# Doubly Strong Equilibrium

Scalzo, Vincenzo

28 March 2020

# DOUBLY STRONG EQUILIBRIUM

VINCENZO SCALZO

Abstract. We present a new concept for (generalized) strategic form games, called *doubly strong equilibrium*, and give an existence result when the players have non-ordered and discontinuous preferences. Since a doubly strong equilibrium is a strong equilibrium in the sense of Aumann, we get the existence of strong equilibria in discontinuous games. The result has been obtained by using the *quasi-Ky Fan minimax inequality*. Applications to exchange economies are given. We prove the existence of *doubly strong allocations*, which maximize consumers' preferences on the set of feasible allocations. The doubly strong allocations belong to the core of the economy. When consumers' preferences are selfish, we have that the doubly strong allocations are fair in the sense of Schmeidler and Yaari. So, we get the existence of fair allocations in the setting of non-ordered and discontinuous preferences.

Keywords. Generalized games.Discontinuous and non-ordered preferences. Doubly strong equilibrium. Quasi-Ky Fan minimax inequality. Exchange economies. Core allocations. Fair allocations.

## 1. Introduction

Consider a strategic form game where the players have non-necessarily complete and transitive preference relations defined on the set of strategy profiles (non-ordered preferences). If cooperation among the players is assumed, the strategy profiles that are not refused by any coalition of players get a particular interest. In this framework, Aumann (1959, 1961) introduced the *strong equilibrium* and the *alpha-core*. A strong equilibrium is a strategy profile $x^*$ for which no coalition $S$ of players has a joint deviation $x_S$ such that the strategy profile $(x_S, x^*_{-S})$ is preferred to $x^*$ for all members of $S$. A strategy profile $\overline{x}$ belongs to the alpha-core if it is not true that there exists a coalition $S$ and $x_S$ such that, for every reaction $z_{-S}$ of the other players, each member of $S$ prefers $(x_S, z_{-S})$ to $\overline{x}$. General results on the existence of strong equilibria have been proved for games where, among other properties, players' preferences are represented by continuous and concave utility functions: see Ichiishi (1981), Nessah and Tian (2014) and references therein. With respect to these results, the conditions which guarantee the non-emptiness of the alpha-core are less restrictive: see Scarf (1971) and Uyanik (2015) for games with payoff functions and Border (1984) and Kajii (1992) for non-ordered preferences.

Concerning the existence of strong equilibria, the mentioned papers do not apply to many situations of interest in economics, where the players have either discontinuous payoff functions or non-ordered preferences.[1]

A strong equilibrium of a game is not necessarily the best situation that the players can obtain: a strategy profile which is the maximal element of every player's preference is better. We call *doubly strong equilibrium* such a strategy profile. A doubly strong equilibrium is a strong equilibrium, but there are strong equilibria which are not doubly strong (an example is shown in a following section).

Our aim is to investigate the existence of doubly strong equilibria in generalized games where, denoted by $X$ the set of strategy profiles, the feasible strategies are given by means of a mapping $K : X \rightrightarrows X$.[2] If $P_i(x) \subseteq X$ is the subset of strategy profiles that player $i$ strictly prefers to $x$, we say that an element $x^* \in X$ is a *doubly strong equilibrium* if $x^* \in K(x^*)$ and $P_i(x^*) \cap K(x^*) = \emptyset$ for each player $i$. In order to identify sufficient conditions which guarantee the existence of doubly strong equilibria, we follow a recent paper on the existence of Nash equilibria in games with discontinuous and non-ordered preferences.[3] In particular, given a generalized game $G$ with non-ordered preferences, we define a real-valued function $\Theta_G$ such that the doubly strong equilibria of $G$ coincide with the solutions to the so-called *quasi-Ky Fan minimax inequality*: (q-KF) *find* $x^* \in X$ *such that* $x^* \in K(x^*)$ *and* $\Theta_G(x, x^*) \leq 0$ *for all* $x \in K(x^*)$. We identify new sufficient conditions for the existence of solutions to (q-KF). So, we give properties on $G$ that allow the function $\Theta_G$ and mapping $K$ to satisfy these conditions.
The properties that we introduce are the *generalized deviation property* and the *uniform quasi-concavity*. The first one requires that, if a strategy profile $z$ is not a doubly strong equilibrium, there exists an upper semicontinuous mapping $\xi_z$ defined on an open neighborhood $O_z$ of $z$ such that, for every $z' \in O_z \setminus \{\text{doubly strong equilibria}\}$, one has $\xi_z(z') \subseteq K(z')$ and, for every $x' \in \xi_z(z')$, at least one player ranks $x'$ to be better than $z'$. This property is a generalization of the *single deviation property* introduced by Nessah and Tian (2008) and Reny (2009) in order to investigate the existence of Nash equilibria in discontinuous games.[4] Let us remark that the single

[1] We refer to the economic examples which have been the source of inspiration for the literature on discontinuous games: afterwards the early papers by Dasgupta and Maskin (1986), Baye et al. (1993), Simon and Zame (1990), Reny (1999), among the others, see Bagh and Jofre (2006), Carmona (2009), Bagh (2010), Mc Lennan et al. (2011), Reny (2011, 2016), Barelli and Meneghel (2013), Prokopovych (2013, 2016), Scalzo (2013, 2019a,b), Carmona and Podczeck (2014, 2016, 2018), He and Yannelis (2015, 2016), Nassah and Tian (2016).

[2] For instance, suppose a standard situation in games where the set $K_i(x_{-i})$ of available strategies of player $i$ depends on the choices $x_{-i}$ of the other players. So, if $N$ denotes the set of players, $K(x) = \prod_{i \in N} K_i(x_{-i})$ is the set of feasible strategy profiles when $x \in X$ is given.

[3] Scalzo (2019a).

[4] See also Nessah and Tian (2016) and Reny (2016).

deviation property is not enough to guarantee the existence of Nash equilibria in games which satisfy the standard quasi-concavity properties: see Reny (2009) and Scalzo (2019a). However, Nessah and Tian (2016) identified a new quasi-concavity property which characterizes the existence of Nash equilibria in games satisfying the single deviation property; see also Scalzo 2019a, where this new property has been called *transfer uniform quasi-concavity*. The uniform quasi-concavity here introduced strengthens the transfer uniform quasi-concavity. A generalized game is uniformly quasi-concave if, given a finite subset $A$ of strategy profiles, for each strict convex combinations $z$ of all elements of $A$, all players are uniform in identifying $x \in A$ so that no one ranks $x$ to be better than $z$. This property implies the following one, which is a standard condition in games and exchange economies: $x$ does not belong to the convex hull of $P_i(x)$, for each $i$ and for each $x$. Let us note that the uniform quasi-concavity and the generalized deviation property hold in discontinuous games.

When $K(x) = X$ for all $x \in X$, as a corollary of our result, we obtain the existence of strong equilibria in the setting of games with discontinuous and non-ordered preferences. We show that the result of the present paper is different from a recent one, where necessary and sufficient conditions for the existence of strong equilibria have been given (see Scalzo 2019c).

We apply our result to exchange economies with a finite number of consumers and non-ordered preferences. Economies where the consumers' preferences are interdependent (for example, economies with externalities) or selfish are considered. In the setting of discontinuous preference relations, we obtain the existence of feasible allocations $x^*$ such that there are no consumers which strictly prefer other allocations to $x^*$. Such an element $x^*$ is called *doubly strong allocation*. In particular, given an economy $\mathcal{E}$, we define a generalized game $G$ so that the set of doubly strong allocations of $\mathcal{E}$ coincides with the set of doubly strong equilibria of $G$. So, we introduce the generalized deviation property and the uniform quasi-concavity on $\mathcal{E}$.
Doubly strong allocations belong to the core of the economy. More precisely, if the preferences of consumers are interdependent, a doubly strong allocation $x^*$ have the following property: $(\alpha_Y)$ it is not possible that a coalition $S$ of consumers can redistribute their initial endowments in a way $x_S$ such that all members of $S$ strictly prefer $(x_S, z_{-S})$ to $x^*$, for every redistribution $z_{-S}$ of the endowments of the others consumers. Property $(\alpha_Y)$ has been introduced by Yannelis (1991), and the set of feasible allocations which satisfy $(\alpha_Y)$ is *Yannelis' alpha-core*. We obtain the non-emptiness of Yannelis' alpha-core in economies with a finite number $n \geq 2$ of consumers and preferences with non necessarily open lower sections (the result given

by Yannelis 1991 holds in economies with 2 consumers and preferences with open lower sections). When the preferences of consumers are selfish, the doubly strong allocations belongs to the standard core of the economy; so, we obtain the non-emptiness of the core in discontinuous preferences case. It is interesting to note that the doubly strong allocations are *fair*, that is: they are Pareto optimal and envy-free (see Foley 1967, Schmeidler and Yaari 1971). So, our result allows to obtain the existence of fair allocations in economies where the commodities can be infinite-dimensional and the consumers' preferences can be non-ordered and discontinuous, differently from the previous literature (see Varian 1974, Svensson 1983, Thomson 2007).

