaining and comparing their work, as well as their solutions. He encouraged them to also consider how their representations were similar and different. For example, Jasmine who had drawn a diagram compared her work with Kenneth who had used equations (see reverse for copies of their work). Jasmine noted that they had gotten the same answer and Kenneth said they both had the number 20 written down seven times. Molly, in particular, was a student who benefited from this sharing process because she was able to acknowledge how confused she had gotten in drawing all those squares (see reverse side) and had lost track of her counting. Her partner helped her mark off the chairs in each row in groups of ten and recount them. The teacher repeated this process once more as students found another classmate and held another sharing and comparing session.

Written by DeAnn Huinker (University of Wisconsin-Milwaukee), drawing on experiences with teachers and students in the Milwaukee area. This case is intended to support the Guiding Principle for Teaching and Learning in *Principles to Actions: Ensuring Mathematical Success for All* (Reston, Va.: National Council of Teachers of Mathematics 2014).

During the whole class discussion, Mr. Harris asked the presenting students to explain what they had done and why and to answer questions posed by their peers. He asked Jasmine to present first since her diagram accurately modeled the situation and it would likely be accessible to all students. Kenneth went next as his approach was similar to Jasmine's but without the diagram. Both clearly showed the number 20 written seven times. Then Teresa presented. Her approach allowed the class to discuss how skip counting by twenties was related to the task and to multiplication, a connection not apparent for many students. Below is an excerpt from this discussion.

Mr. H: So, Teresa skip counted by twenties. How does this relate to the Band Concert situation?

Connor: She counted seven times like she wrote on her paper.

Mr. H: I'm not sure I understand, can someone add on to what Connor was saying?

Grace: Well each time she counted it was like adding 20 more chairs, just like what Kenneth did.

Mr. H: Do others agree with what Grace is saying? Can someone explain it in their own words?

Mason: Yeah, the numbers on top are like the 7 rows and the numbers on the bottom are the total number of chairs for that many rows.

Mr. H: This is interesting. So what does the number 100 mean under the 5?

Mason: It means that altogether five rows have 100 total chairs, since there are 20 chairs in each row.

Mr. H: Then what does the 140 mean?

Mason: It means that seven rows would have a total of 140 chairs.

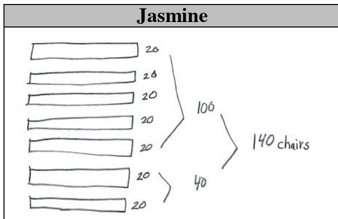
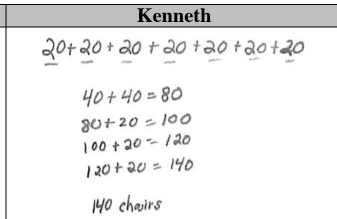
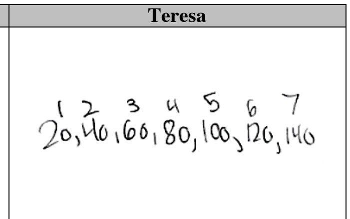
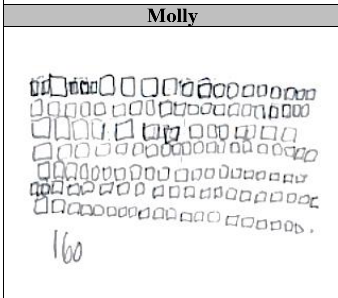
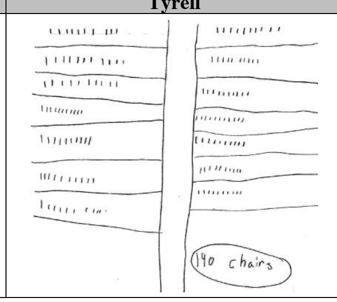
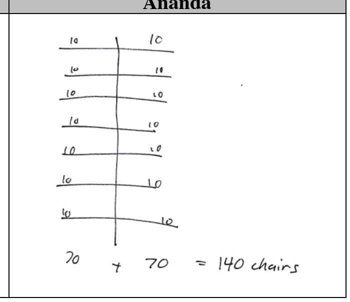
[Mr. Harris paused to write this equation on the board: 7 x 20 = 140.]

