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Pre-service Elementary Teachers' Use of Spatial Diagrams: Investigations of Unbounded Shearing on Solid Figures

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PRE-SERVICE ELEMENTARY TEACHERS' USE OF SPATIAL DIAGRAMS:

INVESTIGATIONS OF UNBOUNDED SHEARING ON SOLID FIGURES

By

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An Abstract of the Thesis Presented
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Mithala and Balacheff (2019) describe three difficulties with two-dimensional representations of three-dimensional geometrical objects: "it is no longer possible to confuse the representation with the object itself," visually observed relationships can be misleading, and analysis of the representation requires the use of lower-dimensional theoretical properties. Despite these difficulties, students are routinely expected to learn about three-dimensional figures through interacting with two-dimensional inscriptions. Three-dimensional alternatives include diagrams realized through various spatial inscriptions (e.g., Dimmel & Bock, 2019; Gecu-Parmaksiz & Delialioglu, 2019; Lai, McMahan, Kitagawa & Connolly, 2016; Ng and Sinclair, 2018). Such diagrams are three-dimensional in the sense that they occupy real (e.g., 3D pen drawings) or rendered (e.g., Virtual Reality/Augmented Reality environments) spaces as opposed to being inscribed or displayed on surfaces. Digital spatial diagrams can be grasped and transformed by gestures (e.g., stretching, pinching, spinning), even though they can't be physically touched (Dimmel & Bock, 2019). Spatial diagrams make it possible to use natural movements of one's head or body to explore figures from new perspectives (e.g., one can step

inside a diagram), as they natively share the three-dimensional space. In this study I ask: How do learners use perspective to make arguments while exploring spatial diagrams? In particular, how do participants use perspectives outside and within geometric figures to make arguments while exploring spatial diagrams?

To investigate this question, I designed a large-scale spatial diagram of a pyramid whose apex and base were confined to parallel planes. The diagram was rendered in an apparently unbounded spatial canvas that was accessible via a head-mounted display. The pyramid was roughly 1 meter in height and the parallel planes appeared to extend indefinitely when viewed from within the immersive environment. I created this diagram as a mathematical context for exploring shearing, a “continuous and temporal” measure-preserving transformation of plane and solid figures (Ng & Sinclair, 2015, p.85).

I report on pairs of pre-service elementary teachers’ arguments about shearing of pyramids, using Pedemonte and Balacheff’s (2016) ck¢-enriched Toulmin model of argument. Shearing is a mathematical context that is likely novel to pre-service elementary teachers and provides an opportunity to connect transformations of plane and solid figures. Participants used perspectives outside and within the diagram to make arguments about the shearing of pyramids that would not be practicable with rigid three-dimensional models or dynamic two-dimensional representations. The results of this study suggest that the dimensionality of the spatial diagrams supported participants’ arguments about three-dimensional figures without mediation through projection or lower-dimensional components. The findings of this study offer a case that challenges the constraints of two-dimensional representations of three-dimensional figures, while maintaining theoretical constraints in a spatiographically accurate representation.

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INTRODUCTION

In school geometry, diagrams are a common form of representation. Students' interactions with diagrams vary based on the affordances of the diagram, which shape their interactions with mathematical concepts (Herbst, 2004). Common diagrams in school geometry include stylus-based inscriptions on plane surfaces (e.g. paper, chalkboard) and dynamic geometry environments (e.g. Geogebra, Cabri). These mediums restrict diagrammatic representations of solid (three-dimensional) figures to two-dimensional projections (Mithila & Balacheff, 2019). Both stylus-inscribed diagrams and projected dynamic geometry environments favor representing geometric relations over spatiographic accuracy. That is, while they represent properties of geometric figures and relationships between those figures, these representations are either iconic or distorted by projection and thus the properties of the figure are not always consistent with a learner's visual or tactile observations of the figure.

Not all representations of solid figures are constrained to projections. Physical manipulatives (e.g. blocks, tangrams) are commonly used to represent geometric figures, especially in the lower grades. These representations give tangible and spatiographically accurate feedback to the learner, but show only one instance of a class of figures and thus limit the attention to geometric relations. The representation is then either iconic, representing the properties of a class (e.g. a right triangle with two perpendicular segments representing many triangles sharing this property), or a non-iconic, specific case of a figure (e.g. a triangle with perpendicular segments of length 3 cm and 4 cm) (Mithalal & Balacheff, 2019). In the iconic case, the learner cannot necessarily use the observed spatiographic features of the representation, and must reason about a class from its properties. For example, the iconic representation of a right triangle with perpendicular sides of equal length might be drawn to represent any right triangle, but it requires the learner's theoretical attention to the independent variability of a right triangle's side lengths. In the non-iconic case, the properties of a class of figures may not be salient. For example, a right triangle drawn with perpendicular segments of length 3 cm and 4 cm might be used in an exercise where students empirically calculate the length of the remaining side (completing the Pythaogrean triple), which could be accomplished without theoretical attention to the relationships between the three sides of a right triangle.

Laborde (2005) notes that dynamic geometry environments offer both affordances of spatio-graphic accuracy and attention to geometric relations, which is not practicable with physical manipulatives or stylus-based inscriptions. Dynamic spatial diagrams offer an opportunity to unite the affordances of spatiographic accuracy and attention to geometric relationships for solid figures. Dynamic geometry environments unite these affordances for plane figures; spatial diagrams offer a novel context for solid figures. In this study, I consider projected spatial inscriptions; projected spatial inscriptions are inscriptions in a three-dimensional space that "fill space but do not take up space" (Dimmel, Paniscio, Bock, submitted) and can be rendered with immersive spatial displays (Dimmel & Bock, 2019). Google's Tilt Brush is one example of a tool for constructing projected spatial inscriptions which "oppose the conventional encounter of a painting [inscribing] as a flat rectangular plane" (Chittenden, 2018, p. 389), where the representation of a three-dimensional figure is not distorted by projection or constrained to a specific point of view.

Previous work has investigated the affordance of perspective in dynamic spatial diagrams (Bock & Dimmel, submitted). In this study, I ask: How do learners use perspective to make arguments while exploring spatial diagrams? In particular, how do participants use perspectives outside and within geometric figures to make arguments while exploring spatial diagrams? To investigate this question, I designed two spatial diagrams: a triangle and a pyramid which could each be sheared without bound (see discussion of shearing in Background) by grasping and throwing their vertices (see Environment Design). The pyramid bound to parallel planes presents a case where the dimensionality of its representation may be significant; the pyramid, a three-dimensional figure, is represented in an immersive, three-dimensional space.

As a spatial diagram, the triangle is a plane figure, embedded in a three-dimensional working space (Bock & Dimmel, submitted). The pyramid is a solid figure rendered in a three-dimensional working-space. Pre-service elementary teacher participants made arguments about the effects of shearing on the area and volume measures of the triangle and pyramid, respectively. These arguments were analyzed using Pedemonte and Balacheff's (2016) ck¢-enriched Toulmin model of student argumentation to describe the relationship between participants' use of the dimensionality of the spatial diagrams and their mathematical knowings about the area and volume preserving properties of the shearing transformation. By dimensionality, I refer to spatial diagrams' properties as inscriptions on a three-dimensional space rather than inscriptions on a two-dimensional surface, which enables learners to take a variety of perspectives through movement of their bodies around and through the diagram. The results of this study suggest that learners use the dimensionality of the spatial diagram to interact with the diagram by taking perspectives within the diagram, observing their own bodily interactions with the diagram (e.g. reaching inside a cube), and passively manipulating the diagram with their body (e.g. dragging a vertex without explicitly planning the manipulation).

These interactions supported arguments that both did and did not use lower-dimensional components of the solid geometric figure to make inferences about the figure's geometric properties. The findings of this study help to explain how novel representations, in particular spatial diagrams, shape learner's arguments about geometric figures.

BACKGROUND

Mathematical concepts are mediated through representations (Duval, 2006). The design of a representation offers a set of affordances and constraints; these affordances and constraints shape how learners engage with concepts that are realized in the representations. Different representations can thus offer access to different ways of knowing about a mathematical concept. Laborde (2005) described how learners engage with theoretical and spatiographic properties of a geometric figure, mediated through a diagram[1]. Theoretical properties of figures include those that are attended to in formal mathematical arguments. For example, that the diagonals of a rectangle are congruent is a theoretical property. Spatiographic properties of a figure are observed from its graphical representation in a diagram and may or may not be necessary consequences of the theoretical properties. For example, that opposite sides of a rectangle are oriented in the same direction is a spatiographic property. Both spatiographic and theoretical properties are used by students to construct geometric arguments(Laborde, 2005).

Mithala and Balacheff (2019) describe three difficulties with two-dimensional representations of three-dimensional geometrical objects: "it is no longer possible to confuse the representation with the object itself," spatiographical relationships can be misleading, and the analysis of the representation requires the use of lower-dimensional theoretical properties. Much of the existing literature uses representations of the volume of solid figures that are bounded in this two-dimensional representation space (Clements, Samara, & Van Dine, 2017; Cullen, Barrett, Kara, Eames, Miller & Klanderman, 2017; Dorko & Speer, 2015, 2013; Huang, 2012; Kulm, 1975; Stevens, Hobson, Moore, Paoletti, LaForest & Mauldin, 2015; Van Dine, Clements,

[1] A mathematical figure is a set of points that satisfy a set of constraints defining a figure and a geometric diagram is a graphical representation of such a figure (Dimmel & Herbst, 2015).

Barrett, Samara, Cullen & Kara , 2017). These representations require students to mediate their interactions with a mathematical figure through a projection, which places constraints on their reasoning.

Spatial Diagrams

Ng and Sinclair (2018) represented three-dimensional figures using 3D pens that allow for making inscriptions in space. Ng and Sinclair (2018) found that the 3D pens offered an opportunity for students to construct solid figures using lower-dimensional reconstructions (e.g. constructing a pyramid by drawing its edges), and noted temporal and motion-based aspects of students' drawings of functions in a plane. While these pens address some of the issues described by Mithala and Balacheff (2019) with three-dimensional representations, the inscriptions are static and do not allow students to easily construct diagrams that are spatiographically and theoretically aligned for a set of figures. That is, the static constraints of 3D pens limit their ability to represent a set of figures that satisfy a set of theoretical constraints while also allowing the learner to make accurate visual observations about the set of figures.

The temporal and motion-based entanglements of the 3D pens are limited to the process of inscription and do not easily transfer to temporal, non-rigid transformations of figures. Ng and Sinclair (2015) noted the importance of representing shearing of plane figures as "a continuous and temporal geometric transformation that preserves area;" current implementations of 3D pens do not make practicable representations of shearing of solid figures as a continuous and temporal transformation.

Other researchers have challenged the constraints of two-dimensional representations of solid figures with three-dimensional digital representations. Lai, McMahan, Kitagawa, and

Connolly (2016) designed a virtual reality environment where rectangular prisms were labeled with lengths along three of the edges (e.g. "H = 1") in each cartesian direction and displayed a symbolic representation of volume as "V = W * H * L" (p. 706). The environment supported a task where students used a cursor to modify the dimensions of the rectangular prism to reach a target volume (Lai et al., 2016). While this representation of volume is both three-dimensional and dynamic, it used a fixed-perspective diagram (i.e. learners cannot walk around the diagram to take different perspectives) with markings similar to those available in two-dimensional print and dynamic geometry environments. While the diagram makes accessible both the theoretical (e.g. measures of edges) and spatiographic properties of the figure (e.g. the space occupied by the figure, perpendicular relationship between faces), learners can reason without these properties as measurements are rendered as values that variables take (e.g. H=1, W=2, L =3), which can be used to procedurally calculate volume without considering the geometry of the figure (e.g. V = W*H*L = 2*1*3). The representations of volume accessible with this spatial diagram are constrained by the actions on the diagram available to the learner and the perspectives that the user is able to take. This limits the opportunities for students to engage with theoretical properties of the triangle, focusing their investigation on execution of empirical procedures.

In another challenge of the dimensionality of two-dimensional representations, Gecu-Parmaksiz and Delialioglu (2019) developed a set of augmented reality manipulatives that rendered solid figures in an augmented reality space, where students could make rigid transformations of the figures through a multi-touch interface. These digital inscriptions had similar transformational capabilities to wooden blocks, but Gecu-Parmaksiz and Delialioglu

(2019) found a significant difference in learners' shape recognition abilities who had engaged with the digital manipulatives. However, Gecu-Parmaksiz and Delialioglu (2019) did not theorize what affordances of the representation contributed to this difference. These manipulatives partially address the three constraints described by Mithala and Balacheff (2019) of two-dimensional displays. While the diagrams are mediated by projection, the projection dynamically adapts to the learner's perspective, and thus spatiographic observations may be made more accurately. However, this implementation is limited to static representations where the properties of the figures are fixed.

Finally, learners can interact with some spatial diagrams through bodily movement. Dimmel and Bock (2017) describe a gesture-based construction environment where learners engage with figures through a grasping gesture, in a virtual reality environment rendered by immersive spatial displays. The revision of the environment presented in this study extends the grasping gesture to include a throwing gesture. Grasping and throwing serve both as an analog to dragging and dropping in dynamic geometry environments (Steinicke & Hinrichs, 2006) and as an analog to interactions with material objects in daily life. The grasping and throwing interaction with a spatial diagram enables the continuous and temporal dynamic transformations that are seen in dynamic geometry environments (Bock & Dimmel, 2017; Bock & Dimmel, submitted; Dimmel, Pandiscio, Bock, submitted). The rendering of a spatial diagram in an immersive spatial display allows for native control over perspective (Bock & Dimmel, submitted; Dimmel, Pandiscio, Bock, submitted), allowing the user to walk around and inside a diagram as it natively shares their three-dimensional space. The combination of the dimensionality of the spatial diagram and the continuous and temporal interactions offer a novel environment where both spatiographic and theoretical properties of the figure are accessible.

Shearing as a Mathematical Concept

To investigate how learners use the dimensionality of spatial diagrams to make arguments, I consider shearing as a mathematical context. A transformation of a plane figure is called shearing and preserves area if a figure can be bound between two parallel lines such that the length of a parallel cross-section of the figure is preserved (Ng & Sinclair, 2015). Consider a triangle *(*Figure 1*: ABC)* bound to two parallel lines (Figure 1*: AB, CD*), where one line passes through two vertices of the triangle (the connecting edge is called the base, Figure 1: *AB*) and a second line runs parallel through the remaining vertex (the apex, Figure 1*: C*). Any non-empty intersection of a line parallel to the base of the triangle and the triangle yields either a point (the apex of the triangle) or a line segment bound by the two edges of the triangle *(*Figure 1*: EF)*. If the apex of the triangle is moved along the line (Figure 1: *ABD*), the cross-section is translated (Figure 1*: HI)*, remaining fixed in length. Thus, moving a vertex of a triangle along a line parallel to the opposite side is a shearing of the triangle and preserves the triangle's area.

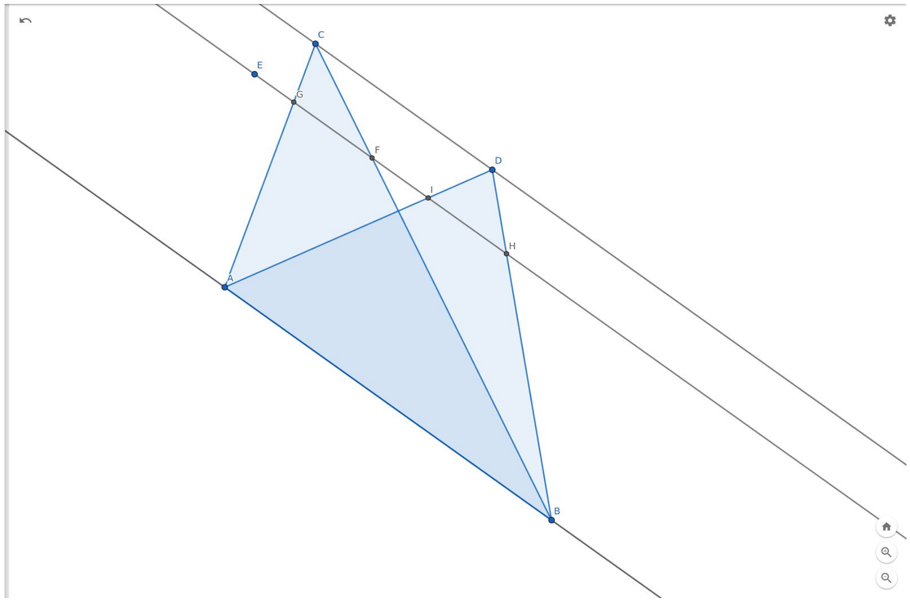

Figure 1. Triangle ABC can be sheared to form Triangle ABD. This figure illustrates the shearing as an area-conserving transformation of Triangle ABC.

Shearing can be extended as a volume-preserving transformation of an *n*-dimensional polytope whose height is bound between two parallel, *(n-1)*-dimensional hyperplanes. As a solid (3-dimensional) figure, consider a pyramid whose apex is bound to a plane parallel to its base (Figure 2*: ABCDEF)*. Any non-empty intersection of a pyramid and a plane parallel to its base is either the apex of the pyramid or a dilated copy of the base of the pyramid. In the case of a dilated copy of the base *(*Figure 2*: IJKLM)*, this cross-section is translated as the apex of the pyramid moves along a plane parallel to its base *(*Figure 2*: OPQRS)*. Then the area of the cross-section and the volume of the pyramid are preserved under the shearing transformation.

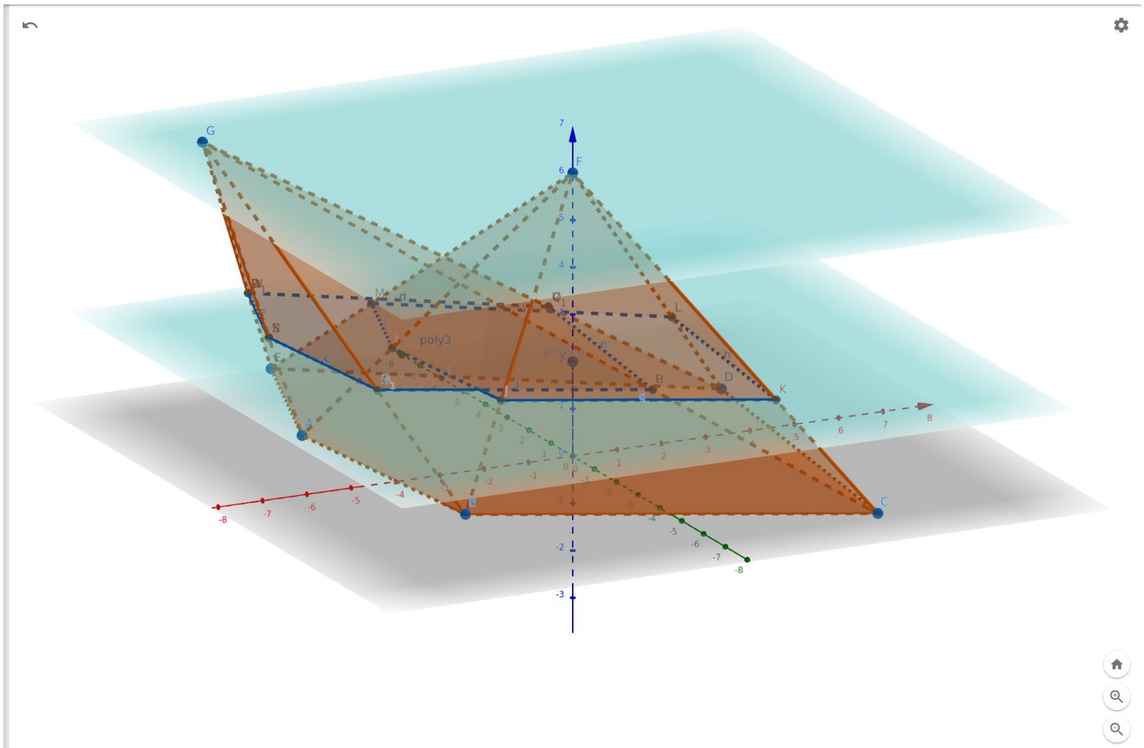

Figure 2. Pyramid ABCDEF can be shared to form Pyramid ABCDEG. This figure illustrates shearing as a volume-preserving transformation of Pyramid ABCDEF.

