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Melissa S. Keranen
Michigan Technological University, msjukuri@mtu.edu

Donald L. Kreher
Michigan Technological University, kreher@mtu.edu

Salvatore Milici
Universit`a di Catania

Antoinette Tripodi
Universit`a di Messina

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Melissa S. Keranen   Donald L. Kreher

Department of Mathematical Sciences
Michigan Technological University, Houghton
Michigan, 49931 U.S.A.
msjukuri@mtu.edu   kreher@mtu.edu

Salvatore Milici*

Dipartimento di Matematica e Informatica
Università di Catania, Catania
Italy
milici@dmi.unict.it

Antoinette Tripodi

Dipartimento di Scienze Matematiche e Informatiche
Scienze Fisiche e Scienze della Terra
Università di Messina, Messina
Italy
atripodi@unime.it

Abstract

If X is a connected graph, then an X-factor of a larger graph is a spanning subgraph in which all of its components are isomorphic to X. A *uniformly resolvable* $\{X, Y\}$*-decomposition* of the complete graph K_v is an edge decomposition of K_v into exactly r X-factors and s Y-factors. In this article we determine necessary and sufficient conditions for when the complete graph K_v has a uniformly resolvable decompositions into 1-factors and $K_{1,4}$-factors.

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1 Introduction and definitions

For any graph G, let $V(G)$ and $E(G)$ be the vertex-set and the edge-set of G, respectively. Throughout the paper K_v will denote the complete graph on v vertices, while $K_v \setminus K_h$ will denote the graph with $V(K_v)$ as vertex-set and $E(K_v) \setminus E(K_h)$ as edge-set (this graph is sometimes referred to as a complete graph of order v with a *hole* of size h).

Given a set $\mathcal{H}$ of pairwise non-isomorphic graphs, an *$\mathcal{H}$-decomposition* (or *$\mathcal{H}$-design*) of a graph G is a decomposition of the edge-set of G into subgraphs (called *blocks*) isomorphic to some element of $\mathcal{H}$. An *$\mathcal{H}$-factor* of G is a spanning subgraph of G whose components are isomorphic to a members of $\mathcal{H}$. If $X \in \mathcal{H}$, then an X-factor is a spanning subgraph whose components are isomorphic to X. An $\mathcal{H}$-decomposition of G is *resolvable* if its blocks can be partitioned into $\mathcal{H}$-factors and is called an *$\mathcal{H}$-factorization* of G. An $\mathcal{H}$-factorization $\mathcal{F}$ of G is called *uniform* if each factor of $\mathcal{F}$ is an X-factor for some graph $X \in \mathcal{H}$. A K_2-factorization of G is known as a 1-*factorization* and its factors are called 1-*factors*; it is well known that a 1-factorization of K_v exists if and only if v is even ([18]).

An $\mathcal{H}$-isofactorization of G is an $\mathcal{H}$-factorization with isomorphic factors. If $\mathcal{H}$ is the set of all possible cycles of K_v, then determining the existence of possible $\mathcal{H}$-isofactorizations of K_v, v odd is known as the *Oberwolfach Problem*. It was first posed in 1967 by Gerhard Ringel and asks whether it is possible to seat an odd number v of mathematicians at n round tables in $(v-1)/2$ meals so that each mathematician sits next to everyone else exactly once. If the n round tables are of sizes $p_1, p_2, \ldots, p_n$ (with $p_1 + p_2 + \cdots + p_n = v$), the Oberwolfach Problem asks for an isofactorization of K_v with factors isomorphic to the 2-factor with components isomorphic to cycles of length $p_1, p_2, \ldots, p_n$. The uniform Oberwolfach problem (all cycles of the 2-factor have the same size) has been completely solved by Alspach and Häggkvist [4] and Alspach, Schellenberg, Stinson and Wagner [5] .

Additional existence problems for $\mathcal{H}$-factorizations of K_v have been studied and many results have been obtained, especially on uniformly resolvable $\mathcal{H}$-decompositions: when $\mathcal{H}$ is a set of two complete graphs of order at most five in [8, 21, 22, 24]; when $\mathcal{H}$ is a set of two or three paths on two, three or four vertices in [11, 12, 17]; for $\mathcal{H} = \{P_3, K_3 + e\}$ in [10]; for $\mathcal{H} = \{K_3, K_{1,3}\}$ in [14]; for $\mathcal{H} = \{C_4, P_3\}$ in [19]; for $\mathcal{H} = \{K_3, P_3\}$ in [20]. And most famous is the variation of the Oberwolfach problem known as the Hamilton-Waterloo problem. In this problem the meals for the dinning mathematicians take place at two different venues. Hence a decomposition of K_v is sought where the factors can be either one of two types. In particular the uniform case asks for a decomposition of K_v into C_p-factors and C_q-factors. Thus the round tables in one venue sit p mathematicians whereas the tables in the other venue each sit q. Of course in this case p and q must divide v, v must be odd and $\mathcal{H} = \{C_p, C_q\}$.

A uniformly resolvable $\{X, Y\}$-decomposition of K_v into exactly r X-factors and s Y-factors, is abbreviated (X, Y)-$\mathsf{URD}(v; r, s)$. The uniform case of the Hamilton-Waterloo problem is the existence problem for (C_p, C_q)-$\mathsf{URD}(v; r, s)$.

In this paper, we focus on the case $\mathcal{H} = \{K_2, K_{1,n}\}$. The resulting uniformly resolvable problem, affectionally known as the stars and stripes problem, can be seen as the attendance at a conference of v participants that has $v/(n+1)$ parallel sessions and in which during the breaks the participants pair up for one on one discussions. The parallel sessions are $K_{1,n}$-factors and are also known as star-factors: the one on one discussions are K_2-factors and are the stripes.

The existence of a $(K_2, K_{1,n})$-$\mathsf{URD}(v; r, s)$ was studied and completely solved for $n = 3$ in [6] and [13]. Here we concentrate on the case $n = 4$ and, because the results for the extremal cases $s = 0$ and $r = 0$ are known, i.e.:

- a $(K_2, K_{1,n})$-$\mathsf{URD}(v; r, 0)$ exists if and only if v is even;
- if n is even, a $(K_2, K_{1,n})$-$\mathsf{URD}(v; 0, s)$ exists is and only $v \equiv 1 \pmod{2n}$ and $v \equiv 0 \pmod{n+1}$ ([25]);

we deal with $(K_2, K_{1,4})$-$\mathsf{URD}(v; r, s)$ where $r, s > 0$ and so $v \equiv 0 \pmod{10}$ and $r = v - 1 - \frac{8s}{5}$.

For $v \equiv 0 \pmod{10}$, define $J(v)$ according to the following table:

v	$J(v)$
$0 \pmod{40}$	$\{(v-1-8x, 5x), x = 0, 1, \dots, \frac{v-8}{8}\}$
$10 \pmod{40}$	$\{(v-1-8x, 5x), x = 0, 1, \dots, \frac{v-2}{8}\}$
$20 \pmod{40}$	$\{(v-1-8x, 5x), x = 0, 1, \dots, \frac{v-4}{8}\}$
$30 \pmod{40}$	$\{(v-1-8x, 5x), x = 0, 1, \dots, \frac{v-6}{8}\}$

Table 1: The set $J(v)$

In this paper we completely solve the existence problem of a $(K_2, K_{1,4})$-$\mathsf{URD}(v; r, s)$ by proving the following result.

Main Theorem. *For any $v \equiv 0 \pmod{10}$, there exists a $(K_2, K_{1,4})$-URD$(v; r, s)$ if and only if $(r, s) \in J(v)$.*

2 Necessary conditions

In this section we will give necessary conditions for the existence of a $(K_2, K_{1,4})$-$\mathsf{URD}(v; r, s)$.

