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Rank reduction of string C-group representations

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RANK REDUCTION OF STRING C-GROUP REPRESENTATIONS

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Abstract. We show that a rank reduction technique for string C-group representations first used in [3] for the symmetric groups generalizes to arbitrary settings. The technique permits us, among other things, to prove that orthogonal groups defined on d-dimensional modules over fields of even order greater than 2 possess string C-group representations of all ranks $3 \leqslant n \leqslant d$. The broad applicability of the rank reduction technique provides fresh impetus to construct, for suitable families of groups, string C-groups of highest possible rank. It also suggests that the alternating group Alt(11)—the only known group having 'rank gaps'—is perhaps more unusual than previously thought.

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1. Introduction

In a recent joint work of the second author and Fernandes [3], a certain "rank reduction" technique was used to show that the symmetric group Sym(m) acts as the group of automorphisms of an abstract regular polytope of rank n for every $3 \leqslant n \leqslant m$. The technique was used again by those same to authors in [4] to prove a similar result for the alternating group Alt(m). Both applications of rank reduction seemed at the time to depend crucially on their particular setting, namely Sym(m) or Alt(m) acting on the natural permutation domain $\{1, \ldots, m\}$. In this paper we show, to the contrary, that it is a substantially more general technique.

Abstract regular polytopes have an equivalent formulation in terms of quotients of Coxeter groups with string diagrams, and it is helpful to frame our discussion in these terms. We say that $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ is a *string group generated by involutions*—or *sggi* for short—if $G = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, each ρ_i is an involution, and the sequence $\rho_0, \ldots, \rho_{n-1}$ satisfies the *commuting property*

$$(1) \qquad \forall i, j \in \{0, \ldots, n-1\} \qquad |i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1.$$

If an sggi $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ additionally satisfies the *intersection property*

$$(2) \qquad \forall I, J \subseteq \{0, \ldots, n-1\} \qquad \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle$$

then it is a *string C-group*, and n is its *rank*. If $|\rho_i \rho_{i+1}| > 2$ for all $0 \leqslant i \leqslant n-2$ then $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ is *irreducible*; unless G is directly decomposable, the string C-group must be irreducible. Our main result is the following.

Theorem 1.1 (Rank Reduction). *Let $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ be an irreducible string C-group of rank $n \geqslant 4$. If $\rho_0 \in \langle \rho_0\rho_2, \rho_3 \rangle$, then $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \ldots, \rho_{n-1}\})$ is a string C-group of rank $n-1$.*

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The condition $\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle$ is an easy one to verify, making the Rank Reduction Theorem a powerful tool in the search for new polytopes. For example, suppose that $\rho_2\rho_3$ has odd order $2k+1$. Then

$$((\rho_0\rho_2)\rho_3)^{2k+1} = (\rho_0(\rho_2\rho_3))^{2k+1} = \rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle,$$

so we obtain the following immediate and useful consequence of Theorem 1.1.

Corollary 1.2. *Let $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ be an irreducible string C-group of rank $n \geqslant 4$. If $\rho_2\rho_3$ has odd order, then $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \ldots, \rho_{n-1}\})$ is a string C-group of rank $n-1$.*

The integer sequence $[p_1, \ldots, p_{n-1}]$, where p_i is the order of $\rho_{i-1}\rho_i$ is called the *Schläfli type* of the string C-group $(G; \{\rho_0, \ldots, \rho_{n-1}\})$. So, Corollary 1.2 tells us to look at the third integer p_3 in the Schläfli type to see if we can apply the rank reduction mechanism once. Suppose we can, and suppose that $n \geqslant 4$. Then we obtain a new string C-group $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \ldots, \rho_{n-1}\})$ with Schläfli type $[q_1, \ldots, q_{n-2}]$, where q_1 is the order of $\rho_1\rho_0\rho_2$, q_2 is the order of $\rho_0\rho_2\rho_3$, and $q_i = p_{i+1}$ for $3 \leqslant i \leqslant n-2$. Thus, if $n > 4$ and p_4 is also odd, we can repeat the rank reduction to obtain a string C-group of rank $n-2$. Iterating, we obtain the following result.