Shearing is a measure-preserving process that can be investigated in both plane and solid figures. As a continuous and temporal transformation (Ng & Sinclair, 2015), shearing lends itself to representation in dynamic diagrams. Further, dynamic spatial diagrams offer a space to represent the shearing of three-dimensional figures without projection.

THEORETICAL FRAMEWORK

In this study, I used Pedemonte and Balacheff’s (2016) conceptions-knowing-concept (ck¢) enriched Toulmin model to analyze learners’ arguments in terms of their conceptions. This model is a hybrid of Balacheff and Gaudin’s (2002; 2010) conceptions-knowing-concept (ck¢) model of student conceptions and the Toulmin model of argumentation (Toulmin, 1958). The ck¢ model of student conceptions has been used to describe conceptions in terms of observable components of the interactions between learners and their environment (Balacheff & Gaudin, 2002; Balacheff & Gaudin, 2010; DeJarnette, 2018; Herbst, 2005; Mithalal & Balacheff, 2019). In the ck¢ model, the learners’ conception is understood to be temporally emergent and situated within the learners' environment. The Toulmin model of argumentation describes a learners’ argument in terms of an inference where the learner transforms data from their environment into a claim. The ck¢-enriched Toulmin model composes these two models to describe how a learner uses their mathematical knowings to transform data from their environment into a claim. In the investigation of an open ended task contextualized within a novel environment, this hybrid model connects learners’ inferences to their interactions with their environment while maintaining a rich description of the learners’ mathematical knowings.

Conceptions-Knowing-Concept Model

The ck¢ model of student conceptions describes four observable components of learners’ interactions with their environment: a problem, a representation system, a set of operators, and a control structure. The problem is a mathematical description of a situation that perturbs the learners’ understanding of the mathematical concept; it is a conflict between their knowing of a mathematical concept and the feedback from the learners’ environment. The representation system is the set of semiotic resources that mediate feedback from the learners' environment. The learners operate on the representation system with operators, mathematical actions that make an attempt to resolve the problem. Finally, the control structure is the system that the learners use to evaluate the appropriateness of an operator and whether a sequence of operators resolves the problem.

To illustrate the components of the model, I describe a hypothetical conception of area equality below using a two-dimensional dynamic geometry environment.

Problem.

> Consider the following description of a problem related to partitioning of a rectangle
> *Does the diagonal of a rectangle necessarily partition the rectangle into two triangles of equal area?*

For this to be a problem for a learner, the learner must not immediately know the solution or have routine procedures for finding the solution. In this hypothetical case, the learner may have investigated several instances of such a partitioning of a rectangle but is looking for a more general explanation using relationships between geometric properties.

Representation System.

A learner might investigate this phenomenon in a dynamic geometry environment (e.g. Geogebra), where they construct a rectangle, with dynamic vertices constrained to maintain a parallel relationship for opposite edges and a perpendicular relationship for adjacent edges and a line drawn through opposite vertices. The learner might drag the vertices of the rectangle through space to observe a set of instances of rectangles.

Operators.

For the representation above, the learner might take several steps to construct and modify the representation to solve the problem. Each of the steps is relevant to solving the problem within some control structure (described below). In this hypothetical example, the learner is given a construction of a rectangle ABCD with a line through its diagonal. The first steps serve to construct the dynamic diagram, the later steps make an inference from observations of the dynamic diagram.

Observe that AB is congruent to CD. Observe that AC is congruent to BD. Observe that AD is congruent to itself. Infer that triangle ABD is congruent to triangle ACD. Since the triangles are congruent, they're areas are equal.

Drag the vertices of the rectangle through space. Observe that all of the segment congruence relationships are maintained. Conclude that diagonal of a rectangle partitions the rectangle into two triangles of equal area.

Control Structure.

For the hypothetical operators and representation system described above, the control structure needs to validate the appropriateness of each operator. The first statement situates the operators that construct the dynamic diagram as relevant to the problem. The second statement relates observed properties of the segments to properties of the triangle. The third statement relates two properties of the triangle. Together, these statements validate the appropriateness each of the hypothetical learner's operators and, in sequence, their resolution of the problem.

If the observed properties of a figures do not change as its vertices are manipulated to space, they are inherent properties of the figure.

Two triangles are congruent if all of their segments are congruent.

Congruent triangles have equal area.

Toulmin Model of Argumentation

The second component of the ck¢-enriched Toulmin model is the Toulmin model of argumentation. The Toulmin model of argumentation (Toulmin, 1958) represents an argument as an inference that transforms data into a claim; in this study, Toulmin models are applied to learners' mathematical arguments. The learner justifies the inference with a warrant, which is situated within a backing; this backing situates the warrant within a set of norms for argument within a particular domain. The learner may offer a rebuttal to hedge their claim by acknowledging cases where their warrant may not hold. Finally, the researcher assesses the mathematical accuracy of the learner’s claims with the qualifier which reflects both the truth of the claim and the validity of the warrant.

Below, I extend the same hypothetical scenario of a student exploring the unbounded shearing of a triangle to be examined through the Toulmin model.

Data.

The data in the learners’ argument comes from the dynamic diagram. In particular, the learner observes that opposite sides of a rectangle are congruent and the diagonal of a rectangle is congruent to itself. In relation to Balacheff and Gaudin’s (2010) ck¢ model of student conceptions, the data is the feedback from the representation system.

Claim.

The learners' claim that the area of the triangles of a rectangle partitioned by its diagonal are equal. In relation to Balacheff and Gaudin's (2010) ck¢ model of student conceptions, the claim is the resolution of the problem. While the problem is a formal mathematical statement of the perturbation of the learners' knowing of a mathematical concept, the claim may be characterized in vivo as an informal mathematical statement.

Warrant.

The triangles are congruent since all of their edges are observed to be congruent. Congruent triangles have equal areas.

In relation to Balacheff and Gaudin's (2010) ck¢ model of student conceptions, the warrant is composed of an operator – the observation of congruent height and base segments – and part of the control structure – the implication that a constant height and a constant base would result in a constant area.

Backing.

The learners construct a dynamic diagram and attend to relationships between quantities in the diagram. The norms for the learners' argument include the use of a dynamic diagram to investigate a set of cases of a mathematical phenomena as representative of the properties of a class of figures. In relation to Balacheff and Gaudin's (2010) ck¢ model of student conceptions, the backing is contained within the control structure. The backing is the axioms about how a problem is investigated that support the appropriateness of a given operator.

Rebuttal.

The learners hedge their claim by noting that they might not have correctly observed the congruence of opposite sides without measuring. This hedging is an acknowledgement of a case where their warrant may fail to support their claim. In Balacheff and Gaudin's (2010) ck¢ model of student conceptions the rebuttal isn't described as part of the students' conception.

Qualifier.

The researchers assign a qualifier of "surely" to the learners' argument. The learners made a mathematically correct claim with an appropriate warrant and sufficient data to support the claim. In Balacheff and Gaudin's (2010) ck¢ model of student conceptions the qualifier is not described, as the ck¢ model recognizes conceptions as situated within a context and allows for limits on the mathematical accuracy or generalizability without explicit evaluation.

Ck¢-enriched Toulmin Model

Pedemonet and Balacheff (2016) offer a model of learners' argumentation which composes the ck¢ model of student conceptions with Toulmin's model of argumentation. This model uses the mathematical richness of the ck¢ model to situate the Toulmin model's description of the learners' inference. Figure 3 shows a graphical representation of the model. The inference transforming the data into the claim is represented with an arrow. The arrow is supported by a warrant, which is contextualized within a backing. The qualifier is applied to inference with respect to both the warrant and the claim, but is situated within the context of the backing. Finally, the warrant is partitioned into two components – the operator and the control – which share the warrant's oval. From the perspective of the ck¢ model, the control is partitioned into two components – the warrant and the backing – which lie adjacent.

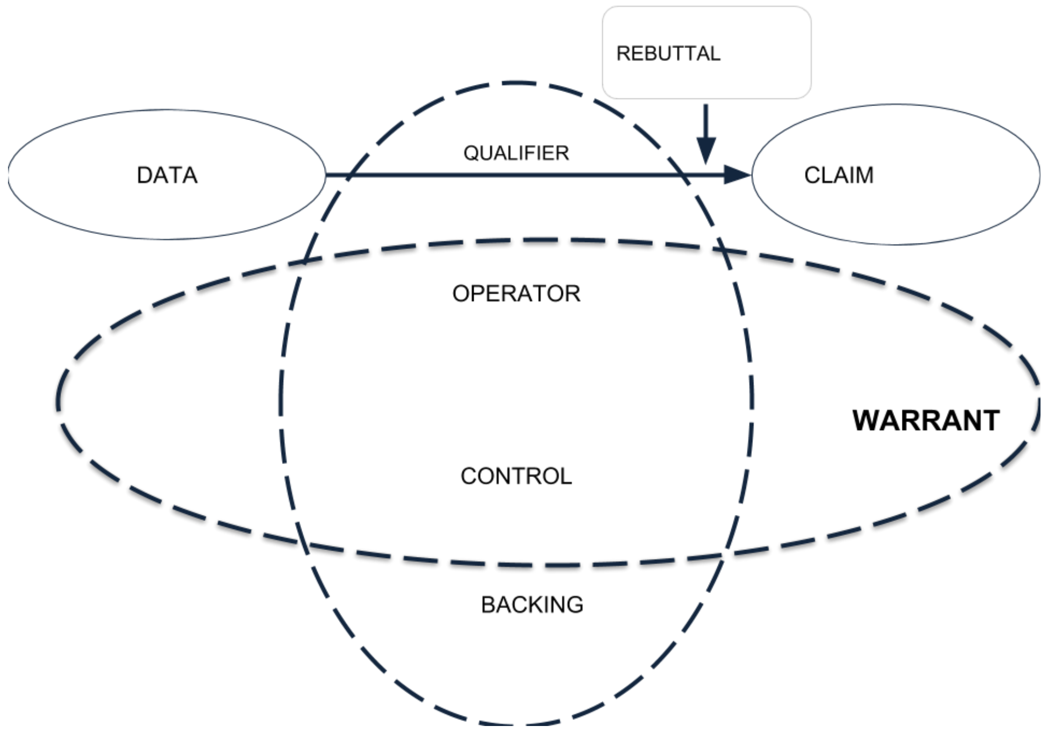

Figure 3. Conception-knowing-concept enriched Toulmin model. This figure is a recreation of an image that appears in Pedemonte & Balacheff (2016, p. 108). This figure illustrates the ck¢-enriched Toulmin Model, structuring components of the model around an inference (arrow) transforming data into a claim.

Both the Toulmin model of argumentation (Toulmin, 1958) and Pedemonte and Balacheff's (2016) ck¢-enriched Toulmin Model sequence components of argument logically with respect to the inference. However, the chronological sequence of learners' construction of an argument may not be linear. For example, a claim may be made and later revised after a warrant is developed or additional data may be acquired as it is needed within the middle of an argument. In this thesis, I describe learners' arguments with respect to the components of the ck¢-enriched Toulmin Model within a narrative that follows the learners chronologically. This choice of representation of the learners' arguments is intended to support the readers’ understanding of the learners’ interactions within a novel environment.

Interactions within the Environment.

The ck¢-enriched Toulmin Model highlights connections between learners' interactions with the spatial diagrams that were developed for this study (see ‘Environment Design’, below) their knowings of shearing of plane and solid figures while they are constructing mathematical arguments. The use of spatial diagrams can be seen within the data used to make an inference and the mathematical knowings appear both in the claim that the learners make and in the operators and control structure used by the learners to support that claim. In this study, I use a rich description of the learners’ mathematical knowings and the interactions with the spatial diagrams to identify features of the interactions that depend on the dimensionality of the spatial diagrams and the related features of the learners’ knowings of shearing. These relationships can be compared with other representations of area and volume to understand whether the learners’ interactions with spatial diagrams are unique within this context or could be constructed in other contexts.

ENVIRONMENT DESIGN

For this study, I designed and developed a virtual environment with the intention of making salient the dimensionality affordances of dynamic spatial diagrams. Dynamic spatial diagrams share a three-dimensional space with the learner and the potential for non-rigid transformations. Shearing provides a context where the affordances of dynamic diagrams allow for students to relate a transformation to the family of figures that it generates (Ng and Sinclair, 2015) and extends to cases in solid geometry where the shared three-dimensional working-space is relevant. Dynamic spatial diagrams provide a context where learners are immersed in the three-dimensional space of the spatial diagram where learners might reason about solid geometric figures without deconstruction into lower-dimensional components.

In this chapter, I first describe the diagrams that are available to the participants. Second, I describe the interactions with the diagram that were accessible to the immersed participant. Third, I describe the perspectives accessible to the immersed and non-immersed participants. Finally, I describe the opportunities that the environment may offer for participants to explore and make arguments about shearing.

Diagrams

The virtual environment included two dynamic spatial diagrams: a triangle and a pyramid. The triangle was constructed such that a line ran through one side of the triangle and a parallel line ran through the opposite vertex (Figure 4)[2]. The triangle's lines were oriented in planes parallel to the floor, so that the participants could easily reach the vertices as they walked

[2]One vertex of the triangle shared the plane of the pyramid's apex. This was referred to as the apex of the triangle. Two of the vertices of the triangle shared the plane of the pyramid's base, they were referred to as the base of the triangle.

and dragged the vertices. Each of the vertices could be moved by pinching and dragging, but were bound to their respective lines (Figure 8). The pyramid was constructed with a plane through its base and a parallel plane through its apex (Figure 5). The pyramid's planes were oriented parallel to the floor. Only the apex of the pyramid could be moved by pinching and dragging; the apex was constrained to its plane while it was dragged (Figure 9). Each of the figures could be sheared without bound by grasping and throwing, which was made possible by the unbounded rendering of the diagram. The rendering of unbounded shearing allowed learners to access a representation of the limit case of a geometric transformation. The constraints on each of the figures required any manipulation of the apex of the pyramid or triangle to result in a shearing transformation.

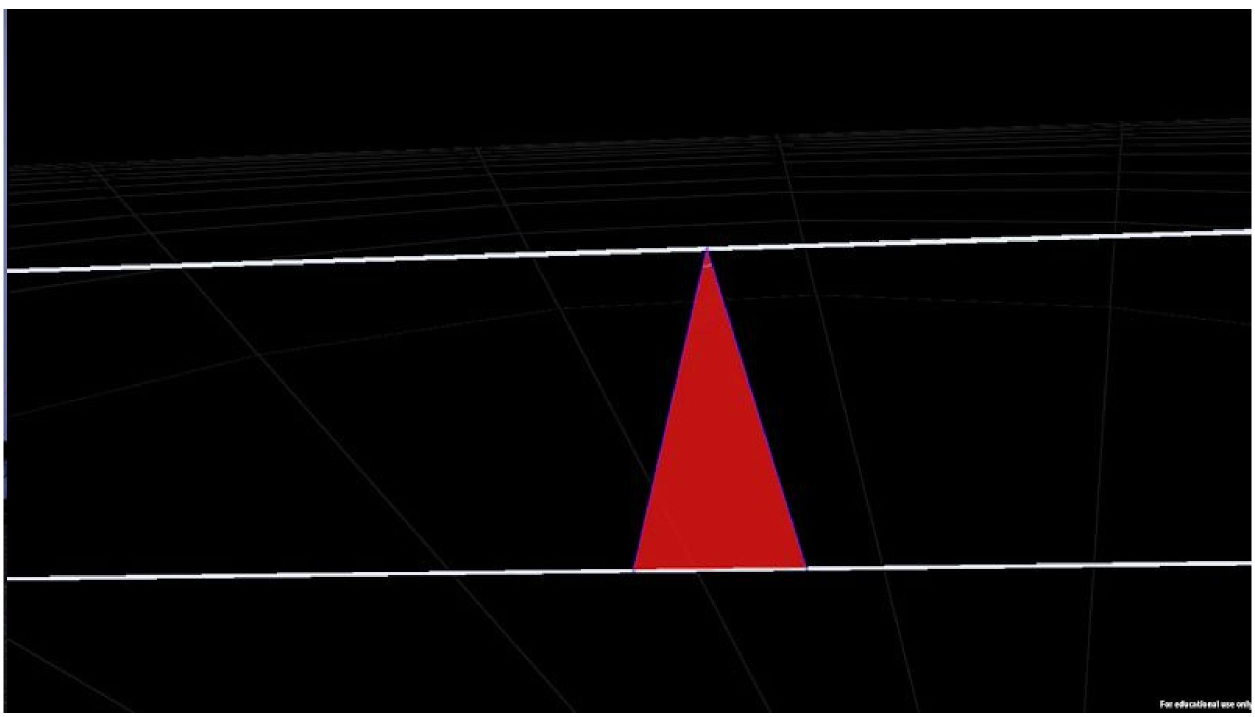

Figure 4. A triangle bound to parallel lines. This figure shows a euclidean perspective of the triangle diagram, in the first person (immersed user's) perspective.

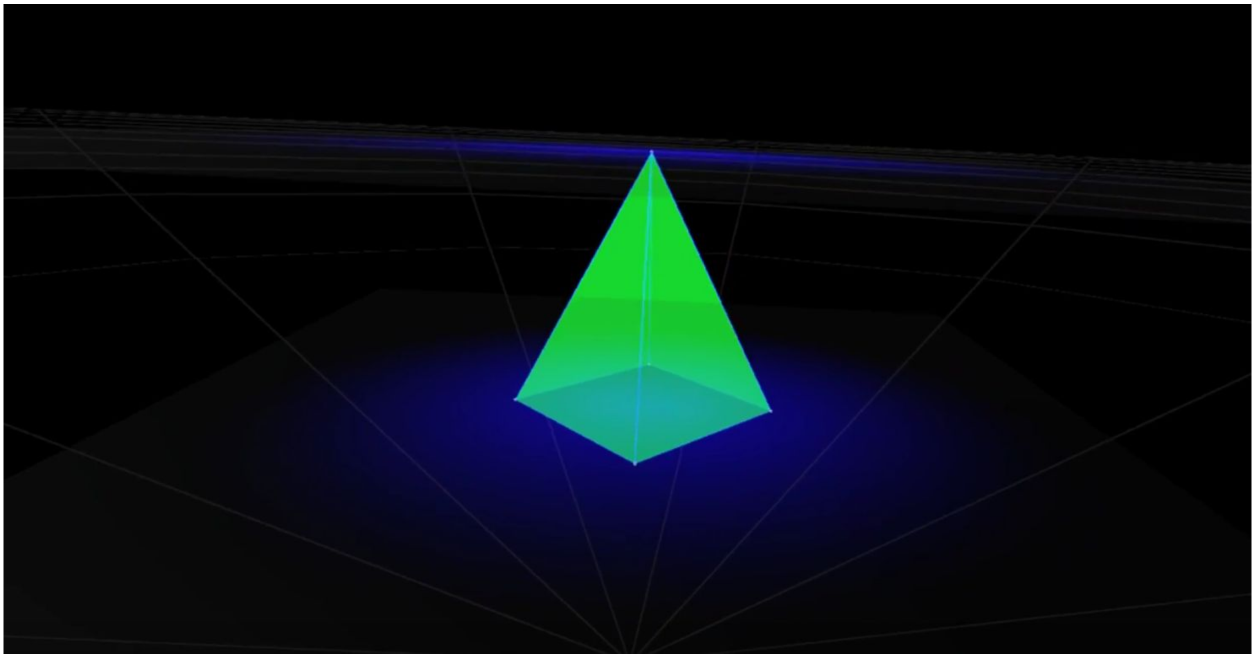

Figure 5. A pyramid bound to parallel planes. This figure shows a first person (immersed participant's) perspective of the pyramid diagram from a short distance away.