Lemma 2.1. *Let $v \equiv 0 \pmod{10}$. If there exists a $(K_2, K_{1,4})$-URD$(v; r, s)$, then $(r, s) \in J(v)$.*

Proof. Assume that there exists a $(K_2, K_{1,4})$-URD$(v; r, s)$. By resolvability, it follows that

$$\frac{rv}{2} + \frac{4sv}{5} = \frac{v(v-1)}{2}$$

and hence

$$5r + 8s = 5(v-1). \tag{1}$$

Denote by R the set of r K_2-factors and by S the set of s $K_{1,4}$-factors. Since the factors of R are regular of degree 1, every vertex of K_v is incident to r edges in R and $(v-1) - r$ edges in S. Assume that the any fixed vertex appears in x $K_{1,4}$-factors with degree 4 and in y $K_{1,4}$-factors with degree 1. Since

$$x + y = s \quad \text{and} \quad 4x + y = v - 1 - r,$$

the equality (1) gives

$$5(v - 1 - 4x - y) + 8(x + y) = 5(v - 1),$$

which implies $y = 4x$ and so $s = 5x$. Further, replacing $s = 5x$ in Equation (1) provides $r = v - 1 - 8x$, where $x \le \frac{v-1}{8}$ (because r is a non-negative integer). □

3 General constructions and related structures

An $\mathcal{H}$-decomposition of $K_{u(g)}$, the complete multi-partite graph with u parts of size g, is known as a *group divisible decomposition* ($\mathcal{H}$-GDD, in short) of type g^u; the parts of size g are called the *groups.* (If $\mathcal{H}$ consists of complete subgraphs, then a GDD is called a *group divisible design.*) When $\mathcal{H} = \{H\}$ we simply write H-GDD and when $H = K_n$ we refer to such a group divisible design as an n-GDD. We denote a (uniformly) resolvable $\mathcal{H}$-GDD by $\mathcal{H}$-(U)RGDD. Specifically, a (X, Y)-URGDD with r X-factors and s Y-factors is denoted by (X, Y)-URGDD(r, s). It is easy to deduce that the number of H-factors of a H-RGDD is $\frac{g(u-1)|V(H)|}{2|E(H)|}$.

If the blocks of an n-GDD of type g^u can be partitioned into *partial* factors, each of which contains all vertices except those of one group, we refer to such a decomposition as a *n-frame.* It is easy to deduce that the number of partial factors missing a specified group is $\frac{g}{n-1}$ (see [9]). It is well known that a 2-frame of type g^u exists if and only if $u \ge 3$ and $g(u-1) \equiv 0 \pmod 2$; and a 3-frame of type g^u exists if and only if $u \ge 4$, g is even and $g(u-1) \equiv 0 \pmod 3$ (see [7]).

An $\mathcal{H}$-decomposition of $K_{v+h} \setminus K_h$ is known as an *incomplete $\mathcal{H}$-design of order $v + h$ with a hole of size h.* We are interested in incomplete resolvable $\mathcal{H}$-designs, which will be used in the "Filling" and "Frame"-Constructions of this section. These designs have two types factors: *partial* factors, which cover every vertex except the ones in the hole; and *full* factors, which cover every vertex of K_{v+h}.

Specifically, a (X, Y)-IURD$(v+h, h; [r', s'], [r, s])$ is a uniformly resolvable (X, Y)-decomposition of $K_{v+h} \setminus K_h$ with r' partial X-factors and s' partial Y-factors which cover every vertex not in the hole, and r X-factors and s Y-factors which cover every point of K_{v+h}.

Given a graph G and a positive integer t, then $G_{(t)}$ will denote the graph on $V(G) \times \mathbb{Z}_t$ with edge-set $\{\{x_i, y_j\} : \{x, y\} \in E(G), i, j \in \mathbb{Z}_t\}$, where the subscript notation a_i is used to denote the pair (a, i). The graph $G_{(t)}$ is said to be obtained from G by expanding each vertex t times. When $G = K_n$, the graph $G_{(t)}$ is the complete equipartite graph $K_{\underbrace{t,t,\ldots,t}_{n \text{ times}}}$ with n parts of size t and will be denoted by $K_{n(t)}$; while $C_{n(t)}$ will denote the graph $G_{(t)}$ where G is an n-cycle.

Remark 3.1. Note that the graph $G_{(t)}$ admits t 1-factors corresponding to each 1-factor of G; for instance, because a $2m$-cycle has two 1-factors, $C_{2m(t)}$ admits $2t$ 1-factors.

For any two pairs of non-negative integers (r, s) and (r', s'), define $(r, s)+(r', s') = (r + r', s + s')$. If X and X' are two sets of pairs of non-negative integers and a is a positive integer, then $X + X'$ will denote the set $\{(r, s) + (r', s') : (r, s) \in X, (r', s') \in X'\}$ and $a * X$ will denote the set of all pairs of non-negative integers which can be obtained by adding any a pairs of X together (repetitions of elements of X are allowed).

Construction 3.2. (GDD-construction) Let t be a positive integer and $\mathcal{G}$ be an H-RGDD of type g^u, where H is a graph with $n \geq 2$ vertices and m edges. If there exists a (X, Y)-URD$(\bar{r}, \bar{s})$ of $H_{(t)}$ for each $(\bar{r}, \bar{s}) \in J$, then so does a (X, Y)-URGDD(r, s) of type $(gt)^u$ for each $(r, s) \in \alpha * J$, where $\alpha = \frac{ng(u-1)}{2m}$.

Proof. Let G_i, $i = 1, 2, \ldots, u$, be the groups and $F_1, F_2, \ldots, F_\alpha$ an H-factorization of $\mathcal{G}$, where $\alpha = \frac{ng(u-1)}{2m}$. Expand each vertex t times, and for each block B of the H-factor F_j, for $j = 1, 2, \ldots, \alpha$, place a copy of a (X, Y)-URD(r_j, s_j) of $H_{(t)}$ with $(r_j, s_j) \in J$ on $V(B) \times \mathbb{Z}_t$. Thus we obtain a (X, Y)-URGDD(r, s) of type $(gt)^u$ with $r = \sum_{j=1}^{\alpha} r_j$ and $s = \sum_{j=1}^{\alpha} s_j$, and so $(r, s) \in \alpha * J$. □

Construction 3.3. (Filling Construction) Suppose there exists a (X,Y)-URGDD(r,s) of type g^u for each $(r, s) \in J$. If there exists a (X, Y)-URD$(g; r', s')$, for each $(r', s') \in J'$, then so does:

(i) a (X, Y)-IURD$(ug, g; [r', s'], [r, s])$ for each $(r', s') \in J'$ and $(r, s) \in J$;

(ii) a (X, Y)-URD$(ug; \bar{r}, \bar{s})$, for each $(\bar{r}, \bar{s}) \in J' + J$.

Proof. Fix any pairs $(r, s) \in J$ and $(r', s') \in J'$, and start with a (X, Y)-URGDD(r, s) with u groups of size g, G_i, $i = 1, 2, \ldots, u$. For every $i = 2, 3, \ldots, u$, place a copy of a (X, Y)-URD$(g; r', s')$ on G_i to obtain a (X, Y)-IURD$(gu, g; [r', s'], [r, s])$ with G_1 as the hole. Finally, on G_1 place a copy of a (X, Y)-URD$(g; r', s')$ to obtain a (X, Y)-URD$(gu; r' + r, s' + s)$. □

Remark 3.4. Note that the "filling" technique allows us to construct a (X,Y)-URD$(v+h; r'+r, s'+s)$ whenever a (X,Y)-IURD$(v+h, h; [r', s'], [r, s])$ and a (X,Y)-URD$(h; r', s')$ are given.

Construction 3.5. (Frame-construction) Let v, g, t, h and u be positive integers such that $v = gtu + h$. If there exists

(i) a n-frame $\mathcal{F}$ of type g^u, $n \geq 2$;

(ii) a (X,Y)-URGDD$(\bar{r}, \bar{s})$ of type t^n for each $(\bar{r}, \bar{s}) \in J$;

(iii) a (X,Y)-IURD$(gt+h, h; [r', s'], [\bar{\bar{r}}, \bar{\bar{s}}])$ for each $(r', s') \in J'$ and $(\bar{\bar{r}}, \bar{\bar{s}}) \in \alpha * J$, where $\alpha = \frac{g}{n-1}$;

(iv) a (X,Y)-URD$(h; r', s')$ for each $(r', s') \in J'$;

then so does a (X,Y)-URD$(v; r, s)$ for each $(r, s) \in J' + u\alpha * J$.