The paper is organized as follows. Section 2 introduces the quasi-Ky Fan minimax inequality and gives new sufficient conditions for the existence of solutions. The doubly strong equilibrium is presented in Section 3, while Section 4 is devoted to the existence of doubly strong equilibria in generalized games with discontinuous and non-ordered preferences. The applications to exchange economies are given in Section 5. Section 6 concludes the paper.

## 2. A mathematical tool: the quasi-Ky Fan minimax inequality

In this Section, we assume that $X$ is a non-empty and convex subset of a metrizable subset of a locally convex Hausdorff topological vector space. Let $\Theta$ be a real-valued function defined on $X \times X$ and let $K$ be a mapping (set-valued function) from $X$ to $X$. The problem:

$$\text{(q-KF)} \quad \begin{cases} \text{find } x^* \in X \text{ such that} \\ x^* \in K(x^*) \ \text{ and} \\ \Theta(x, x^*) \leq 0 \quad \forall\, x \in K(x^*)\ . \end{cases}$$

is the so-called *quasi-Ky Fan minimax inequality*. Element $x^*$ is a *solution* to the inequality. We aim to provide very general conditions which guarantee the existence of solutions. First, we need to recall some definitions and to give a preliminary result.

**Definition 1.** (Scalzo 2013) The function $\Theta$ is said to be *generalized* 0*-quasi-transfer continuous* if $\Theta(x,z) > 0$ implies that there exists an open neighborhood $O_z$ of $z$ and a well-behaved mapping $\xi : O_z \rightrightarrows X$ such that $\Theta(x', z') > 0$ for all $z' \in O_z$ and all $x' \in \xi(z')$.[5]

[5]A *well-behaved* mapping is an upper semicontinuous set-valued function with non-empty, convex and compact values.

**Definition 2.** (Zhou and Chen 1988) The function $\Theta$ is said to be 0-*diagonally quasi-concave* if, for every $\{x^1, ..., x^k\} \subset X$ and for every $z \in \text{sco}\{x^1, ..., x^k\}$, there exists $x \in \{x^1, ..., x^k\}$ such that $\Theta(x, z) \leq 0$.[6]

**Remark 1.** When $K(x) = X$ for all $x \in X$, the problem above is the classical *Ky Fan minimax inequality* (Ky Fan 1972). The properties recalled in Definitions 1 and 2 allow the existence of solutions: see Scalzo (2013, Proposition 2).

**Definition 3.** (Scalzo 2015) A mapping $F : W \rightrightarrows X$ is said to be *generalized transfer open lower sections* if $F(z) \neq \emptyset$ implies that there exists an open neighborhood $O_z$ of $z$ and a well-behaved mapping $\xi_z : O_z \rightrightarrows X$ such that $\xi_z(z') \subseteq F(z')$ for all $z' \in O_z$.

We need the following result (see the Appendix for the proof and comments).

**Theorem 1.** *Let $F : W \rightrightarrows D$ be a generalized transfer open lower sections mapping with non-empty and convex values. Assume that $W$ is a paracompact subset of a Hausdorff space and $D$ is a convex and compact subset of a locally convex Hausdorff topological vector space. Then, $F$ admits a well-behaved selection, that is a well-behaved mapping $\xi$ defined on $W$ such that $\xi(z) \subseteq F(z)$ for all $z \in W$.*[7]

Now, we can prove the existence of solutions to the inequality (q-KF) under general assumptions.

**Theorem 2.** *Assume that $X$ is compact and:*

i) $\Theta$ *is* 0-*diagonally quasi-concave;*
ii) $K : X \rightrightarrows X$ *is well-behaved;*
iii) *the mapping $F : X \rightrightarrows X$ defined by $F(z) = \{x \in K(z) : \Theta(x, z) > 0\}$ for each $z \in X$ is generalized transfer open lower sections.*

*Then, the solution set to the inequality* (q-KF) *is non-empty and compact.*

*Proof.* Define $W = \{z \in X : F(z) \neq \emptyset\}$ and let $z \in W$. From assumption iii), for some open neighborhood $O_z$ of $z$ and a well-behaved mapping $\xi_z : O_z \rightrightarrows X$, we get $\emptyset \neq \xi_z(z') \subseteq F(z')$ for all $z' \in O_z$. So, the open neighborhood $O_z$ is included in $W$, which proves that $W$ is an open subset of a metrizable space. Then $W$ is paracompact (see Michael 1953). Now, consider the mapping $T$ defined by $T(z) = \text{co}F(z)$ for each $z \in W$; in the light of iii), one has that $T$ is generalized transfer

[6] Given a finite subset $A$ of a vector space, we denote by sco$A$ the subset of strict convex combinations of all elements of $A$.

[7] We recall that a set $W$ is *paracompact* if every open covering $\mathcal{C}$ of $W$ admits an open and locally finite refinement $\mathcal{U}$, that is: for each $U \in \mathcal{U}$ there is $O \in \mathcal{C}$ such that $U \subseteq O$, and for each $z \in W$ there is an open neighborhood of $z$ which intersects only finitely many elements of $\mathcal{U}$ (see Michael 1953).

open lower sections. So, Theorem 1 applies on $T$ and we obtain the existence of a well-behaved mapping $\xi : W \rightrightarrows X$ such that $\xi(z) \subseteq T(z)$ for all $z \in W$. Define the set-valued function $Q : X \rightrightarrows X$ as below:

$$Q(z) = \begin{cases} \xi(z) & \text{if } z \in W \\ K(z) & \text{otherwise} . \end{cases}$$

From assumption ii), we get that $Q$ is well-behaved and, in the light of Kakutani-Fan-Glicksberg fixed point theorem, there exists at least one fixed point $x^*$ of $Q$. If $x^* \in W$, we have $x^* \in T(x^*) = \text{co}F(x^*)$. So, $x^* \in \text{sco}\{x^1, ..., x^k\}$, where $\{x^1, ..., x^k\} \subseteq K(x^*)$ and $\Theta(x^h, x^*) > 0$ for $h = 1, ..., k$. But assumption i) implies that $\Theta(x, x^*) \leq 0$ for at least one $x \in \{x^1, ..., x^k\}$, and we get a contradiction. Hence, $x^* \notin W$, which proves that $x^*$ is a solution to the inequality (q-KF). Finally, from iii) it follows that the solution set to (q-KF) is a closed subset of the compact space $X$. This concludes the proof. □

**Remark 2.** When $K(z) = X$ for all $z \in X$, the mapping $F$ is generalized transfer open lower sections if and only if $\Theta$ is generalized 0-quasi-transfer continuous. So, Theorem 2 includes Scalzo (2013, Proposition 2) as a special case. Moreover, in this case, $\Theta$ is 0-diagonally quasi-concave if and only if $z \notin \text{co}F(z)$ for all $z \in X$.

**Remark 3.** Zhou and Chen (1988) and Tian and Zhou (1991) obtained the existence of solutions to the inequality (q-KF) assuming that $\Theta(x, \cdot)$ is a lower semicontinuous function. Castellani et al. (2018), in finite dimensional spaces, supposed that the mapping $F$ is lower semicontinuous on the set of fixed points of $K$. The following example presents an inequality where the assumptions of Theorem 2 are satisfied, but the conditions of the mentioned papers fail.