Interactions

Users could interact with the diagrams by pinching and dragging, throwing, or moving their bodies. To translate a vertex, users reached out and pinched the vertex with three fingers and dragged the vertex through space (Figure 8; Figure 9). To throw a vertex, users reached out and pinched the vertex with three fingers, dragged the vertex and released the vertex while making a throwing motion (Figure 10; Figure 11). Finally, participants could manipulate cross-sections of the triangle (Figure 6) and pyramid (Figure 7) by making a downward facing open palm in a plane parallel to the floor. The cross-secting plane (for the pyramid or the triangle) would move to the plane of the user's hand until the user interrupted the gesture by closing their hand, turning their hand, or looking away from their hand.

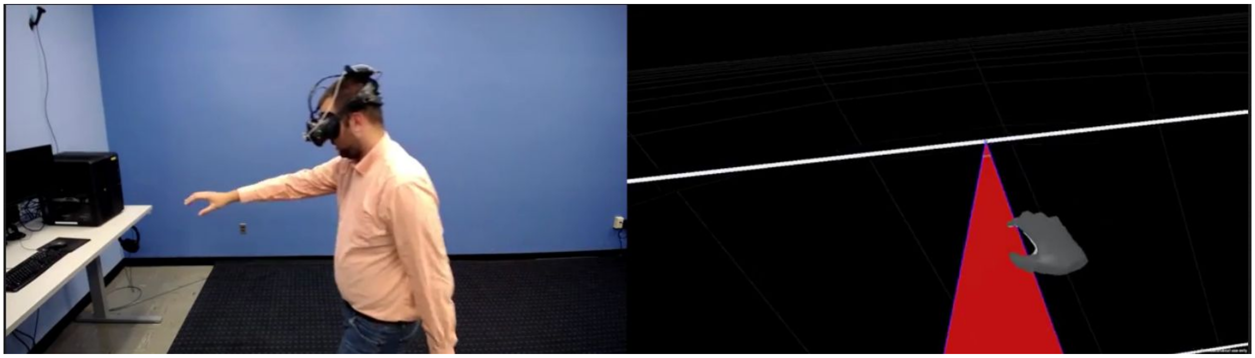

Figure 6. Triangle cross-section. This figure shows the user [left] manipulating the cross-section [right, grey line segment] of the triangle with a downward facing open palm gesture. The triangle is cross-sected in the plane of the user's hand.

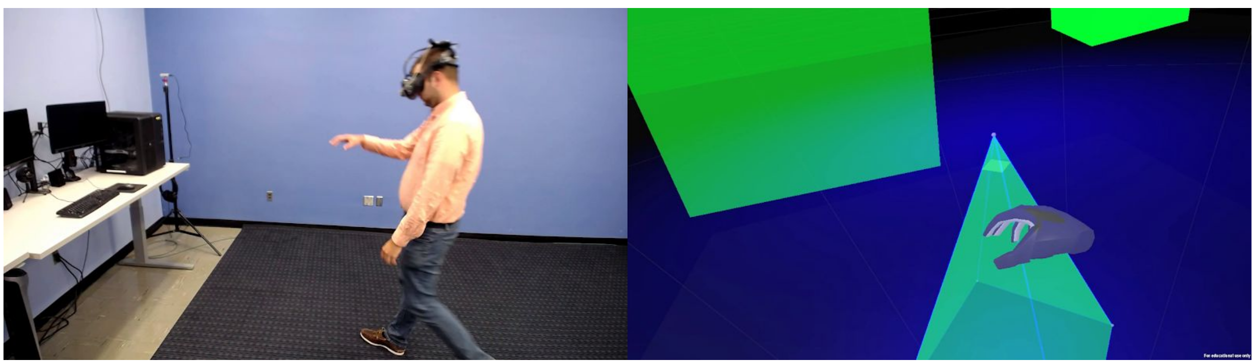

Figure 7. Pyramid cross-section. This figure shows the user [left] manipulating the cross-section [right, light green square] of the pyramid with a downward facing open palm gesture. The pyramid is cross-sected in the plane of the user's hand.

Perspective

In Figure 12, users controlled their perspective in the environment by moving their body (e.g. turning their head, walking, bending; see Bock & Dimmel, submitted; Dimmel, Bock & Pandiscio, submitted).

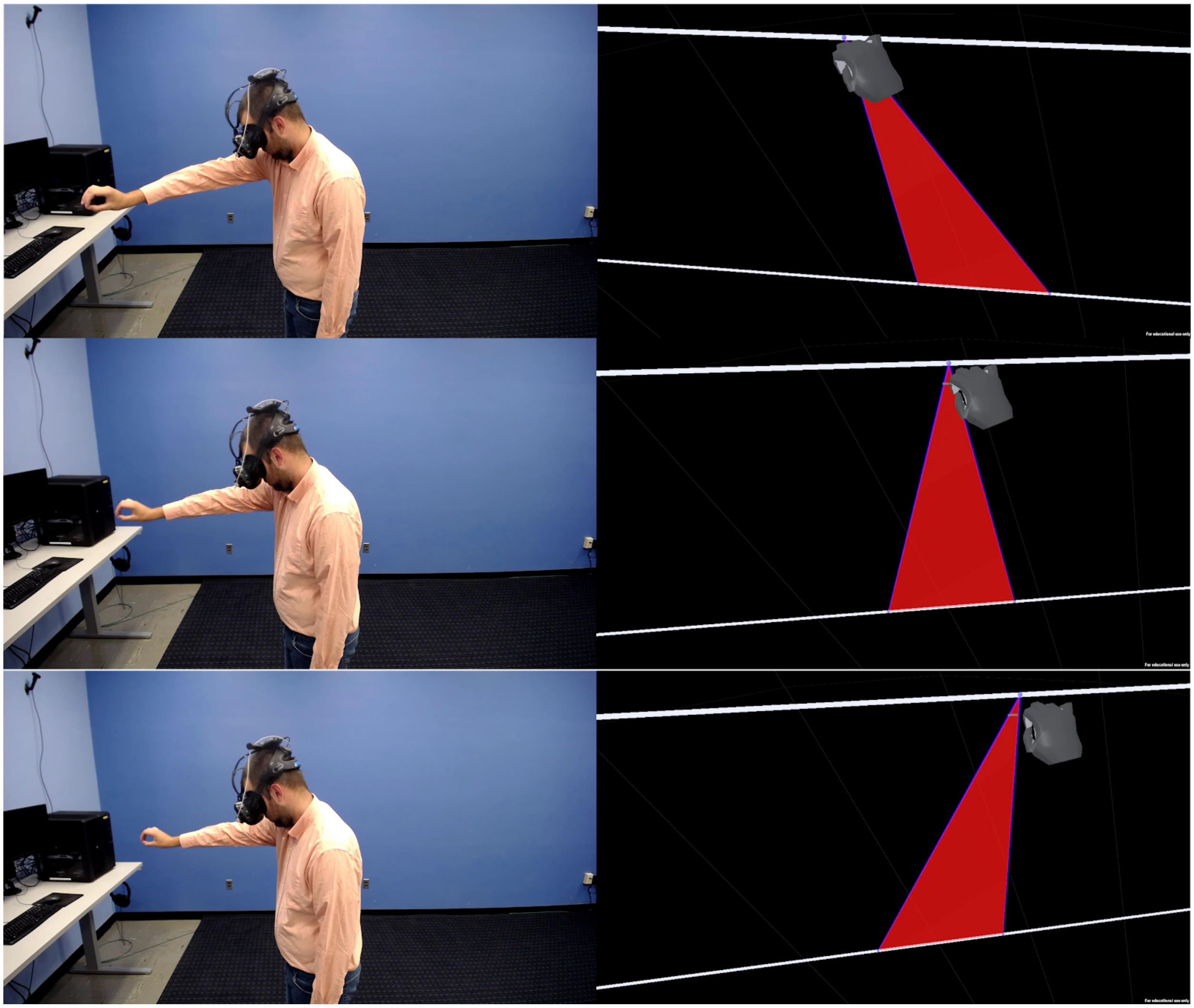

Figure 8. Grasping and Dragging a Vertex of the Triangle. This figure illustrates a user [left] dragging the apex of the triangle [right] along its line.

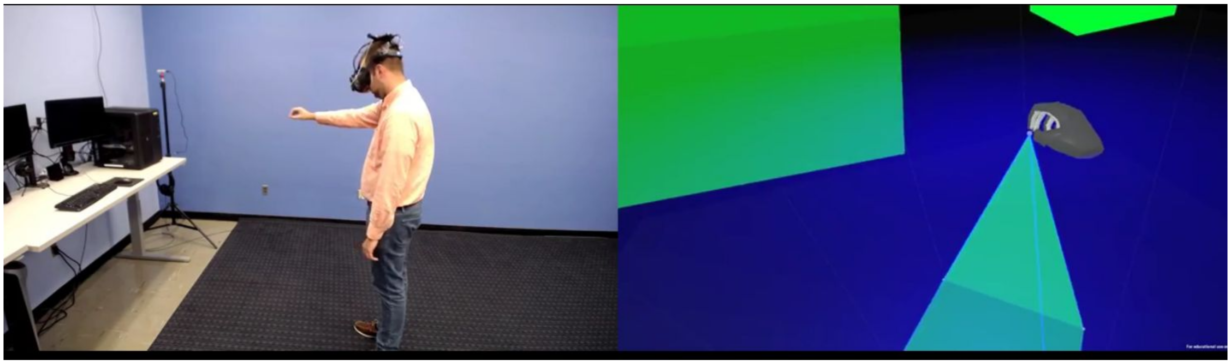

Figure 9. Grasping and Dragging a Vertex of the Pyramid. This figure illustrates a user [left] dragging the apex of the pyramid [right] within its' plane.

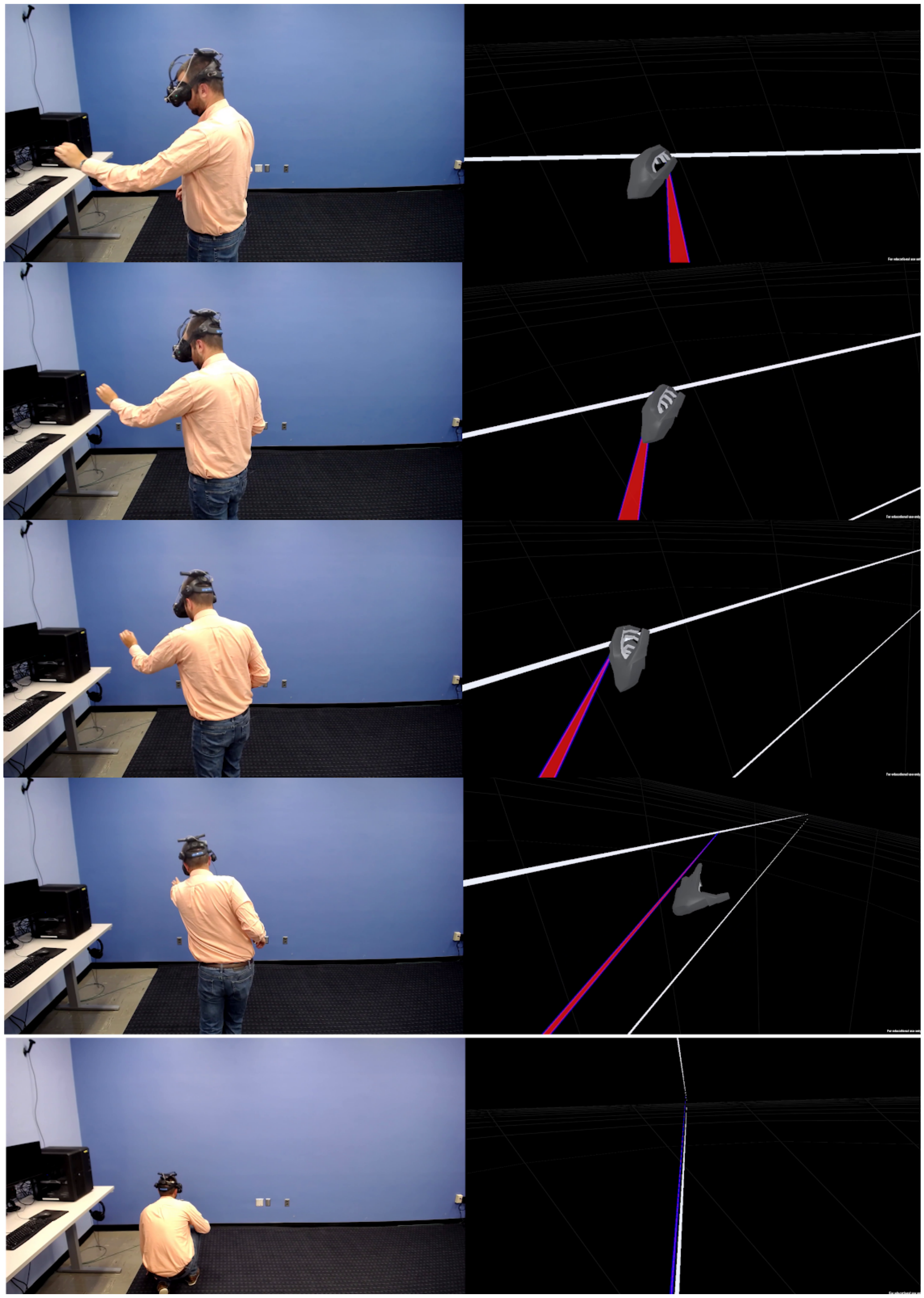

Figure 10. Throwing the apex of the triangle into the distance. This figure illustrates a user [right] throwing the apex of the triangle along its' line [top, middle]. The user kneels [right, bottom] to view the triangle extend indefinitely into the distance as it is sheared [left, bottom]

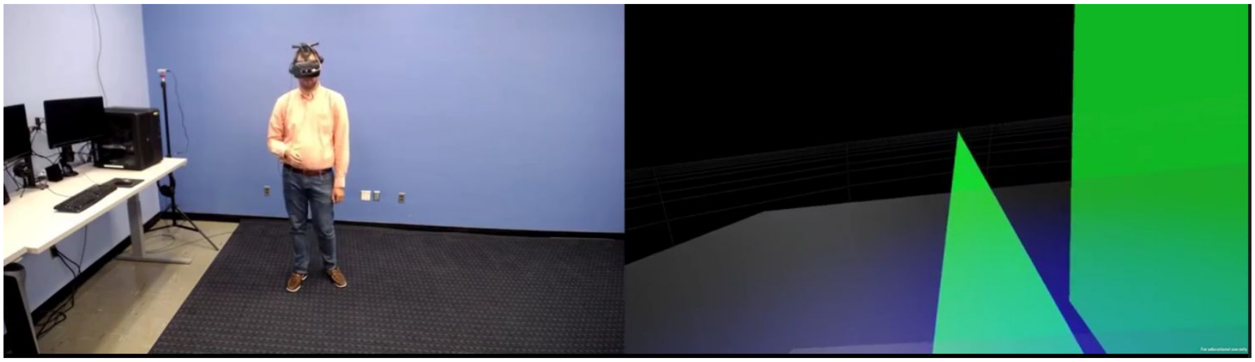

Figure 11. Throwing the apex of the pyramid into the distance. The user [right] has thrown the apex of the pyramid. The pyramid [right] moves into the distance, without bound. Right of the pyramid [right], the surface area square is shown, which is increasing in size without bound.

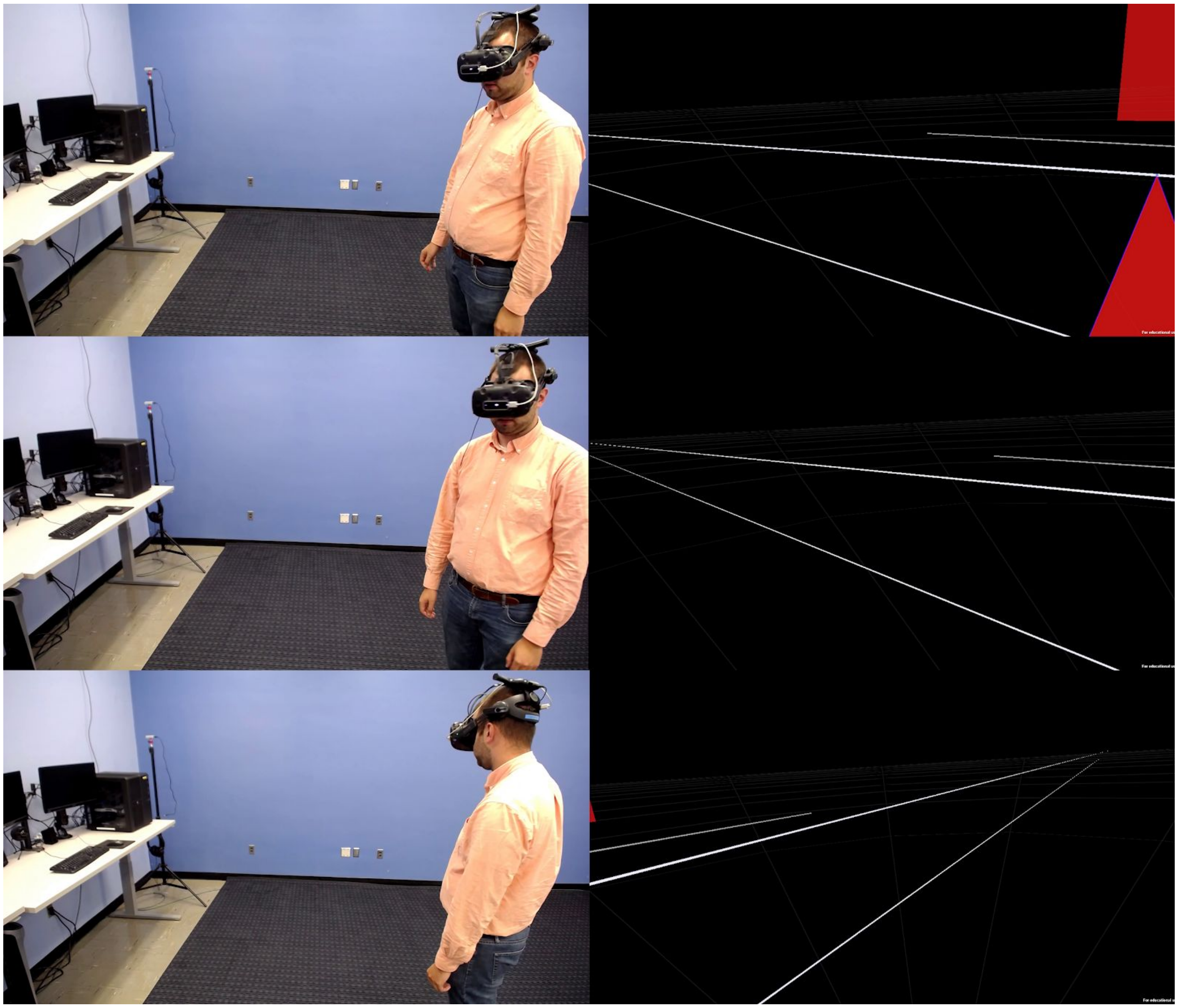

Figure 12. Turning one's head to get three different projective views of the parallel lines. This figure illustrates the user [right] turning their head to view the triangle [left]. Shown are three projective perspectives: Looking towards the triangle [Top], looking along the parallel lines to the left [Middle], looking along the parallel lines to the right [Bottom].

Diagrams for Measurement

If users asked for tools to take measurements, diagrams were added to the environment by the interviewers. A line segment and a square were available to represent measures of the triangle. The length of the line segment was dynamic and reflected the perimeter of the triangle (Figure 13); the dynamic square maintained an area equal to the area of the triangle. Similarly, a square with an area equal to the surface area of the pyramid and a cube with a volume equal to the volume of the pyramid were available (Figure 14). Each of these figures could be translated and rotated through space by pinching and grasping (Figure 15).

Figure 13. Perimeter segment and area square. This figure shows the triangle diagram with the perimeter diagram [line segment, right] and the area diagram [square, right].