Corollary 1.3. *Let $(G; \{\rho_0, \ldots, \rho_{n-1}\})$ be an irreducible string C-group of rank $n \geqslant 4$. Let $[p_1, \ldots, p_{n-1}]$ be its Schläfli type, and put*

$$t = \max\{j \in \{0, \ldots, n-3\} \colon \forall i \in \{0, \ldots, j\},\ p_{2+i} \text{ is odd}\}.$$

Then G is a string C-group of rank $n-i$ for each $i \in \{0, \ldots, t\}$.

Section 2 is devoted to the proof of the Rank Reduction Theorem. While the condition $\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle$ is convenient to ensure the success of the process, it is not essential; we illustrate this in Section 2. In Section 3, we show that some of the striking results proved in [3] and [4] for the groups $\mathrm{Sym}(m)$ and $\mathrm{Alt}(m)$ are, in fact, immediate consequences of Corollary 1.3. Indeed, the potential to iterate rank reduction on string C-groups via Corollary 1.3 impresses on us the importance of obtaining *some* general high rank construction for suitable families of groups. To emphasize that point, in Section 4 we examine recent constructions of high rank polytopes for symplectic and orthogonal groups through the lens of rank reduction, proving in particular the following new result.

Theorem 1.4. *Let $k \geqslant 2$ and $m \geqslant 2$ be integers.*

(a) *The symplectic group $\mathrm{Sp}(2m, \mathbb{F}_{2^k})$ is a string C-group of rank n for each $3 \leqslant n \leqslant 2m+1$.*
(b) *The orthogonal groups $\mathrm{O}^+(2m, \mathbb{F}_{2^k})$ and $\mathrm{O}^-(2m, \mathbb{F}_{2^k})$ are string C-groups of rank n for each $3 \leqslant n \leqslant 2m$.*

2. The Rank Reduction Theorem

Let $(G;\ \{\rho_0, \ldots, \rho_{n-1}\})$ be an sggi of rank n. If G is a string C-group, and $I \subseteq \{0, \ldots, n-1\}$, it readily follows that $\langle \rho_i \colon i \in I\rangle$ is also a string C-group on its defining generating sequence. The following result, proved in [6, Proposition 2E16], facilitates an inductive approach to verifying string C-group representations.

Proposition 2.1. *Let* $(G; \{\rho_0, \dots, \rho_{n-1}\})$ *be an sggi. If the subgroups* $\langle \rho_0, \dots, \rho_{n-2}\rangle$ *and* $\langle \rho_1, \dots, \rho_{n-1}\rangle$ *are both string C-groups, and*

$$\langle \rho_0, \dots, \rho_{n-2}\rangle \cap \langle \rho_1, \dots, \rho_{n-1}\rangle = \langle \rho_1, \dots, \rho_{n-2}\rangle,$$

then $(G; \{\rho_0, \dots, \rho_{n-1}\})$ *is a string* C*-group.*

The next elementary result follows from [2, Classification Theorem 1.2].

Lemma 2.2. *The dihedral group of order* $2k \geqslant 6$ *is an irreducible string C-group of rank* n *if, and only if,* $n = 2$.

Proof of Theorem 1.1. From the hypothesis $\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle$ it is immediate that rank reduction on the generators of G does not yield a proper subgroup. It remains to show that the new generating sequence is again a string C-group representation of G. For this we induct on the rank $n \geqslant 4$.

In the base case we have an irreducible string C-group $(G; \{\rho_0, \rho_1, \rho_2, \rho_3\})$ of rank 4, and must show that $(G; \{\rho_1, \rho_0\rho_2, \rho_3\})$ is a string C-group. Evidently, both $\langle \rho_1, \rho_0\rho_2\rangle$ and $\langle \rho_0\rho_2, \rho_3\rangle$ are string C-groups, so it suffices to show that the intersection of these two dihedral groups is the cyclic group $\langle \rho_0\rho_2\rangle$. Note,

$$\langle \rho_1, \rho_0\rho_2\rangle \cap \langle \rho_0\rho_2, \rho_3\rangle \leqslant \langle \rho_0, \rho_1, \rho_2\rangle \cap \langle \rho_0, \rho_2, \rho_3\rangle = \langle \rho_0, \rho_2\rangle,$$

since G is a string C-group. Thus, if $\langle \rho_1, \rho_0\rho_2\rangle \cap \langle \rho_0\rho_2, \rho_3\rangle$ properly contains $\langle \rho_0\rho_2\rangle$, it must contain ρ_0. But then $\langle \rho_0, \rho_1, \rho_2\rangle = \langle \rho_1, \rho_0\rho_2\rangle$ is dihedral, contrary to Lemma 2.2.