**Example 1.** Let $\Theta$ be the function defined on $[0, 1] \times [0, 1]$ as below:

$$\Theta(x, z) = \begin{cases} 0 & \text{if } (x, z) \in [0, 1[ \times [0, 1/2[ \\ 1 & \text{if } (x, z) \in \{1\} \times [0, 1/2[ \\ 0 & \text{if } (x, z) \in [0, 2/3[ \times \{1/2\} \\ 1 & \text{if } (x, z) \in [2/3, 1] \times \{1/2\} \\ 0 & \text{if } (x, z) \in [0, 1] \times ]1/2, 1] \end{cases}$$

and $K(z) = [0, 1]$ for all $z \in [0, 1]$. The mapping $F$ defined by iii) of Theorem 2 is the following one:

$$F(z) = \begin{cases} \{1\} & \text{if } z \in [0, 1/2[ \\ [2/3, 1] & \text{if } z = 1/2 \\ \emptyset & \text{otherwise} . \end{cases}$$

$F$ is generalized transfer open lower sections and $z \notin \text{co}F(z)$ for each $z \in [0,1]$, which means that $\Theta$ is 0-diagonally quasi-concave (see Remark 2). So, all the assumptions of Theorem 2 are met. On the other hand, it is clear that $F$ is not lower semicontinuous on the set of fixed points of $K$ (which coincides with $[0,1]$): more precisely, $F$ is not lower semicontinuous at $z = 1/2$. Moreover, $\Theta(2/3, \cdot)$ is not lower semicontinuous at $z = 1/2$. Hence, the results from Zhou and Chen (1988), Tian and Zhou (1991) and Castellani et al. (2018) do not apply.

## 3. Doubly strong equilibrium in generalized games

Let $N$ be a finite set of players and, for each $i \in N$, assume that $X_i$ is a (non-empty) convex and compact subset of a metrizable subset of a locally convex Hausdorff topological vector space. For every $i \in N$, let $P_i$ be a mapping from $X = \prod_{j \in N} X_j$ to $X$; $P_i(x)$ is the set of strategy profiles that player $i$ strictly prefers to $x$. For each $x \in X$, we denote by $K(x)$ the set of strategy profiles which become feasible to the players when $x$ is chosen; let $K$ be the mapping $x \in X \longrightarrow K(x)$. We call *generalized game with non-ordered preferences* (in short *generalized game*) the list $G = \langle X_i, K, P_i \rangle_{i \in N}$. When $K(x) = X$ for all $x \in X$, $G$ is a *game with non-ordered preferences* (in short *game*). Assume that the players cooperate in order to reach agreements on the strategy profiles. Every non-empty subset of $N$ is called coalition; given a coalition $S$, a strategy profile $x$ is also denoted by $x = (x_S, x_{-S})$, where $x_S \in X_S = \prod_{i \in S} X_i$ and $x_{-S} \in X_{-S} = \prod_{j \notin S} X_j$.

Let us consider games. Because of the cooperation, the strategy profiles which cannot be refused (blocked) by any coalition get a particular interest. Aumann (1959) and (1961) proposed, respectively, the strong equilibrium and the alpha-core:

- a strategy profile $x^*$ is said to be a *strong equilibrium* if there are no coalitions $S$ and $x_S \in X_S$ such that $(x_S, x^*_{-S}) \in P_i(x^*)$ for all $i \in S$;
- a strategy profile $x^*$ belongs to the *alpha-core* if there are no coalitions $S$ and $x_S \in X_S$ such that $(x_S, z_{-S}) \in P_i(x^*)$ for all $i \in S$ and all $z_{-S} \in X_{-S}$.

The early literature on the existence of strong equilibria and non-emptiness of the alpha-core concerns games with continuous preferences; more precisely, preferences endowed with continuous utility functions in Ichiishi (1981) - also in Nessah and Tian (2014) - and open graph preferences in Kajii (1992). About the strong equilibrium, we note that the previous results have been given only for normal form games.[8] However, discontinuities in games arise in several remarkable economic situations such as the oligopolies of Bertrand (1883) and Hotelling (1929). Moreover,

[8]We recall that in a *normal form game* the mappings $P_i$ are given by means of the payoff functions $u_i$, and $P_i(z) = \{x \in X : u_i(x) > u_i(z)\}$.

there are oligopolies where the players have discontinuous utility functions and the set of strong equilibria, as well as the alpha-core, is non-empty: among the others, see Baye, Tian and Zhou (Example 1, 1993) and Uyanik (Example 1, 2015). The paper by Uyanik (2015) introduces sufficient conditions for the non-emptiness of the alpha-core in the setting of discontinuous normal form games with finite dimensional Euclidean spaces of strategies. More recently, in infinite dimensional spaces, necessary and sufficient conditions for the existence of strong equilibria have been given for games where discontinuous and non-ordered preferences are allowed: see Scalzo (2019c).

The definitions of strong equilibrium and alpha-core do not require that $P_i(x^*) = \emptyset$ for all $i \in N$; in other words, a game can have strong equilibria, as well as alpha-core elements, even if none of these strategy profiles is a maximal elements of all players' preferences.[9] For instance, consider the 2-player normal form game where $X_1 = X_2 = [-1, 1]$, $u_1(x_1, x_2) = 3x_1 - x_2^2 + 4x_2$ and $u_2(x_1, x_2) = -x_1^2 + x_1 - 2x_2$ (see Nessah and Tian 2014, Example 4.1); note that $u_1$ and $u_2$ are continuous functions. The strategy profile $(1, -1)$ is a strong equilibrium but $P_1(1, -1) \neq \emptyset$. On the other hand, it is clear that if $P_i(x^*) = \emptyset$ for all $i \in N$, then $x^*$ is a strong equilibrium.
So, it would be interesting to find reasonable conditions which guarantee the existence of a strategy profile satisfying all the players, in the sense that it is a maximal element on the set of feasible strategies for everyone. Note that such elements exist even in discontinuous oligopolies, such as that presented by Baye, Tian and Zhou (1993, Example 1) (see the following Example 4).
Our aim is to identify a class of generalized games, with possibly discontinuous and non-ordered preferences, where such kind of elements exist. First, we formalize these elements in the following definition.

**Definition 4.** Let $G = \langle X_i, K, P_i \rangle_{i \in N}$ be a generalized games. A strategy profile $x^*$ is said to be a *doubly strong equilibrium* of $G$ if $x^* \in K(x^*)$ and $P_i(x^*) \cap K(x^*) = \emptyset$ for each $i \in N$. $\mathfrak{S}_G$ denotes the set of doubly strong equilibria of $G$.

## 4. Existence of doubly strong equilibria

Given a generalized game $G = \langle X_i, K, P_i \rangle_{i \in N}$, for every $i \in N$, let $X_i$ be a convex and compact subset of a metrizable subset of a locally convex Hausdorff topological vector space. We set $\widehat{P}_i(x, z) = 1$ if $x \in P_i(z)$ and $\widehat{P}_i(x, z) = 0$ otherwise and define

[9]Given an asymmetric preference relation $\succ$ (strict preference) on $X$, define the mapping $P : X \rightrightarrows X$ by $P(z) = \{x \in X : x \succ z\}$. We recall that $x^* \in X$ is said to be a *maximal element* of $\succ$ if $P(x^*) = \emptyset$.

the real-valued function $\Theta_G$ as below:

$$\Theta_G(x,z) = \sum_{i\in N} \widehat{P}_i(x,z) \quad \text{for all } \ (x,z) \in X \times X \ . \tag{1}$$

It is easy to see that a strategy profile $x^*$ is a doubly strong equilibrium of $G$ *if and only if* $x^*$ is a solution to the inequality (q-KF) corresponding to the function $\Theta_G$ defined by (1) and the mapping $K$. So, in order to obtain the existence of doubly strong equilibria, we identify conditions on $G$ which allow the function $\Theta_G$ and the mapping $K$ to satisfy the assumptions of Theorem 2. We give the following definitions.

**Definition 5.** We say that $G$ is *uniformly quasi-concave* if, for each $\{x^1,...,x^k\} \subset X$ and each $z \in \text{sco}\{x^1,...,x^k\}$, there exists $x \in \{x^1,...,x^k\}$ such that $x \notin P_i(z)$ for all $i \in N$.

The uniform quasi-concavity holds in discontinuous generalized games, as the following example shows.