Proof. Let $\mathcal{F}$ be an n-frame of type g^u with groups G_i, $i = 1, 2, \ldots, u$. Expand each vertex t times and add a set $H = \{a_1, a_2, \ldots, a_h\}$. For $j = 1, 2, \ldots, \alpha = \frac{g}{n-1}$, let F_{ij} be the j-th partial factor which misses the group G_i. For each block $B \in F_{ij}$, on $B \times \mathbb{Z}_t$ place a copy, $\mathcal{D}_{ij}(B)$, of a (X,Y)-URGDD(r_{ij}, s_{ij}) of type t^n with $(r_{ij}, s_{ij}) \in J$. For $i = 1, 2, \ldots, u$, on $H \cup (G_i \times \mathbb{Z}_t)$ place a copy $\mathcal{D}_i$ of a (X,Y)-IURD$(gt+h, h; [r', s'], [r_i, s_i])$ with $(r', s') \in J'$ and $(r_i, s_i) = \sum_{j=1}^{\alpha} (r_{ij}, s_{ij}) \in \alpha * J$. For every $i = 1, 2, \ldots, u$, combine all together the factors of $\mathcal{D}_{ij}(B)$, $B \in F_{ij}$, along with the full factors of $\mathcal{D}_i$ so to obtain $\bar{r}$ X-factors and $\bar{s}$ Y-factors, where $(\bar{r}, \bar{s}) = \sum_{i=1}^{u} (r_i, s_i) \in u\alpha * J$. Now, fill the hole H with a copy $\mathcal{D}$ of a (X,Y)-URD$(h; r', s')$ with $(r', s') \in J'$. Combine the factors of $\mathcal{D}$ with the partial factors of $\mathcal{D}_i$ to obtain further r' X-factors and s' Y-factors with $(r', s') \in J'$. The result is a (X,Y)-URD$(v; r, s)$ where $(r, s) = (r' + \bar{r}, s' + \bar{s}) \in J' + u\alpha * J$. ☐

4 Small cases

In what follows, we will denote by $(a_1; a_2, a_3, a_4, a_5)$ the graph $K_{1,4}$ on the vertex-set $\{a_1, a_2, a_3, a_4, a_5\}$ with edge-set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}\}$; and by $(a_1, a_2, \ldots, a_n)$ the n-cycle on $\{a_1, a_2, \ldots, a_n\}$ with edge-set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_1\}\}$. If the vertices of $B = (a; b, c, d, e)$ belong to $\mathbb{Z}_n$, then we will say *orbit of B under $\mathbb{Z}_n$* the set $\{(a+i; b+i, c+i, d+i, e+i) : i \in \mathbb{Z}_n\}$.

For any positive integer n, let $I(n)$ be the set of pairs of non-negative integers

$$I(n) = \{(n - 8x, 5x) : x = 0, 1, \ldots, \left\lfloor \frac{n}{8} \right\rfloor\}.$$

By induction it is easy to prove the following lemma.

Lemma 4.1. *If $n \equiv 0 \pmod 8$, then $\alpha * I(n) = I(\alpha n)$ for any positive integer α.*

Lemma 4.2. *A $(K_2, K_{1,4})$-URD(r, s) of $C_{2m(5)}$ exists for every $(r, s) \in I(10)$.*

Proof. The case $(r,s) = (10,0)$ follows by Remark 3.1. For the case $(r,s) = (2,5)$, let $C_{2m(5)}$ be the graph obtained by starting with the cycle $C = (0,1,\ldots,2m-1)$ on $\mathbb{Z}_{2m}$ and taking the five $K_{1,4}$-factors

$$F_j = \{(i_j; (1+i)_{j+1}, (1+i)_{j+2}, (1+i)_{j+3}, (1+i)_{j+4}) : i \in \mathbb{Z}_{2m}\}, \quad j \in \mathbb{Z}_5.$$

The two 1-factors are easily obtainable by decomposing the remaining set of edges, which can be considered as the disjoint union of the five $2m$-cycles $C_j = (0_j, 1_j, \ldots, (2m-1)_j)$, $j \in \mathbb{Z}_5$. □

Lemma 4.3. *A $(K_2, K_{1,4})$-URGDD(r,s) of type 2^5 exists for every $(r,s) \in I(8)$.*

Proof. The case $(r,s) = (8,0)$ corresponds to a 1-factorization of $K_{5(2)}$, which is known to exist ([7]). To settle the case $(r,s) = (0,5)$, take the orbit of $B = (0;1,2,3,4)$ under $\mathbb{Z}_{10}$, which can be decomposed into the five $K_{1,4}$-factors:

$$F_j = \{B + j + 5i : i = 0,1\}, \quad j = 0,1,2,3,4.$$

The groups are the cosets $H, H+1, H+2, H+3, H+4$ of $H = 5\mathbb{Z}_{10}$ in $\mathbb{Z}_{10}$. □

Lemma 4.4. *A $(K_2, K_{1,4})$-URD$(10; r,s)$ exists for every $(r,s) \in J(10)$.*

Proof. The case $(r,s) = (9,0)$ corresponds to a 1-factorization of the complete K_{10}, which is known to exist ([7]). For the case $(r,s) = (1,5)$, apply the Filling Construction to a $(K_2, K_{1,4},)$-URGDD$(0,5)$ of type 2^5, which is given by Lemma 4.3. □

Lemma 4.5. *A $(K_2, K_{1,4})$-URGDD(r,s) of type 10^2 exists for every $(r,s) \in I(10)$.*

Proof. Apply the GDD-construction with $t = 5$ to a trivial C_4-RGDD of type 2^2, where $\alpha = 1$. The input designs are given by Lemma 4.2. □

Lemma 4.6. *A $(K_2, K_{1,4})$-URD$(20; r,s)$ exists for every $(r,s) \in J(20)$.*

Proof. The Filling Construction applied to a $(K_2, K_{1,4})$-URGDD$(\bar{r}, \bar{s})$ of type 10^2 from Lemma 4.5 (with input designs given by Lemma 4.4) gives a $(K_2, K_{1,4})$-URD$(20; r,s)$ for each $(r,s) \in J(10) + I(10) = J(20)$.

□

Lemma 4.7. *A $(K_2, K_{1,4})$-URD$(40; r,s)$ exists for every $(r,s) \in J(40)$.*

Proof. Applying the GDD-construction with $t = 10$ to a 2-RGDD of type 2^2 (where $\alpha = 2$) gives a $(K_2, K_{1,4})$-URGDD$(\bar{r}, \bar{s})$ of type 20^2 for each $(\bar{r}, \bar{s}) \in 2{*}I(10)$ (the input designs are given by Lemma 4.5). Now filling the groups with designs given by Lemma 4.6 gives a $(K_2, K_{1,4})$-URD$(40; r,s)$ for each $(r,s) \in J(20) + 2 * I(20) = J(40)$. □

Lemma 4.8. *A $(K_2, K_{1,4})$-URD$(0,25)$ of $C_{m(20)}$ exists for every $m \geq 3$.*

Proof. Let $C_m = (1,2,\ldots,m)$. For $i = 1,2\ldots,m$, let $X^{(i)} = \{i\} \times \mathbb{Z}_{20} = \bigcup_{k=0}^{4} X_k^{(i)}$, where $X_k^{(i)} = \{i_{4k}, i_{4k+1}, i_{4k+2}, i_{4k+3}\}$, and for every $r,s \in \mathbb{Z}_5$ let $R_{rs}^{(i)}$ denote the set of the following four copies of $K_{1,4}$