Next, suppose $(G; \{\rho_0, \dots, \rho_{n-1}\})$ is an irreducible string C-group of rank $n > 4$, with $\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle$. Assume the result holds for string C-groups of smaller ranks. In particular, since $H := \langle \rho_0, \dots, \rho_{n-2}\rangle$ is an irreducible string C-subgroup of G of rank $n-1$, by induction it follows that $(H;\ \{\rho_1, \rho_0\rho_2, \rho_3, \dots, \rho_{n-2}\})$ is a string C-group (of rank $n-2$). Further, since $\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle$, it follows that

$$K := \langle \rho_0\rho_2, \rho_3, \dots, \rho_{n-1}\rangle = \langle \rho_0, \rho_2, \dots, \rho_{n-1}\rangle.$$

As $(K;\ \{\rho_0, \rho_2, \rho_3, \dots, \rho_{n-1}\})$ is a string C-group, an easy induction shows that $(K;\ \{\rho_0\rho_2, \rho_3, \dots, \rho_{n-1}\})$ is also a string C-group. Finally,

$$\begin{aligned} H \cap K &= \langle \rho_1, \rho_0\rho_2, \rho_3, \dots, \rho_{n-2}\rangle \cap \langle \rho_0\rho_2, \rho_3, \dots, \rho_{n-1}\rangle \\ &= \langle \rho_0, \dots, \rho_{n-2}\rangle \cap \langle \rho_0, \rho_2, \dots, \rho_{n-1}\rangle \\ &= \langle \rho_0, \rho_2, \rho_3, \dots, \rho_{n-2}\rangle \\ &= \langle \rho_0\rho_2, \rho_3, \dots, \rho_{n-2}\rangle, \end{aligned}$$

and it now follows from Proposition 2.1 that $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \dots, \rho_{n-1}\})$ is a string C-group, as required. □

The $\boldsymbol{\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle}$ criterion. The condition $\rho_0 \in \langle \rho_0\rho_2, \rho_3\rangle$, while easy both to state and verify, is by no means essential for the rank reduction trick to work. We conclude this section with an example that can readily be checked on a computer. Consider the group G preserving the symmetric form

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

defined over $\mathbb{F}_3$. The Witt index of F is 1, so $G = \mathrm{O}^-(4, \mathbb{F}_3) \cong \mathrm{SL}(2, \mathbb{F}_9){:}\, 2$. For $v \in V = \mathbb{F}_3^4$ nonsingular, let $\tau(v)$ denote the reflection in the 1-space $\langle v\rangle$ relative

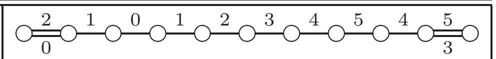
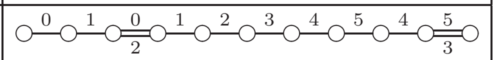
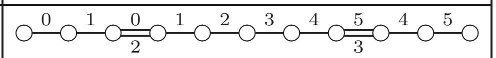
Type	CPR graph
{5,3,6,3,5}	
{5,5,6,3,5}	
{5,5,6,5,5}	

FIGURE 1. CPR graphs of rank six regular polytopes for Alt(11).