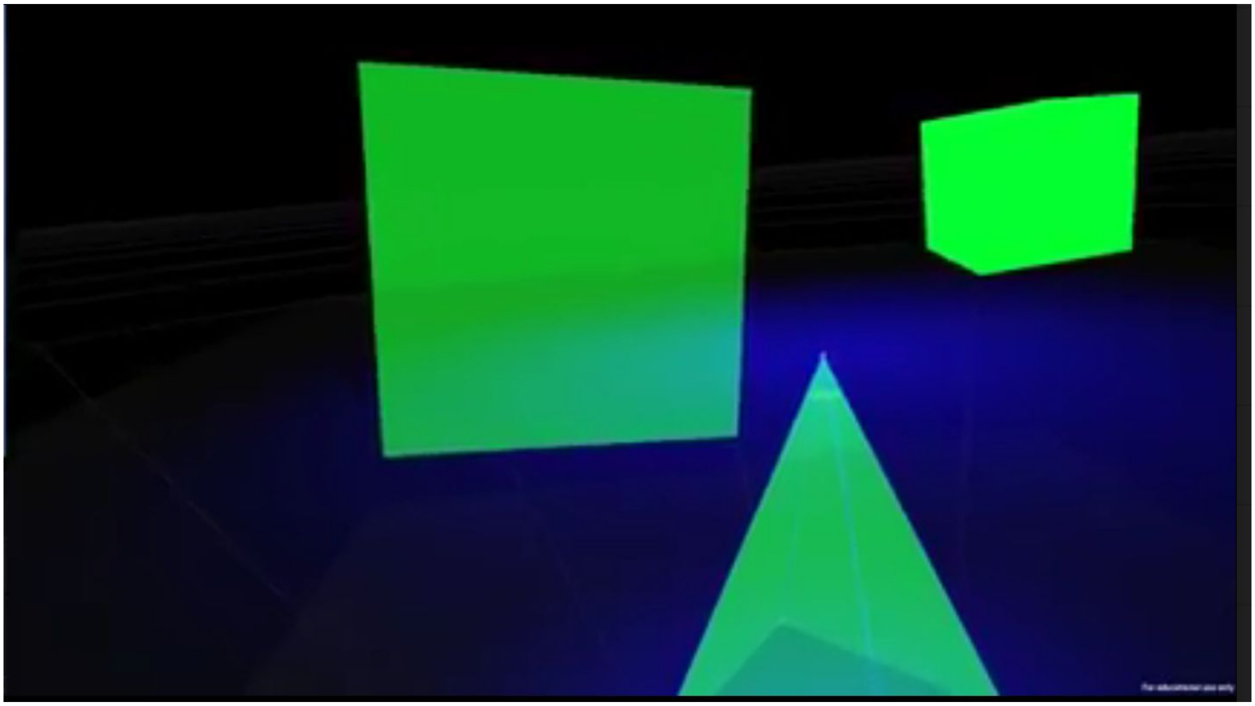

Figure 14. Surface area square and volume cube. This figure shows the pyramid diagram with the surface area diagram [square] and the volume diagram [cube].

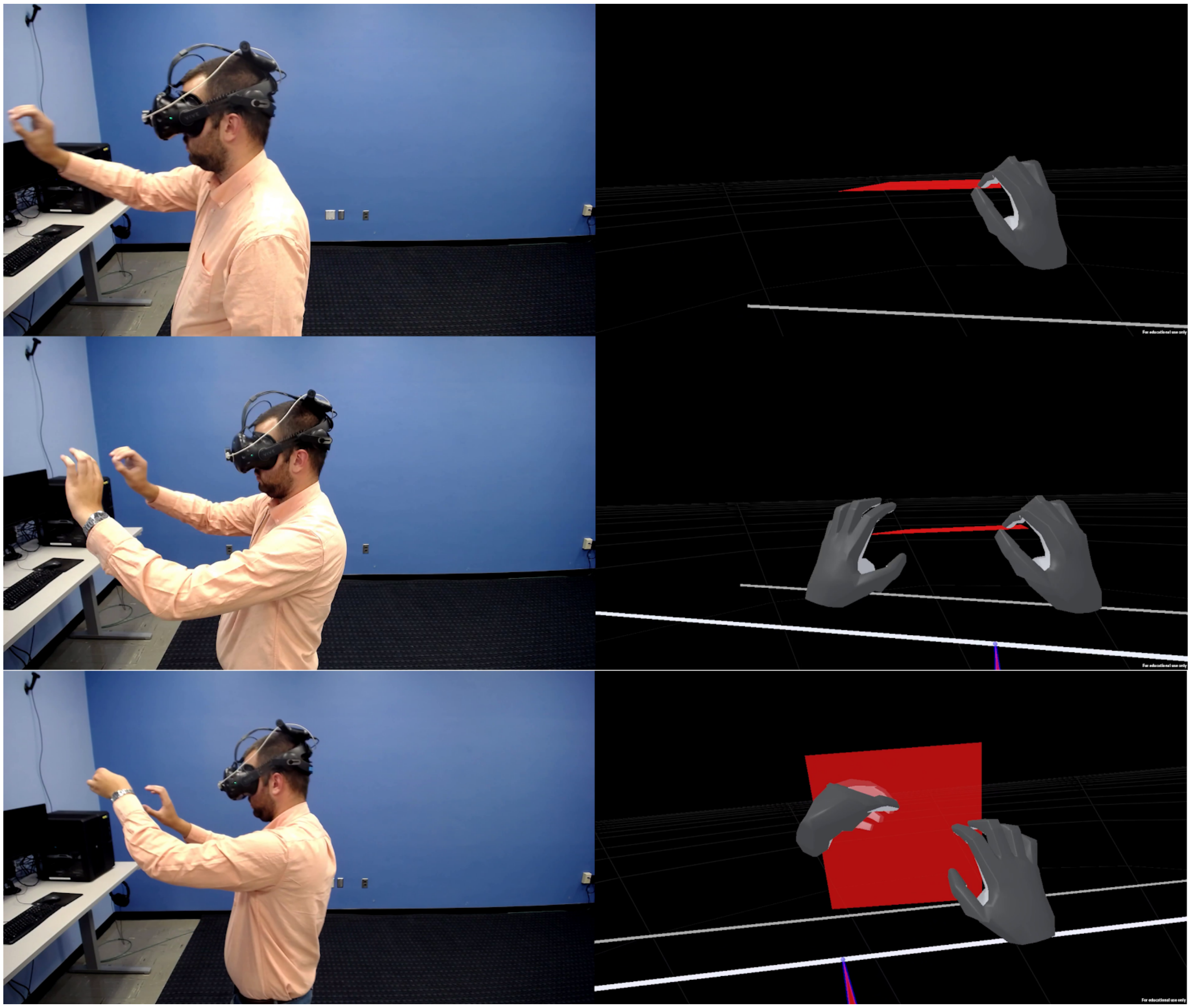

Figure 15. Manipulating the area square. This figure illustrates the one-handed [top] and two-handed [middle, bottom] grasping gestures that allow for translations and rotations of the perimeter diagram, area diagram, surface area diagram and volume diagram.

This environment was designed to capitalize on affordances of dimensionality which are uniquely available in spatial diagrams. The triangle bound to parallel lines is a case of a plane figure embedded in a three-dimensional working space (Bock & Dimmel, submitted). The pyramid is a case of a dynamic three-dimensional diagram embedded in a three-dimensional working space. This allows users to interact with the diagram without mediating through a projection or cross-section and maintain the native control over perspective afforded by the triangle (Bock & Dimmel, submitted). As a three-dimensional diagram, the native control over perspective allows users to enter the pyramid with their bodies. This is an affordance that is not practicable with two-dimensional dynamic diagrams or physical three-dimensional manipulatives. For example, users can move their head inside the pyramid to inspect its interior. The affordances of dynamic, three-dimensional renderings of diagrams in geometry have been largely unexplored in the literature. In this study, I designed, implemented and used a virtual environment that made available dynamic spatial diagrams that might draw attention to both spatiographic and theoretical properties of plane and solid figures. The spatial diagrams embedded within the environment offer participants the ability to take perspectives within and outside solid figures and to represent solid figures without mediation through projection or deconstruction into lower-dimensional components.

METHODOLOGY

Case studies investigate a single case or a set of cases in a bounded, real-life context (Creswell & Poth, 2018; Yin, 2006). Case study research develops a rich description of a case through situated analysis of multiple data sources. Case study research supports the exploration of new research areas where the reasons for future investigations in a topic are not known (Yin, 2006).

In this study, I consider a set of arguments made using a diagram as the unit for a case. The study is of a set of arguments constructed by pre-service elementary teachers who used a spatial diagram as a representation, which are situated within the physical contexts of the IMRE Lab[3] and the virtual context of the designed environment. Since participants were working collaboratively to investigate the diagrams, I consider the learners' co-constructed arguments as the unit of analysis. In the episodes below, participants in both immersed and non-immersed roles contributed to the development of each argument. The learners' co-constructed arguments are a unique case of learners' arguments about measure in geometry within the context of a set of available diagrammatic representations.

The study of learners' arguments can help identify how the affordances and constraints of the spatial diagrams are used in meaning making. The practicability of similar use with other classes of diagrams can then be discussed in reference to existing literature. A case study design allows for a rich description of learners' interactions with spatial diagrams. This rich description

[3] The Immersive Mathematics in Rendered Environments Laboratory (IMRE Lab) is a laboratory classroom that is optimized for the use of immersive spatial displays. It is divided into two halves - an open play space for using immersive spatial displays and two eight-person tables for debrief and discussion.

can make the novel context more accessible to unfamiliar readers and describe more opportunities for the direction of future inquiry.

Context

I describe the context of the study in three sections. First, the interviewers and participants experience of the virtual environment was mediated through three different perspectives. Second, the virtual environment was designed to take advantage of specific affordances of immersive spatial displays. Finally, I describe the phases of data collection and participant selection.

Participants worked in pairs in semi-structured task-based interviews (see Methods), in immersed and non-immersed roles. Notably, the non-immersed participant shared the same physical space as the virtual manipulatives. Though the non-immersed participant couldn't directly see the virtual manipulatives in the three-dimensional space, they could see the immersed participant's physical interactions in the shared physical space and watch the results of the interaction through projections on the monitors.

Participants completed three phases of semi-structured interviews. First participants completed the *SteamVR* orientation to virtual reality, in order to familiarize the participants with the perspective controls and diminish the novelty of the representation in subsequent interviews Second, the first of two semi-structured task-based interviews introduced participants to shearable triangle and pyramid manipulatives. Third, the final semi-structured task-based interview included manipulatives where participants might relate quantities (e.g. volume of a pyramid, volume of a cube) using relationships discovered in the second interview.

Physical Context of the Virtual Environment.

Participants were interviewed in a laboratory setting, in a classroom designed for use with immersive spatial displays, outside of the context of regular instruction. Participants had two roles in the interview: (a) one participant was immersed in the virtual environment, wearing a head mounted display (HMD) and (b) the other participant viewed one of two perspectives on the immersed participant's experience (described below). Participants were free to switch roles at any time during the interview. As part of the interview protocol, participants switched protocol when each new manipulative was loaded into the environment. In the data, participants switched roles at least once while investigating a given manipulative. However participants did not switch roles during any of the episodes presented in this analysis.

Immersed Stereoscopic View. The immersed participant had a stereoscopic view[4] of the virtual environment while wearing the head-mounted display. This view appears in three-dimensions (with depth) to the immersed participant, and was positionally tracked, so that the diagrams (e.g. triangle bound to parallel lines, pyramid bound to parallel lines) are fixed in space as the participant traverses the environment. The immersed participant was free to move within an open area (5m X 3m) without obstacles in the physical (and by extension virtual) environment.

First-Person Projected View. A composite of the stereoscopic view was projected onto a traditional two-dimensional display to approximate the immersed participant's perspective for

[4] An image was rendered for each eye and displayed on a monitor in the head mounted display, providing the illusion of digital renderings with depth that adapt to the user's orientation and position (see Robinett & Rolland, 1992)

non-immersed participants. This view was displayed on a 22-inch monitor to the interviewers and participants during the interviews.

Third-Person Mixed-Reality View. MixCast (Blueprint Reality Inc, 2017) renders a mixed-reality view of the environment using synchronized physical and virtual third-person cameras. This image was composed with a foreground projection of the virtual objects in front of the participant and a background projection of the virtual objects behind the participant. This composite was displayed to the immersed participant within the environment, displayed live on the 70-inch television for the non-immersed participant and interviewers.

This view is a different view than is shown in the previous sections. The mixed-reality view first appears in Figure 24. Due to a corrupt file, the mixed-reality view was not captured for the first interview for one group of participants. An episode from this interview is analyzed in the first section of the results.

Design of the Virtual Environment.

As described in the previous chapter, a virtual environment was designed and developed for this study to capitalize on the dimensionality of dynamic diagrams. The virtual environment included two scenes: a triangle bound to two parallel lines and a pyramid bound to two parallel planes. Within each spatial diagram, two additional manipulatives can be loaded: a square representing the area of the triangle and a line segment representing the perimeter of the triangle or a cube representing the volume of the pyramid and a square representing the surface area of the pyramid, respectively. According to the interview protocol, these manipulatives were only loaded if the participants asked for ways to measure these quantities.

Participants

I recruited pre-service elementary teachers who were enrolled in or had completed their mathematics content courses at a large, public New England research university. Participants were offered $15.00 for each of the sessions that they attended, and matched by scheduling constraints into pairs. Fifteen participants completed the orientation, eight completed the first task-based interview, and six continued to complete the second task-based interview. Two pairs of participants (four participants in total) appear in the episodes selected for analysis, since only two pairs of participants made arguments about the effects of unbounded shearing on the measures of a pyramid. The design of the diagrams was intended to provoke these arguments with each pair of participants. The case was bounded to arguments about unbounded shearing since the diagrams were designed to highlight this phenomenon with the expectation that participants would use the dimensionality of the diagram in their arguments about unbounded shearing.

All four participants identified as female. The first pair of participants were two first-year pre-service elementary teachers without selected concentrations. The second pair of participants were a junior and senior pre-service elementary teacher with concentrations in mathematics and art, respectively. In order to highlight the participants' different access to the diagrams in the episodes, the results describe the participants in terms of their roles in the semi-structured interview. Below, I introduce each of the four participants with pseudonyms.

Abigail was a junior elementary education major who did not report a concentration. She described mathematics as "expanding, connected, and logical," and described proof as "logical" and "comprehensive." Based on scheduling, Abigail was paired to work with Madison during the interviews. Madison was a sophomore art education major who did not report a concentration. She described mathematics as "complex, strategic, numerical and insightful" and described proof as "evidently" and "evidence."

Olivia was a freshman elementary education major with a concentration in child development and family relations. She described mathematics as "exciting, challenging, and rewarding" and described proof as "difficult" and "tedious." Based on scheduling, Olivia was paired to work with Abigail during the interviews. Emily was a freshman elementary education major with a concentration in child development and family relations. Emily described mathematics as "number-oriented" and "calculation-based" and described proof as "truth, definite, shows why something is true."

Data collection

Three video feeds were recorded to capture the immersed and non-immersed participants' explorations of the virtual environment: (a) the first person composite view, (b) the mixed-reality composite view, and (c) a traditional third-person physical camera. Additionally, a microphone was used to record dialogue between participants and interviewers with noise reduction in post production. The three video captures and audio recording were synchronized and recorded into a master video recording in post production. Each of the three video feeds within the episode were used to identify the gestures made by each participant that related to either interactions with the diagrams or their argument. Dialogue was transcribed verbatim.

Semi-Structured Task-Based Interviews

A set of semi-structured task-based interviews with pairs of pre-service elementary teachers is the source of evidence for the case study. Participants first completed a brief background survey and an orientation to immersive virtual reality. In two subsequent interviews, participants were asked to make observations and claims about virtual manipulatives.

Two semi-structured task-based interviews were conducted with pairs of pre-service elementary teachers. Participants worked on a series of five tasks over two or three two-hour interviews. Each task introduced a new manipulative and participants were asked to make observations and test conjectures about the manipulatives. Manipulatives from previous tasks in the interview remained in the environment and participants could return to them at any time.

Only the second task is considered in this study[5]. In the second task, participants had access to the triangle bound to parallel lines and the square-based pyramid bound to two parallel planes, with their respective cross-sections. The interviewers asked the following questions to the participants:

- What do you observe about the figure?
- The apex of the pyramid can be moved by pinching and dragging. What do you observe when you move the apex?
- Similar to the triangle, a cross-section can be generated with an open palm. What do you observe about the cross-section?
- What claims can you make about the figure?

[5] All five tasks are available in Appendix A.

Participants were encouraged to make claims about the diagram. In the course of making claims, I expected that the participants would engage with each other to construct arguments in support of their claims.

Data Analysis

Within each interview, I identified where the participants made arguments about the unbounded[6] shearing on the measures of a pyramid using the video recordings. The case was bounded around unbounded shearing, since the diagrams were intended to make this phenomenon accessible to students in both a triangle and the pyramid and provoke participants to make arguments about the novel mathematical context. Arguments about the measures of a pyramid offer a three-dimensional figure that can share the learner's three-dimensional space in a spatial diagram but must be projected in two-dimensional diagrams. Each argument defined an episode, which was partitioned from the master video recording and transcribed. This partitioning embeds subcases of individual arguments made while using the same diagram by different pairs of participants within the larger case of a set of arguments made by all participants while using a diagram.

Within each episode, the participants' claims, warrants and data were characterized from the transcripts. This initial phase of analysis focused on the main elements of the Toulmin model of argumentation. In a second phase, the warrants were re-characterized as operators and control structures and the backing was characterized. This phase integrates components of the ck¢ model

[6] The unbounded constraint or the selection of arguments about the effects unbounded shearing on the measure of a triangle or pyramid criteria was used to select episodes for analysis of both dimensionality and perspective. While the results of this study don't suggest a connection between the unbounded diagrams and the dimensionality of the diagram, the unboundedness of the diagram was a significant part of the analysis of the use of perspective.

of into the Toulmin model of argumentation. In the final phase of the characterization of each argument, the participants rebuttals were characterized and a qualifier was assigned by the researcher. This final phase modifies the inference described by the ck¢-enriched Toulmin model, by the participants with a rebuttal and by the reviewer with an evaluation of mathematical accuracy.

In a second round of analysis, the ck¢-enriched Toulmin models were described within the narrative of the participants' construction of the argument for each episode. These descriptions were used to identify where the representations in the participants' data sources related to the dimensional affordances of the spatial diagrams. Finally, each of these instances was considered with respect to existing literature on measurement of plane and solid figures to assess with the mathematical knowing that was supported by the affordance might be practicable to construct using another medium.

RESULTS

In the results, two examples where a pair of participants engaged with the dimensionality of the pyramid while making arguments about unbounded shearing are described. Two groups are represented in these results, one in each episode. In each of these episodes, the participants engaged with the spatial diagram of a pyramid as a three-dimensional object in a three-dimensional working space. The episodes contrast one element of the participants' perspectives – inside and outside of the diagram – where the inside perspective is made accessible by the three-dimensional representation in a spatial diagram.

Within each argument, I refer to the participants by both their roles and their pseudonyms in order to make the collaborative efforts to engage with the spatial diagram more clear. I use the abbreviation ImP for immersed participant and NP for non-immersed participant. For example, Abigail [ImP] is a participant with the pseudonym Abigail acting in the role of the immersed participant during the episode. Each of the arguments is described in a narrative that highlights the participant's interactions with the spatial diagrams in terms of the components of the ck¢-enriched Toulmin model.

Perspectives Outside the Spatial Diagram

Abigail [ImP] and Madison [NP] made an argument about the unbounded shearing of a pyramid where the only perspectives they used were outside the spatial diagram while working on the second task (see Methodology). Abigail [ImP] introduced the argument by describing the effects of shearing the pyramid on a cross-section of the pyramid parallel to its' base: "it really doesn't look like the [cross-section] itself...it's still a square – its just – its just the lengths are all the same. And it is interesting to know that if you manipulate the [apex] then the [cross-section] will stay the same – which would make sense." As Abigail [ImP] described the effects of shearing on a cross-section of the pyramid, she dragged the vertex through its plane (Figure 16).