$\big(i_{4r}; (i+1)_{4s}, (i+1)_{4s+1}, (i+1)_{4s+2}, (i+1)_{4s+3}\big),$

$\big(i_{4r+1}; (i+1)_{4s+4}, (i+1)_{4s+5}, (i+1)_{4s+6}, (i+1)_{4s+7}\big),$

$\big(i_{4r+2}; (i+1)_{4s+8}, (i+1)_{4s+9}, (i+1)_{4s+10}, (i+1)_{4s+11}\big),$

$\big(i_{4r+3}; (i+1)_{4s+12}, (i+1)_{4s+13}, (i+1)_{4s+14}, (i+1)_{4s+15}\big),$

where $m+1=1$. If $m=2n$, $n \geq 2$, take the five $K_{1,4}$-factors

$$F_1 = \bigcup_{i=0}^{n-1} \left(R_{01}^{(2i+1)} \cup R_{01}^{(2i+2)}\right),$$
$$F_2 = \bigcup_{i=0}^{n-1} \left(R_{02}^{(2i+1)} \cup R_{11}^{(2i+2)}\right),$$
$$F_3 = \bigcup_{i=0}^{n-1} \left(R_{03}^{(2i+1)} \cup R_{21}^{(2i+2)}\right),$$
$$F_4 = \bigcup_{i=0}^{n-1} \left(R_{04}^{(2i+1)} \cup R_{31}^{(2i+2)}\right),$$
$$F_5 = \bigcup_{i=0}^{n-1} \left(R_{00}^{(2i+1)} \cup R_{41}^{(2i+2)}\right),$$

while if $m=2n+1$, $n \geq 1$, take the five $K_{1,4}$-factors

$$F_1' = \left(R_{01}^{(1)} \cup R_{01}^{(2)} \cup R_{01}^{(3)}\right) \cup \left[\bigcup_{i=2}^{n} \left(R_{01}^{(2i)} \cup R_{01}^{(2i+1)}\right)\right],$$
$$F_2' = \left(R_{02}^{(1)} \cup R_{13}^{(2)} \cup R_{21}^{(3)}\right) \cup \left[\bigcup_{i=2}^{n} \left(R_{02}^{(2i)} \cup R_{11}^{(2i+1)}\right)\right],$$
$$F_3' = \left(R_{03}^{(1)} \cup R_{20}^{(2)} \cup R_{41}^{(3)}\right) \cup \left[\bigcup_{i=2}^{n} \left(R_{03}^{(2i)} \cup R_{21}^{(2i+1)}\right)\right],$$
$$F_4' = \left(R_{04}^{(1)} \cup R_{32}^{(2)} \cup R_{11}^{(3)}\right) \cup \left[\bigcup_{i=2}^{n} \left(R_{04}^{(2i)} \cup R_{31}^{(2i+1)}\right)\right],$$
$$F_5' = \left(R_{00}^{(1)} \cup R_{44}^{(2)} \cup R_{31}^{(3)}\right) \cup \left[\bigcup_{i=2}^{n} \left(R_{00}^{(2i)} \cup R_{41}^{(2i+1)}\right)\right].$$

The required 25 star factors are

$$F_{k,j} = (F_k) + j = \{R_{r+j,s+j}^{(i)} : R_{rs}^{(i)} \in F_k,\ r,s \in \mathbb{Z}_5\},\ j \in \mathbb{Z}_5,\ k = 1,2,3,4,5$$

when $m=2n$ and

$$F_{k,j}' = (F_k') + j = \{R_{r+j,s+j}^{(i)} : R_{rs}^{(i)} \in F_k',\ r,s \in \mathbb{Z}_5\},\ j \in \mathbb{Z}_5,\ k = 1,2,3,4,5$$

when $m=2n+1$. □

Lemma 4.9. *A C_{2m}-decomposition of $C_{m(2)}$ exists for any $m \geq 3$.*

Proof. Let $C_{m(2)}$ be the graph obtained by expanding twice the vertices of the cycle $(1, 2, \ldots, m)$, $m \geq 3$. If m is even, take the two $2m$-cycles

$C_1 = (1_0, 2_1, 3_0, 4_1, \ldots, m_1, 1_1, 2_0, 3_1, 4_0, \ldots, m_0)$,

$C_2 = (1_0, 2_0, 3_0, 4_0, \ldots, m_0, 1_1, 2_1, 3_1, 4_1, \ldots, m_1)$,

while if m is odd, take the following ones:

$C'_1 = (1_0, 2_1, 3_0, 4_1, \ldots, m_0, 1_1, m_1, (m-1)_0, (m-2)_1, (m-3)_0, \ldots, 2_0)$,

$C'_2 = (1_0, m_0, (m-1)_0, (m-2)_0, (m-3)_0, \ldots, 2_0, 1_1, 2_1, 3_1, 4_1, \ldots, m_1)$. □

Lemma 4.10. *A $(K_2, K_{1,4})$-URD(r, s) of $C_{m(20)}$, $m \geq 3$, exists for every $(r, s) \in I(40)$.*

Proof. The case $(r, s) = (0, 25)$ follows by Lemma 4.8. For any $(r, s) \in I(40) \setminus \{(0, 25)\}$, start from the C_{2m}-decomposition of $C_{m(2)}$ of Lemma 4.9, which admits $\alpha = 4$ 1-factors (each $2m$-cycle gives two 1-factors). Expand each vertex 10 times. For each edge e of a given 1-factor, place on $e \times \mathbb{Z}_{10}$ a copy of a $(K_2, K_{1,4})$-URGDD$(\bar{r}, \bar{s})$ of type 10^2 with $(\bar{r}, \bar{s}) \in I(10)$ (given by Lemma 4.5) so to obtain a $(K_2, K_{1,4})$-URD(r, s) of $C_{m(20)}$ with $(r, s) \in 4 * I(10) = I(40) \setminus \{(0, 25)\}$. □

Lemma 4.11. *A $(K_2, K_{1,4})$-URD(r, s) of $C_{m(10)}$, $m \geq 3$, exists for every $(r, s) \in I(20)$.*

Proof. Start with a C_{2m}-decomposition of $C_{m(2)}$, which is given by Lemma 4.9 and is trivially resolvable with $\alpha = 2$ factors (i.e., the two $2m$-cycles). Expand each vertex 5 times. For each cycle C, place on $V(C) \times \mathbb{Z}_5$ a copy of a $(K_2, K_{1,4})$-URD$(\bar{r}, \bar{s})$ of $C_{2m(5)}$ with $(\bar{r}, \bar{s}) \in I(10)$ given by Lemma 4.2 so to obtain a $(K_2, K_{1,4})$-URD(r, s) of $C_{m(10)}$ with $(r, s) \in 2 * I(10) = I(20)$. □

Lemma 4.12. *A $(K_2, K_{1,4})$-URGDD(r, s) of type 40^2 exists for every $(r, s) \in I(40)$.*

Proof. Apply the GDD-construction with $t = 20$ to a trivial C_4-RGDD of type 2^2, where $\alpha = 1$. The input designs are given by Lemma 4.10 for $m = 4$. □

Lemma 4.13. *A $(K_2, K_{1,4})$-URD$(30; r, s)$ exists for every $(r, s) \in J(30)$.*

Proof. The Filling Construction applied to a $(K_2, K_{1,4})$-RGDD$(\bar{r}, \bar{s})$ of type 10^3 with $(\bar{r}, \bar{s}) \in I(20)$ (from Lemma 4.11) gives a $(K_2, K_{1,4})$-URD$(30; r, s)$ for each $(r, s) \in J(10) + I(20) = J(30)$. The input designs are given by Lemma 4.4. □

Lemma 4.14. *There exists a $(K_2, K_{1,4})$-URGDD$(0, 25)$ of type 10^5.*

Proof. The union of the orbits of $B_i = (0; 1+5i, 2+5i, 3+5i, 4+5i)$, $i = 0, 1, 2, 3, 4$, under $\mathbb{Z}_{50}$ gives the block set of a GDD of type 10^5, whose groups are the cosets $H, H+1, H+2, H+3, H+4$ of $H = 5\mathbb{Z}_{50}$ in $\mathbb{Z}_{50}$. For every $i = 0, 1, 2, 3, 4$, the orbit of B_i can be decomposed into five $K_{1,4}$-factors:

$$F_{ij} = \{B_i + j + 5k, k = 0, 1, \ldots, 9\}, \quad j = 0, 1, 2, 3, 4.$$

□

Lemma 4.15. *A* $(K_2, K_{1,4})$*-URGDD*(r, s) *of type* 10^5 *exists for every* $(r, s) \in I(40)$.