**Example 2.** Consider the normal form game $G$ where $X_i = [0,1]$, with $i = 1,2$, $u_i(x_i,x_{-i}) = 1$ if $x_i > x_{-i}$, $u_i(x_i,x_{-i}) = 0$ if $x_i < x_{-i}$, $u_i(x_i,x_{-i}) = 1$ if $x_i = x_{-i} > 0$ and $u_i(0,0) = 0$. We have $P_i(x) = \{z \in X : z_i \geq z_{-i}\}\setminus\{(0,0)\}$ if $x_i < x_{-i}$, $P_i(x) = \emptyset$ if $x_i \geq x_{-i} > 0$ and $P_i(0,0) = \{z \in X : z_i \geq z_{-i} > 0\}$. The mapping $K$ is constant-valued and $K(x) = [0,1] \times [0,1]$. Note that $\mathfrak{S}_G = \{x \in X : x_1 = x_2 > 0\}$.
This situation can interpreted as two individuals that make bids in order to obtain a facility, which is allocated to both of them, if the bids coincide and are non-zero, or to the individual whose bid is greater.[10]
We prove that $G$ is uniformly quasi-concave. We proceed by contradiction: assume that, for some $z \in \text{sco}\{x^1,...,x^k\}$ and for each $x^h \in \{x^1,...,x^k\}$, there is a player $i_h$ for whom $x^h \in P_{i_h}(z)$. This implies that: *a*) $z \in \text{sco}\bigcup_{i=1,2} P_i(z)$. But, if $z_i > z_{-i}$, we have $\text{sco}\bigcup_{j=1,2} P_j(z) = \{z' \in X : z'_{-i} \geq z'_i\}\setminus\{(0,0)\}$. If $z_i = z_{-i} > 0$, we have $\text{sco}\bigcup_{j=1,2} P_j(z) = \emptyset$, and $\text{sco}\bigcup_{j=1,2} P_j(0,0) = \text{sco}\bigcup_{j=1,2}\{z' \in X : z'_j \geq z'_{-j} > 0\}$. In all cases we get $z \notin \text{sco}\bigcup_{i=1,2} P_i(z)$, which contradicts *a*). This proves that $G$ is uniformly quasi-concave.

Another case of preference relations satisfying the uniform quasi-concavity property is given below.

**Example 3.** Consider two individuals that choose alternatives in $\mathbb{R}^2_+$ through the preference relations given below:

$P_1(x) = \big\{z \in \mathbb{R}^2_+ : \min\{z_1,z_2\} > \min\{x_1,x_2\}\big\}$ and $P_2(x) = x + \mathbb{R}^2_{++}$ for all $x \in \mathbb{R}^2_+$ .[11]

[10] A generalization of this game has been investigated by Reny (1999, Example 5.2).
[11] $\mathbb{R}^2_{++}$ denotes the interior of $\mathbb{R}^2_+$.

These preferences are well known in the literature: in particular, $P_1$ defines the asymmetric part of Leontief's order. One has that $P_2(x) \subset P_1(x)$ and $P_1(x)$ is convex for each $x \in \mathbb{R}^2_+$. Using the arguments of Example 2, we obtain that $P_1$ and $P_2$ satisfy the uniform quasi-concavity property.

It is easy to check the proposition below.

**Proposition 1.** *A generalized game $G$ is uniformly quasi-concave if and only if the function $\Theta_G$ defined by* (1) *is* 0*-diagonally quasi-concave.*

The following definition introduces the other property we need for the doubly strong equilibrium existence result.

**Definition 6.** We say that $G$ satisfies the *generalized deviation property* if $z \notin \mathfrak{S}_G$ implies that there exists an open neighborhood $O_z$ of $z$ and a well-behaved mapping $\xi_z : O_z \rightrightarrows X$ such that: i) $\xi_z(z') \subseteq K(z')$ for each $z' \in O_z \backslash \mathfrak{S}_G$; ii) for each $z' \in O_z \backslash \mathfrak{S}_G$ and for each $x' \in \xi_z(z')$, there is a player $i$ for whom $x' \in P_i(z')$.

The game presented in Example 2 (as well as any game obtained with the preferences given in Example 3) satisfies the generalized deviation property. Indeed, given an open neighborhood $O_z$ of $z \notin \mathfrak{S}_G$, it is sufficient to set $\xi_z(z') = \{(1,1)\}$ for all $z' \in O_z$ (when $z = (0,0)$, the player who profitably deviates depends on $z'$) An other situation satisfying the generalized deviation property is the following one (see Baye, Tian and Zhou 1993, Example 1).

**Example 4.** Two individuals sell the same good in a market and set prices in $[0,T]$. The profit functions are the following ones, where $i \in \{1,2\}$: $u_i(x_i, x_{-i}) = x_i$ if $x_i \leq x_{-i}$ and $u_i(x_i, x_{-i}) = x_i - c$ otherwise, where $c \in ]0,T[$ - we assume $K(x) = [0,T] \times [0,T]$ for all $x \in [0,T] \times [0,T]$ and $P_i(x) = \{z \in [0,T] \times [0,T] : u_i(z) > u_i(x)\}$. In this situation, the individual that fixes the higher price has to pay a tax (namely $c$) in order to remain in the market. We have $\mathfrak{S}_G = \{(1,1)\}$. Let $z \neq (1,1)$. If $z_1 > z_2$, we fix $\varepsilon > 0$ such that $z'_1 > z'_2$ for all $z' \in O_z = ]z_1 - \varepsilon, z_1 + \varepsilon[ \times ]z_2 - \varepsilon, z_2 + \varepsilon[$ and $z_1 - \varepsilon > z_2 + \varepsilon$. Set $\xi_z(z') = (z_1 - \varepsilon, z_1 - \varepsilon)$ for each $z' \in O_z$, we obtain $\xi_z(z') \in P_2(z')$. Similarly if $z_1 < z_2$: in this case, player 1 deviates. If $z_1 = z_2$, one can set $\xi_z(z') = (1,1)$. Finally, we see that the generalized deviation property holds true.[12]

The generalized deviation property is implied by a classical property on mappings. In fact, we have:

[12]Obviously, one can define $\xi_z(z') = (1,1)$ also in the case $z_i > z_{-i}$. Nevertheless, the function $\xi_z(z') = (z_i - \varepsilon, z_i - \varepsilon)$ allows to show the local character of the generalized deviation property.

**Proposition 2.** *Let $G$ be a generalized game where $x \in K(x)$ for all $x \in X$. Then, $G$ satisfies the generalized deviation property whether $P_i$ and $K$ are open lower sections, for each $i \in N$.*[13]

*Proof.* Let $z \notin \mathfrak{S}_G$. Since $z \in K(z)$, for at least one player $i$ and a strategy profile $x$, we have $z \in K^{-1}(x) \cap P_i^{-1}(x)$. So, there is an open neighborhood $O_z$ of $z$ such that $x \in K(z') \cap P_i(z')$ for all $z' \in O_z$. Finally, it is sufficient to set $\xi_z(z') = x$ and the thesis follows. □

The property by Definition 6 is given in the spirit of the *single deviation property* introduced by Nessah and Tian (2008) and Reny (2009) for what concerns the existence of Nash equilibria.[14] The single deviation property requires that, if a strategy profile $z$ is not a Nash equilibrium, there exists a strategy profile $x'$ such that, for each $z'$ which belongs to a suitable open neighborhood of $z$, at least one player can use his strategy in $x'$ in order to get a profitable unilateral deviation with respect to $z'$. So, the deviating player depends on $z'$. Let us remark that, even if this property seems to be very natural, it is not a sufficient conditions for the existence of Nash equilibria in games which satisfy the usual quasi-concavity like properties: see Reny (2016, Counterexample 6.1) and Scalzo (2019a, Example 3). However, there exist conditions which allow the existence of Nash equilibria in games satisfying the single deviation property. In particular, when the single deviation property holds true, the existence of Nash equilibria is characterized by means of a new quasi-concavity like property: see Nessah and Tian (2016, Theorem 6) and Scalzo (2019a, Theorem).[15]

Now, we can state the existence of doubly strong equilibria in generalized games with discontinuous and non-ordered preferences.

**Theorem 3.** *Let $G$ be a generalized game where $K$ is a well-behaved mapping. Assume that $G$ is uniformly quasi-concave and satisfies the generalized deviation property. Then, the set of doubly strong equilibria of $G$ is non-empty.*

*Proof.* By contradiction, assume $\mathfrak{S}_G = \emptyset$. Consider the function $\Theta_G$ given in (1) and the mapping $F : X \rightrightarrows X$ defined by $F(z) = \{x \in K(z) : \Theta_G(x, z) > 0\}$ for all $z \in X$. Let $F(z) \neq \emptyset$. Since $G$ satisfies the generalized deviation property and $z \notin \mathfrak{S}_G$, there exists an open neighborhood $O_z$ of $z$ and a well-behaved mapping $\xi_z : O_z \rightrightarrows X$ such that: $\xi_z(z') \subseteq K(z')$ for all $z' \in O_z$; $\Theta_G(x', z') > 0$ for all $x' \in \xi_z(z')$ and all $z' \in O_z$. So, $\xi_z(z') \subseteq F(z')$ for every $z' \in O_z$, that is: the mapping $F$ is generalized transfer

[13] This means that $K^{-1}(z)$ and $P_i^{-1}(z)$ are open sets for all $z \in X$.