Mr. H: Some of you wrote this equation on your papers. How does this equation relate to each of the strategies that we have discussed so far? Turn and talk to a partner about this equation.

[After a few minutes, the whole class discussion continued and Grace shared what she talked about with her partner.]

Grace: Well, we talked about how the 7 means seven rows like Jasmine showed in her picture and how Teresa showed. And the 20 is the number of chairs that go in each row like Jasmine showed, and like how Kenneth wrote down. Teresa didn't write down all those twenties but we know she counted by twenty.

Toward the end of the lesson, Mr. Harris had Tyrell and Ananda present their representations because they considered the aisle and worked with tens rather than with twenties. After giving the students a chance to turn and talk with a partner, he asked them to respond in writing whether it was okay to represent and solve the task using either of these approaches and to justify their answers. He knew this informal experience with the distributive property would be important in subsequent lessons and the student writing would provide him with some insight into whether or not his students understood that quantities could be decomposed as a strategy in solving multiplication problems.

Jasmine	Kenneth	Teresa
		
Molly	**Tyrell**	**Ananda**
		

Appendix B: Story Problems and Solution Strategies of the videos shown to PTs (Carpenter et al., 2015).

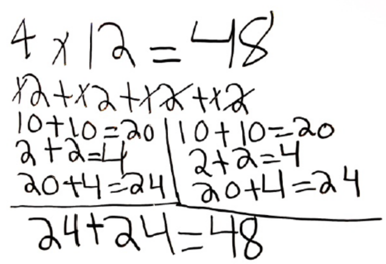
Problem	Solution Strategies
Grandma has 3 plates with 6 cookies on each plate. How many cookies does Grandma have? (Video 4.1)	Child used a Direct Modeling strategy and used three unifix cubes to represent each group and then placed six unifix cubes under each of the three initial cubes to represent the number of cookies in each group. Child then counted all of the cubes that represented the cookies by ones.
Grandma has 4 plates with 3 cookies on each plate. How many cookies does Grandma have? (Video 4.5)	Child used a Counting strategy and drew four circles and wrote the numeral 3 in each circle. She then said she knew that 3 + 3 = 6, plus 3 more was 9, and 3 more was 12. She then skip counted by threes, "3, 6, 9, 12."
Number Fact: 4 × 5 (Video 4.6)	Child used a Derived Fact strategy and said that she knew 5 x 5 = 25 and take away one group of five to get 20.
The teacher has 7 boxes of crayons. Each box has 8 crayons in it. How many crayons does the teacher have in all? (Video 4.8)	Child used a Derived Fact strategy and decomposed 8 into 5 and 3. She then wrote 7 x 5=35 and 7 x 3 = 21. She then added these partial products by place value (20 + 30 = 50 and 5 + 1 = 6) to arrive at 56 for her answer.

Appendix C: Problem and First Piece of Student Work

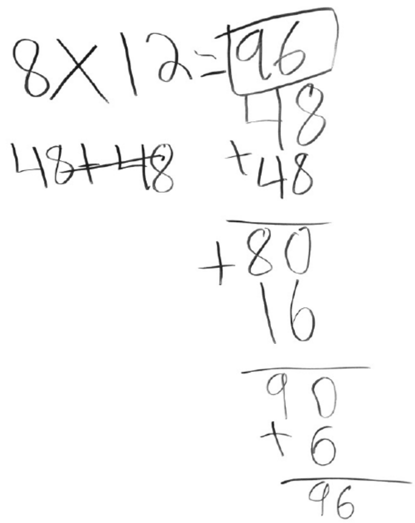
Name: Olivia

Sam had ____ fish bowls. He had ____ fish in each bowl.
How many fish did Sam have?

(4, 12)



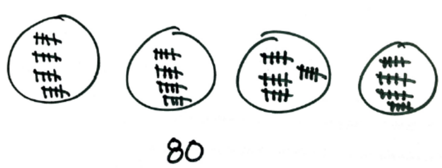
(8, 12)



Name: Kyla

Sam had ____ fish bowls. He had ____ fish in each bowl.
How many fish did Sam have?

(4, 20)



(8, 20)