Proof. The case $(r, s) = (0, 25)$ follows by Lemma 4.14. For any $(r, s) \in I(40) \setminus \{(0, 25)\}$, the GDD-Construction applied with $t = 10$ to a trivial C_5-RGDD of type 1^5 (where $\alpha = 2$) gives a $(K_2, K_{1,4})$-URGDD(r, s) of type 10^5 for each $(r, s) \in 2 * I(20) = I(40) \setminus \{(0, 25)\}$. The input designs are given by Lemma 4.11. □

Lemma 4.16. *A* $(K_2, K_{1,4})$*-IURD*$(50, 10; [r', s'], [r, s])$ *exists for every* $(r', s') \in J(10)$ *and* $(r, s) \in I(40)$.

Proof. Apply the Filling Construction to a $(K_2, K_{1,4})$-URGDD(r, s) of type 10^5 with $(r, s) \in I(40)$ from by Lemma 4.15 (the input designs are given by Lemma 4.4). □

Let $S \subset \mathbb{Z}_n$ be such that if $s \in S$, then $-s \notin S$ and set $B = \big\{\{0, s\} : s \in S\big\}$, then the orbit of B is the circulant graph with edges $\{x, y\}$ where either $x - y$ or $y - x \in S$. The edge $\{x, y\}$ has even order if $s = y - x$ has even additive order modulo n. In the next Lemma we use the following famous result of Stern and Lenz.

Theorem 4.17. *(Theorem of Stern and Lenz [23]) Every circulant graph containing an edge of even order has a one-factorization.*

Lemma 4.18. *A* $(K_2, K_{1,4})$*-URGDD*(r, s) *of type* 10^9 *exists for every* $(r, s) \in \{(8, 45), (0, 50)\}$

Proof. On $\mathbb{Z}_{90}$ let:

$$\begin{aligned} F = \{&(89; 0, 1, 18, 19), (52; 53, 54, 71, 72), (2; 40, 67, 87, 85),\\ &(3; 41, 68, 88, 86), (4; 65, 73, 77, 81), (5; 66, 74, 78, 82),\\ &(6; 46, 48, 50, 61), (7; 47, 49, 51, 62), (8; 38, 42, 59, 83),\\ &(9; 39, 43, 60, 84), (10; 32, 34, 36, 63), (11; 33, 35, 37, 64),\\ &(12; 20, 69, 24, 79), (13; 21, 70, 25, 80), (14; 28, 30, 55, 57),\\ &(15; 29, 31, 56, 58), (16; 44, 22, 26, 75), (17; 45, 23, 27, 76)\}. \end{aligned}$$

and $B = (0; 3, 4, 11, 32)$. Take the forty-five $K_{1,4}$-factors $F + 2i$, for $i = 0, 1, \ldots, 44$, and partition the orbit of B under $\mathbb{Z}_{90}$ into the five $K_{1,4}$-factors:

$$F_j = \{B + j + 5k, k = 0, 1, \ldots, 17\}, \quad j = 0, 1, 2, 3, 4.$$

The resulting design is a $(K_2, K_{1,4})$-URGDD$(0, 50)$ of type 10^9, whose groups are the cosets of $H = 9\mathbb{Z}_{90}$ in $\mathbb{Z}_{90}$, i.e., $H + h$, for $h = 0, 1, \ldots, 8$.
For the case $(r, s) = (8, 45)$, remove the $K_{1,4}$-factors obtained from the orbit of B and decompose the graph whose edges cover the differences of B into 1-factors by using the theorem of Stern and Lenz. □

Lemma 4.19. *A* $(K_2, K_{1,4})$*-URGDD*(r, s) *of type* 10^9 *exists for every* $(r, s) \in I(80)$.

Proof. The cases $(r, s) = (0, 50)$, $(8, 45)$ follow by Lemma 4.18. To settle the remaining cases, apply the GDD-construction with $t = 10$ to a C_9-RGDD of type 1^9 (where $\alpha = 4$) to get a $(K_2, K_{1,4})$-URGDD(r, s) of type 10^9 for each $(r, s) \in 4 * I(20) = I(80) \setminus \{(0, 50), (8, 45)\}$. The input designs are given by Lemma 4.11. □

Lemma 4.20. *A* $(K_2, K_{1,4})$-*IURD*$(90, 10; [r', s'], [r, s])$ *exists for every* $(r', s') \in J(10)$ *and* $(r, s) \in I(80)$.

Proof. Apply the Filling Construction to a $(K_2, K_{1,4})$-URGDD(r, s) of type 10^9 with $(r, s) \in I(80)$ from by Lemma 4.19 (the input designs are given by Lemma 4.4). □

Lemma 4.21. *A* $(K_2, K_{1,4})$-*IURD*$(70, 30; [r', s'], [0, 25])$ *exists for every* $(r', s') \in J(30)$.

Proof. Let $V = \mathbb{Z}_{40} \cup \{a_1, a_2, \ldots, a_{30}\}$ be the vertex set, where $\{a_1, a_2, \ldots, a_{30}\}$ is the hole. Consider the five $K_{1,4}$-factors on V:

$$\begin{aligned}F_1 = \{&(a_1; 16, 17, 18, 19), (a_2; 20, 21, 22, 23), (a_3; 24, 25, 26, 27),\\ &(a_4; 28, 29, 30, 31), (a_5; 32, 33, 34, 35), (a_6; 36, 37, 38, 39),\\ &(0; 8, a_{14}, a_{16}, a_{18}), (1; 9, a_{26}, a_{28}, a_{30}), (2; 10, a_{20}, a_{22}, a_{24}),\\ &(3; 11, a_8, a_{10}, a_{12}), (4; 12, a_{13}, a_{15}, a_{17}), (5; 13, a_{25}, a_{27}, a_{29}),\\ &(6; 14, a_{19}, a_{21}, a_{23}), (7; 15, a_7, a_9, a_{11})\};\end{aligned}$$

$$\begin{aligned}F_2 = \{&(a_7; 8, 9, 10, 11), (a_8; 12, 13, 14, 15), (a_9; 24, 25, 26, 27),\\ &(a_{10}; 28, 29, 30, 31), (a_{11}; 32, 33, 34, 35), (a_{12}; 36, 37, 38, 39),\\ &(0; 16, a_{26}, a_{28}, a_{30}), (1; 17, a_2, a_4, a_6), (2; 18, a_{14}, a_{16}, a_{18}),\\ &(3; 19, a_{20}, a_{22}, a_{24}), (4; 20, a_1, a_3, a_5), (5; 21, a_{19}, a_{21}, a_{23}),\\ &(6; 22, a_{13}, a_{15}, a_{17}), (7; 23, a_{25}, a_{27}, a_{29})\};\end{aligned}$$

$$\begin{aligned}F_3 = \{&(a_{13}; 8, 9, 10, 11), (a_{14}; 20, 21, 22, 23), (a_{15}; 24, 25, 26, 27),\\ &(a_{16}; 28, 29, 30, 31), (a_{17}; 32, 33, 34, 35), (a_{18}; 36, 37, 38, 39),\\ &(0; 12, a_8, a_{10}, a_{12}), (1; 13, a_{20}, a_{22}, a_{24}), (2; 14, a_2, a_4, a_6),\\ &(3; 15, a_{26}, a_{28}, a_{30}), (4; 16, a_7, a_9, a_{11}), (5; 17, a_1, a_3, a_5),\\ &(6; 18, a_{25}, a_{27}, a_{29}), (7; 19, a_{19}, a_{21}, a_{23})\};\end{aligned}$$