[14] See also Nessah and Tian (2016) and Reny (2016).

[15] Scalzo (2019a) considers a condition more general than the single deviation property. See also this paper for a comparison between the single deviation property and other sufficient conditions for the existence of Nash equilibria in discontinuous games.

open lower sections. Moreover, from Proposition 1, we have that $\Theta_G$ is 0-diagonally quasi-concave. Finally, Theorem 2 applies and we obtain the existence of solutions to the inequality (q-KF) corresponding to $\Theta_G$ and $K$. But such solutions are doubly strong equilibria of $G$, and we get a contradiction. □

**Remark 4.** A recent paper introduces necessary and sufficient conditions for the existence of strong equilibria - in the sense of Aumann (1959) - in games (not generalized games), where the players have non necessarily continuous and non-ordered preferences (see Scalzo 2019c). In particular, given a game $G = \langle X_i, P_i \rangle_{i \in N}$, the following conditions have been provided:

- *SE-deviation property*: for each $z \notin \{\text{strong equilibria of } G\}$ there exists an open neighborhood $O_z$ of $z$ and $x' \in X$ such that: for all $z' \in O_z \setminus \{\text{strong equilibria of } G\}$ there is $\emptyset \neq S \subseteq N$ so that $(x'_S, z'_{-S}) \in \bigcap_{i \in S} P_i(z')$;
- *SE-transfer uniform quasi-concavity*: for each $\{x^1, ..., x^k\} \subset X$ there is $\{z^1, ..., z^k\} \subset X$ ($z^h$ is associated with $x^h$, $h = 1, ..., k$) such that $z \in \text{sco}\{z^{h_1}, ..., z^{h_l}\}$ implies that there exists $x \in \{x^{h_1}, ..., x^{h_l}\}$ so that: for each $\emptyset \neq S \subseteq N$, at least one player $i \in S$ gets $(x_S, z_{-S}) \notin P_i(z)$.

Then, it has been proved that *if a game G satisfies the SE-deviation property, the set of strong equilibria of G is non-empty if and only if G is SE-transfer uniformly quasi-concave* (Scalzo 2019c, Theorem 2).

In the case where $K(x) = X$ for each $x \in X$, Theorem 3 and the result recalled above are not comparable. In particular, the following Example 5 shows a game which satisfies the assumptions of Scalzo (2019c, Theorem 2), but Theorem 3 does not apply. On the other hand, Example 6 proves that Theorem 3 cannot be deduced from Scalzo (2019c, Theorem 2).

**Example 5.** Let $G$ be the 2-person game where $X_1 = X_2 = [-1, 1]$ and the payoff functions are given by $u_1(x_1, x_2) = 3x_1 - x_2^2 + 4x_2$ and $u_2(x_1, x_2) = -x_1^2 + x_1 - 2x_2$ (see Nessah and Tian 2014, Example 4.1). Set $P_i(x) = \{z \in X : u_i(z) > u_i(x)\}$ for each $x \in X$ and $i = 1, 2$. Since the functions $u_1$ and $u_2$ are continuous, the *SE*-deviation property holds true. Moreover, $(1, -1)$ is a strong equilibrium; so, $G$ is *SE*-transfer uniformly quasi-concave. The game satisfies the assumptions of Scalzo (2019c, Theorem 2). On the other hand, we get: $P_1(1, 1) = \emptyset$ and $P_1(x) \neq \emptyset$ for each $x \neq (1, 1)$; $P_2(1/2, -1) = \emptyset$ and $P_2(x) \neq \emptyset$ for each $x \neq (1/2, -1)$. So, there is no $x^* \in X$ such that $P_i(x^*) = \emptyset$ for $i = 1, 2$. Finally, we deduce that Theorem 3 does not apply on $G$.

**Example 6.** Consider a 2-person game $G$ with $X_1 = X_2 = [0, 1]$. For $i = 1, 2$, the mapping $P_i$ is defined by $P_i(x) = \{x_{-i}\} \times [0, 1]$ if $x_i > x_{-i}$ and $P_i(x) = \emptyset$ otherwise (see Basile and Scalzo 2019, Example 4.2). The set of doubly strong equilibria is $\mathfrak{S}_G = \{x \in X : x_1 = x_2\}$. Let us prove that $G$ satisfies the generalized deviation property. Suppose that $z \notin \mathfrak{S}_G$. For some player $i$, one has $z_i > z_{-i}$; so, let $O_z$ be an open neighborhood of $z$ such that $z'_i > z'_{-i}$ for all $z' \in O_z$. Define the mapping $\xi_z$ by $\xi_z(z') = \{z'_{-i}\} \times [0, 1]$ for each $z' \in O_z$. We obtain $x' \in P_i(z')$ for every $x' \in \xi_z(z')$ and every $z' \in O_z$; so, the generalized deviation property is satisfied. Note that the strategy profile $x' \in \xi_z(z')$ which belongs to $P_i(z')$ depends on $z'$, and there is no way to find any strategy profile $x''$ for which $x'' \in P_i(z')$ for every $z' \in O_z$. So, the game does not satisfy the *SE*-deviation property, that is: Scalzo (2019c, Theorem 2) does not apply on $G$. Now, if $x_i > x_{-i}$, one has $\bigcup_{j=1}^2 P_j(x) = \{x_{-i}\} \times [0, 1]$ and $x \notin \mathrm{co} \bigcup_{j=1}^2 P_j(x)$. So, using the argument given in Example 2, one gets that $G$ is uniformly quasi-concave. Finally, the assumptions of Theorem 3 hold on $G$.

**Remark 5.** Recently, Basile and Scalzo (2019) have given new results on the non-emptiness of the Aumann's alpha-core in the setting of games with non-ordered preferences. The authors have introduced the sets of assumptions recalled below, where $G = \langle X_i, P_i \rangle_{i \in N}$ is a game and $\mathcal{C}$ denotes the alpha-core of $G$:

I) $G$ satisfies the *coalitional deviation property* if, for each $z \notin \mathcal{C}$, there exists an open neighborhood $O_z$ of $z$ and $x' \in X$ such that, for every $z' \in O_z \backslash \mathcal{C}$, some coalition $S$ gets $\{x'_S\} \times X_{-S} \subseteq P_i(z')$ for each $i \in S$;

$G$ is *coalitional transfer quasi-concave* if, for each $\{x^1, ..., x^k\} \subset X$ there exists $\{z^1, ..., z^k\} \subset X$, where $x^h \mapsto z^h$, such that, for every $z \in \mathrm{sco}\{z^{h_1}, ..., z^{h_l}\}$, with $\{z^{h_1}, ..., z^{h_l}\} \subseteq \{z^1, ..., z^k\}$, one can find $x \in \{x^{h_1}, ..., x^{h_l}\}$ so that no coalitions can block $z$ by using $x$, that is: for all coalition $S$, there is $w_{-S} \in X_{-S}$ and $i \in S$ such that $(x_S, w_{-S}) \notin P_i(z)$.

II) $G$ satisfies the *generalized coalitional deviation property* if, for each $z \notin \mathcal{C}$, there exists an open neighborhood $O_z$ of $z$ and a well-behaved mapping $\xi_z : O_z \rightrightarrows X$ such that, for each $z' \in O_z \backslash \mathcal{C}$ and for each $x' \in \xi_z(z')$, there exists a coalition $S$ for which $\{x'_S\} \times X_{-S} \subseteq P_i(z')$ for every $i \in S$;

$G$ is *coalitional quasi-concave* if, for each $\{x^1, ..., x^k\} \subset X$ and for each $z \in \mathrm{sco}\{x^1, ..., x^k\}$, there exists $x \in \{x^1, ..., x^k\}$ so that no coalitions can block $z$ by using $x$.