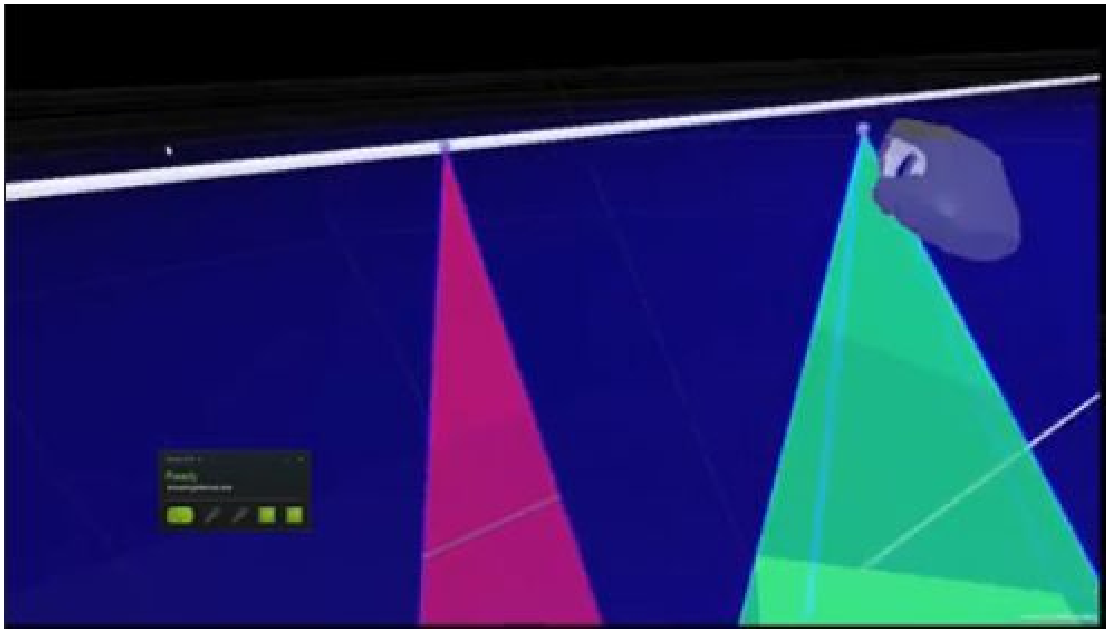

Figure 16. Dragging the apex of the pyramid. In this figure, Abigail [ImP] dragged the apex of the pyramid as she made a claim about the volume-preserving properties of the shearing transformation.

As Abigail [ImP] continued to move the apex in a circular path in its plane, she made a claim about the volume of the pyramid: "You could talk about it in terms of volume. Just in like no matter how I manipulate [the apex] … so I was saying as I move [the apex] around, the volume inside the pyramid is not changing." The interviewer prompted the participants to explain how they know the volume is not changing. Madison [NP] responds, contradicting the first participant's claim: "I feel like the volume would be changing though - the more you slant it along the plane, the longer the sides get 'cause it has to tilt for them to reach it." Viewing the argument as co-constructed between the participants, Madison [NP] is still engaging with the positive framing of the claim - although she is challenging it and offering a warrant in opposition. This action can also be seen as provoking the pair of participants to construct a stronger warrant in support of their claim.

In response, Abigail [ImP] says "tilt it like that" (Figure 17, top pane) and then pulled the apex so that it is centered above the base of the pyramid (Figure 17, middle pane). Abigail [ImP] then drags the apex within a neighborhood of its centered position (Figure 17, lower pane). Madison [NP] responded "when you tilt it, it will change the length of the sides [faces] because it changes the distance between the point at the bottom [segment of base of face] and the point at the top [apex]." Madison [NP] explained that the height of the faces will change as the pyramid is sheared; she made a spatiographic observation about the figure that she could not directly observe or measure.

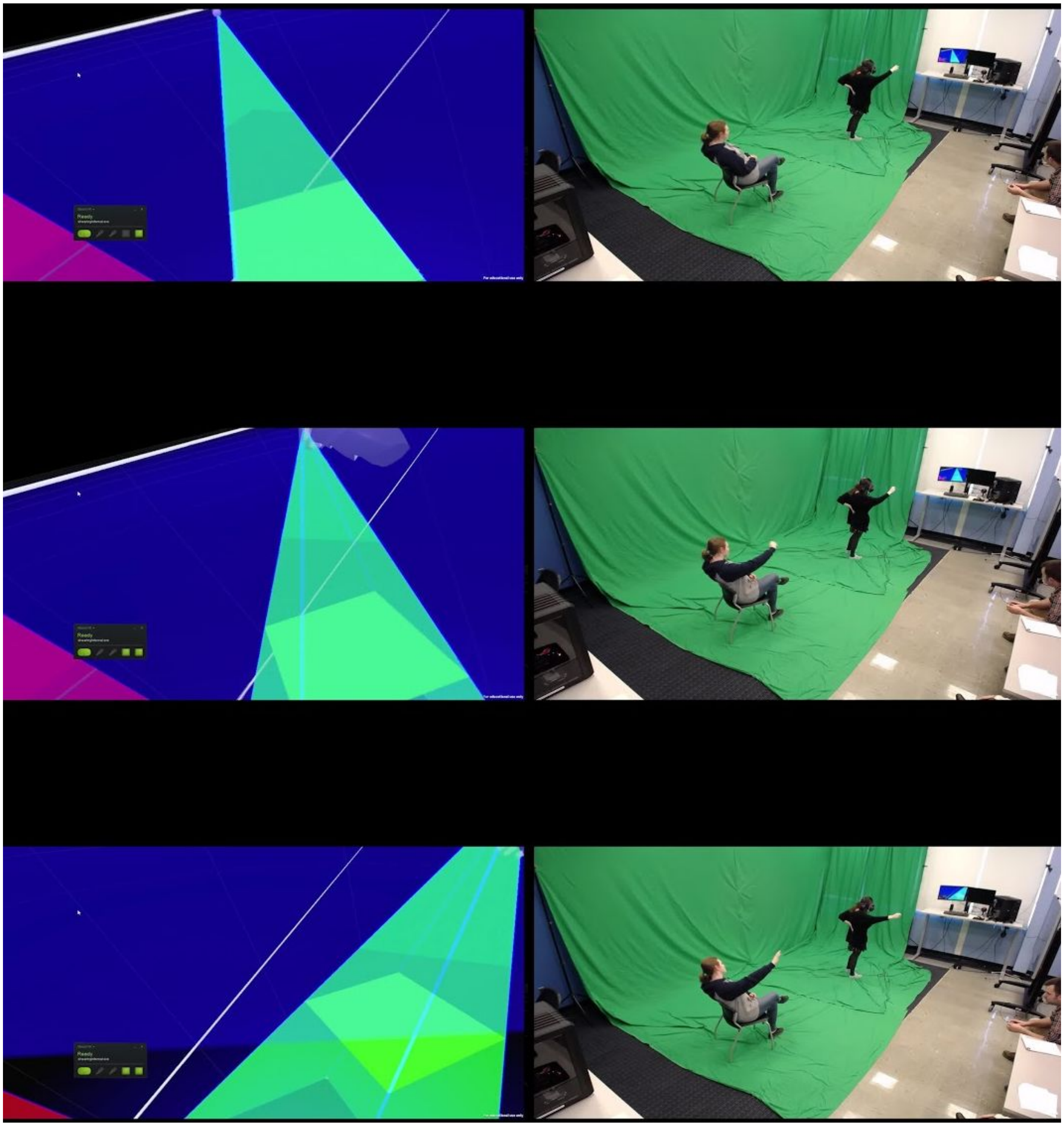

Figure 17. "Tilt it like that?" In this figure, Abigail [ImP] dragged the apex of the pyramid to the right [left panes] and asked if this is the manipulation that Madison [NP] requested.

Madison [NP] continued and said "so then it would be changing the volume – well no, because the height doesn't change." This statement highlights a tension between the participants' spatiographic and theoretical understandings of the volume of the pyramid: their control was contextualized in a theoretical backing while they negotiated data from spatiographic observations. Abigail [ImP] agreed with Madison [NP]'s correction and Madison [NP] questioned whether the surface area would change. Here, Madison [NP] is taking a lead role in constructing the argument while Abigail [ImP] is still engaged in both transforming the representation and the construction of the warrant.

Madison [NP] responded "Then it changes the surface area then, Right?"; which could be a claim in a new argument, but Abigail [ImP] then says "I feel like the volume would be changing" and returns the focus of the argument to volume. She began to explain how the perimeter of the triangles on the faces of the pyramids might not change. Abigail [ImP] then moved to make an argument about the effects of shearing on the area of a triangle to support her warrant for the argument about the pyramid; she walked to the triangle, grasps its apex and drags it to the left - shearing the triangle (Figure 18).

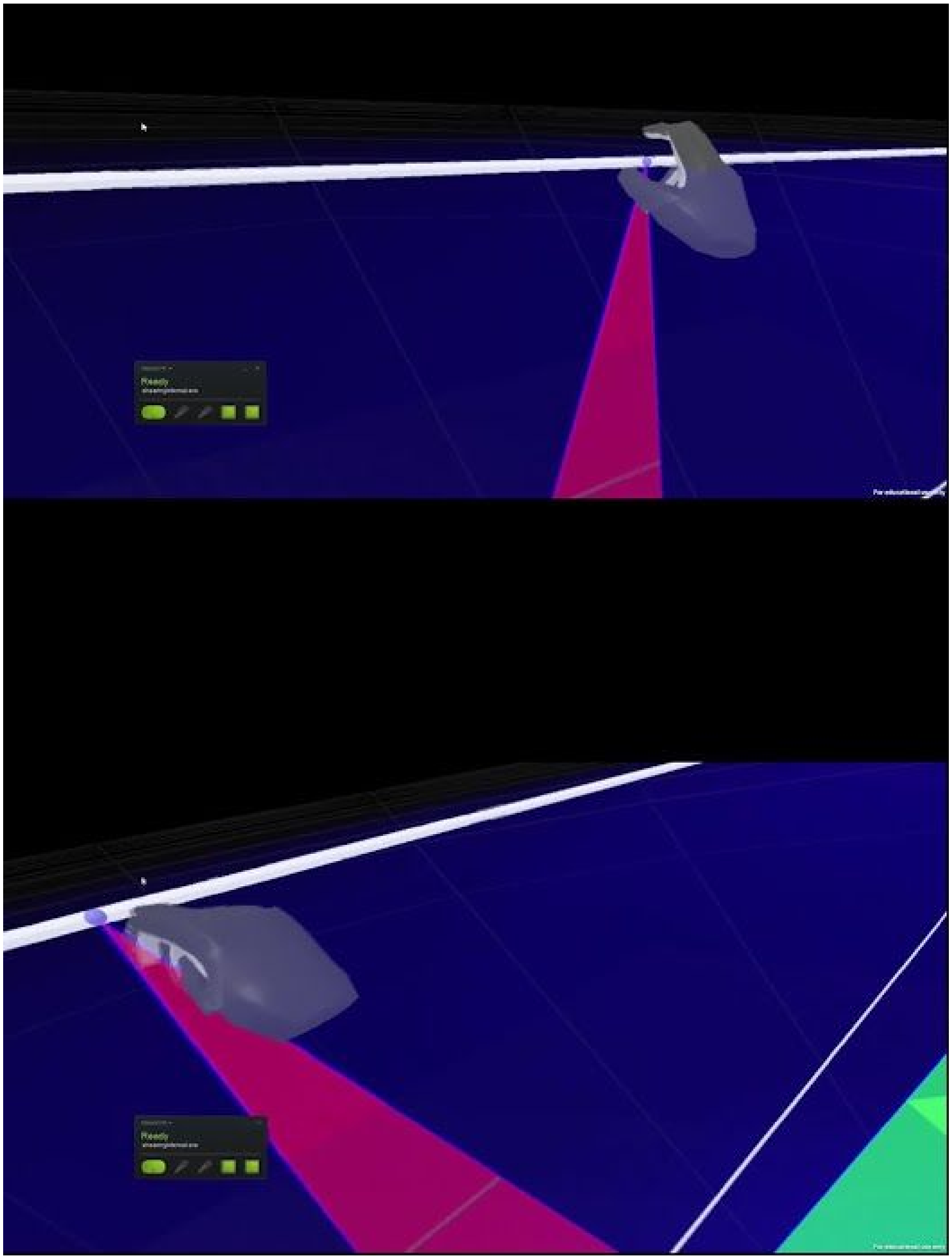

Figure 18. "Think about over here." In this figure, Abigail [ImP] walked to the

After dragging the triangle, Abigail [ImP] explained that the perimeter of the triangle is constant while the triangle is sheared. "Shorten this [edge] and make this [edge] longer, its at the same rate." This statement functioned as a warrant for the triangle's area equality under shearing. While the participants' statement is not mathematically accurate, Madison [NP] validated the Abigail [ImP] when she responded "that makes sense."

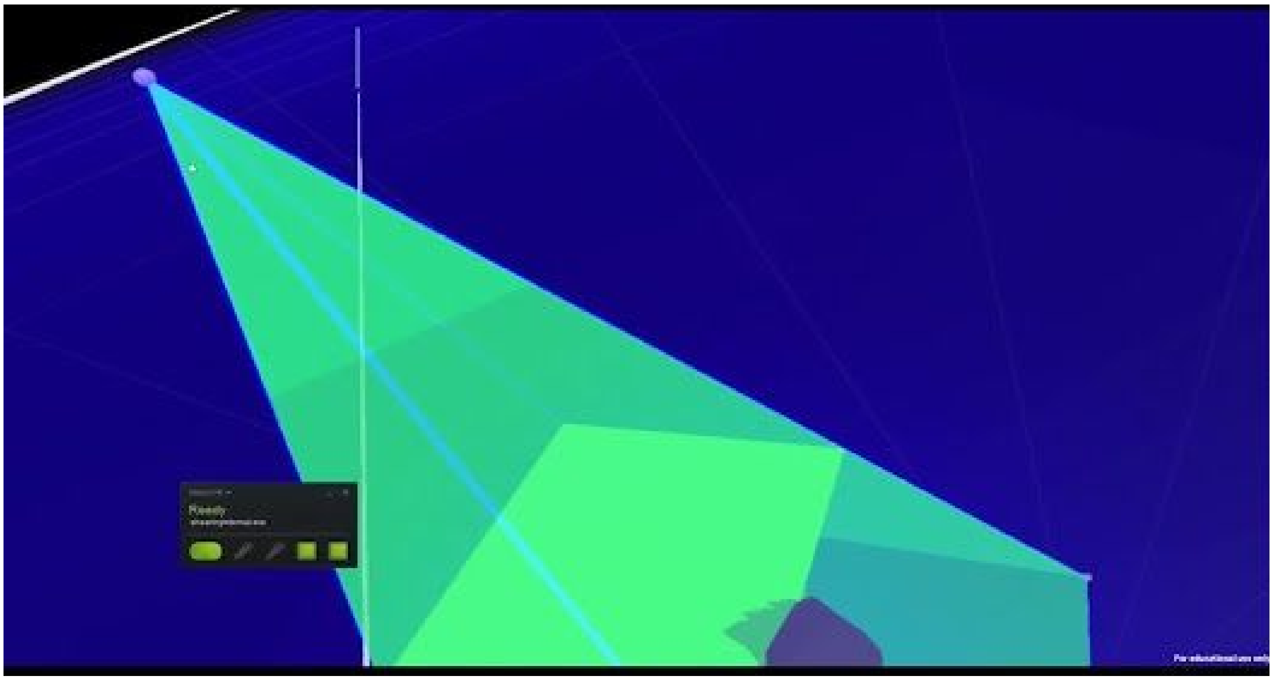

Figure 19. "The volume and surface area are going to stay the same." In this figure, tAbigail [ImP] moved an open palm through the pyramid, moving the cross-section plane within the diagram.

Abigail [ImP] returned to the pyramid and claimed "I think the volume and the surface area are going to stay the same no matter how I manipulate them." This is an expansion of the participants' earlier claim that "the volume of the pyramid is not changing." As she made this claim, Abigail [ImP] quickly moved the cross-section up and down within the pyramid by moving her open palm (Figure 19). Notably, Abigail [ImP] also engaged with the cross-section when she initially claimed that the volume of the pyramid is constant. However, a relationship between the cross-section and the volume was not described by either of the participants in the episode.

Abigail [ImP] continued to explain that "as I take [the apex] over, this angle (Figure 20, top panel) goes past 90 and becomes obtuse and at the same rate that this angle (Figure 20, top panel) wide, this angle (Figure 20, lower panel) collapses," pointing to the vertices at the base of the triangle. In Figure 20, Abigail [ImP] has access to a spatial diagram including both the triangle bound to two parallel planes and the triangle bound to two parallel lines. Abigail [ImP] looks past the green pyramid, to point at the vertices of the red triangle. She looks through the blue plane that passes through the apex of the pyramid and towards the blue plane that the base of the pyramid and triangle lie in.

Madison [NP] emphasized "so it stays the same." Here, the participants made a spatiographic observation that is mathematically neither true nor relevant to their argument about the triangle. The participants used the apparent preservation of perimeter and the sum of the base angles as a warrant for the preservation of the area of the triangle under shearing. In terms of the ck¢-enriched Toulmin model, the participants' spatiographic observation that the perimeter is constant serves as an operator. A relationship between area and perimeter serves as the control, where a fixed perimeter implies a fixed area.

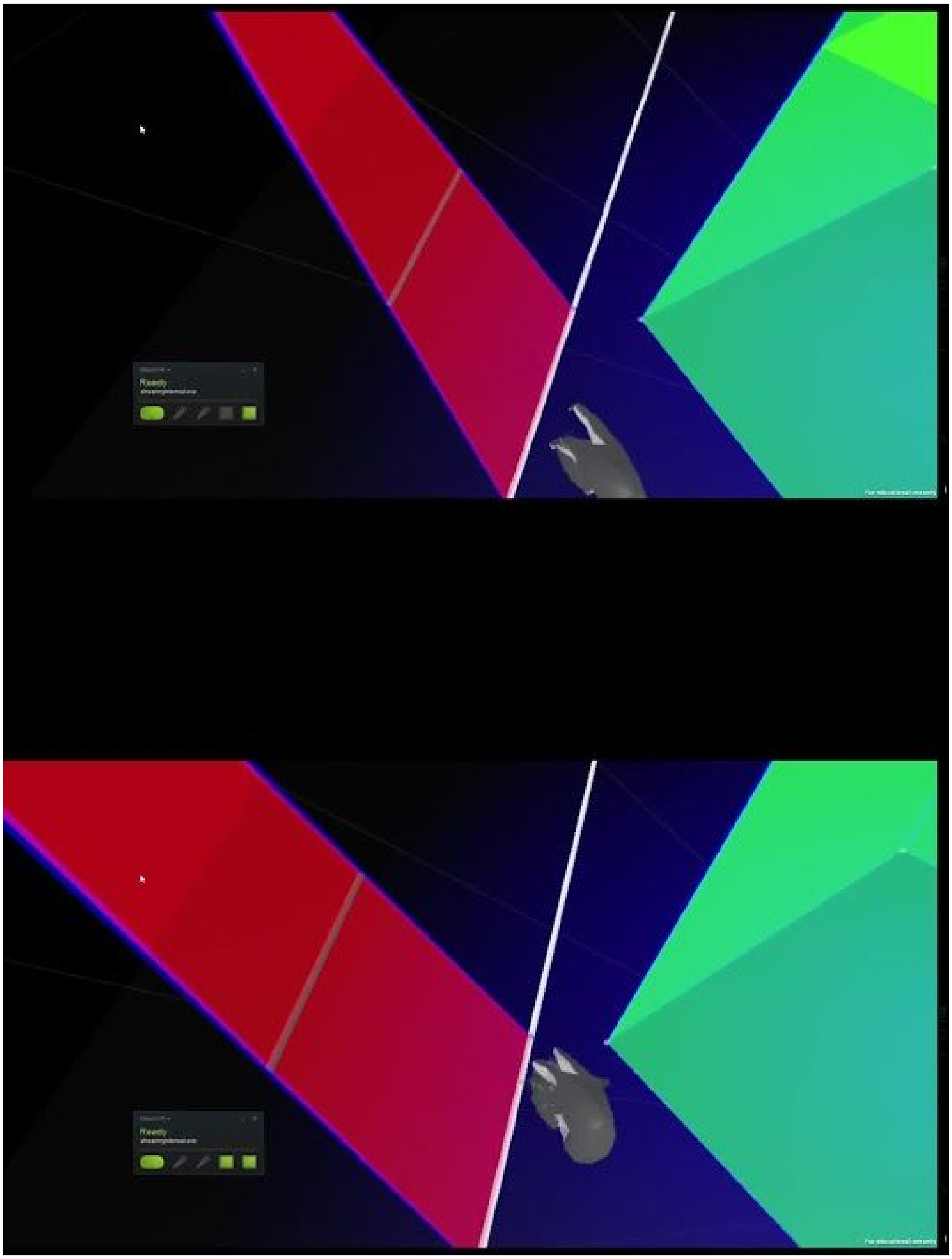

Figure 20. "These two angles are changing." This figure shows the (red) triangle and (green) pyramid. Abigail [ImP] pointed towards the two vertices of the triangle and observed that the measure of the angle at each vertex is changing.