$$\begin{aligned}F_4 = \{&(a_{19}; 8, 9, 10, 11), (a_{20}; 12, 13, 22, 23), (a_{21}; 24, 25, 26, 27),\\ &(a_{22}; 28, 29, 30, 31), (a_{23}; 32, 33, 34, 35), (a_{24}; 36, 37, 38, 39),\\ &(0; 14, a_2, a_4, a_6), (1; 15, a_8, a_{10}, a_{12}), (2; 16, a_{26}, a_{28}, a_{30}),\\ &(3; 17, a_{14}, a_{16}, a_{18}), (4; 18, a_{25}, a_{27}, a_{29}), (5; 19, a_7, a_9, a_{11}),\\ &(6; 20, a_1, a_3, a_5), (7; 21, a_{13}, a_{15}, a_{17})\};\end{aligned}$$

$$\begin{aligned}F_5 = \{&(a_{25}; 8, 9, 10, 11), (a_{26}; 12, 13, 14, 15), (a_{27}; 16, 17, 26, 27),\\ &(a_{28}; 28, 29, 30, 31), (a_{29}; 32, 33, 34, 35), (a_{30}; 36, 37, 38, 39),\\ &(0; 18, a_{20}, a_{22}, a_{24}), (1; 19, a_{14}, a_{16}, a_{18}), (2; 20, a_8, a_{10}, a_{12}),\\ &(3; 21, a_2, a_4, a_6), (4; 22, a_{19}, a_{21}, a_{23}), (5; 23, a_{13}, a_{15}, a_{17}),\\ &(6; 24, a_7, a_9, a_{11}), (7; 25, a_1, a_3, a_5)\}.\end{aligned}$$

For each $j = 1, 2, 3, 4, 5$, take the five $K_{1,4}$-factors $F_j + 8i$, for $i = 0, 1, 2, 3, 4$, where $a_k + x = a_k$ for every $x \in \mathbb{Z}_{40}$ and for every $k = 1, 2, \ldots, 30$. Let $D = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 17, 19, 20\}$, i.e. the set of the 15 differences not covered by the 25 above factors, and decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{40}$ and $d \in D$, as follows.

For $(r', s') = (29, 0)$, apply the theorem of Stern and Lenz to decompose the graph consisting of the edges $\{i, d+i\}$, $i \in \mathbb{Z}_{40}$ and $d \in D$, into 29 partial 1-factors on $\mathbb{Z}_{40}$.

For $(r', s') = (21, 5)$, take the base block $B = (0; 7, 9, 11, 13)$, whose orbit modulo 40 can be decomposed into five $K_{1,4}$-factors on $\mathbb{Z}_{40}$:

$$F'_j = \{B + j + 5k : k = 0, 1, \ldots, 7\}, \quad j = 0, 1, 2, 3, 4.$$

Decompose the graph whose edges cover the remaining differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (13, 10)$, take the base blocks

$$B = (0; 7, 9, 11, 13) \text{ and } B_1 = (0; 1, 2, 3, 4),$$

which give in total ten $K_{1,4}$-factors on $\mathbb{Z}_{40}$. Then decompose the graph whose edges cover the differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (5, 15)$, take the base blocks $B = (0; 7, 9, 11, 13)$, $B_2 = (0; 3, 4, 5, 19)$ and $B_3 = (6; 7, 8, 12, 21)$. Obtain five $K_{1,4}$-factors on $\mathbb{Z}_{40}$ from B, and ten $K_{1,4}$-factors from B_2 and B_3 as follows:

$$F''_j = \{B_2 + j + 10k, B_3 + j + 10k : k = 0, 1, 2, 3\}, \quad j = 0, 1, \ldots, 9.$$

Decompose the graph whose edges cover the remaining differences 10, 17 and 20 of D into 1-factors by using the theorem of Stern and Lenz. □

Lemma 4.22. *A $(K_2, K_{1,4})$-IURD$(70, 30; [r', s'], [r, s])$ exists for every $(r', s') \in J(30)$ and $(r, s) \in I(40)$.*

Proof. The case $(r, s) = (0, 25)$ follows by Lemma 4.21. For any $(r, s) \in I(40) \setminus \{(0, 25)\}$, start from the decomposition of the graph $K_7 \setminus K_3$ on $X = \{x, y, z\} \cup \{a_1, a_2, a_3, a_4\}$ into one 4-cycle $C_0 = (a_1, a_2, a_3, a_4)$ and two hamiltonian cycles $C_1 = (a_1, a_3, y, a_2, z, a_4, x)$ and $C_2 = (a_1, z, a_3, x, a_2, a_4, y)$. Expand each vertex 10 times and on $V(C_j) \times \mathbb{Z}_{10}$, for $j = 0, 1, 2$, place a copy of a $(K_2, K_{1,4})$-URD(r_j, s_j) of $C_{m(10)}$ ($m = 4$ or 7) with $(r_j, s_j) \in I(20)$ (given by Lemma 4.11). It follows that, corresponding to the hamiltonian cycles C_1 and C_2, there are r full 1-factors and s full $K_{1,4}$-factors, where $(r, s) = (r_1 + r_2, s_1 + s_2) \in 2 * I(20) = I(40) \setminus \{(0, 25)\}$; while C_0 provides r_0 partial 1-factors and s_0 partial $K_{1,4}$-factors missing the hole $\{x, y, z\} \times \mathbb{Z}_{10}$. Now, placing on each set $\{a_i\} \times \mathbb{Z}_{10}$, $i = 1, 2, 3, 4$, a copy of a $(K_2, K_{1,4})$-URD$(10, r'', s'')$ with $(r'', s'') \in J(10)$ (from Lemma 4.4) gives further r'' partial 1-factors and s'' partial $K_{1,4}$-factors so that the resulting design is a $(K_2, K_{1,4})$-IURD$(70, 30; [r', s'], [r, s])$, where $(r', s') = (r_0 + r'', s_0 + s'') \in I(20) + J(10) = J(30)$. □

Lemma 4.23. *A $(K_2, K_{1,4})$-IURD$(110, 30; [r', s'], [r, s])$ exists for every $(r', s') \in J(30)$ and $(r, s) \in \{(8, 45), (0, 50)\}$.*

Proof. Let $V = \mathbb{Z}_{80} \cup \{a_1, a_2, \ldots, a_{30}\}$ be the vertex set, where $\{a_1, a_2, \ldots, a_{30}\}$ is the hole. Consider the two $K_{1,4}$-factors on V:

$$\begin{aligned} F_1 = \{&(a_{25}; 16, 18, 20, 22), (a_{26}; 24, 26, 28, 30), (a_{27}; 32, 34, 36, 38),\\ &(a_{28}; 17, 19, 21, 23), (a_{29}; 25, 27, 29, 31), (a_{30}; 33, 35, 37, 39),\\ &(0; 14, a_{22}, a_{23}, a_{24}), (1; 15, a_{19}, a_{20}, a_{21}), (2; 8, a_{16}, a_{17}, a_{18}),\\ &(3; 9, a_{13}, a_{14}, a_{15}), (4; 12, a_{10}, a_{11}, a_{12}), (5; 13, a_7, a_8, a_9),\\ &(6; 10, a_4, a_5, a_6), (7; 11, a_1, a_2, a_3),\\ &(40; 67, 68, 69, 70), (41; 62, 64, 66, 72), (42; 63, 65, 73, 78),\\ &(43; 54, 58, 60, 79), (44; 55, 56, 59, 61), (45; 48, 50, 52, 57),\\ &(46; 49, 51, 53, 71), (47; 74, 75, 76, 77)\}; \end{aligned}$$

$$\begin{aligned} F_2 = \{&(a_{25}; 4, 14, 24, 34), (a_{26}; 2, 12, 22, 32), (a_{27}; 0, 10, 20, 30),\\ &(a_{28}; 5, 15, 25, 35), (a_{29}; 3, 13, 23, 33), (a_{30}; 1, 11, 21, 31),\\ &(46; 56, a_{22}, a_{23}, a_{24}), (47; 57, a_{19}, a_{20}, a_{21}), (44; 54, a_{16}, a_{17}, a_{18}),\\ &(45; 55, a_{13}, a_{14}, a_{15}), (42; 52, a_{10}, a_{11}, a_{12}), (43; 53, a_7, a_8, a_9),\\ &(40; 50, a_4, a_5, a_6), (41; 51, a_1, a_2, a_3),\\ &(6; 8, 48, 64, 66), (7; 9, 49, 65, 67), (16; 18, 74, 76, 62),\\ &(17; 19, 75, 77, 63), (26; 28, 68, 60, 72), (27; 29, 69, 61, 73),\\ &(36; 38, 78, 70, 58), (37; 39, 79, 71, 59)\}; \end{aligned}$$

and the sets of pairs $A_1 = \{\{0, 20\}, \{1, 21\}, \{6, 26\}, \{7, 27\}\}$ and $A_2 = \{\{2, 40\}, \{3, 41\}, \{6, 28\}, \{7, 29\}\}$.