The sets of assumptions I) and II) guarantee the non-emptiness of the alpha-core: see, respectively, Theorem 4.1 and Theorem 4.2 of the mentioned paper. In particular, in the setting of games which satisfy the coalitional deviation property, the non-emptiness of the alpha-core is characterized by means of the coalitional transfer

quasi-concavity.
The *generalized deviation property* and the *uniform quasi-concavity* introduced in the present paper are given in the same spirt of the assumptions of the set II). Since the set of doubly strong equilibria is included in the alpha-core, Theorem 3 and Theorem 4.2 by Basile and Scalzo (2019) can be compared only in the case where every alpha-core element is a doubly strong equilibrium. So, let us consider games where the alpha-core coincides with set of doubly strong equilibria. It is easy to see that *the generalized coalitional deviation property implies the generalized deviation property*, while the converse does not hold. Similarly, it is clear that *the uniform quasi-concavity implies the coalitional quasi-concavity*, but the viceversa is not true. Therefore, even if Theorem 3 is concerning the existence of a concept of equilibrium included in the alpha-core, the assumptions of Theorem 3 are not included in the set II). Finally, Example 6 shows that the generalized deviation property is not connected with the coalitional deviation property.

Remarks 4 and 5 have pointed out the differences between Theorem 3 and recent results on the existence of strong equilibria and non-emptiness of the alpha-core. We highlight that the differences between these results are not only on the sets of assumptions. Theorem 3 deals with the existence of the new concept of doubly strong equilibrium, which is a refinement of both the strong equilibrium and the alpha-core. Moreover, the existence of doubly strong equilibria has been obtained for generalized games. So, Theorem 3 allows to study situations where the strategies that are feasible to the players are subject to constraints. Finally, as the following Section shows, the existence of doubly strong equilibria finds applications in exchange economies.

## 5. Applications to exchange economies

In this Section, we apply the doubly strong equilibrium existence result to exchange economies with discontinuous and non necessarily ordered preferences. First, we consider the case where consumers' preferences are interdependent, that is: the preferences are defined on the set of allocations (this situation occurs, for example, in economies with externalities). We obtain the existence of feasible allocations $x^*$, called *doubly strong allocations*, such that there are no consumers that strictly prefer other feasible allocations to $x^*$ (these allocations belong to the core of the economy). Then, we deal with consumers that have selfish preferences. We obtain that the doubly strong allocations satisfy a fairness rule.

5.1. **Exchange economies with interdependent preferences.** Consider an exchange economy $\mathcal{E}$ with a finite number of consumers ($N$ denotes the set of consumers) and let $Y$ be the space of bundle of goods, that we assume to be included in a metrizable subset of a locally convex Hausdorff topological vector space. For each $i \in N$, $X_i \subseteq Y$ is the consumption set of consumer $i$ and $e_i \in X_i$ is the initial endowment of $i$; we assume that every consumption set is convex and compact. The elements of $X = \prod_{j\in N} X_j$ are called *allocations*. An allocation $x$ is said to be *feasible* if $\sum_{i\in N} x_i = \sum_{i\in N} e_i$; the set of feasible allocations of $\mathcal{E}$ is denoted by $\mathcal{F}_{\mathcal{E}}$. We assume that the preferences of consumers are interdependent and not necessarily ordered. So, for each $i \in N$, we have a mapping $P_i : X \rightrightarrows X$ where, for all $x \in X$, $P_i(x)$ is the set of allocations that consumer $i$ strictly prefers to $x$. We set $\mathcal{E} = \langle X_i, P_i, e_i\rangle_{i\in N}$.

Let us focus on a cooperative equilibrium concept, where the consumers cooperate in order to identify the feasible allocations that cannot be refused by any coalition. In this setting, Yannelis (1991) introduced a concept of alpha-core. More precisely, Yannelis considered the feasible allocations $\bar{x}$ for which it is not true that there exists a coalition $S$ and $x_S \in \prod_{i\in S} X_i$ such that: i) $\sum_{i\in S} x_i = \sum_{i\in S} e_i$; ii) $(x_S, z_{-S}) \in \bigcap_{i\in S} P_i(\bar{x})$ for all $z_{-S}$ with $\sum_{j\notin S} z_j = \sum_{j\notin S} e_j$. If i) and ii) hold true, we say that the coalition $S$ *Y-blocks* the allocation $\bar{x}$. So, the *Yannelis' alpha-core* of the economy is the set of feasible allocations that are not $Y$-blocked.

The concept proposed by Yannelis seems to be very natural for an exchange economy. Of course, the best situation for the economy is the existence of feasible allocations $x^*$ such that $P_i(x^*) \cap \mathcal{F}_{\mathcal{E}} = \emptyset$ for all consumer $i$; let us call such elements *doubly strong allocations*. The set of doubly strong allocations of the economy $\mathcal{E}$ is denoted by $\mathfrak{S}_{\mathcal{E}}$. Obviously, every doubly strong allocation belongs to Yannelis' alpha-core. So, the following questions arise: *Are there reasonable conditions which guarantee the existence of doubly strong allocations? What about these conditions, if any, and the sufficient conditions for the existence of alpha-core allocations in the sense of Yannelis (1991)?*

For exchange economies with 2 consumers, Yannelis (1991) obtained the existence of alpha-core allocations providing that the following assumptions are satisfied ($i = 1, 2$): *a*) $P_i$ is open lower sections; *b*) $x \notin \mathrm{co}P_i(x)$ for all $x \in X$. Using counterexamples, Holly (1994) proved that Yannelis' result cannot be extended to economies with more than 2 consumers. This would seem to show that an affirmative answer to the first of the questions above must involve conditions more restrictive than *a*) and *b*). But the existence of doubly strong allocations in economies with more than 2 consumers can be obtained under conditions which do not imply both *a*) and *b*). More precisely, we relax condition *a*) and strengthen condition *b*); we show that our strengthening of *b*) is not connected with *a*) (see Remark 6). So, we obtain the existence of doubly strong allocations, and therefore of alpha-core allocations in the sense of Yannelis (1991), for exchange economies with a finite number $n \geq 2$ of consumers and discontinuous and non-ordered preferences: see the following Theorem 4. First, we give some definitions.

**Definition 7.** We say that the economy $\mathcal{E}$ satisfies the *generalized deviation property* if, for each allocation $z \notin \mathfrak{S}_{\mathcal{E}}$, there exists an open neighborhood $O_z$ of $z$ and a well-behaved mapping $\zeta_z : O_z \rightrightarrows X$ such that: i) $\zeta_z(z') \cap \mathcal{F}_{\mathcal{E}} \neq \emptyset$ for all $z' \in O_z \backslash \mathfrak{S}_{\mathcal{E}}$; ii) for each $z' \in O_z \backslash \mathfrak{S}_{\mathcal{E}}$ and each $x' \in \zeta_z(z')$, there exists a consumer $i$ for whom $x' \in P_i(z')$.

Using the arguments of the proof of Proposition 2, one can prove:

**Proposition 3.** *Given $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$, if $P_i$ is open lower sections for all $i \in N$, then $\mathcal{E}$ satisfies the generalized deviation property.*

We define the *uniform quasi-concavity* for an exchange economy $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ as given in Definition 5 for generalized games. We have:

**Proposition 4.** *If an exchange economy is uniformly quasi-concave, then condition b) holds true.*

*Proof.* By contradiction, suppose that $z \in \mathrm{co}P_i(z)$ for at least one $z \in X$ and $i \in N$. So, $z \in \mathrm{sco}\{x^1, ..., x^k\}$ where $\{x^1, ..., x^k\} \subseteq P_i(z)$. Since the economy is uniformly quasi-concave, for at least one $x \in \{x^1, ..., x^k\}$, we have $x \notin P_j(z)$ for all $j \in N$, and we get a contradiction. □

The existence of doubly strong allocations is obtained below.

**Theorem 4.** *Assume that an exchange economy satisfies the generalized deviation property and is uniformly quasi-concave. Then, the set of doubly strong allocations is non-empty.*

*Proof.* Let $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$ be an economy satisfying the generalized deviation property and the uniform quasi-concavity. Consider the generalized game $G = \langle X_i, K, P_i \rangle_{i \in N}$ where $K(x) = \mathcal{F}_{\mathcal{E}}$ for every $x \in X$. It is clear that $\mathfrak{S}_{\mathcal{E}} = \mathfrak{S}_G$. Suppose that $z \notin \mathfrak{S}_G$. Since $\mathcal{E}$ satisfies the generalized deviation property, for some well-behaved mapping $\zeta_z : O_z \rightrightarrows X$, we have that: $\zeta_z(z') \cap \mathcal{F}_{\mathcal{E}} \neq \emptyset$ for all $z' \in O_z \backslash \mathfrak{S}_G$; for each $z' \in O_z \backslash \mathfrak{S}_{\mathcal{E}}$ and each $x' \in \zeta_z(z')$, there exists $i \in N$ such that $x' \in P_i(z')$. Now, define $\xi_z(z') = \zeta_z(z') \cap \mathcal{F}_{\mathcal{E}}$ for every $z' \in O_z$. One has that $G$ satisfies the generalized deviation property. Moreover, $G$ is obviously uniform quasi-concave. So, the assumptions of Theorem 3 hold, and we obtain that $\mathfrak{S}_G = \mathfrak{S}_{\mathcal{E}} \neq \emptyset$. □

As a corollary of Theorem 4, we obtain a non-emptiness result for Yannelis' alpha-core in economies with more than 2 consumers. The following example shows a 3-consumer and one commodity exchange economy satisfying the assumptions of Theorem 4; in particular, the preference relations are given by means of discontinuous utility functions and the mappings $P_i$ are not open lower sections.