to the form F. Let v_0, v_1, v_2, v_3 denote the standard basis of V relative to which F is written and, for $i, j \in \{0, 1\}$, put $\rho_{2i+j} := (-1)^j \tau(v_{2i+j})$. As matrices,

$$\rho_0 = \begin{bmatrix} 2&0&0&0\\ 1&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix},\ \rho_1 = \begin{bmatrix} 2&1&0&0\\ 0&1&0&0\\ 0&1&2&0\\ 0&0&0&2 \end{bmatrix},\ \rho_2 = \begin{bmatrix} 1&0&0&0\\ 0&1&1&0\\ 0&0&2&0\\ 0&0&2&1 \end{bmatrix},\ \rho_3 = \begin{bmatrix} 2&0&0&0\\ 0&2&0&0\\ 0&0&2&1\\ 0&0&0&1 \end{bmatrix}.$$

Then $(G; \{\rho_0, \rho_1, \rho_2, \rho_3\})$ is a string C-group with Schläfli type $[4, 4, 6]$. Applying rank reduction, $(G; \{\rho_1, \rho_0\rho_2, \rho_3\})$ is also a string C-group of Schläfli type $[6, 6]$. Finally, ρ_0 is not in the dihedral group $\langle \rho_0\rho_2, \rho_3 \rangle$.

3. Symmetric and alternating groups

The $(m-1)$-simplex of $\mathrm{Sym}(m)$ has Schläfli type consisting entirely of 3's. Hence, applying Corollary 1.3 gives a family of string C-group representations for $\mathrm{Sym}(m)$ of every rank from $m - 1$ down to 3. This was already proved in [3, Theorem 3].

Recently, Fernandes and Leemans [4] showed for $m \geqslant 12$ that $\mathrm{Alt}(m)$ has string C-group representations of every rank $\lfloor (m - 1)/2 \rfloor$ down to 3. However, only in the case $m \equiv 1 \pmod 4$ can one apply Corollary 1.3 to the string C-group representation of (highest) rank $\lfloor (m-1)/2 \rfloor$ to produce representations of all ranks down to 3. We remark, though, that for n even [4, Theorem 4.1] is a consequence of Corollary 1.3.

These findings raise the question, for a family of groups, of whether one can find a highest rank string C-group representation that, via rank reduction, generates all other permissible ranks of string C-group representations. We have an affirmative answer for the symmetric groups and for subfamilies of alternating groups.

At the time of writing, the only group that has gaps in its set of possible ranks of string C-group representations is the alternating group Alt(11). This group has representations of ranks 3 and 6 but none of ranks 4 or 5. Figure 1 gives the Schläfli types and CPR graphs[1], extracted from [5], of the three pairwise non-isomorphic rank 6 string C-group representations of Alt(11). The reader will immediately notice that third integer (from the left or right!) in the Schläfli type is 6, so one cannot hope to use Corollary 1.2. Further, one can readily see from the graphs in Figure 1 that applying our rank reduction technique (again, from the left or from the right) will produce graphs that are not connected. This means that our rank reduction mechanism always generates proper (intransitive) subgroups of Alt(11).

[1] A CPR graph is a special type of Schreier coset graph that encodes the defining involutions of a string C-subgroup of $\mathrm{Sym}(m)$ as a labelled graph on the points of its domain $\{1, \ldots, m\}$.

4. Orthogonal and symplectic groups

The previous section provided a striking illustration of the power of rank reduction in situations where we can get our hands on at least one string C-group representation of high rank. Unfortunately, other than the symmetric and alternating groups, very few families of groups are known to have such representations. In this final section, however, we revisit two constructions for orthogonal and symplectic groups, applying our rank reduction technique to obtain new results.

Modular reduction of string Crystallographic groups. In a series of three papers, Monson and Schulte [7, 8, 9] conducted a comprehensive investigation of string C-groups that arise under modular reduction of crystallographic Coxeter groups. Their setting is as follows. Let $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be an abstract Coxeter group with string diagram and Schläfli type $[p_1, \dots, p_{n-1}]$. Suppose Γ has a faithful representation $\Gamma \to \mathrm{GL}(n, \mathbb{Z})$ as a group of reflections, and let $G = \langle r_0, \dots, r_{n-1} \rangle$ be its image in $\mathrm{GL}(n, \mathbb{Z})$. For each odd prime p, one can consider the group $\overline{G} = \langle \overline{r}_0, \dots, \overline{r}_{n-1} \rangle \leqslant \mathrm{GL}(n, \mathbb{F}_p)$ obtained by reducing matrix entries modulo p. Evidently, $\overline{G}$ remains an sggi. Further, although $\overline{G}$ does not automatically inherit the intersection property from its parent group G, Monson and Schulte show that in many cases it does.