Take the 50 full $K_{1,4}$-factors $F_1 + 2i$, for $i = 0, 1, \ldots, 39$, and $F_2 + 8i$, for $i = 0, 1, \ldots, 9$, where $a_k + x = a_{k+3x}$ for every $x \in \mathbb{Z}_{80}$ and $k = 1, 2, \ldots, 30$; and the two partial 1-factors $(A_j) = \{\{x + 8i, y + 8i\} : \{x, y\} \in A_j, i = 0, 1, \ldots, 9\}$, $j = 1, 2$. Let $D = \{1, 9, 13, 16, 18, 19, 24, 26, 32, 33, 35, 37, 39, 40\}$, i.e. the set of the 14 differences not covered by (A_1), (A_2) and the above 50 full $K_{1,4}$-factors. Decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{80}$ and $d \in D$, as follows:

For $(r', s') = (29, 0)$, apply the theorem of Stern and Lenz to decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{80}$ and $d \in D$, into 27 partial 1-factors on $\mathbb{Z}_{80}$ so to obtain 29 partial 1-factors along with (A_1) and (A_2).
For $(r', s') = (21, 5)$, take the base block $B = (0; 26, 32, 33, 39)$, whose orbit can be decomposed into five $K_{1,4}$-factors on $\mathbb{Z}_{80}$:

$$F'_j = \{B + j + 5k : k = 0, 1, \ldots, 15\}, \quad j = 0, 1, 2, 3, 4.$$

Decompose the graph whose edges cover the remaining differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (13, 10)$, take the base blocks $B = (0; 26, 32, 33, 39)$ and $B_1 = (0; 1, 13, 24, 37)$, which give in total ten $K_{1,4}$-factors on $\mathbb{Z}_{80}$, while decomposing the graph whose edges cover the differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (5, 15)$, take the base blocks $B = (0; 26, 32, 33, 39)$, $B_2 = (0; 1, 16, 19, 35)$ and $B_3 = (4; 13, 17, 22, 28)$. Obtain five $K_{1,4}$-factors on $\mathbb{Z}_{80}$ from B, and ten

$K_{1,4}$-factors from B_2 and B_3 as follows:

$$F''_j = \{B_2 + j + 10k, B_3 + j + 10k : k = 0, 1, \ldots, 7\}, \quad j = 0, 1, \ldots, 9.$$

Decompose the graph whose edges cover the remaining differences 37 and 40 of D into 1-factors by using the theorem of Stern and Lenz.

Finally, to prove the existence of a $(K_2, K_{1,4})$-IURD$(110, 30; [r', s'], (8, 45))$, with $(r', s') \in J(30)$, it will be sufficient to start from the constructed $(K_2, K_{1,4})$-IURD $(110, 30; [r, s], (0, 50))$. Destroy the 5 full $K_{1,4}$-factors $F_1 + 16i$, for $i = 0, 1, 2, 3, 4$, and rearrange the resulting edges into the 8 full 1-factors $(M_j) = \{\{x+16i, y+16i\} : \{x, y\} \in M_j, i = 0, 1, 2, 3, 4\}$, $j = 1, 2, \ldots, 8$, where:

$$M_1 = \left\{ \begin{array}{l} \{7, a_3\}, \{6, a_5\}, \{5, a_7\}, \{2, a_{18}\}, \{1, a_{20}\}, \{0, a_{22}\}, \\ \{4, 12\}, \{40, 67\}, \{41, 62\}, \{43, 58\}, \{47, 77\} \end{array} \right\};$$

$$M_2 = \left\{ \begin{array}{l} \{6, a_6\}, \{5, a_8\}, \{4, a_{10}\}, \{1, a_{21}\}, \{0, a_{23}\}, \{18, a_{25}\}, \\ \{7, 11\}, \{42, 63\}, \{44, 56\}, \{45, 57\}, \{46, 51\} \end{array} \right\};$$

$$M_3 = \left\{ \begin{array}{l} \{5, a_9\}, \{4, a_{11}\}, \{3, a_{13}\}, \{0, a_{24}\}, \{30, a_{26}\}, \{17, a_{28}\}, \\ \{6, 10\}, \{41, 72\}, \{43, 79\}, \{44, 55\}, \{45, 50\} \end{array} \right\};$$

$$M_4 = \left\{ \begin{array}{l} \{7, a_1\}, \{4, a_{12}\}, \{3, a_{14}\}, \{2, a_{16}\}, \{38, a_{27}\}, \{29, a_{29}\}, \\ \{40, 69\}, \{41, 64\}, \{44, 59\}, \{46, 49\}, \{47, 74\} \end{array} \right\};$$

$$M_5 = \left\{ \begin{array}{l} \{7, a_2\}, \{6, a_4\}, \{3, a_{15}\}, \{2, a_{17}\}, \{1, a_{19}\}, \{37, a_{30}\}, \\ \{0, 14\}, \{40, 68\}, \{42, 73\}, \{44, 61\}, \{47, 75\} \end{array} \right\};$$

$$M_6 = \left\{ \begin{array}{l} \{20, a_{25}\}, \{28, a_{26}\}, \{32, a_{27}\}, \{23, a_{28}\}, \{31, a_{29}\}, \{33, a_{30}\}, \\ \{2, 8\}, \{3, 9\}, \{5, 13\}, \{42, 78\}, \{43, 54\} \end{array} \right\};$$

$$M_7 = \left\{ \begin{array}{l} \{16, a_{25}\}, \{26, a_{26}\}, \{34, a_{27}\}, \{19, a_{28}\}, \{25, a_{29}\}, \{39, a_{30}\}, \\ \{1, 15\}, \{40, 70\}, \{45, 52\}, \{46, 53\}, \{43, 60\} \end{array} \right\};$$

$$M_8 = \left\{ \begin{array}{l} \{22, a_{25}\}, \{24, a_{26}\}, \{36, a_{27}\}, \{21, a_{28}\}, \{27, a_{29}\}, \{35, a_{30}\}, \\ \{41, 66\}, \{42, 65\}, \{45, 48\}, \{46, 71\}, \{47, 76\} \end{array} \right\}.$$

□

Lemma 4.24. *A* $(K_2, K_{1,4})$*-IURD*$(110, 30; [r', s'], [r, s])$ *exists for every* $(r', s') \in J(30)$ *and* $(r, s) \in I(80)$.

Proof. The cases $(r, s) = (8, 45), (0, 50)$ follow by Lemma 4.21. For any $(r, s) \in I(80) \setminus \{(8, 45), (0, 50)\}$, start from the decomposition of the graph $K_{11} \setminus K_3$ on $X = \{x, y, z\} \cup \{a_1, a_2, \ldots, a_8\}$ into one 8-cycle $C_0 = (a_1, a_2, , \ldots, a_8)$ and four hamiltonian cycles

$$C_1 = (a_1, x, a_3, y, a_5, z, a_7, a_2, a_6, a_8, a_4),$$

$$C_2 = (a_2, x, a_4, y, a_6, z, a_8, a_3, a_1, a_7, a_5),$$

$$C_3 = (a_5, x, a_7, y, a_1, z, a_3, a_6, a_4, a_2, a_8),$$

$$C_4 = (a_6, x, a_8, y, a_2, z, a_4, a_7, a_3, a_5, a_1).$$

Expanding each vertex 10 times and using similar arguments to the proof of Lemma 4.22 gives a $(K_2, K_{1,4})$-IURD$(110, 30; [r', s'], [r, s])$ for each $(r', s') \in I(20) + J(10) = J(30)$ and $(r, s) \in 4 * I(20)) = I(80) \setminus \{(8, 45), (0, 50)\}$. □

Lemma 4.25. *Let* $v = 50, 70, 90, 110$. *A* $(K_2, K_{1,4})$-*URD*(v, r, s) *exists for every* $(r, s) \in J(v)$.