**Example 7.** Let $\mathcal{E}$ be the exchange economy where the set of consumers is $N = \{1, 2, 3\}$, $X_i = [0, 1]$ for each $i \in N$ and the preference relations are represented by the utility functions $v_1$, $v_2$ and $v_3$ defined as follows: $v_i(x) = u_i(x_1, x_2)$ for $i = 1, 2$ and $x \in X\backslash\{(0, 0, t) : t > 0\}$, where $u_1$ and $u_2$ are the functions given by Example 2; $v_1(x) = v_2(x) = 1$ if $x \in \{(0, 0, t) : t > 0\}$ and $v_1(0, 0, 0) = v_2(0, 0, 0) = 0$; $v_3(x) = 1$ if $x_1 = x_2 = x_3 > 0$ and $v_3(x) = 0$ otherwise. For each $i \in N$ and for each $x \in X$, let $P_i(x) = \{z \in X : v_i(z) > v_i(x)\}$. The initial endowments are any positive numbers $e_1$, $e_2$ and $e_3$ such that $\sum_{i=1}^{3} e_i = 1$.
Suppose that $x \notin \{(0, 0, t) : t \geq 0\}$: if $x_1 \neq x_2$, we have $P_i(x) = \emptyset$ for some $i \in \{1, 2\}$ and $\bigcup_{j \neq i} P_j(x) = \{z \in X : (z_1, z_2) \neq (0, 0) \text{ and } z_j \geq z_i \text{ with } j \neq 3\}$, which implies $x \notin \text{co} \bigcup_{h \in N} P_h(x)$; if $x_1 = x_2 > 0$, we get $x \notin \text{co} \bigcup_{h \in N} P_h(x) = \{z \in X : z_1 = z_2 = z_3 > 0\}$ when $x_3 \neq x_1 = x_2$ and $\bigcup_{h \in N} P_h(x) = \emptyset$ otherwise. If $x = (0, 0, t)$ with $t > 0$, one has $x \notin \text{co} \bigcup_{h \in N} P_h(x) = \{z \in X : z_1 = z_2 = z_3 > 0\}$, while $\text{co} \bigcup_{h \in N} P_h((0, 0, 0)) = X\backslash\{(0, 0, 0)\}$. Now, using the arguments given in Example 2, one can see that $\mathcal{E}$ is uniformly quasi-concave.
There is only one doubly strong allocation of $\mathcal{E}$, that is: $x^* = (1/3, 1/3, 1/3)$. Let $z$ be an allocation such that $z \neq x^*$ and let $O_z$ be an open neighborhood of $z$. In order to check whether the generalized deviation property holds true, for each $z' \in O_z$, one can set $\zeta_z(z') = (1/2, 1/2, 0)$ if $z_1 \neq z_2$ and $\zeta_z(z') = x^*$ otherwise. Finally, $\mathcal{E}$ satisfies the assumptions of Theorem 4.

**Remark 6.** Note that the mappings $P_i$ $(i = 1, 2, 3)$ given in the example above are not open lower sections; for instance, $(0, 0, 0) \in P_3^{-1}(1, 1, 1)$ but $(t, t, t) \notin P_3^{-1}(1, 1, 1)$ for all $t > 0$ (we have similar situations on $P_1$ and $P_2$). So, even if the uniform quasi-concavity of an exchange economy is a condition more restrictive than *b)* $x \notin P_i(x)$ for all $x \in X$ and all $i \in N$, Example 7 shows that the family of exchange economies satisfying the assumptions of Theorem 4 is not included in the family of economies where the mappings $P_i$ are open lower sections.

5.2. **Exchange economies with selfish preferences.** In this section, we consider exchange economies $\mathcal{E} = \langle X_i, P_i, e_i \rangle_{i \in N}$, where the setting is the same of the previous subsection except for consumers' preferences, that here are assumed to be selfish (and non necessarily complete or transitive): $P_i : X_i \rightrightarrows X_i$ for every $i \in N$; $P_i(x_i)$ is the set of bundles of goods that consumer $i$ strictly prefers to $x_i$. Accordingly to the section above, we define a feasible allocation $x^*$ to be a *doubly strong allocation* if there are no feasible allocations $x$ and no consumers $i$ such that $x_i \in P_i(x_i^*)$ ($\mathfrak{S}_{\mathcal{E}}$ denotes the set of doubly strong allocations of $\mathcal{E}$). If we define $\widetilde{P}_i(x) = P_i(x_i) \times X_{-\{i\}}$ for all $x \in X$ and all $i \in N$, then the doubly strong allocations of $\mathcal{E}$ coincide with the doubly strong allocations of $\widetilde{\mathcal{E}} = \left\langle X_i, \widetilde{P}_i, e_i \right\rangle_{i \in N}$. We say that $\mathcal{E}$ satisfies the generalized deviation property, as well as that $\mathcal{E}$ is uniformly quasi-concave, if $\widetilde{\mathcal{E}}$ is. So, from Theorem 4, we have:

**Theorem 5.** *Assume that $\mathcal{E}$ satisfies the generalized deviation property and is uniformly quasi-concave. Then, the set of doubly strong allocations of $\mathcal{E}$ is non-empty.*

**Remark 7.** In the setting of economies with selfish preferences, Border (1984) and Yannelis (1991) introduced the alpha-core as the set of allocations $\bar{x}$ for which there are no coalitions of consumers $S$ and $x_S \in X_S$ such that $\sum_{i \in S} x_i = \sum_{i \in S} e_i$ and $x_i \in P_i(\bar{x}_i)$ for each $i \in S$. Obviously, every doubly strong allocation belongs to the alpha-core as recalled above. So, Theorem 5 implies the non-emptiness of the alpha-core in economies where the generalized deviation property end uniform quasi-concavity holds. We note that the assumptions of Theorem 5 are not more restrictive than those given by the previous results. Indeed, Border (1984) and Yannelis (1991) consider mappings $P_i$ with open graph, which is a condition more restrictive than the open lower sections property. Now, using the arguments of the examples given in the previous subsection, one can easily find economies with non-open graph and selfish preferences satisfying both the generalized deviation property and the uniform quasi-concavity (see also the following Example 8).

A issue of interest in exchange economies is the existence of allocations that are *fair* (see Schmeidler and Yaari 1971), in the sense that they are Pareto optimal[16] and envy-free. We recall that an allocation $x$ is said to be *envy-free* if $x_j \notin P_i(x_i)$ for all $\{i, j\} \subseteq N$ (see Foley 1967). Theorem 5 implies the existence of fair allocations in exchange economies with discontinuous and non-ordered preference relations. In fact, we have the following result.[17]

**Proposition 5.** *Assume that the consumption sets of the consumers are identical. Then, every doubly strong allocation is fair.*

*Proof.* Let $x^*$ be a doubly strong allocation. From Remark 7, we know that $x^*$ is Pareto optimal. Suppose that $x^*$ is not envy-free; so, there exists $\{i, j\} \subseteq N$ such

[16] We mean that, given an allocation $x$, there are no allocations $x'$ such that $x'_i \in P_i(x_i)$ for all $i \in N$ (see, for example, Yannelis 1991).