While the participants described a warrant for the area of the triangle in terms of the triangles' angles and edge lengths, when Abigail [ImP] said that the triangle "still has the same area," she swept a left-facing open palm in line with the altitude of the triangle (Figure 21). However, Abigail [ImP] did not make any verbal reference to the height or altitude of the triangle.

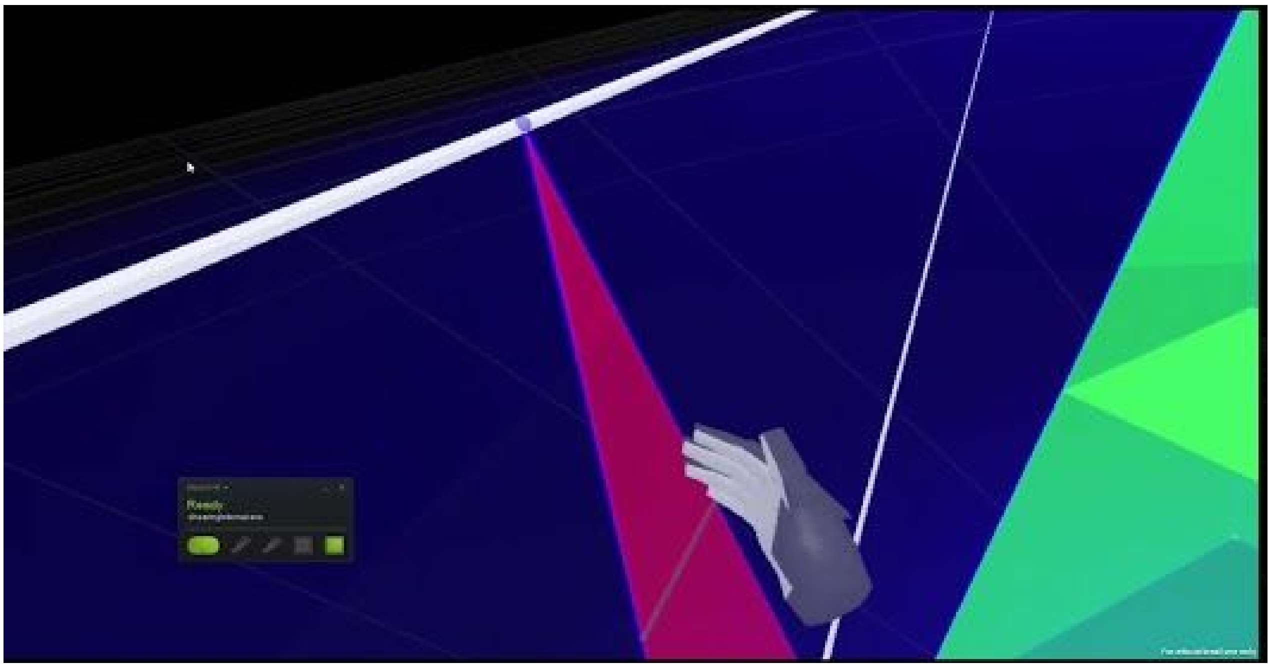

Figure 21. "Still has the same area." Abigail [ImP] recalled that the triangle (red) had the same area. She simultaneously moved an open palm along the length of the triangle's altitude, but did not reference the triangle's altitude.

Figure 22 composes each of the components of the participants' arguments about the triangle and the pyramid into a nested ck¢-enriched Toulmin model. The participants' argument about the area of a triangle serves as a control for and was constructed to support the participants' argument about the surface area and volume of a pyramid.

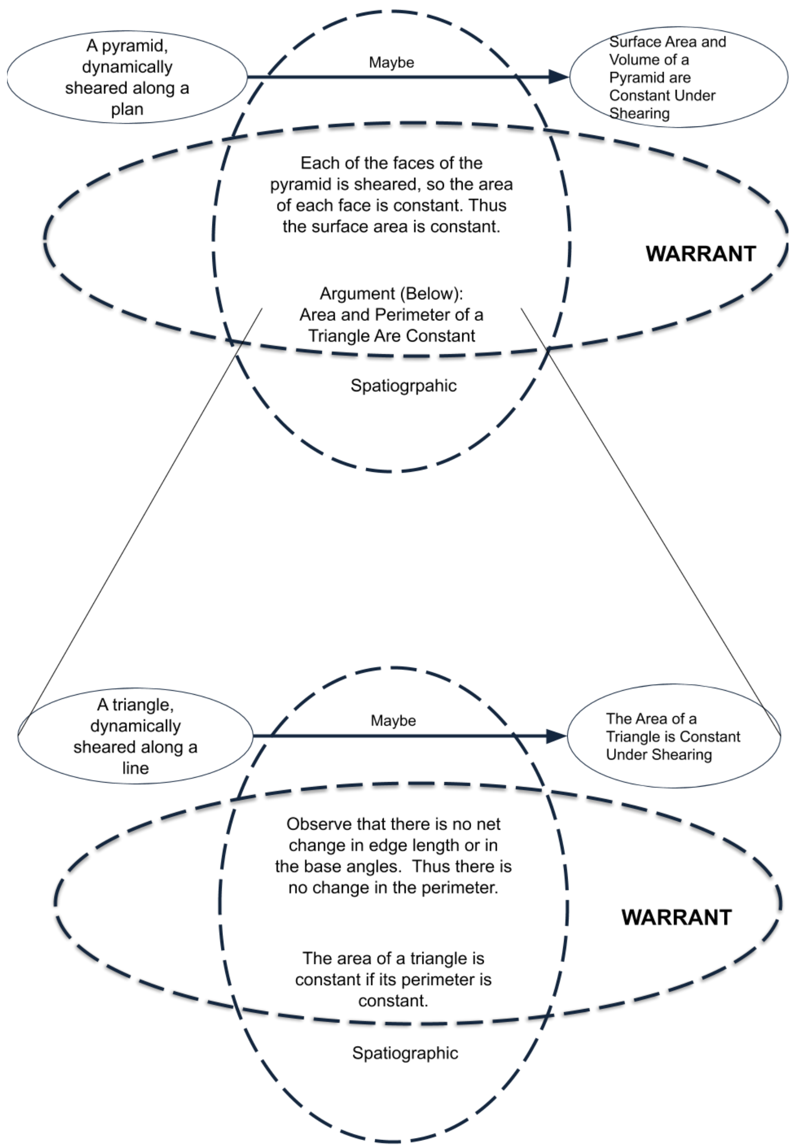

Figure 22. Ck¢-enriched Toulmin model of the first episode. This figure illustrates the components of the ck¢-enriched Toulmin model in the analysis of the first episode, arranged around an arrow representing the inference as a transformation of data into a claim.

The interviewer then prompted the participants to "keep going in that direction" as they sheared the triangle. After throwing the apex of the triangle into the distance (Figure 23), Abigail [ImP] says "we're never getting that back." The participants observed the area of the triangle under unbounded shearing – "our area is like miniscule" and ``it's not that we're wrong, it's just that it's in incredibly stretched" – and the perimeter – "oh yeah, the perimeter is different."

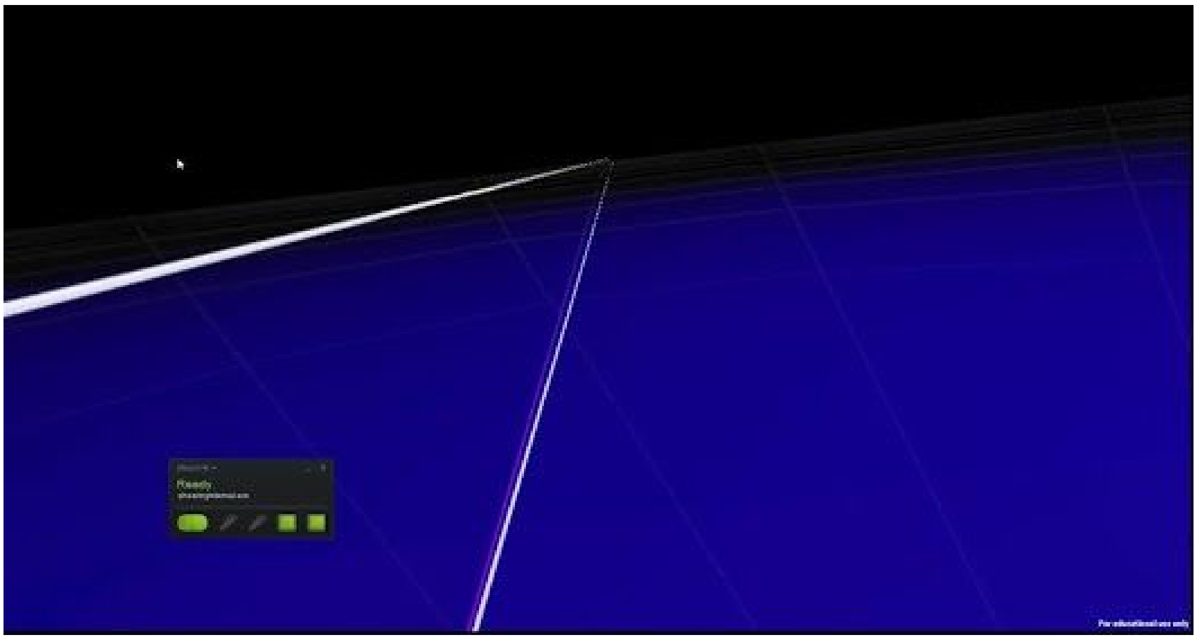

Figure 23. "Never getting that back." This figure shows the triangle (red) as it was sheared without bound. The edges (blue) of the triangle approach the slope of the parallel lines.

The participants then concluded that "yes you can have the same area even if the perimeters are completely widely different. But I don't know how to work this, I just know that it's a thing that can happen." Before the participants revisited their argument about the pyramid, the interview protocol was broken[7] and the participants were redirected by the interviewers.

[7] One of the interviewers asked questions outside of the interview protocol, which prompted participants to engage with a formula that they had not described. The episode was terminated when the interview protocol was broken in this analysis.

Perspectives Within the Spatial Diagram

The first episode offers an example of a pair of learners constructing an argument about shearing exclusively with perspectives outside a spatial diagram. The second episode offers and example of a pair of learners using perspectives from within and outside a spatial diagram to construct an argument about shearing

In the second episode, Emily [ImP] and Olivia [NP] engaged with the same spatial diagrams. Emily [ImP] began the argument by conjecturing that "I don't know how different volume would be to area" noting that they concluded that "area stayed the same when we sent [the triangle]." As the participant made the connection between the shearing of the triangle and the pyramid, she faced and dragged the pyramid's apex to the right (Figure 24). She then turned to face the triangle and drag the triangle's apex as she referenced "sending" the triangle. The participants had previously made arguments about the unbounded shearing of the triangle and had experimented with dragging the triangle's apex and with throwing the triangle's apex along its line. The participants statement established the participants' claim that the volume of the pyramid is conserved under shearing and begins to situate that claim within an analogy between the shearing of a solid figure and the shearing of a plane figure.

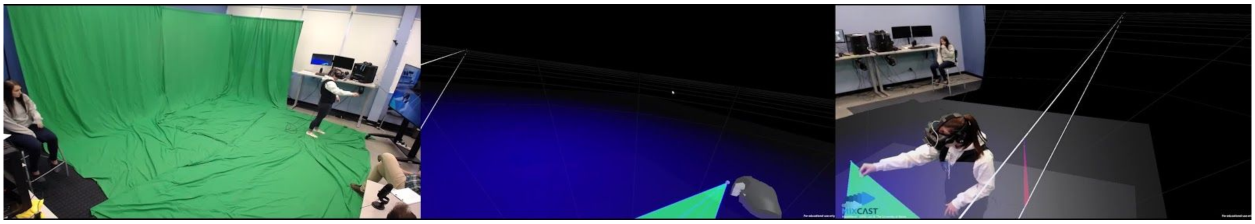

Figure 24. "I don't know how different volume would be to area." In this figure, Emily [ImP] dragged the apex of the pyramid as she made a conjecture about the relationship between the shearing of the pyramid and the shearing of the triangle.

Emily [ImP] continued to describe how the volume of the pyramid might be conserved in spatiographic terms. She continued the analogy with the triangle by referencing her previous argument that a triangle's area could be preserved under shearing if it was represented by "grains of sand" which could be "spread out in one little grain at a time" (see Bock & Dimmel, submitted) and described the pyramid in analogous terms.

Figure 25 shows the participant's gestures as she described that the pyramid could be represented by a "square of blocks, like 10 by 10." Emily [ImP] described how the "square of blocks" would "slowly get shorter" but "it would get longer." In the first frame, the participant had her palms facing vertically, and moved to slant them facing in parallel to represent this transformation. Emily [ImP] then described the transformation in terms of moving the blocks within the representation: "I'd take one and put it here; one and put it here." In the third frame, the participant is pinching one of the imagined blocks, in the fourth frame she dragged the block down and to her right. In the final two frames, the participant repeats her earlier gestures to show the transformation of the figure as a whole.

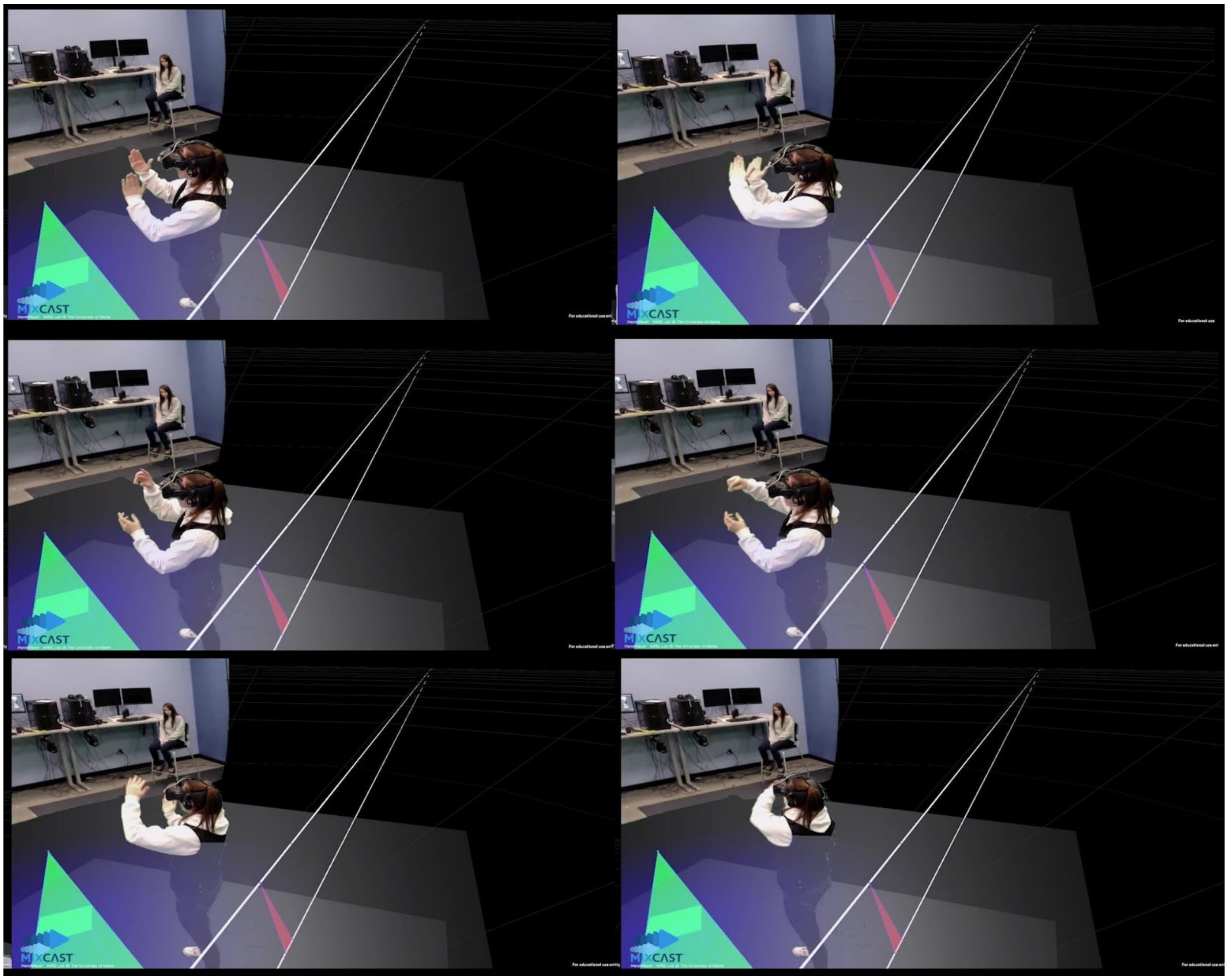

Figure 25. "If I had a square of blocks." Emily [ImP] describes how a square of blocks can be transformed through space while preserving their measure. The figure can be tiled [top, bottom] and component blocks [middle] can be rearranged to preserve its measure.

Emily [ImP]'s description of the transformation of a "square of blocks" served as warrant for the claim "I don't know how different volume would be to area." The warrant is composed of a mathematical operation – an analogy between the shearing of a pyramid and the shearing of a triangle – and a control – conservation of volume or surface area can be shown by rearranging units of volume. In standard school mathematics, this warrant is insufficient as it does not use theory or empirical verification to show that the volume of the pyramid is constant. However, it is situated within a spatiographic context and argues that it is plausible that the volume of the pyramid *could* be constant. When considering the argument in the context of a backing which allows for alignment of spatiographic warrants of plausibility with spatiographic observations, the participants’ warrant is internally consistent.

Emily [ImP] then “sent” the apex of the pyramid into the distance by pinching, grasping and throwing the vertex. As she drags the apex of the pyramid (Figure 26), Emily [ImP] described the pyramid as a “three dimensional figure.” As the apex is thrown into the distance, Emily [ImP] waves goodbye (Figure 27).

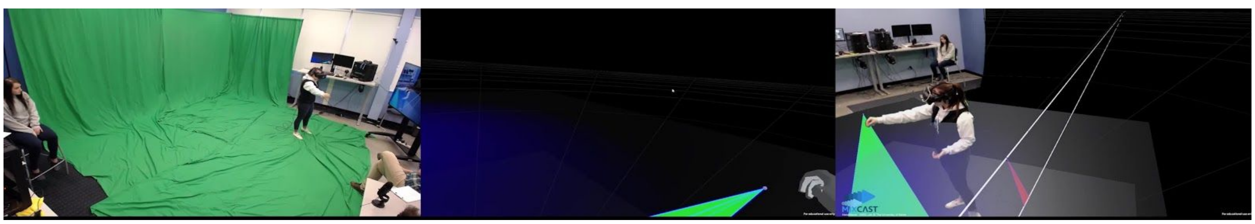

Figure 26. “With the three dimensional figure.” Emily [ImP] dragged the apex of the pyramid [green, middle & left] as she described how the “square of blocks” related the the pyramid.