Proof. Apply the Filling Construction to the IURDs of Lemmas 4.16, 4.20, 4.22 and 4.24. The input designs are given by Lemmas 4.4 and 4.13. □

The next three results are all obtained by applying the Frame-Construction with various parameters. We leave them as separate results so that it is easier for the reader to find each case.

Lemma 4.26. *A* $(K_2, K_{1,4})$-*URD*$(190; r, s)$ *exists for every* $(r, s) \in J(190)$.

Proof. Apply the Frame-Construction with $t = 20$ and $h = 30$ to a 3-frame of type 2^4 (where $\alpha = 1$) to obtain a $(K_2, K_{1,4})$-URD$(190; r, s)$ for each $(r, s) \in J(30)+4*I(40) = J(30) + I(160) = J(190)$ (where the first equality follows by Lemma 4.1). The input designs are given by Lemmas 4.10, 4.13 and 4.22. □

Lemma 4.27. *Let* $v = 690, 930$. *A* $(K_2, K_{1,4})$-*URD*$(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Apply the Frame-Construction with $t = 40$ and $h = 10$ to a 2-frame of type $1^{\frac{v-10}{40}}$ (where $\alpha = 1$) to obtain a $(K_2, K_{1,4})$-URD$(v; r, s)$ for each $(r, s) \in J(10) + \frac{v-10}{40} * I(40) = J(10) + I(v - 10) = J(v)$ (where the first equality follows by Lemma 4.1). The input designs are given by Lemmas 4.4, 4.12 and 4.16. □

Lemma 4.28. *A* $(K_2, K_{1,4})$-*URD*$(1290; r, s)$ *exists for every* $(r, s) \in J(1290)$.

Proof. Apply the Frame-Construction with $t = 40$ and $h = 10$ to a 2-frame of type 2^{16} (where $\alpha = 2$) to obtain a $(K_2, K_{1,4})$-URD$(1290; r, s)$ for each $(r, s) \in J(10) + 32 * I(40) = J(10) + I(1280) = J(1290)$ (where the first equality follows by Lemma 4.1). The input designs are given by Lemmas 4.4, 4.12 and 4.20. □

5 The main result

In the proof of the following lemmas we make use of the equality $\alpha * I(n) = I(\alpha n)$, which holds by Lemma 4.1 when n is a multiple of 8.

Lemma 5.1. *Let* $v \equiv 0 \pmod{40}$. *Then a* $(K_2, K_{1,4})$-*URD* $(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Let $v = 40k$, $k \geq 1$. The case $v = 40$ follows by Lemma 4.7. For $k > 1$, applying the GDD-Construction with $t = 20$ to a C_{2k}-RGDD of type 2^k, i.e., a decomposition of $K_{k(2)}$ into $\alpha = k-1$ hamiltonian cycles (see [15]), gives a $(K_2, K_{1,4})$-URGDD$(\bar{r}, \bar{s})$ of type 40^k for each $(\bar{r}, \bar{s}) \in (k-1) * I(40)$. (The input designs are given by Lemma 4.10.) Filling each group with a $(K_2, K_{1,4})$-URD$(40; r', s')$ with $(r', s') \in J(40)$ (from Lemma 4.7) gives a $(K_2, K_{1,4})$-URD$(v; r, s)$ for each $(r, s) \in J(40) + (k-1) * I(40) = J(40) + \frac{v-40}{40} * I(40) = J(40) + I(v-40) = J(v)$. □

Lemma 5.2. *Let* $v \equiv 10 \pmod{40}$. *Then a* $(K_2, K_{1,4})$-*URD*$(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Let $v = 40k + 10$, $k \geq 0$. The cases $v = 10, 50, 90, 690, 930, 1290$ follow by Lemmas 4.4, 4.25, 4.27, and 4.28. For $k > 2$, $k \neq 17, 23, 32$, start from a 5-RGDD of type 1^{20k+5} (where $\alpha = 5k + 1$, see [1, 2, 3, 9, 26]). Applying the GDD-construction with $t = 2$ gives a $(K_2, K_{1,4})$-URGDD$(\bar{r}, \bar{s})$ of type 2^{20k+5} for each $(\bar{r}, \bar{s}) \in (5k + 1) * I(8)$ (the input designs are given by Lemma 4.3). Now fill the groups with a trivial $(K_2, K_{1,4})$-URD$(2; 1, 0)$ to get a $(K_2, K_{1,4})$-URD$(v; r, s)$ for each $(r, s) \in \{(1, 0)\} + (5k+1) * I(8) = \{(1, 0)\} + \frac{v-2}{8} * I(8) = \{(1, 0)\} + I(v-2) = J(v)$. □

Lemma 5.3. *Let* $v \equiv 20 \pmod{40}$. *Then a* $(K_2, K_{1,4})$-*URD*$(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Let $v = 40k + 20$, $k \geq 0$. The case $v = 20$ follows by Lemma 4.6. For $k > 0$, applying the GDD-Construction with $t = 20$ to a C_{2k+1}-cycle system of order $2k + 1$, i.e. a decomposition of K_{2k+1} into $\alpha = k$ hamiltonian cycles (see [3, 16]), gives a $(K_2, K_{1,4})$-URGDD$(\bar{r}, \bar{s})$ of type 20^{2k+1} for each $(\bar{r}, \bar{s}) \in k * I(40)$. (The input designs are given by Lemma 4.10.) Filling each group with a copy of a $(K_2, K_{1,4})$-URD$(20; r', s')$, with $(r', s') \in J(20)$ (from Lemma 4.6) gives a $(K_2, K_{1,4})$-URD$(v; r, s)$ for each $(r, s) \in J(20) + k * I(40) = J(20) + \frac{v-20}{40} * I(40) = J(20) + I(v-20) = J(v)$. □

Lemma 5.4. *Let* $v \equiv 30 \pmod{80}$. *Then a* $(K_2, K_{1,4})$-*URD*$(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Let $v = 80k + 30$, $k \geq 0$. The cases $v = 30, 110, 190$ follow by Lemmas 4.13, 4.25 and 4.26. For $k > 2$, applying the Frame-Construction with $t = 40$ and $h = 30$ to a 2-frame of type 2^k (where $\alpha = 2$) gives a $(K_2, K_{1,4})$-URD$(v; r, s)$ for each $(r, s) \in J(30) + 2k * I(40) = J(30) + \frac{v-30}{40} * I(40) = J(30) + I(v-30) = J(v)$. (The input designs are given by Lemmas 4.12, 4.13 and 4.24.) □

Lemma 5.5. *Let* $v \equiv 70 \pmod{80}$. *Then a* $(K_2, K_{1,4})$-*URD*$(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. Let $v = 80k + 70$, $k \geq 0$. The case $v = 70$ follows by Lemma 4.25. For $k > 0$, apply the Frame-Construction with $t = 40$ and $h = 30$ to a 2-frame of type 1^{2k+1} (where $\alpha = 1$) to obtain a $(K_2, K_{1,4})$-URD$(v; r, s)$ for each $(r, s) \in J(30) + (2k + 1) * I(40) = J(30) + \frac{v-30}{40} * I(40) = J(30) + I(v-30) = J(v)$. (The input designs are given by Lemmas 4.12, 4.13 and 4.22.) □

As consequence of Lemmas 2.1, 5.1–5.5, our main result immediately follows.

Theorem 5.6. *A* $(K_2, K_{1,4})$-*URD*$(v; r, s)$, *with* $r, s > 0$, *exists if and only if* $v \equiv 0 \pmod{10}$ *and* $(r, s) \in J(v)$.

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