[17] The author is grateful to Marialaura Pesce that have proved this property.

that $x_j^* \in P_i(x_i^*)$. Define the allocation $\bar{x}$ by $\bar{x}_{-\{i,j\}} = x^*_{-\{i,j\}}$, $\bar{x}_i = x_j^*$ and $\bar{x}_j = x_i^*$. We have that $\bar{x} \in \mathcal{F}_{\mathcal{E}}$ and $\bar{x}_i \in P_i(x_i^*)$. So, we get a contradiction. □

**Remark 8.** The previous literature on the existence of fair allocations considers exchange economies with a finite number of commodities and consumers' preferences that are strongly monotonic and represented by continuous utility functions: see Varian (1974), Svensson (1983), Thomson (2007) and references therein. Theorem 5 and Proposition 5 ensure the existence of fair allocations when the preferences are neither necessarily ordered nor strongly monotonic. An example of economy satisfying the assumptions of Theorem 5 but not those of the mentioned papers is given below.[18]

**Example 8.** Consider the 2-consumer exchange economy $\mathcal{E}$, where $X_1 = X_2 = [0,1] \times [0,1]$ and the preferences are given by means of the functions $u_1$ and $u_2$ introduced in Example 2. Let $e_1$ and $e_2$ belonging to $[0,1] \times [0,1]$ such that $e_1 + e_2 = (1,1)$. It is easy to see that the set of doubly strong allocations is $\mathfrak{S}_{\mathcal{E}} = \{((t,t),(s,s)) : t, s \in ]0,1] \text{ and } t + s = 1\}$. The generalized deviation property and the uniform quasi-concavity are satisfies (so, Theorem 5 applies). For instance, given $(z^1, z^2) \notin \mathfrak{S}_{\mathcal{E}}$ and $z_1^1 < z_2^1$, one can set $\zeta_{(z^1,z^2)}(z'^1, z'^2) = \{(1,1)\} \times ([0,1] \times [0,1])$ for all $(z'^1, z'^2)$ which belongs to an open neighborhood of $(z^1, z^2)$ where $z_1'^1 < z_2'^1$. The uniform quasi-concavity follows from the arguments given in Example 2. Moreover, it is clear that $u_1$ and $u_2$ are neither continuous nor strongly monotonic. Nevertheless, $\mathcal{E}$ has fair allocations.

## 6. Conclusions

In the setting of generalized games with non-ordered and discontinuous preferences defined on the set of strategy profiles, we have proved the existence of a strategy profile, called *double strong equilibrium*, which is a maximal element for the preferences of all players. The conditions which guarantee the existence of doubly strong equilibria are the *generalized deviation property* and the *uniform quasi-concavity*. Examples have been given in order to compare the properties and results with the previous literature on strategic form games. In particular, it has been showed that the generalized deviation property is a very general condition and it is implied by the previous ones. The uniform quasi-concavity extends the standard convexity on preference relations. More precisely, the uniform quasi-concavity requires that, if a strategy profile $z$ is a strict convex combinations of a finite number of strategy

[18]A utility function $u$ defined on $\mathbb{R}^\ell$ is *strongly monotonic* if $x \leq y$ and $x \neq y$ implies $u(x) > u(y)$ (among the others, see Svensson 1983 and Aliprantis et al. 1989). The meaning of inequality $x \leq y$ is the standard one: $x_t \leq y_t$ for all $t \in \{1, ..., \ell\}$.

profiles, all players are uniform in choosing the same $x$ among the latter so that no one ranks $x$ to be better than $z$. The doubly strong equilibrium existence result has been obtained by using the *quasi-Ky Fan minimax inequality*; we have given new sufficient conditions for the existence of solutions of the inequality. In the case of games (where there is no mapping $K$), a doubly strong equilibrium is a strong equilibrium in the sense of Aumann (1959), and therefore it is an alpha-core element (Aumann 1961). So, our result has provided new sufficient conditions for the existence of strong equilibria and non-emptiness of the alpha-core. Also in this case, examples have showed that our result is not connected with the previous ones. Then, we have applied our results to exchange economies where the consumers have interdependent or selfish discontinuous and non-ordered preferences. We have obtained the existence of *doubly strong allocations*. In the case of interdependent preferences (for example, economies with externalities), the doubly strong allocations belongs to the alpha-core in the sense of Yannelis (1991). So, we have obtained the non-emptiness of the Yannelis' alpha-core in discontinuous economies with more than 2-consumers (the previous result has guaranteed the existence of alpha-core allocations in the case of 2 consumers). When consumers' preferences are selfish, the doubly strong allocations belongs to the core and are envy-free (Foley 1967). So, we have obtained the existence of fair allocations (Schmeidler and Yaari 1971) in exchange economies with discontinuous and non-ordered preferences (the previous results have required economies with a finite number of commodities and preferences represented by continuous utility functions).

## Appendix

**Proof of Theorem 1.** We proceed as in the proof of Scalzo (2015, Theorem 1) and define the mapping $\xi : W \rightrightarrows X$ by

$$\xi(z) = \sum_{a \in \mathcal{I}(z)} \beta_a(z)\xi_a(z) \quad \forall\, z \in W \,,$$

where: $\{\beta_a : a \in A\}$ is a partition of the unity subordinate to an open and locally finite covering of $W$ (see Michael 1953); $\mathcal{I}(z) = \{a \in A : \beta_a(z) > 0\}$ (which is finite); the mappings $\xi_a$ are well-behaved and derived from the assumption on $F$ to be generalized transfer open lower sections. Given the proof of the mentioned result, we only need to prove that $\xi$ is a closed mapping under the new assumption on $D$. Let $(z_t)_t$ be a net converging to $z$ in $W$ and let $(s_t)_t$ be a net converging to $s$ in $D$, with $s_t \in \xi(z_t)$ for all $t$ (we have $\mathcal{I}(z) \subseteq \mathcal{I}(z_t)$ for $t$ sufficiently large). We get:

$$s_t = \sum_{a \in \mathcal{I}(z)} \beta_a(z_t)s_t^a \;\; + \sum_{b \in \mathcal{I}(z_t)\setminus\mathcal{I}(z)} \beta_b(z_t)s_t^b \,, \tag{2}$$

where, for each $t$, $s_t^a \in \xi_a(z_t)$ for all $a \in \mathcal{I}(z)$ and $s_t^b \in \xi_b(z_t)$ for all $b \in \mathcal{I}(z_t)\backslash\mathcal{I}(z)$. Fixed $b_t \in \mathcal{I}(z_t)\backslash\mathcal{I}(z)$, for every $t$, one has

$$0 \leq \lim_t \beta_{b_t}(z_t) \leq \lim_t \sum_{b\in\mathcal{I}(z_t)\backslash\mathcal{I}(z)} \beta_b(z_t) = 0 \ . \tag{3}$$

Since $D$ is compact, Tychonoff's Theorem guarantees that the net $\mathfrak{n} = (s_t^b)_{(t,b\in\mathcal{I}(z_t))}$ admits a converging subnet (see, for example, Aliprantis and Border 1999). Let us assume that $\mathfrak{n}$ converges. From (3), we get

$$\lim_t \beta_{b_t}(z_t) s_t^{b_t} = \text{ null vector} \ . \tag{4}$$

So, in the light of (4) and (2), one has:

$$s = \lim_t \sum_{a\in\mathcal{I}(z)} \beta_a(z_t) s_t^a = \sum_{a\in\mathcal{I}(z)} \beta_a(z) s^a \ , \tag{5}$$

Finally, since $\xi_a$ is closed for each $a \in \mathcal{I}(z)$, we have that $s^a \in \xi_a(z)$ for all $a$, that is: $s \in \xi(z)$ (upper semicontinuity and closeness are equivalent properties in our setting). This concludes the proof.

**Remark 9.** A previous version of the result above was provided by Scalzo (2015, Theorem 1), where the set $D$ was assumed to be a convex and compact subset of a Banach space. However, the improvement here presented can be proved by using the same arguments of the proof of the previous result. Theorem 1 was also obtained by He and Yannelis (2016) with a different proof. We point out that Corson and Lindenstrauss (1966) proved the existence of one-to-one continuous selections from a mapping through a strengthening of the property introduced in Definition 3: more precisely, they assumed that $\xi_z$ is a one-to-one continuous function.

**Remark 10.** It is clear that if a mapping admits a well-behaved selection, then it is generalized transfer open lower sections. So, Theorem 1 identifies a general setting where the mappings having well-behaved selections are characterized by means of the generalized transfer open lower sections property.

**Remark 11.** Michael (1956) proved the existence of one-to-one continuous selections from lower semicontinuous mappings. It is interesting to point out that the generalized transfer open lower sections property is not connected with the lower semicontinuity. Indeed, the mapping $F$ defined by $F(x) = \{0\}$ if $x \in [0, 1/2[$, $F(1/2) = [0, 1]$ and $F(x) = \{1\}$ if $x \in ]1/2, 1]$ is clearly generalized transfer open lower sections and not lower semicontinuous.

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Department of Economics and Statistics (DISES), University of Naples Federico II, via Cinthia 21, 80126 Napoli, Italy, e-mail: scalzo@unina.it