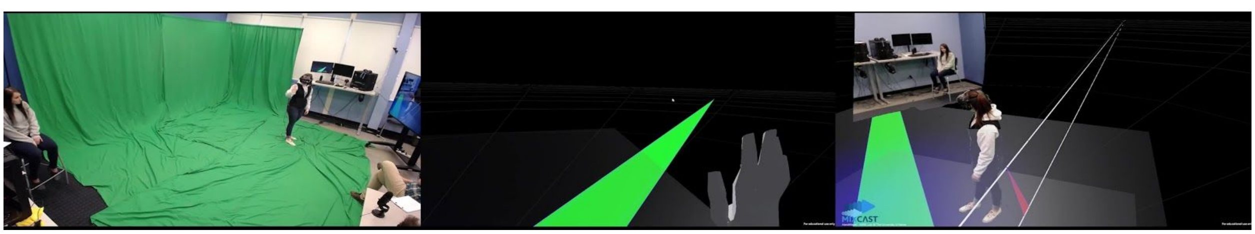

Figure 27. Immersed participant waves goodbye. Emily [ImP] threw the apex of the pyramid [green, middle & left] into the distance.

As the apex moves indefinitely into the distance, Emily [ImP] remarked that the pyramid “looks like its getting bigger” while her gaze was approximately aligned with altitudes of the pyramid’s faces. Emily [ImP] then walked beside the pyramid, bent down, and remarked that “if you look at it...then it's getting so thin?" (Figure 28). Staying in that position, Emily [ImP] explained that "whatever space was being taken up this way [parallel to a line between the base of the pyramid and its apex] it's just being taken up this way [perpendicular to a line between the base of the pyramid and its apex]." As Emily [ImP] described how the space inside the pyramid could be accounted for, she moved her gaze to cross a line between the base of the pyramid and its apex. Here, the participant is engaging with feedback from the spatial diagram which served as data to support their argument, through spatiographic observations.

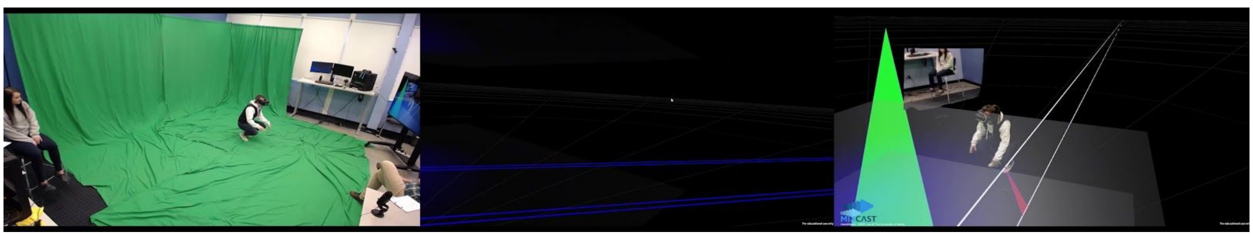

Figure 28. "Looks like its getting bigger." Emily [ImP] describes how the pyramid [right: green, middle: space between blue line segments] could look like it is increasing in size when viewed from a perspective beside the pyramid. The green fill of the faces of the pyramid were not shown [middle] because of an error in the shaders used in the virtual environment.

Emily [ImP] summarized that the "space being taken up this way (Figure 29, top pane) is just being taken up this way (Figure 29, bottom pane)". As Emily [ImP] said this, she described the effects of the shearing transformation on the space inside the pyramid in order to support a claim about the preservation of volume. While this appears as a summary of the participants warrant, it also makes the participants' spatiographic observations more specific – the volume of the pyramid is the space inside the pyramid.

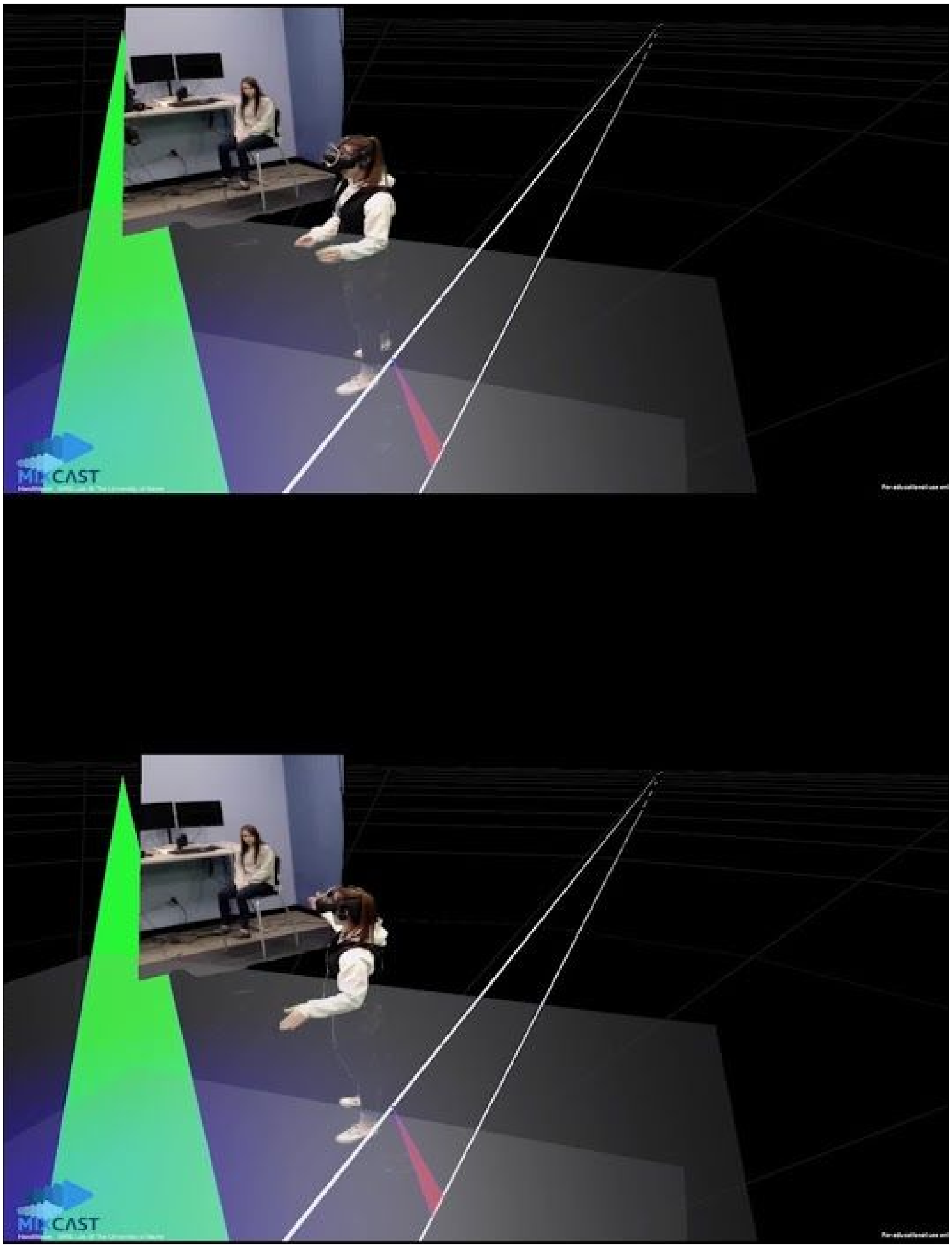

Figure 29. “Space being taken up.” In this figure Emily [ImP] stretched her arms vertically [top] to describe how the space inside the pyramid was being “taken up” before it was sheared and then stretches her hands horizontally [bottom] to describe how the space inside the pyramid was being “taken up” after it was sheared.

Emily [ImP] then stepped into the pyramid, so that the pyramid cross-sected her legs and said "this kind of looks like a road, I feel like Dorothy" (Figure 30). Emily [ImP] began by looking along the length of the pyramid from outside the pyramid in a standing position. In the top panel of Figure 30, the participants gaze captures part of the pyramid from beside. As Emily [ImP] steps towards the pyramid, she looks down at the pyramid near her feet from above and outside the pyramid (Figure 30, second panel). Emily [ImP] paused next to the pyramid, aligning her gaze with the apex of the pyramid, capturing most of the pyrmaid's length (Figure 30, third panel). Emily [ImP] then entered the pyramid, and turned to face her body and her gaze in the direction of the apex (Figure 30, fourth pane). Finally, Emily [ImP] looked down towards her feet while still facing the apex (Figure 30, fifth pane) and remarked, "I wish I could see my legs and seeing them being chopped off by the planes." This statement is notable because the participant engaged with the spatial diagram with her body, and imagined her body's intersection with the diagram. However, only the non-immersed participant can actually see the participants' legs intersect the diagram (Figure 30, fifth pane, right); Emily [ImP] simply sees a part of the pyramid below her (Figure 30, fifth pane, center).

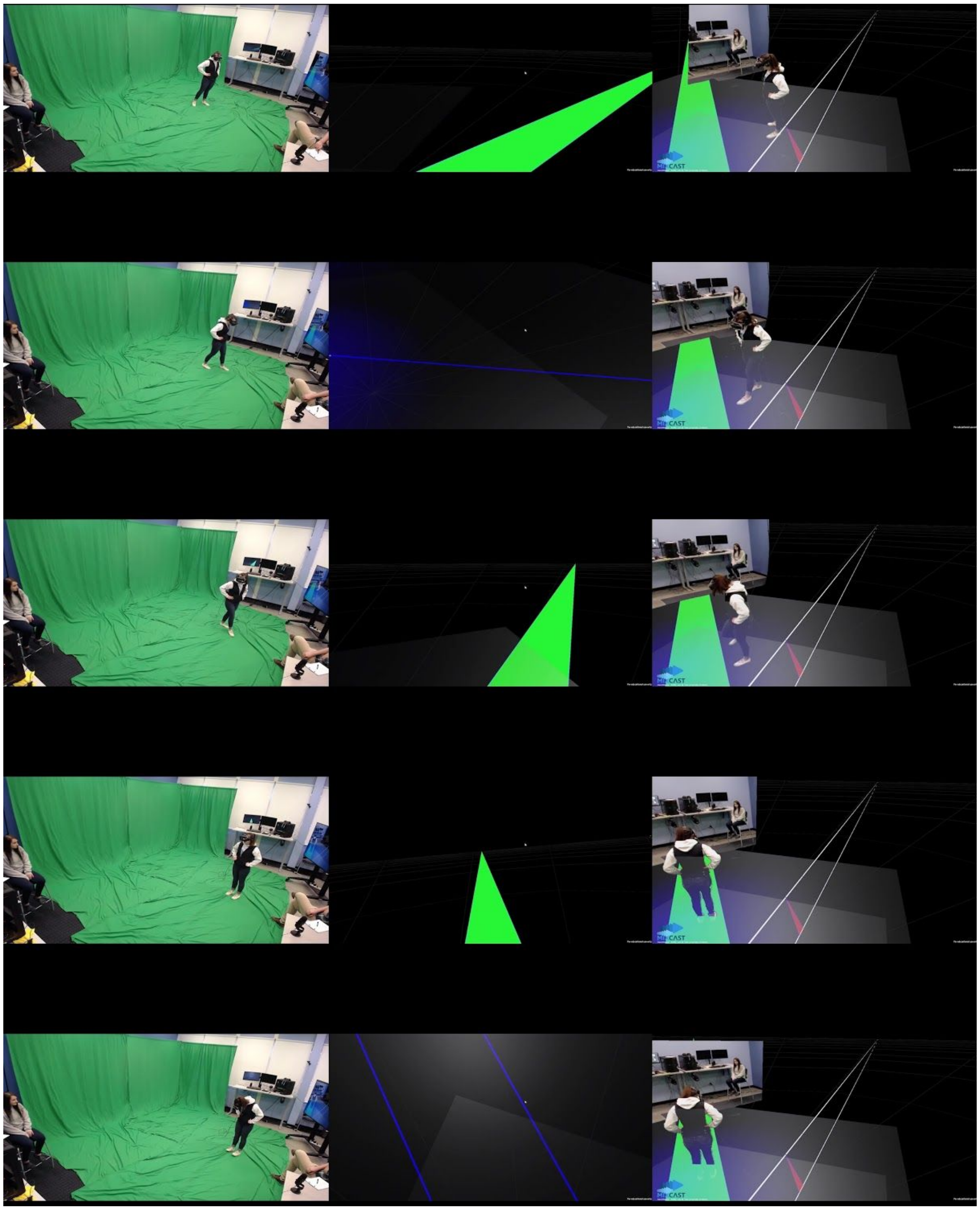

Figure 30. "I feel like Dorothy." Emily [ImP] steps looked at the pyramid from beside [first row], walked towards the pyramid [second row], looked along the pyramid from immediately beside the pyramid [third row], stepped into the pyramid and looked along the pyramid [fourth row], and looked towards her legs while within the pyramid [fifth row].

This concludes the first iteration of Emily [ImP]'s argument, as Emily [ImP] continued to ask if the interviewers "got any more questions?" and seemed content with her argument's resolution. While the arguments in this study were analyzed as co-constructed by pairs of participants, Emily [ImP] has ownership of this argument as she was directly interacting with the spatial diagram and was the only speaker. Figure 31 represents this argument in a ck¢-enriched Toulmin model.

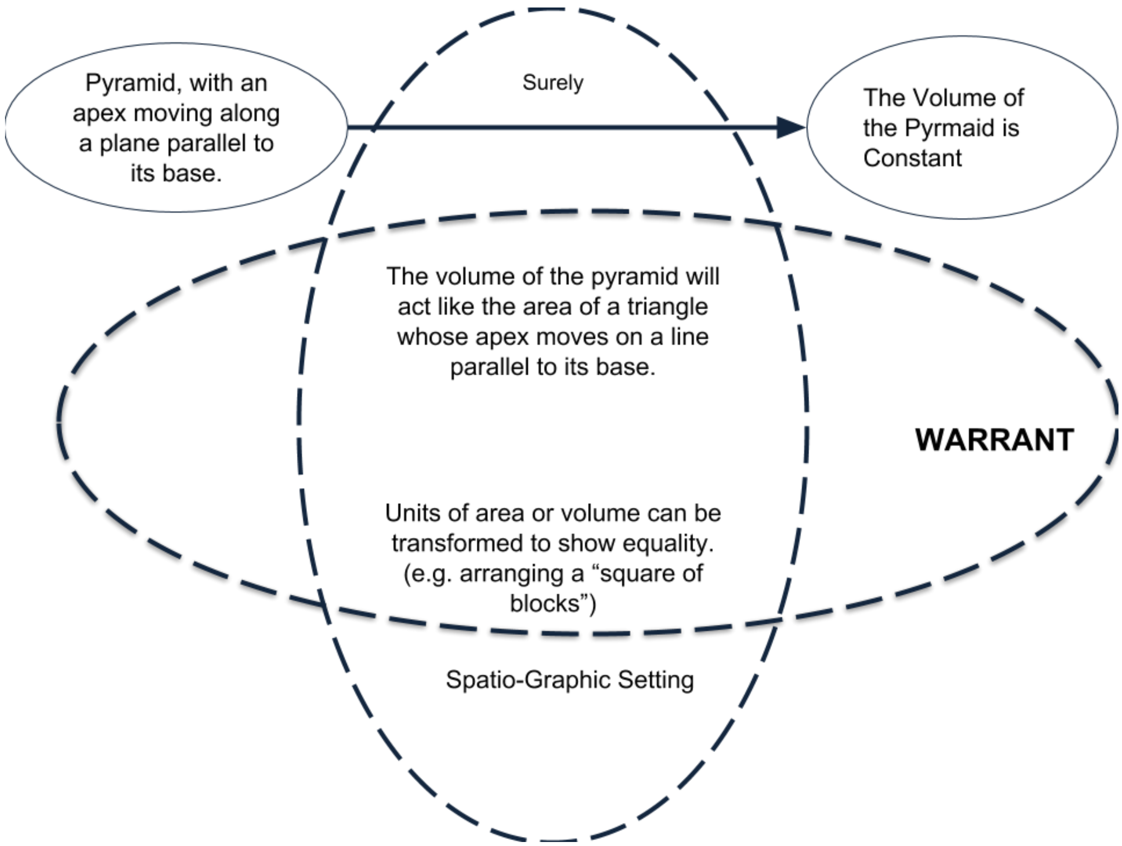

Figure 31. Ck¢-enriched Toulmin model of the second episode. This figure illustrates the components of the ck¢-enriched Toulmin model in the analysis of the second episode, arranged around an arrow representing the inference as a transformation of data into a claim.

The interviewer prompted the participants to consider if "there [is] anything else about the pyramid?...is there anything else about the pyramid changing?" In response, Emily [ImP] acknowledged that the length [altitude] of the sides is becoming "super, super long...this looks infinitely long." The interviewer the prompted "so are you saying that a pyramid with infinitely long sides can have a finite [volume]?" The participants then co-constructed revisions to their initial argument.

Emily [ImP] responded first; she began "if I had an infinitely thin area [volume] between [the faces]...well I don't know how that works because from here I can't tell if its at the same height just because it looks flat." This statement addressed a constraint of the spatial diagram – as the representation extends into the distance some aspects cannot be easily observed. Additionally, the statement touched on a relevant theoretical property of the triangle – its height – through a desire to spatiographically observe the property. The interviewers tell the participants that they can assume that the height of the pyramid is constant, and the participants do not engage with the height of the pyramid again. This is notable, because the height of the pyramid could be a relevant theoretical property to construct a new warrant for their claim in a different backing. However, the question of the pyramid's constant height serves as a rebuttal because it seems that it acknowledges a case where the participants do not expect their argument to hold. The reasoning for this rebuttal is not clear from the participants' dialogue.

Olivia [NP] continued the argument, and reiterated how the space inside the pyramid is accounted for during the shearing transformation; she said "it would have to also get infinitely thin [as it is sheared], if it's not flattening out then I don't know where the space inside would like go." As Olivia [NP] said this, she made a gesture pushing palms facing together (Figure 32, top pane) and then pushed her palms together while tilting horizontally (Figure 32, middle and bottom panes]. She repeated this gesture seven times, as she said, “it needs to continuously change this way (Figure 32, top pane) for it to continuously change that way (Figure 32) because the volume has to go somewhere – it can’t just disappear.” Here, Olivia [NP] described the shearing of the pyramid as a continuous and temporal process, reframing the warrant and control to be in terms of continuous transformations.

Figure 32. "It needs to continuously change this way for it to continuously change that way."

Olivia [NP] [sitting, left] pushed her hands together vertically and then horizontally as she explained that the transformation the immersed participant described must be continuous in both directions to preserve volume.

Olivia [NP] concluded that "if the pyramid isn't getting bigger, it has to be getting infinitely thinner". This last statement makes it clear that the warrant in the participant's argument still supports the plausibility of the shearing transformation preserving volume. Figure 33 shows a ck¢-enriched Toulmin model for the participants' final argument.