Of particular interest to us are the constructions of high rank. In [9, Section 3] the authors study the family of so-called '3–infinity' groups, namely those having Schläfli type $[3^k, \infty^l, 3^m]$. Setting $m = 0$, they show in particular that for any prime $p \geqslant 5$, and any positive integers k, l, the reduction $\overline{G}$ modulo p of the string crystallographic Coxeter group G of Schläfli type $[3^k, \infty^l]$ is again a string C-group. Furthermore, $\overline{G}$ is a maximal subgroup of $\mathrm{O}(k + l + 1, \mathbb{F}_p)$ and has Schläfli type $[3^k, p^l]$. The residue class of p modulo 4 determines which maximal subgroup we generate—the generating reflections all have the same spinor norm relative to the quadratic form preserved by $\overline{G}$, and that norm varies according to p. (Note, we ignore the reduction modulo 3 since that just gives a string C-group representation of $\mathrm{Sym}(k + l + 1)$ as permutation matrices.)

A direct application of Corollary 1.3 now yields the following result:

Theorem 4.1. *For each $d \geqslant 3$ and each odd prime $p \geqslant 5$, there is a subgroup M of $\mathrm{O}(d, \mathbb{F}_p)$ of index 2 such that M is a string C-group of rank n for each $3 \leqslant n \leqslant d$.*

Proof of Theorem 1.4. More recently, the authors showed in joint work with Ferrara that, provided $k \geqslant 2$, the groups $\mathrm{O}(2m, \mathbb{F}_{2^k})$ and $\mathrm{Sp}(2m, \mathbb{F}_{2^k})$ also have string C-group representations of high rank [1, Corollary 1.5]: rank $2m$ for orthogonal groups, and rank $2m+1$ for symplectic groups, since $\mathrm{Sp}(2m, \mathbb{F}_{2^k}) \cong \mathrm{O}(2m+1, \mathbb{F}_{2^k})$. The approach—in some sense dual to that of Monson and Schulte—was to construct a quadratic form φ as a matrix relative to a basis of nonsingular vectors in such a way that the sequence of *symmetries* (generalizations of reflections) determined by these vectors gives a generating sequence for the desired orthogonal group as a string C-group. (The example given at the end of Section 2 was based on this construction.) It was further shown, in even dimension, how to control the Witt index of the orthogonal space determined by φ. It is a natural consequence of the construction that the product of successive symmetries in the generating sequence is always an element of order $2^k + 1$ [1, Section 5]. In particular, the Schläfli types of the string C-group representations arising from this construction are sequences of odd integers. Hence, Theorem 1.4 is an immediate consequence of Corollary 1.3 and [1, Corollary 1.5].

5. Final thoughts

Our rank reduction theorem shows the real interest in trying to find, for a given infinite family of groups, the string C-group representations of highest possible ranks. Indeed, given a 'highest rank' string C-group representation for a group G, we can attempt to use the rank reduction technique to produce new string C-group representations of lower ranks.

We have seen how this works almost effortlessly for the family of symmetric groups $\mathrm{Sym}(m)$, orthogonal groups $\mathrm{O}^{\pm}(2m, \mathbb{F}_{2^k})$, and symplectic groups $\mathrm{Sp}(2m, \mathbb{F}_{2^k})$. We have also seen how, with substantially more effort, it can be used to fill in the 'rank gaps' in the alternating groups $\mathrm{Alt}(m)$. The single exception in this regard is the group $\mathrm{Alt}(11)$, which was already identified in [4] as being special. In view of how broadly successful the rank reduction technique appears to be (albeit from our somewhat limited experience using it) the group $\mathrm{Alt}(11)$ strikes us now as an anomaly, and prompts us to conclude with the following conjecture.

Conjecture 5.1. The group $\mathrm{Alt}(11)$ is the only finite simple group whose set of ranks of string C-group representations is not an interval in the set of integers.

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