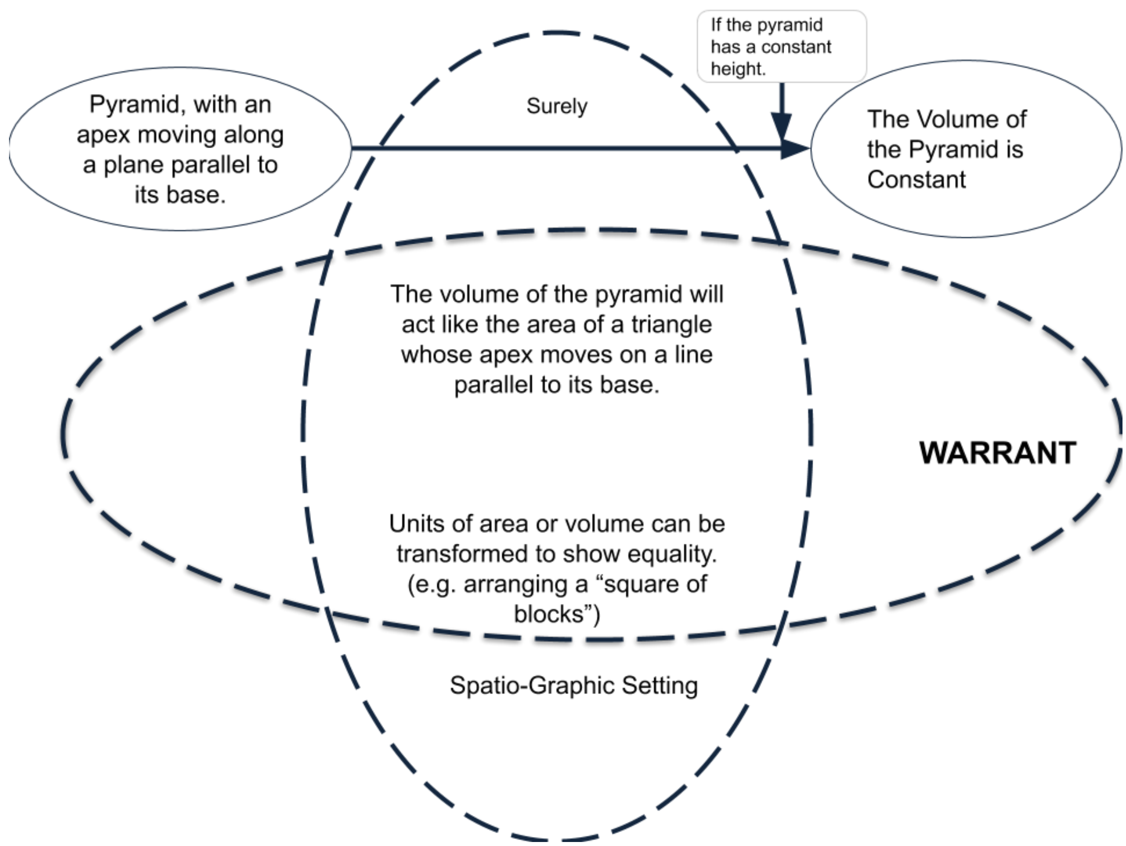

Figure 33. Ck¢-enriched Toulmin model of the second episode, final argument. This figure illustrates the components of the ck¢-enriched Toulmin model in the analysis of the second episode, arranged around an arrow representing the inference as a transformation of data into a claim. In this figure, the participants have added a rebuttal to their previous argument.

DISCUSSION

In the two episodes analysed in this study, one pair of participants engaged with the spatial diagrams using perspectives where their body was outside the diagram while another participant also engaged with perspectives where the body was intersecting the spatial diagram. These two episodes describe two different ways that students used the dimensionality of dynamic spatial diagrams to make mathematical arguments. Participants engaged with the spatial diagrams with their bodies, both in cases where the immersed participants' body parts were visible and invisible to the immersed participant. Both the perspectives taken and the bodily engagement with the diagram are only practicable with spatial diagrams.

Perspectives Inside the Spatial Diagram

I expected participants to engage in perspectives inside the spatial diagram by putting their head inside the pyramid, consistent with participants entering spheres described by Dimmel et al. (submitted). However, Emily [ImP] stepped into the figures, allowing the figure to cross-sect their body, while their head remained outside. While a similar perspective of the unbounded shearing of the pyramid could be achieved through a two-dimensional rendering, the bodily engagement of the participants may contribute to their use of the representation. When engaging with a bodily intersection with the spatial diagram, Emily [ImP] and Olivia [NP] described volume as "space inside" and the transformation of volume as a movement of space: "whatever space was being taken up this way is just being taken up this way." While Emily [ImP] and Olivia [NP] made rich spatiographic observations, their argument did not attend well to theoretical properties of the pyramid.

Emily [ImP] described mathematics as "number-oriented" and "calculation-based," but in her role as the immersed participant did not have easy access to numerical measures to use in empirical calculations. Emily [ImP] was the only participant of these two pairs to relate mathematics and calculation. The lack of familiar mathematical operations might have provoked Emily to take a novel perspective in the environment.

In contrast, Abigail [ImP] and Madison [NP] made claims that attended to the relationship of theoretical properties based on data from spatiographic observations. Abigail [ImP] and Madison [NP] only described the measures of the pyramid and triangle in theoretical terms and described the relationship between perimeter and area and a relationship between surface area and volume that they later realized were inconsistent with spatiographic observations of the limit case. Due to a break in the interview protocol, the resolution of this realization was not observed.

Bodily Engagement

In the second episode, Emily [ImP] seemed to have an understanding of her body cross-secting the the pyramid; she described that her legs would be "chopped of by" the faces of the pyramid. Emily [ImP] described the faces as "planes" in plural, which suggests that she imagined multiple faces cutting through her legs. This would likely result in a three-dimensional cross-section of her legs. While Emily [ImP] could not see her legs cross-secting the diagram, the participants' imagined cross-section may be related to the Emily [ImP] and Olivia's [NP] understanding of volume as the "space inside" the pyramid in their argument. Emily [ImP] and Olivia's [NP] description of volume as "space inside" contrasts with Emily's [ImP] description of mathematics as "number-based" which calls into question the earlier explanation of Emily [ImP] stepping into the figure as a result of a lack of familiar mathematical operators.

In the first episode, Abigail [ImP] manipulated the cross-section of the pyramid each time she described the volume of the pyramid by moving an open, upward facing palm through the interior of the pyramid. However, neither Abigail [ImP] nor Madison [NP] made reference to this cross-section in their argument. While the manipulation of a similar cross-section would be possible in a two-dimensional rendering, it would not be practicable. The learner would need to choose a direction of movement through a projected interface and provide a series of commands through keyboard and mouse input to move the cross-secting plane. The coincidence of Abigail [ImP] and Madison [NP]'s description of volume and manipulation of the cross-section suggests that the cross-section has some relationship to volume for the participant. While further research is needed to explore how this representation might be use to participants understanding of volume and if it is connected to dimensional deconstruction, the spatial diagram allowed the participant to passively engage with the volume without explicit use in the argument.

Implications

The dimensionality of the diagram allowed the participants to take perspectives within and outside the diagram and to use their bodies to engage with the diagram. The dimensionality of the diagram allows participants to freely choose their perspective by turning their head or walking through space (Bock & Dimmel, submitted; Dimmel, Pandiscio, & Bock, submitted). This includes perspectives within, inside and outside the diagram.

Bodily engagement with diagrams has also been explored in two-dimensional contexts. Ng and Sinclair (2015) describe participants dragging points on a multitouch tablet to explore the shearing of plane figures. Here, the learner can manipulate the point with a movement of their body, which might emphasize the "continuous and temporal" (p. 85) transformations described by Ng and Sinclair (2015). However, the learner moves their body into the plane of the figure's display, which bounds the figures working space. The working space is bounded not only within the plane, but in a small partition of the plane (Bock & Dimmel, submitted). Spatial diagrams offer an opportunity for diagrams to have an unbounded (Bock & Dimmel, submitted) working space, where its dimensionality allows the learner to be entirely immersed within the diagram. The efficiency and affordability of immersive spatial displays make accessible a novel opportunity for the representation of solid figures in dynamic spatial diagrams.

Spatial diagrams ease the constraints described by Mithala and Balacheff (2019). The diagrams allow for solid figures to be represented in a way that allows for reasonably accurate spatiographic observations and does not require analysis through dimensional deconstruction. In this study, one pair of participants used dimensional deconstruction to make an argument about the volume of the pyramid while another only used spatiographic observations of the "space inside" the pyramid. Spatial diagrams also make it possible to confuse the figure with its representation.

Finally, the background demographic information collected in this study was not rich enough to conclusively connect participant's mathematical backgrounds to their use of the diagrams. A revised study would include participants' descriptions of their mathematical histories, as well as more detailed demographic information including the mathematics courses they have taken, plan to take and what level of mathematics they plan to teach. A revised study might also include stimulated recall interviews to elicit the connections that participants see between their experiences with other types of diagrams and their use of spatial diagrams.

Future Inquiry

More research is needed into how the constraints of two-dimensional representations of three-dimensional figures apply to spatial diagrams. Further research is needed to understand what role dimensional deconstruction - the use of lower dimensional components to reason about higher dimensional figures - plays in students' use of spatial diagrams. Also, further research could investigate whether three-dimensional figures are confused for their representations in spatial diagrams. If so, do these confusions present similarly to their two-dimensional analogues?

Further research is needed to develop connections between participants' mathematical backgrounds (e.g. experiences with empirical arguments, spatiographic arguments and formal mathematical proof) and the ways that participants interact with diagrams. In particular, further research is needed to investigate how participants engage with spatial diagrams in comparison to other diagrams they may be more familiar with as representations of the same mathematical concept.

Additionally, the affordances of spatial diagrams offer opportunities to change how learners interact with diagrams. In this study, learners used some active bodily engagement (e.g. dragging the vertex, stepping into the figure) and some passive bodily engagement (e.g. moving the cross-section of the pyramid while describing its' volume. Further research is needed to understand how to design human-computer interfaces that intuitively support these intersections and the effect of these bodily interactions on the learner's agency over the diagram, the figure and the mathematical concept.

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APPENDIX A: INTERVIEW PROTOCOL

This interview will be composed of three sessions:

- Session 1 [60 min]:
  - Background Survey
  - Orientation to virtual reality VR

- Session 2 [120 min]:
  - Task 1: Triangle Shearing
  - Task 2: Pyramid Shearing
  - Task 3: Pyramid Shearing Generalization

- Session 3 [120 min]:
  - Task 4: Pyramid-Prism Volumes
  - Task 5: Euclid's proposition

The first session will be conducted with a group of all participants. The written background survey will be conducted individually.

The VR orientation serves two purposes. First, it provides an opportunity to capture participant's initial experiences in Virtual Reality as they become familiar with immersive environments and their ability to physically navigate a virtual space. While this orientation uses physical controllers instead of gesture-based controls, we anticipate that it will help set expectations for how to work in a shared space along with virtual objects. Second, the orientation will help to separate participants' initial reactions to virtual reality from their reactions to the virtual learning environments that are the settings for the study. Our hope is that providing an initial scene for participants to get their bearings in virtual reality will help focus the task-based interview sessions on the mathematical elements of the virtual environment, rather than on the immersive spatial display technology itself.

The second and third sessions will be conducted with groups of three participants. The purpose of the second and third sessions is to investigate how groups of students reason about mathematical tasks when immersed in a virtual world.

Overview

Hi (participant name). Thank you for agreeing to participate in our study. We will be asking you to explore figures in a a Virtual Reality environment. We are interested in your experience in the virtual environment. To help us to understand that, please narrate your thinking as you explore the environment. We may ask you to describe what you are thinking as you explore and as you work on specific mathematical tasks.

The interview will start with an orientation to Virtual Reality, since you may not have used this before. After you are comfortable with the controls within the environment, a set of virtual manipulatives will be loaded into the environment for you to explore. After this is loaded, we will ask you to solve 4 or 5 mathematical tasks. The interview will be conducted in 3 sessions: a 60 minute orientation, 120 minutes for the first three tasks and 90 minutes for the remaining two tasks. Breaks will be provided between tasks.

We will only use the visual immersion of the Virtual Reality environment, so we will be able to verbally communicate throughout the interview. You will maintain control over the virtual environment, and can exit the software or remove the HMD at any time.

Your view of the virtual world will be recorded on screen. We will also make an audio recording of your conversation. Additionally, video recording will be used to capture a 3rd person Mixed Reality view of the environment.

At any point, you may withdraw from participating in the study. Do you have any questions before we begin? Do you object to us making an audio recording of the interview or a screen recording of your work on the computer?

Background Survey

Participants will be asked to individually complete a paper-and pencil demographic survey. *For the orientation and tasks, participants will work in small groups.*

Begin Audio Recording Interview. Begin Screen Recording.

Orientation

Each participant will complete the SteamVR tutorial to orient them to the Virtual Reality interface. This introduces the spatial display technologies as well as virtual representations of hand controllers.

Participants will begin the think-aloud protocol, describing what they think and see. The following questions will be used, as needed, to probe for addition information and confirmation of participant thinking.

- Why did you say/do...?
- What are you thinking?
- What were you thinking when...?
- What kind of tool would you like to...?
- What do you mean by….?
- How would you explain…?
- What does...do?
- How would you describe (volume, surface area, area, perimeter) to a student?
- What can you do to test your conjecture?
- What representation helps to test your conjecture?
- How would you change the visualization to better represent your thinking?

For each of the tasks, one participant will be immersed in VR, while the other participants in the group will be able to view the 3rd person MixedReality video capture on a screen and verbally communicate with the immersed participant.

Task 1: Triangle on Parallel Planes

A triangle with vertices on two parallel lines is loaded into the environment. Participants are able to manipulate each of the vertices by pinching and dragging, and construct cross-sections by gesturing with an open palm at a given height. Before introducing the tasks, the pinch-to-drag and open-palm cross-section gestures will be explained, and participants will be given a short time to practice the gestures.

Participants may be asked:

- What do you observe about the figure?
- You can manipulate the triangle by pinching and moving each vertex.
    - What happens to the triangle as you move one of its vertices?
- You can cross-sect the triangle with an open palm parallel to the floor.
    - What do you observe about the cross-section?
- What claims can you make about the figure?

Scaffolding and Clarification

- What claims can you make about the figure's area? If participants need direction.
- If a participant references measures of area or perimeter:
    - What is area?
    - What is perimeter?
    - How are area and perimeter related?
    - What is the cross-section of a polygon?
- If a participant discusses extreme values for the vertices, or after 10 minutes:
    - How far can you move the vertex?

    - Reflect participants discussion of infinity as a large value or as a limit:
        - How does the perimeter change when you move a vertex to a large value/towards infinity?
        - How does the area change when you move a vertex to a large value/towards infinity?
        - How does the cross-section length change when you move a vertex to a large value/towards infinity?
- If a participant discusses a relationship between the length of the cross-section and the area or perimeter?
    - How are the cross-section and the area related?
    - How are the cross-section and the perimeter related?
    - Is this true for all cross-section heights?
    - Is this true for all vertex positions?
- If a participant asks for measurement tools or discusses formulas:
    - Are there any other ways that you can compare their measures?
    - Interviewers may add a panel displaying numeric measurement output on request of the participants.

Break for 5 minutes between tasks.

Task 2: Square-Based Pyramid on Parallel Planes

A pyramid with a square base is loaded into the environment. The apex is restricted to movement in a plane parallel to the base of the pyramid. Participants are able to manipulate each of the vertices by pinching and dragging, and construct cross-sections by gesturing with an open palm at a given height.

Participants may be asked:

- What do you observe about the figure?
- The apex of the pyramid can be moved by pinching and dragging. What do you observe when you move the apex?
- Similar to the triangle, a cross-section can be generated with an open palm. What do you observe about the cross-section?
- What claims can you make about the figure?

Scaffolding and Clarification

- What claims can you make about the figure's volume? If participants need direction.
- If a participant references measures of surface area or volume:
    - What is volume?
    - What is surface area?
    - How does volume relate to area?
    - How does volume relate to surface area?
- If a participant discusses the triangle while working on this case?
    - How does this relate to the triangle?
- If a participant discusses extreme values for the apex, or after 15 minutes:

    - How far can you move the apex?
    - Reflect participants discussion of infinity as a large value or as a limit:
        - How does the surface area change when you move a vertex to a large value/towards infinity?
        - How does the volume change when you move a vertex to a large value/towards infinity?
        - How does the cross-section change when you move a vertex to a large value/towards infinity?
- If a participant discusses a relationship between the length of the cross-section and the area or perimeter?
    - How are the cross-section and the volume related?
    - How are the cross-section and the surface area related?
- If a participant asks for measurement tools or discusses formulas:
    - Are there any other ways that you can compare their measures?
    - Interviewers may add a panel displaying numeric measurement output on request of the participants.

Task 3: Heptagon-based pyramid on parallel planes

A pyramid with a regular heptagon base is loaded into the environment. The apex is restricted to movement in a plane parallel to the base of the pyramid. Participants are able to manipulate each of the vertices by pinching and dragging, and construct cross-sections by gesturing with an open palm at a given height.

Participants may be asked:

- A new pyramid has been loaded. What do you observe about this pyramid?
- What claims can you make about the figure?

- How does your reasoning about the square-based pyramid apply to this pyramid?
- How does your reasoning about the Triangle apply to this pyramid?

Scaffolding and Clarification

- If a participant discusses relations between measurements of plane and solid figures:
    - What measurements of the triangle are analogous to measurements of the pyramid? Why?

- We call these transformations of figures on parallel lines or planes *shearing?*
    - What set of plane figures will behave like the triangle under shearing?
    - What set of solid figures will behave like the pyramid under shearing?

Task 4: Unit Cube with Square-based pyramid partitions

A unit cube is loaded into the environment, with a point at the center of one of it's faces. Six pyramids are constructed, using the faces of the cubes for their base and the centered point as their apex.

If participants have not concluded from the previous tasks that shearing does not affect volume, that may be explained at the interviewers' discretion.

Participants may be asked:

- What do you observe about the figure?
- How does the [color] pyramid relate to the cube?
- How many pyramids fill the cube?
  - Is that the maximum number?
  - Is that the minimum number?

Scaffolding and Clarification

- How does the volume of the [color] pyramid relate to the volume of the cube?
  - Can you make the volume of all of the pyramids equal?
  - What happens when the apex is on the perimeter of the face?
- What happens when you bring the apex to the edge of the cube?
  - How many pyramids remain?
  - What is the volume of that pyramid?
- What happens when you bring the apex to the corner of the cube?
  - How many pyramid are there in the cube?

- If a participant discusses congruent pyramids:
    - How do you know if the pyramids are congruent/the same?
    - What does this tell you about the relationship of the [color] pyramid to the prism?
- If a participant discusses volume equality:
    - How do you know that their volumes are equal?
    - Are they congruent?
        - If no, can their volumes be equal if they aren't congruent?
    - Can you use your conclusions about shearing to support your reasoning?

Break for 5 minutes between tasks.

Task 5: Triangular prism with Triangular Pyramid Partitions

A prism is loaded into the environment, with equilateral triangle bases, and three pyramids within the prism according to Euclid's construction.

Participants may be asked:

- What do you observe about this figure?
- We've generated a visualization of Euclid's Proposition XII.7. He claims that the area of each of the triangular pyramids that compose the triangular prism are exactly ⅓ the volume of the prism.
- Does the total volume of the three pyramids equal that of the prism? How do you know?
    - Is there any volume that isn't account for? Double counted?
- Are you convinced by Euclid's proposition that all of the pyramids volumes are equal?
    - If so, explain?
    - If not, why are you skeptical?
- How can you relate the volume of two of the pyramids?

Scaffolding and Clarification

- Euclid claims that Pyramid 1 [color] and Pyramid 2 [color] are congruent.
    - What does it mean for two pyramids to be congruent?
    - How do you know if Pyramid 1 and Pyramid 2 are congruent?
- Euclid claims that Pyramid 2 [color] and Pyramid 3 [color] equal in volume.
    - Are they congruent?
    - Are there other ways to relate their volumes?
    - Can you use your conclusions about shearing to relate their volumes?

- Finally, Euclid claims that Pyramid 1 [color] and Pyramid 3 [color] are equal in volume.
    - How can you relate their volumes?
    - Can you repeat the reasoning from a previous example?
    - Can you use your conclusions about shearing to relate their volumes?
    - Is volume equality transitive?

End Audio Recording Interview. End Screen Recording.

BIOGRAPHY OF THE AUTHOR

Camden Bock is a research assistant in the College of Education and Human Development at the University of Maine, pursuing a MST and PhD in STEM Education with concentrations in mathematics education. Camden works in the IMRE Lab, under the supervision of Professor Justin Dimmel to develop HandWaver, a room-scale gesture-based virtual environment for geometric manipulation. Prior to joining the IMRE Lab, Camden completed a B.S. in mathematics and certification in secondary mathematics education at Bates College in Lewiston, Maine. Camden grew up in New Boston, New Hampshire and attended Goffstown High School in Goffstown, New Hampshire. Camden is a candidate for the Master of Science in Teaching in STEM Education from The University of Maine in December of 2019.